8.323 Problem Set 1 Solutions

February 15, 2023

Question 1: Quantum harmonic oscillator in the Heisenberg picture (25 points)
Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω,

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2)$$

In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

$$[\hat{x}(t), \hat{p}(t)] = i$$

Here and below we set $\hbar = 1$.

(a) Obtain the Heisenberg evolution equations for $\hat{x}(t)$ and $\hat{p}(t)$.
We use Heisenberg's equations of motion for $\hat{x}(t)$ and $\hat{p}(t)$:

$$\frac{d\hat{x}(t)}{dt} = i[H, \hat{x}(t)] \qquad\qquad \frac{d\hat{p}(t)}{dt} = i[H, \hat{p}(t)]$$

The right hand sides can be computed using $H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2)$, the commutator $[\hat{x}, \hat{p}] = i$, and the Heisenberg time evolution $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$. For instance:

$$[H, \hat{x}(t)] = [H, e^{iHt}\hat{x}e^{-iHt}] = e^{iHt}[H, \hat{x}]e^{-iHt} = -ie^{iHt}\hat{p}e^{-iHt} = -i\hat{p}(t)$$

Hence, we have:

$$\frac{d\hat{x}(t)}{dt} = \hat{p}(t) \qquad\qquad \frac{d\hat{p}(t)}{dt} = -\omega^2\hat{x}(t)$$

(b) Suppose the initial conditions at $t = 0$ are given by

$$\hat{x}(0) = \hat{x} \qquad\qquad \hat{p}(0) = \hat{p}$$

Find $\hat{x}(t)$ and $\hat{p}(t)$.
We can decouple the system by converting to second order equations:

$$\ddot{\hat{x}} = -\omega^2\hat{x}(t) \qquad\qquad \ddot{\hat{p}}(t) = -\omega^2\hat{p}(t)$$

Solving with the initial conditions $\hat{x}(0) = \hat{x}$ and $\hat{p}(0) = \hat{p}$, we find

$$\hat{x}(t) = \hat{x}\cos\omega t + \frac{1}{\omega}\hat{p}\sin\omega t \qquad\qquad \hat{p}(t) = \hat{p}\cos\omega t - \omega\hat{x}\sin\omega t$$

(c) It is convenient to introduce operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ defined by:

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}(\hat{a}(t) + \hat{a}^\dagger(t)), \qquad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}(\hat{a}(t) - \hat{a}^\dagger(t))$$

Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy the equal-time commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1$$

We solve for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in terms of $\hat{x}(t)$ and $\hat{p}(t)$:

$$\hat{a}(t) = \sqrt{\frac{\omega}{2}}\hat{x}(t) + i\sqrt{\frac{1}{2\omega}}\hat{p}(t), \qquad \hat{a}^\dagger(t) = \sqrt{\frac{\omega}{2}}\hat{x}(t) - i\sqrt{\frac{1}{2\omega}}\hat{p}(t)$$

Using the commutation relations between position and momentum operators, we have

$$[\hat{a}(t), \hat{a}^\dagger(t)] = -\frac{i}{2}[\hat{x}(t), \hat{p}(t)] + \frac{i}{2}[\hat{p}(t), \hat{x}(t)] = 1$$

(d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

$$\begin{aligned} H = H(t) = e^{iHt}He^{-iHt} &= \frac{1}{2}(\hat{p}(t)^2 + \omega^2\hat{x}(t)^2) \\ &= \frac{\omega}{4}(-(\hat{a}(t) - \hat{a}^\dagger(t))^2 + (\hat{a}(t) - \hat{a}^\dagger(t))^2) = \frac{\omega}{2}(\hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t)) \\ &= \omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right) = \omega\left(N(t) + \frac{1}{2}\right) \end{aligned}$$

where in the last equality we define the number operator, $N(t) = \hat{a}^\dagger(t)\hat{a}(t)$.

(e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
Using the results in parts (c) and (d), we have

$$\begin{aligned} \frac{d\hat{a}(t)}{dt} &= i[H, \hat{a}(t)] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}(t)] = -i\omega\hat{a}(t) \\ \frac{d\hat{a}^\dagger(t)}{dt} &= i[H, \hat{a}^\dagger(t)] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}^\dagger(t)] = i\omega\hat{a}^\dagger(t) \end{aligned}$$

(f) Suppose the initial conditions at $t = 0$ are given by

$$\hat{a}(0) = \hat{a}, \qquad \hat{a}^\dagger(0) = \hat{a}^\dagger$$

Find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
The equations in (e) are decoupled, and first-order linear. We immediately have

$$\hat{a}(t) = \hat{a}e^{-i\omega t}, \qquad \hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t}$$

(g) Express $\hat{x}(t)$, $\hat{p}(t)$, and the Hamiltonian H in terms of $\hat{a}$ and $\hat{a}^\dagger$.
We substitute the expressions obtained in (f) into parts (c) and (d).

$$\begin{aligned} \hat{x}(t) &= \sqrt{\frac{1}{2\omega}}(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \\ \hat{p}(t) &= -i\sqrt{\frac{\omega}{2}}(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}) \\ H(t) &= H = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \end{aligned}$$

Question 2: Lorentz transformations (15 points)

(a) Probe that the 4-dimensional δ-function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3)$$

is Lorentz invariant, i.e.

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p})$$

where $\tilde{p}^\mu$ is a Lorentz transformation of p^μ.

We express the δ-function in integral form, and use that $p \cdot x$ is a Lorentz scalar, i.e. $\Lambda p \cdot \Lambda x = p \cdot x$.

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int d^4 x e^{ip\cdot x} = \frac{1}{(2\pi)^4} \int d^4 x e^{i\Lambda p \cdot \Lambda x}$$

Now we make the change of variables $\tilde{x} = \Lambda x$. Note that $d^4\tilde{x} = d^4 x$. To see this we use $\Lambda^T \eta \Lambda = \eta$, which implies $1 = \det(\Lambda^T)\det(\Lambda) = (\det\Lambda)^2$. Hence, the Jacobian $J = |\det\Lambda| = 1$. One thus has

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int d^4 \tilde{x} e^{i\Lambda p\cdot \tilde{x}} = \frac{1}{(2\pi)^4} \int d^4 x e^{i\Lambda p \cdot x} = \delta^{(4)}(\Lambda p)$$

(b) Show that

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$

is Lorentz invariant, i.e.

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) = \omega_1' \delta^{(3)}(\vec{k}_1' - \vec{k}_2')$$

Here $\vec{k}_1$ and $\vec{k}_2$ are respectively the spatial part of four-vectors $k_1^\mu = (\omega_1, \vec{k}_1)$ and $k_2^\mu = (\omega_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 + k_2^2 = -m^2$$

and $k_1'^\mu = (\omega_1', \vec{k}_1')$, $k_2'^\mu = (\omega_2', \vec{k}_2')$ are related to k_1^μ, k_2^μ by the same Lorentz transformation.

We consider the expression $\delta(k^2 + m^2)$ which imposes the mass-shell constraint. We can simplify this using the δ-function identity $\delta(f(x)) = \sum_{x_i \text{ s.t. } f(x_i)=0} \frac{1}{|f'(x_i)|}\delta(x - x_i)$.

$$\delta(k^2 + m^2) = \delta(-k_0^2 + \vec{k}^2 + m^2) = \frac{1}{2|\omega_{\vec{k}}|}\left(\delta(k_0 - |\omega_{\vec{k}}|) + \delta(k_0 + |\omega_{\vec{k}}|)\right) \tag{1}$$

We will assume $\omega_{\vec{k}_1}, \omega_{\vec{k}_2} > 0$, as is the case for physical 4-momenta.

We can pick out the $k_1^0 = \omega_{\vec{k}_1}$ enforcing δ-function in (1) by multiplying both sides by $\theta(\omega_{\vec{k}_1})$.

$$\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2) = \frac{1}{2\omega_{\vec{k}_1}}\delta(k_1^0 - \omega_{\vec{k}_1})\theta(\omega_{\vec{k}_1}) = \frac{1}{2\omega_{\vec{k}_1}}\delta(k_1^0 - \omega_{\vec{k}_1}) \tag{2}$$

Now we multiply both sides by $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$:

$$\begin{aligned}
\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2)\cdot 2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2) &= \delta(k_1^0 - \omega_{\vec{k}_1})\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
&= \delta(k_1^0 - \omega_{\vec{k}_2})\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
&= \delta(k_1^0 - k_2^0)\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2)\cdot 2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2) &= \delta^{(4)}(k_1^\mu - k_2^\mu)
\end{aligned} \tag{3}$$

In the second equality, we use that the $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ allows us to replace $\omega_{\vec{k}_1}$ with $\omega_{\vec{k}_2}$. For this step, it is crucial that $\mathrm{sign}(\omega_{\vec{k}_1}) = \mathrm{sign}(\omega_{\vec{k}_2})$, which is true since both are positive.

Finally, let us study (3). The right-hand side is Lorentz invariant by part (a). On the left-hand side, $\delta(k_1^2 + m^2)$ is Lorentz invariant since k_1^2 is a Lorentz scalar, and $\theta(\omega_{\vec{k}_1})$ is Lorentz invariant because the energy of a particle does not change under a (proper, orthochronous) Lorentz transformation. It then follows that $\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ is Lorentz invariant.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$, prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k), \qquad \omega_{\vec{k}} = \sqrt{\vec{k^2} + m^2}$$

is Lorentz invariant, in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\tilde{k})$$

where $\tilde{k}^\mu = \Lambda^\mu{}_\nu k^\nu$ is a Lorentz transformation of k^μ.

Since the momentum is on the mass-shell, we write $f(k) = f(\omega_{\vec{k}}, \vec{k})$.

By introducing another δ-function, we may write this expression as a integral over 4-dimensions:

$$\begin{aligned}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\omega_{\vec{k}}, \vec{k}) = \frac{1}{(2\pi)^3}\int d^4k\frac{1}{2\omega_{\vec{k}}}\delta(k^0 - \omega_{\vec{k}})f(k^0, \vec{k}) \\ &= \frac{1}{(2\pi)^3}\int d^4k\,\theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^\mu) \end{aligned} \tag{4}$$

where in the last equality we have used (2).

Now we make a change of variables, $k = \Lambda k'$, for Λ an arbitrary (proper, orthochronous) Lorentz transformation. In parts (a)-(b), we showed that $d^4k = d^4k'$, $\theta(\omega_{\vec{k}}) = \theta(\omega_{\vec{k}'})$, and $\delta(k^2 + m^2) = \delta(k'^2 + m^2)$. Hence,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int d^4k\,\theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^\mu) &= \frac{1}{(2\pi)^3}\int d^4k'\,\theta(\omega_{\vec{k}'})\delta(k'^2 + m^2)f((\Lambda k')^\mu) \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\Lambda k)\end{aligned}$$

where the last equality is obtained using the reverse sequence of operations that led to (4).

Putting everything together, we have the desired result,

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\Lambda k)$$

Question 3: A complex scalar field (20 points)

Consider the field theory of a complex valued scalar field $\phi(x)$ with action

$$S = \int d^4x\big(-\partial_\mu\phi^*\partial^\mu\phi - V(|\phi|^2)\big), \qquad |\phi|^2 = \phi^*\phi$$

One could either consider the real and imaginary parts of ϕ, or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.

(a) Check that the action is Lorentz invariant, and find the equations of motion.

A Lorentz transformation acts as $\phi \to \phi'$, such that $\phi'(x) = \phi(\Lambda^{-1}x)$. The action transforms as:

$$\begin{aligned} S \to S' &= \int d^4x\big(-\partial_\mu\phi'^*(x)\partial^\mu\phi'(x) - V(|\phi'(x)|^2)\big) \\ &= \int d^4x\big(-\partial_\mu\phi^*(\Lambda^{-1}x)\partial^\mu\phi(\Lambda^{-1}x) - V(|\phi(\Lambda^{-1}x)|^2)\big) \end{aligned}$$

Now we make the change of variable $x' = \Lambda^{-1}x$. We showed in 2(a) that $d^4x = d^4x'$. Furthermore, by the chain rule we have $\partial_\mu = (\Lambda^{-1})_\mu{}^\nu\partial'_\nu$. Here ∂_μ and ∂'_μ denote differentiation with respect to x and x'.

$$\begin{aligned} S' &= \int d^4x'\big(-(\Lambda^{-1})_\mu{}^\nu\partial'_\nu\phi^*(x')\ (\Lambda^{-1})^\mu{}_\rho\partial'^\rho\phi(x') - V(|\phi(x')|^2\big) \\ &= \int d^4x'\big(-\delta^\nu_\rho\partial'_\nu\phi^*(x')\partial'^\rho\phi(x') - V(|\phi(x')|^2\big) \\ &= \int d^4x'\big(-\partial'_\nu\phi^*(x')\partial'^\nu\phi(x') - V(|\phi(x')|^2\big) = S \end{aligned}$$

where in the second line we use

$$(\Lambda^{-1})_\mu{}^\nu(\Lambda^{-1})^\mu{}_\rho = ((\Lambda^{-1})^T)^\nu{}_\mu(\Lambda^{-1})^\mu{}_\rho = (\Lambda)^\nu{}_\mu(\Lambda^{-1})^\mu{}_\rho = \delta^\nu_\rho$$

To find the equations of motion, we use the Euler-Lagrange equations, treating ϕ and ϕ^* as independent:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}, \qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \frac{\partial\mathcal{L}}{\partial\phi^*}$$

The left-hand side can be confusing to evaluate due to the contracted indices, so we do one calculation very explicitly:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\left[\frac{\partial}{\partial(\partial_\mu\phi)}(-\partial_\nu\phi^*\partial^\nu\phi)\right] = \partial_\mu[-\partial_\nu\phi^*\delta^\nu_\mu] = -\partial^2\phi^*$$

Hence, the equations of motion are

$$\partial^2\phi^* - V'(|\phi|^2)\phi^* = 0, \qquad \partial^2\phi - V'(|\phi|^2)\phi = 0$$

Note that these are conjugate equations, as expected.

(b) Find the canonical conjugate momenta for ϕ and ϕ^*, and the Hamiltonian H.

We write the Lagrangian density as

$$\mathcal{L} = \partial_t\phi^*\partial_t\phi - \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - V(|\phi|^2)$$

The conjugate momenta are thus

$$\pi := \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi^*, \qquad \pi^* := \frac{\partial\mathcal{L}}{\partial(\partial_t\phi^*)} = \partial_t\phi$$

The Hamiltonian is given by

$$H = \int d^3x(\pi\partial_t\phi + \pi^*\partial_t\phi - \mathcal{L}) = \int d^3x(\pi^*\pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))$$

(c) The action is invariant under the transformation

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^*$$

for arbitrary constant α. When α is small, i.e. for an infinitesimal transformation, this becomes

$$\delta\phi = i\alpha\phi, \qquad \delta\phi^* = -i\alpha\phi^*$$

Use Noether's theorem to find the corresponding conserved current j^μ and conserved charge Q.
By Noether's theorem, the conserved current is given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)}\delta\Phi_a - \mathcal{F}^\mu, \qquad \delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$$

In this case, $\delta\phi = i\alpha\phi$, $\delta\phi^* = -i\alpha\phi^*$, and $\delta\mathcal{L} = 0$. We find

$$j^\mu = -\partial^\mu\phi^*(i\alpha\phi) - \partial^\mu\phi(-i\alpha\phi^*) = i\alpha(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

One may remove the proportionality constant if desired, to get

$$j^\mu = \phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*$$

The corresponding charge is then

$$Q = \int d^3x j^0 = \int d^3x(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$$

(d) Use the equations of motion from part (a) to verify directly that j^μ is conserved.
We compute:

$$\begin{aligned}\partial_\mu j^\mu = \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) &= \phi^*\partial^2\phi - \phi\partial^2\phi^* \\ &= V'(|\phi|^2)\phi^*\phi - V'(|\phi|^2)\phi^*\phi = 0\end{aligned}$$

where in the second line we use the equations of motion from part (a).

Question 4: The energy-momentum tensor (20 points)

In this problem we work out the energy-momentum tensor of the complex scalar theory in Question 3.

(a) Under a spacetime translation

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

a scalar field transforms as

$$\phi'(x') = \phi(x)$$

Show that the action is invariant under the transformation $\phi(x) \to \phi'(x)$.

Under the transformation, the scalar field satisfies $\phi'(x) = \phi(x-a)$. The action transforms as:

$$\begin{aligned} S \to S' &= \int d^4x \big(-\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2) \big) \\ &= \int d^4x \big(-\partial_\mu \phi^*(x-a) \partial^\mu \phi(x-a) - V(|\phi(x-a)|^2) \big) \\ &= \int d^4x \big(-\partial_\mu \phi^*(x) \partial^\mu \phi(x) - V(|\phi(x)|^2) \big) = S \end{aligned}$$

where in the last line we change variables from $x^\mu \to x^\mu + a^\mu$, which does not change the integration measure.

(b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether's theorem to find the corresponding conserved currents $T^{\mu\nu}$.

An infinitesimal translation acts on the fields as:

$$\begin{aligned} \delta\phi &= \phi'(x) - \phi(x) = \phi(x-a) - \phi(x) = -a^\mu \partial_\mu \phi(x) \\ \delta\phi^* &= \phi'^*(x) - \phi^*(x) = \phi^*(x-a) - \phi^*(x) = -a^\mu \partial_\mu \phi^*(x) \end{aligned}$$

We also need the change in the Lagrangian density under translations:

$$\begin{aligned} \delta\mathcal{L} = \mathcal{L}' - \mathcal{L} &= -\partial_\nu(\phi^*(x) - a^\mu \partial_\mu \phi^*(x))\ \partial^\nu(\phi(x) - a^\mu \partial_\mu \phi(x)) \\ &\quad - V\big((\phi^*(x) - a^\mu \partial_\mu \phi^*(x))(\phi(x) - a^\mu \partial_\mu \phi(x))\big) - \mathcal{L} \\ &= a^\mu(\partial_\nu \partial_\mu \phi^*(x) \partial^\nu \phi(x) + \partial_\nu \phi^*(x) \partial^\nu \partial_\mu \phi(x)) + a^\mu V'(\phi^*\phi)\big(\partial_\mu \phi^* \phi + \phi^* \partial_\mu \phi\big) + \mathcal{O}(a^\mu a^\nu) \\ &= -a^\mu \partial_\mu \mathcal{L} = a_\mu \partial_\nu(-\eta^{\mu\nu}\mathcal{L}) := (a_\mu \partial_\nu)\mathcal{F}^{\mu\nu} \end{aligned}$$

The translations are parameterized by a 4-vector a^μ, and we have a Noether current (itself a 4-vector) for each. Hence, we can encode the conserved currents from translations into a rank-2 tensor, $T^{\mu\nu}$. In the following, we let the first index pick out the direction of the translation a^μ.

The Noether current is

$$\begin{aligned} T^{\mu\nu} := (j^\mu)^\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}(\delta\phi)^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^*)}(\delta\phi^*)^\mu - \mathcal{F}^{\mu\nu} \\ &= -\partial^\nu\phi^*(-\partial^\mu\phi) - \partial^\nu\phi(-\partial^\mu\phi^*) + \eta^{\mu\nu}\mathcal{L} \\ &= \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi - \eta^{\mu\nu}\big(\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2)\big) \end{aligned}$$

(c) The conserved charge for a time translation

$$H = \int d^3x T^{00}$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x T^{0i}$$

is identified with the total momentum. Thus $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explict expressions for H and P^i. Compare H obtained here with the Hamiltonian in problem 3(b).

We compute:

$$\begin{aligned}
H &= \int d^3x T^{00} = \int d^3x \left(2\partial^t\phi^*\partial^t\phi + (-\partial^t\phi^*\partial^t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right) \\
&= \int d^3x \left(\partial_t\phi^*\partial_t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right) \\
P^i &= \int d^3x T^{0i} = \int d^3x \left(\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi\right) = -\int d^3x \left(\partial_t\phi^*\partial_i\phi + \partial_i\phi^*\partial_t\phi\right)
\end{aligned}$$

The expression for the Hamiltonian is equal to the Hamiltonian obtained in problem 3(b).

(d) Use the equations of motion of problem 3(a) to verify directly that $T^{\mu\nu}$ is conserved.

Recall that the first index of $T^{\mu\nu}$ picks out the direction of the translation a^μ, so formally Noether conservation should tell us $\partial_\nu T^{\mu\nu} = 0$. However, from part (c) it can be seen that $T^{\mu\nu}$ is symmetric, so we can contract the derivative with respect to either index.

We compute:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi) - \partial^\nu(\partial_\rho\phi^*\partial^\rho\phi) - \partial^\nu V(|\phi|^2) \\
&= \partial^2\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^2\phi - \partial^\nu V(|\phi|^2) \\
&= \phi^*\partial^\nu\phi \; V'(|\phi|^2) + \phi\partial^\nu\phi^* \; V'(|\phi|^2) - \phi^*\partial^\nu\phi \; V'(|\phi|^2) - \phi\partial^\nu\phi^* \; V'(|\phi|^2) = 0
\end{aligned}$$

where we use the equations of motion in the 3rd equality. Thus the Noether currents are conserved.

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 2 Solutions

February 21, 2023

Question 1: A Problem with Relativistic Quantum Mechanics (20 points)
The Schrödfinger equation for a free non-relativistic particle is:

$$i\partial_t \psi(\vec{x}, t) = -\frac{1}{2m}\nabla^2\psi(\vec{x}, t)$$

The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

$$\partial_t^2\psi(\vec{x}, t) - \nabla^2\psi(\vec{x}, t) + m^2\psi(\vec{x}, t) = 0$$

We emphasize that in both these equations, $\psi(\vec{x}, t)$ is interpreted as a wave function for the dynamical variable $\vec{x}(t)$, rather than a dynamical field.

(a) As a reminder, derive from the Schrödinger equation the continuity equation for the probability

$$\partial_t \rho + \nabla \cdot \vec{J} = 0$$

where

$$\rho = |\psi|^2, \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

We compute:

$$\begin{aligned}\partial_t \rho = \psi\partial_t\psi^* + \psi^*\partial_t\psi &= -\frac{i}{2m}\left(\psi\vec{\nabla}^2\psi^* - \psi^*\vec{\nabla}^2\psi\right)\\ &= \frac{i}{2m}\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = -\vec{\nabla}\cdot\vec{J}\end{aligned}$$

where we use the Schrödinger equation in the second equality.

(b) Suppose $\psi(\vec{x}, t)$ has the plane wave form, i.e.

$$\psi(\vec{x}, t) \propto e^{i\vec{k}\cdot\vec{x}}$$

for some real vector $\vec{k}$. Find the solutions to the Klein-Gordon equation above.
We substitute the ansatz $\psi(\mathbf{x}, t) = e^{i\mathbf{k}\cdot\mathbf{x}}\phi(t)$ into the Klein-Gordon equation to get an equation for $\phi(t)$:

$$\partial_t^2\phi + (\mathbf{k}^2 + m^2)\phi = 0$$

This has plane-wave solutions of positive and negative frequencies,

$$\phi(t) = Ae^{-i\omega_{\mathbf{k}}t} + Be^{i\omega_{\mathbf{k}}t}, \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$$

Hence, the Klein-Gordon equation has solutions

$$\psi(\vec{x}, t) = Ae^{i(-\omega_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})} + Be^{i(\omega_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}$$

(c) Show that the Klein-Gordon equation also leads to a continuity equation, with ρ and $\vec{J}$ now given by

$$\rho = \frac{i}{2m}(\psi^*\partial_t\psi - \psi\partial_t\psi^*). \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

In the same way as in part (a), we compute:

$$\partial_t\rho = \frac{i}{2m}\left(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right) = \frac{i}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = -\vec{\nabla}\cdot\vec{J}$$

where we use the Klein-Gordon equation in the second equality.

(d) Argue that this ρ cannot be interpreted as a probability density.
We write

$$\rho = \frac{i}{2m}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) = \frac{1}{m}\mathrm{Im}(\psi\partial_t\psi^*)$$

Any proper probability density must be positive definite, i.e. $\rho \geq 0$. This is not the case here. For instance, for the plane wave solution $A = 0$, B from (b) we compute

$$\rho = \mathrm{Im}\left(Be^{i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}}(-i\omega_{\mathbf{k}})Be^{-i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\right) = -B^2\omega_{\mathbf{k}} < 0$$

Since $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} > 0$, the existence of negative-frequency solutions means that ρ cannot be positive definite, and cannot be interpreted as a probability density.

Question 2: Commutation relations of creation and annihilation operators (20 points)
For the real scalar field theory discussed in lecture,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

we showed that the time-evolution of the quantum operator $\phi(\mathbf{x}, t)$ is given by

$$\phi(\mathbf{x}, t) = \int d\!\!{}^-\!{}^2k \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}u_\mathbf{k}(\mathbf{x}, t) + a_\mathbf{k}^\dagger u_\mathbf{k}^*(\mathbf{x}, t)\right)$$

where

$$\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}, \qquad u_\mathbf{k} = e^{-i\omega_\mathbf{k}t + i\mathbf{k}\cdot\mathbf{x}}$$

We use $\pi(\mathbf{x}, t)$ to denote the momentum density conjugate to ϕ. The canonical commutation relations among ϕ and π are

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0, \qquad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

(a) Show that it is enough to impose the canonical commutation relations at $t = 0$. That is, once we impose them at $t = 0$, then the relations at general t are automatically satisfied.
Note: this statement in fact applies not only to $V(\phi) = \frac{1}{2}m^2\phi^2$, but any potential $V(\phi)$.

In the Heisenberg picture we have:

$$[A(\mathbf{x}, t), B(\mathbf{x}', t)] = [e^{iHt}A(\mathbf{x}, 0)e^{-iHt}, e^{iHt}B(\mathbf{x}', 0)e^{-iHt}] = e^{iHt}[A(\mathbf{x}, 0), B(\mathbf{x}', 0)]e^{-iHt}$$

Now let us impose the canonical commutation relations at $t = 0$. Then, it follows that

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = e^{iHt}0e^{-iHt} = 0$$
$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = e^{iHt}i\delta^{(3)}(\mathbf{x} - \mathbf{x}')e^{-iHt} = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

These are again precisely the canonical commutation relations, now at generic t.

(b) Express $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$ in terms of $\phi(\mathbf{k})$ and $\pi(\mathbf{k})$, where $\phi(\mathbf{k})$ and $\pi(\mathbf{k})$ are Fourier transforms of $\phi(\mathbf{x}, t = 0)$ and $\pi(\mathbf{x}, t = 0)$, e.g.

$$\phi(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}, t = 0)$$

We start with the mode expansions for $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$

$$\phi(\mathbf{x}, t) = \int d\!\!{}^-\!{}^3k \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t + i\mathbf{k}\cdot\mathbf{x}} + a_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}t - i\mathbf{k}\cdot\mathbf{x}}\right)$$
$$\pi(\mathbf{x}, t) = -i\int d\!\!{}^-\!{}^3k \sqrt{\frac{\omega_\mathbf{k}}{2}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t + i\mathbf{k}\cdot\mathbf{x}} - a_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}t - i\mathbf{k}\cdot\mathbf{x}}\right)$$

This is almost of the form of a Fourier transform, and by changing variables of one of the terms from $\mathbf{k} \to -\mathbf{k}$ we have

$$\phi(\mathbf{k}, t) = \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t} + a_{-\mathbf{k}}^\dagger e^{i\omega_\mathbf{k}t}\right)$$
$$\pi(\mathbf{k}, t) = -i\sqrt{\frac{\omega_\mathbf{k}}{2}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t} - a_{-\mathbf{k}}^\dagger e^{i\omega_\mathbf{k}t}\right)$$

Now, observe that the equations are decoupled in $\mathbf{k}$. We can take $t = 0$ and solve this as a regular system of equations for $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.

$$\begin{aligned} a_{\mathbf{k}} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\phi(\mathbf{k}) + i\sqrt{\frac{1}{2\omega_{\mathbf{k}}}}\pi(\mathbf{k}) \\ a^\dagger_{\mathbf{k}} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\phi(-\mathbf{k}) - i\sqrt{\frac{1}{2\omega_{\mathbf{k}}}}\pi(-\mathbf{k}) \end{aligned} \tag{1}$$

(c) Using the expressions derived in part (b), deduce the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}], \qquad [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}], \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$$

from the commutation relations above at $t = 0$.

It is useful to take the Fourier transform $\mathcal{F}$ (from position to momentum space) of the $t = 0$ canonical commutation relations:

$$\begin{aligned} [\phi(\mathbf{k}), \phi(\mathbf{k}')] &= [\pi(\mathbf{k}), \pi(\mathbf{k}')] = \mathcal{F}_{\mathbf{x}\to\mathbf{k}} \circ \mathcal{F}_{\mathbf{x}'\to\mathbf{k}'}(0) = 0 \\ [\phi(\mathbf{k}), \pi(\mathbf{k}')] &= \mathcal{F}_{\mathbf{x}\to\mathbf{k}} \circ \mathcal{F}_{\mathbf{x}'\to\mathbf{k}'}(i\delta^{(3)}(\mathbf{x}-\mathbf{x}')) = i\int d^3\mathbf{x} d^3\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'}\delta^{(3)}(\mathbf{x}-\mathbf{x}') \\ &= i\int d^3\mathbf{x} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} = i(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}') \end{aligned}$$

Now we compute commutators of creation and annihilation operators using the results in (b)

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= +\frac{i}{2}\left([\phi(\mathbf{k}), \pi(\mathbf{k}')] + [\pi(\mathbf{k}), \phi(\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(\delta^{(3)}(\mathbf{k}+\mathbf{k}') - \delta^{(3)}(\mathbf{k}'+\mathbf{k})) &&= 0 \\ [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] &= -\frac{i}{2}\left([\phi(-\mathbf{k}), \pi(-\mathbf{k}')] + [\pi(-\mathbf{k}), \phi(-\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(\delta^{(3)}(-\mathbf{k}-\mathbf{k}') - \delta^{(3)}(-\mathbf{k}'-\mathbf{k})) &&= 0 \\ [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] &= +\frac{i}{2}\left(-[\phi(\mathbf{k}), \pi(-\mathbf{k}')] + [\pi(\mathbf{k}), \phi(-\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(-\delta^{(3)}(\mathbf{k}-\mathbf{k}') - \delta^{(3)}(\mathbf{k}'-\mathbf{k})) \\ &= (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \end{aligned} \tag{2}$$

Question 3: Noether charges in terms of creation and annihilation operators (20 points)
In problem set 1, we obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(a) Express the Hamiltonian H of this theory in terms of $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.
From problem set 1, we quote

$$H = \frac{1}{2}\int d^3x(\pi^2 + (\nabla\phi)^2 + m^2\phi^2)$$

It is convenient to first convert this expression into momentum space, before using the decomposition into creation and annihilation operators. We use the identity:

$$\begin{aligned}\int d^3\mathbf{x} f(\mathbf{x})g(\mathbf{x}) &= \int d^3\mathbf{x} d^3\mathbf{k} d^3\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} f(\mathbf{k})g(\mathbf{k}') \\ &= \frac{1}{(2\pi)^3}\int d^3\mathbf{k} d^3\mathbf{k}' \delta^{(3)}(\mathbf{k}+\mathbf{k}') f(\mathbf{k})g(\mathbf{k}') = \int đ^3\mathbf{k} f(\mathbf{k})g(-\mathbf{k})\end{aligned} \tag{3}$$

More generally if there are derivatives acting on f or g, each derivative acting on f drags down a factor of $i\mathbf{k}$, while each derivative actin on g drags down a factor of $i\mathbf{k}'$, which becomes $-i\mathbf{k}$ after performing the $d^3\mathbf{x}$ integral. We further use the shorthand $đx = dx/2\pi$ ($đ$ is to d as $\hbar$ is to h).

Hence, we can now write

$$\begin{aligned}H &= \frac{1}{2}\int đ^3\mathbf{k}\left(\pi(\mathbf{k},t)\pi(-\mathbf{k},t) + (\mathbf{k}^2+m^2)\phi(\mathbf{k},t)\phi(-\mathbf{k},t)\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}\left(\pi(\mathbf{k},t)\pi(-\mathbf{k},t) + \omega_{\mathbf{k}}^2\phi(\mathbf{k},t)\phi(-\mathbf{k},t)\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}\left(a_{\mathbf{k}}(t)a_{\mathbf{k}}(t)^\dagger + a_{\mathbf{k}}(t)^\dagger a_{\mathbf{k}}(t)\right) = \int đ^3\mathbf{k}\omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2}(2\pi)^3\delta(0)\int đ^3\mathbf{k}\omega_{\mathbf{k}}\end{aligned}$$

In the third equality we use the relations (1) from problem 2(b). In the last equality we use the commutator (2) from problem 2(c), as well as the expressions $a_{\mathbf{k}}(t) = e^{-i\omega_{\mathbf{k}}t}a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}(t) = e^{i\omega_{\mathbf{k}}t}a^\dagger_{\mathbf{k}}$. Note that in the above calculation, we showed that the time-dependence cancels explicitly. We could have also used that H is conserved to remove the time-dependence immediately by evaluating all fields at $t=0$.

This can be written as

$$H = \int đ^3\mathbf{k}\omega_{\mathbf{k}} N_{\mathbf{k}} + E_0$$

for the number operator $N_{\mathbf{k}} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, and zero-point energy $E_0 = \frac{1}{2}(2\pi)^3\delta(0)\int đ^3\mathbf{k}\omega_{\mathbf{k}}$.

(b) Express the conserved charges P^i for spatial translations, in terms of $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.
Again we quote the charges from problem set 1, and use (3) to write it in momentum space.

$$\begin{aligned}P^i &= \int d^3\mathbf{x}\,\pi\partial^i\phi = -i\int đ^3\mathbf{k}\pi(\mathbf{k},t)\phi(-\mathbf{k},t)k^i \\ &= \frac{1}{2}\int đ^3\mathbf{k}k^i\left(a_{\mathbf{k}}(t) - a_{-\mathbf{k}}(t)^\dagger\right)\left(a_{-\mathbf{k}}(t) + a_{\mathbf{k}}(t)^\dagger\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}k^i\left(a_{\mathbf{k}}a_{-\mathbf{k}}e^{-2i\omega_{\mathbf{k}}t} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}a_{-\mathbf{k}} - a^\dagger_{-\mathbf{k}}a^\dagger_{\mathbf{k}}e^{2i\omega_{\mathbf{k}}t}\right)\end{aligned}$$

In the third equality we use the relations (1) from problem 2(b). Observe that due to the k^i factor and the commutation relations (2), the first and fourth terms in our final expression are odd under the change of variables $\mathbf{k} \to -\mathbf{k}$, so they must vanish. Therefore,

$$P^i = \frac{1}{2}\int \bar{d}^3\mathbf{k}k^i\left(a_{\mathbf{k}}a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right) = \int \bar{d}^3\mathbf{k}k^i a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\int \bar{d}^3\mathbf{k}(2\pi)^3\delta^{(3)}(0)k^i = \int \bar{d}^3\mathbf{k}k^i N_{\mathbf{k}}$$

In the second equality we use the commutator (2), and in the third equality we note that the last term vanishes because the integrand is $\mathbf{k}$-odd.

We may combine this with the expression for H in part (a) to write

$$P^\mu = \int \bar{d}^3\mathbf{k}\, k^\mu N_{\mathbf{k}} + \delta^{\mu 0}E_0, \qquad k^0 = \omega_{\mathbf{k}} \tag{4}$$

(c) Starting with

$$\phi(0,0) = \int \bar{d}^3k \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}\right)$$

show that under the action of translation operators,

$$\phi(\mathbf{x},t) = e^{iHt - iP^ix^i}\phi(0,0)e^{-iHt+iP^ix^i}$$

Hint: this problem becomes trivial using the following formula for a harmonic oscillator,

$$e^{i\alpha N}ae^{-i\alpha N} = e^{-i\alpha}a, \qquad\qquad N = a^\dagger a$$

The formula in the hint follows from the Baker-Campbell-Hausdorff (BCH) formula,

$$e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots$$

We check:

$$\begin{aligned}[i\alpha N, a] &= i\alpha[a^\dagger a, a] = -i\alpha a \\ e^{i\alpha N}ae^{-i\alpha N} &= a + (-i\alpha)a + \frac{1}{2!}(-i\alpha)^2 a + \cdots = e^{-i\alpha}a\end{aligned}$$

and similarly, $e^{-i\alpha N}a^\dagger e^{-i\alpha N} = e^{i\alpha}a^\dagger$.

Now we generalize. For $\alpha(\mathbf{k}')$ a real-valued function,

$$\begin{aligned}e^{i\int \bar{d}^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int \bar{d}^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}} &= e^{i\int \bar{d}^3\mathbf{k}'\delta(\mathbf{k}-\mathbf{k}')\alpha(\mathbf{k}')N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int \bar{d}^3\mathbf{k}'\delta(\mathbf{k}-\mathbf{k}')\alpha(\mathbf{k}')N_{\mathbf{k}'}} \\ &= e^{i\alpha(\mathbf{k})N_{\mathbf{k}}}a_{\mathbf{k}}e^{-i\alpha(\mathbf{k})N_{\mathbf{k}}} = e^{-i\alpha(\mathbf{k})}a_{\mathbf{k}}\end{aligned} \tag{5}$$

In the first equality, we use that the 2 sets of operators $\{\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{k}}, N_{\mathbf{k}}\}$ and $\{\alpha_{\mathbf{k}'}, \alpha^\dagger_{\mathbf{k}'}, N_{\mathbf{k}'}\}$ commute with each other for $\mathbf{k} \neq \mathbf{k}'$. This allows us to move all but the $\mathbf{k}' = \mathbf{k}$ exponentials on the left-hand side past the $a_{\mathbf{k}}$ factor, where it cancels out with the exponentials on the right. Note that for this, it is essential that $\alpha(\mathbf{k}')$ is a real-valued function. The last equality follows from the instance of the BCH formula derived above. In the same way, we have that

$$e^{i\int \bar{d}^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}}a^\dagger_{\mathbf{k}}e^{-i\int \bar{d}^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}} = e^{i\alpha(\mathbf{k})}a^\dagger_{\mathbf{k}} \tag{6}$$

Using our expressions for H and P^i in part (a) and (b), identities (5)-(6) allow us to compute

$$\begin{aligned}
e^{i(Ht-P^ix^i)}a_{\mathbf{k}}e^{-i(Ht-P^ix^i)} &= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}+iE_0t}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}-iE_0t} \\
&= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}} = e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}a_{\mathbf{k}} \\
e^{i(Ht-P^ix^i)}a^\dagger_{\mathbf{k}}e^{-i(Ht-P^ix^i)} &= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}}a^\dagger_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}} = e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}a^\dagger_{\mathbf{k}}
\end{aligned}$$

Finally, we get

$$\begin{aligned}
e^{i(Ht-P^ix^i)}\phi(0,0)e^{-i(Ht-P^ix^i)} &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}e^{i(Ht-P^ix^i)}(a_{\mathbf{k}}+a^\dagger_{\mathbf{k}})e^{-i(Ht-P^ix^i)} \\
&= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}(a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+a^\dagger_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}) = \phi(\mathbf{x},t)
\end{aligned}$$

Question 4: Noether charges for Lorentz symmetries of a real scalar (20 points + 10 bonus)
In this problem we work out the conserved currents corresponding to the Lorentz symmetries of a real scalar theory,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(a) Consider an infinitesimal Lorentz transformation

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal numbers. Show that this satisfies

$$\Lambda_\mu{}^\rho\eta_{\rho\lambda}\Lambda_\nu{}^\lambda = \eta_{\mu\nu}$$

to first order in $\omega_{\mu\nu}$, so this does give a Lorentz transformation.

We compute:

$$\begin{aligned}\Lambda_\mu{}^\rho\eta_{\rho\lambda}\Lambda_\nu{}^\lambda &= (\delta_\mu{}^\rho + \omega_\mu{}^\rho)\eta_{\rho\lambda}(\delta_\nu{}^\lambda + \omega_\nu{}^\lambda) = (\eta_{\mu\lambda} + \omega_{\mu\lambda})(\delta^\lambda{}_\nu - \omega^\lambda{}_\nu)\\ &= \eta_{\mu\nu} + \omega_{\mu\nu} - \omega_{\mu\nu} + \omega_{\mu\lambda}\omega^\lambda{}_\mu = \eta_{\mu\nu} + \mathcal{O}(\omega^2)\end{aligned}$$

(b) Write down how ϕ transforms under an infinitesimal Lorentz transformation, and show that the conserved Noether current for this transformation can be written as

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}$$

where $T^{\mu\nu}$ is the conserved energy-momentum tensor derived in problem set 1.

A Lorentz scalar field transforms in a way obeying $\phi'(x') = \phi(x)$. Therefore, under an infinitesimal Lorentz transformation, the scalar field ϕ transforms as

$$\delta\phi = \phi'(x) - \phi(x) = \phi((\Lambda^{-1})^\mu{}_\nu x^\nu) - \phi(x^\nu) = \phi((\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu) - \phi(x^\nu) = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi$$

where in the last equality we Taylor expand $\phi(x^\mu - \omega^\mu{}_\nu x^\nu) = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi$.

Using this, the Lagrangian density transforms as

$$\delta\mathcal{L} = \mathcal{L}[\phi'] - \mathcal{L}[\phi] = \mathcal{L}[\phi - \omega^\lambda{}_\nu x^\nu\partial_\lambda\phi] - \mathcal{L}[\phi] = -\omega^\lambda{}_\nu x^\nu\partial_\lambda\phi\frac{\partial\mathcal{L}}{\partial\phi} = -\omega^\lambda{}_\nu x^\nu\partial_\lambda\mathcal{L} = -\partial_\lambda(\omega^\lambda{}_\nu x^\nu\mathcal{L})$$

We expand only to first order in ω. In the final equality, we use that $\omega_{\mu\nu}$ is antisymmetric, i.e.

$$\partial_\lambda(\omega^\lambda{}_\nu x^\nu f(x)) = \omega^\lambda{}_\nu\delta^\nu_\lambda f(x) + \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x) = \omega^\lambda{}_\lambda f(x) + \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x) = \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x)$$

Hence, $\delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$, for $\mathcal{F}^\mu = -\omega^\mu{}_\nu x^\nu\mathcal{L}$
The Noether current for the transformation parameterized by $\omega_{\lambda\nu}$ is given by

$$\begin{aligned}j^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \mathcal{F}^\mu = \omega^\lambda{}_\nu x^\nu\partial^\mu\phi\partial_\lambda\phi + \omega^\mu{}_\nu x^\nu\mathcal{L}\\ &= \omega_{\lambda\nu}x^\nu(\partial^\mu\phi\partial^\lambda\phi + \eta^{\mu\lambda}\mathcal{L}) = \omega_{\lambda\nu}x^\nu T^{\mu\lambda}\end{aligned}$$

for the energy-momentum tensor from problem set 1 (now with a real scalar):

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2)$$

Note that $\omega_{\lambda\nu}$ is an arbitrary antisymmetric tensor which parameterizes our infinitesimal transformation. In total we have an antisymmetric tensor worth of conserved currents, which we can package in $J^{\mu\lambda\nu}$:

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}, \qquad\qquad j^\mu = -\frac{1}{2}\omega_{\lambda\nu}J^{\mu\lambda\nu}$$

In writing $J^{\mu\lambda\nu}$ we have made the antisymmetry in λ and ν manifest by explicitly antisymmetrizing over these indices.

(c) Using conservation of the energy-momentum tensor, verify that the current in (b) is conserved, i.e.

$$\partial_\mu J^{\mu\lambda\nu} = 0$$

We compute:

$$\begin{aligned}\partial_\mu J^{\mu\lambda\nu} &= \partial_\mu(x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}) = \delta_\mu{}^\lambda T^{\mu\nu} + x^\lambda\partial_\mu T^{\mu\nu} - \delta_\mu{}^\nu T^{\mu\lambda} - x^\nu\partial_\mu T^{\mu\lambda} \\ &= T^{\lambda\nu} - T^{\nu\lambda} = 0\end{aligned}$$

In the 3rd equality we used the conservation law $\partial_\mu T^{\mu\nu} = 0$, and in the 4th equality we used from problem set 1 that $T^{\mu\nu} = T^{\nu\mu}$ is symmetric.

(d) (**Bonus problem**) Consider the conserved charges associated with $J^{\mu\lambda\nu}$,

$$M^{\lambda\nu} = \int d^3x J^{0\lambda\nu}$$

Express the conserved charges $M^{\mu\nu}$ for the Lorentz symmetries of this theory in terms of $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.
From part (b), we have

$$M^{\mu\nu} = \int d^3x J^{0\mu\nu} = \int d^3x(x^\mu T^{0\nu} - x^\nu T^{0\mu})$$

This is antisymmetric in μ and ν, so we need to compute M^{0i} and M^{ij}. To do this, we need to expand $T^{0\mu}$ in terms of creation and annihilation operators:

$$\begin{aligned}T^{0\mu} &= -\pi\partial^\mu\phi - \frac{1}{2}\eta^{0\mu}\left(-\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \\ &= \left(\frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2), -\pi\partial^i\phi\right) = (\mathcal{H}(x), \mathcal{P}^i(x))\end{aligned}$$

Since $M^{\mu\nu}$ are conserved currents, we can compute them at $t = 0$.
First, we need the identity

$$\begin{aligned}\int d^3\mathbf{x} x^i f(\mathbf{x})g(\mathbf{x}) &= \int d^3\mathbf{x} d^3\mathbf{k} d^3\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} x^i f(\mathbf{k})g(\mathbf{k}') \\ &= \frac{1}{(2\pi)^3}\int d^3\mathbf{k} d^3\mathbf{k}'(-i\partial_{k_i}\delta^{(3)}(\mathbf{k}+\mathbf{k}'))f(\mathbf{k})g(\mathbf{k}') = i\int d\!\!{}^-{}^3\mathbf{k}\,\partial_{k_i} f(\mathbf{k})g(-\mathbf{k}) \qquad (7)\end{aligned}$$

where we have used integration by parts in the last equality. More generally if there are derivatives acting on f or g, each derivative acting on f drags down a factor of $i\mathbf{k}$, while each derivative actin on g drags down a factor of $i\mathbf{k}'$, which becomes $-i\mathbf{k}$ after performing the $d^3\mathbf{x}$ integral.

Now we are ready to compute

$$
\begin{aligned}
M^{0i} &= \int d^3\mathbf{x}(t\mathcal{P}^i(x) - x^i\mathcal{H}(x))|_{t=0} = -\frac{1}{2}\int d^3\mathbf{x}x^i(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \\
&= -\frac{i}{2}\int đ^3\mathbf{k}\left(\partial_{k_i}\pi(\mathbf{k})\pi(-\mathbf{k}) + \omega_\mathbf{k}^2\partial_{k_i}\phi(\mathbf{k})\phi(-\mathbf{k})\right) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}\Big(-\partial_{k_i}(\sqrt{\omega_\mathbf{k}}(a_\mathbf{k} - a^\dagger_{-\mathbf{k}}))\cdot(\sqrt{\omega_\mathbf{k}}(a_{-\mathbf{k}} - a^\dagger_\mathbf{k})) \\
&\qquad\qquad + \omega_\mathbf{k}^2\partial_{k_i}(\frac{1}{\sqrt{\omega_\mathbf{k}}}(a_\mathbf{k} + a^\dagger_{-\mathbf{k}}))\cdot(\frac{1}{\sqrt{\omega_\mathbf{k}}}(a_{-\mathbf{k}} + a^\dagger_\mathbf{k}))\Big) \\
&= +\frac{i}{4}\int đ^3\mathbf{k}\Big((k^i + \omega_\mathbf{k}\partial_{k_i})(a_\mathbf{k} - a^\dagger_{-\mathbf{k}})\cdot(a_{-\mathbf{k}} - a^\dagger_\mathbf{k}) + (k^i - \omega_\mathbf{k}\partial_{k_i})(a_\mathbf{k} + a^\dagger_{-\mathbf{k}})\cdot(a_{-\mathbf{k}} + a^\dagger_\mathbf{k})\Big) \\
&= -\frac{i}{2}\int đ^3\mathbf{k}\omega_\mathbf{k}\Big((\partial_{k_i}a_\mathbf{k})a^\dagger_\mathbf{k} + (\partial_{k_i}a^\dagger_{-\mathbf{k}})a_{-\mathbf{k}}\Big) = -\frac{i}{2}\int đ^3\mathbf{k}\omega_\mathbf{k}\Big((\partial_{k_i}a_\mathbf{k})a^\dagger_\mathbf{k} - (\partial_{k_i}a^\dagger_\mathbf{k})a_\mathbf{k}\Big) \\
&= -\frac{i}{2}\int đ^3\mathbf{k}\omega_\mathbf{k}\Big(a^\dagger_\mathbf{k}(\partial_{k_i}a_\mathbf{k}) + (2\pi)^3\partial_{k_i}\delta(\mathbf{k}-\mathbf{k}')|_{\mathbf{k}'=\mathbf{k}} - (\partial_{k_i}a^\dagger_\mathbf{k})a_\mathbf{k}\Big) \\
&= -\frac{i}{2}\int đ^3\mathbf{k}\omega_\mathbf{k}\Big(a^\dagger_\mathbf{k}(\partial_{k_i}a_\mathbf{k}) - (\partial_{k_i}a^\dagger_\mathbf{k})a_\mathbf{k}\Big)
\end{aligned}
$$

Line 2 follows from identity (7). Line 5 is obtained by noting that all but 2 terms in the integrand of line 4 either cancel out or are odd. In line 6 we use the identity obtained by taking the $\mathbf{k}$-derivative of $[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$, and evaluating at $\mathbf{k}' = \mathbf{k}$. Finally, to reach line 7 we use that the $\delta^{(3)}$-term in line 6 is odd.

In a similar way, we can also compute for $i \neq j$ (since $M^{ii} = 0$ by asymmetry)

$$
\begin{aligned}
M^{ij} &= \int d^3\mathbf{x}(x^i\mathcal{P}^j(x) - x^j\mathcal{P}^i(x)) = -\int d^3\mathbf{x}(x^i\pi\partial^j\phi - x^j\pi\partial^i\phi) \\
&= \frac{1}{2}\int đ^3\mathbf{k}\left(k^j\pi(\mathbf{k})\partial_{k_i}\phi(-\mathbf{k}) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}\left(k^j\sqrt{\omega_\mathbf{k}}(a_\mathbf{k} - a^\dagger_{-\mathbf{k}})\cdot\partial_{k_i}(\frac{1}{\sqrt{\omega_\mathbf{k}}}(a_{-\mathbf{k}} + a^\dagger_\mathbf{k})) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}\left(k^j(a_\mathbf{k} - a^\dagger_{-\mathbf{k}})\cdot\left(-\frac{k^i}{\omega_\mathbf{k}} + \partial_{k_i}\right)(a_{-\mathbf{k}} + a^\dagger_\mathbf{k}) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}k^j\left(a_\mathbf{k}\partial_{k_i}a_{-\mathbf{k}} + a_\mathbf{k}\partial_{k_i}a^\dagger_\mathbf{k} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a_{-\mathbf{k}} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a^\dagger_\mathbf{k}\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}k^j\left(a_\mathbf{k}\partial_{k_i}a^\dagger_\mathbf{k} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a_{-\mathbf{k}}\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}k^j\left(-(\partial_{k_i}a_\mathbf{k})a^\dagger_\mathbf{k} - a^\dagger_\mathbf{k}(\partial_{k_i}a_\mathbf{k})\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int đ^3\mathbf{k}k^j\left((2\pi)^3\partial_{k_i}\delta(\mathbf{k}-\mathbf{k}')|_{\mathbf{k}'=\mathbf{k}} - 2a^\dagger_\mathbf{k}(\partial_{k_i}a_\mathbf{k})\right) - (i\leftrightarrow j) \\
&= -\frac{i}{2}\int đ^3\mathbf{k}\left(k^i a^\dagger_\mathbf{k}(\partial_{k_j}a_\mathbf{k}) - k^j(\partial_{k_i}a^\dagger_\mathbf{k})a_\mathbf{k}\right)
\end{aligned}
$$

where we make ample use of integration by parts, drop total-derivative terms, and use $i \neq j$ to completely ignore having to deal with $\propto \partial_{k_i}k^j$ terms.

Physically, the conserved quantities are the center of mass velocities M^{0i} and angular momenta M^{ij}.

Altogether, we may write

$$M^{\mu\nu} = -\frac{i}{2}\int đ^3\mathbf{k}k^\mu(a_\mathbf{k}^\dagger\partial_{k_\nu}a_\mathbf{k} - \partial_{k_\nu}a_\mathbf{k}^\dagger a_\mathbf{k}) - (\mu \to \nu), \qquad k^0 = \omega_\mathbf{k}$$

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 3 Solutions

February 28, 2023

Question 1: Lorentz Transformations for Operators and States (15 points)

(a) From the commutation relation of creation and annihilation operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

and the result of Problem 2(b) of Problem Set 1, argue that $a_{\mathbf{k}}$, $a^\dagger_{\mathbf{k}}$ should transform under a Lorentz transformation as

$$a_{\mathbf{k}} \to \tilde{a}_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad a^\dagger_{\mathbf{k}} \to \tilde{a}^\dagger_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a^\dagger_{\Lambda\mathbf{k}}$$

We start with the commutator

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

The left-hand side transforms to $[\tilde{a}_{\mathbf{k}}, \tilde{a}^\dagger_{\mathbf{k}'}]$. Furthermore, since $\omega_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$ is Lorentz-invariant (Problem 2 of PS1), the right-hand side transforms to

$$\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}} (2\pi)^3 \delta^{(3)}(\Lambda\mathbf{k} - \Lambda\mathbf{k}') = \frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}} [a_{\Lambda\mathbf{k}}, a^\dagger_{\Lambda\mathbf{k}'}]$$

The transformations of the two sides must be equal, which is satisfied by imposing

$$a_{\mathbf{k}} \to \tilde{a}_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad a^\dagger_{\mathbf{k}} \to \tilde{a}^\dagger_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a^\dagger_{\Lambda\mathbf{k}}$$

(b) **Bonus.** Using the expression of $M_{\mu\nu}$ of Problem 4(c) of Problem Set 2, compute

$$\frac{1}{2}[\omega_{\mu\nu} M^{\mu\nu}, a_{\mathbf{k}}], \qquad \frac{1}{2}[\omega_{\mu\nu} M^{\mu\nu}, a^\dagger_{\mathbf{k}}]$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal constants. Show that this indeed generates an infinitesimal Lorentz transformation in $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ which is consistent with part (a).

From Problem Set 2 we quote the result

$$M^{\mu\nu} = -\frac{i}{2} \int đ^3k\, k^\mu \left(a^\dagger_{\mathbf{k}} (\partial_{k_\nu} a_{\mathbf{k}}) - (\partial_{k_\nu} a^\dagger_{\mathbf{k}}) a_{\mathbf{k}} \right) - (\mu \leftrightarrow \nu), \qquad \partial_{k_0} a_{\mathbf{k}} = 0, \qquad k^0 = \omega_{\mathbf{k}}$$

By antisymmetry, we have

$$\omega_{\mu\nu} M^{\mu\nu} = -i \int đ^3k\, k^\mu \left(a^\dagger_{\mathbf{k}} (\partial_{k_\nu} a_{\mathbf{k}}) - (\partial_{k_\nu} a^\dagger_{\mathbf{k}}) a_{\mathbf{k}} \right)$$

Now we are ready to compute

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = -i\omega_{\mu\nu}\int đ^3k' k'^{\mu}\left([a^\dagger_{\mathbf{k}'}, a_{\mathbf{k}}](\partial_{k'_\nu}a_{\mathbf{k}'}) - [\partial_{\mathbf{k}'_\nu}a^\dagger_{\mathbf{k}'}, a_{\mathbf{k}}]a_{\mathbf{k}'}\right)\delta_{\nu\neq 0}$$
$$= i\omega_{\mu\nu}\int đ^3k' k'^{\mu}\left([a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}](\partial_{k'_\nu}a_{\mathbf{k}'}) - [a_{\mathbf{k}}, \partial_{\mathbf{k}'_\nu}a^\dagger_{\mathbf{k}'}]a_{\mathbf{k}'}\right)\delta_{\nu\neq 0}$$

We append the $\delta_{\nu\neq 0}$ for convenience because the integrand is zero when $\nu = 0$ (as we have used the convenient shorthand $\delta_{k_0}a_{\mathbf{k}} := 0$). This does nothing at the moment, but it will prevent confusion when we use the ladder-operator commutators, which will give factors like $\partial_{k'^0}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. Moving on,

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = i\omega_{0i}\int đ^3k'\omega_{\mathbf{k}'}\left(\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_i}a_{\mathbf{k}'}) - (\partial_{k'_i}\delta^{(3)}(\mathbf{k}-\mathbf{k}'))a_{\mathbf{k}'}\right)$$
$$+ i\omega_{ij}\int đ^3k' k'^i\left(\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_j}a_{\mathbf{k}'}) - (\partial_{k'_j}\delta^{(3)}(\mathbf{k}-\mathbf{k}'))a_{\mathbf{k}'}\right) = (\mathrm{I}) + (\mathrm{II})$$

Note that the $\delta_{\nu\neq 0}$ added in the last step kills any ω_{i0} term. We first compute the second term:

$$(\mathrm{II}) = i\omega_{ij}\int đ^3k'\left(k'^i\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_j}a_{\mathbf{k}'}) + \partial_{k'_j}(k'^i a_{\mathbf{k}'})\delta^{(3)}(\mathbf{k}-\mathbf{k}')\right)$$
$$= i\omega_{ij}(k^i\partial_{k_j}a_{\mathbf{k}} + \partial_{k_j}(k^i a_{\mathbf{k}})) = 2i\omega_{ij}k^i\partial_{k_j}a_{\mathbf{k}} + i\omega_{ij}\delta^{ij}a_{\mathbf{k}} = 2i\omega_{ij}k^i\partial_{k_j}a_{\mathbf{k}}$$

where in the last equality, the second term vanishes because ω_{ij} is antisymmetric, but δ^{ij} is symmetric. We also have the first term:

$$(\mathrm{I}) = i\omega_{0i}\int đ^3k'\left(\omega_{\mathbf{k}'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_i}a_{\mathbf{k}'}) + \partial_{k'_i}(\omega_{\mathbf{k}'}a_{\mathbf{k}'})\delta^{(3)}(\mathbf{k}-\mathbf{k}')\right)$$
$$= i\omega_{0i}\left(\omega_{\mathbf{k}}(\partial_{k_i}a_{\mathbf{k}}) + \partial_{k_i}(\omega_{\mathbf{k}}a_{\mathbf{k}})\right) = 2i\omega_{0i}\omega_{\mathbf{k}}\partial_{k_i}a_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a_{\mathbf{k}}$$

Adding these together gives (where again we define $\delta_{k_0}a_{\mathbf{k}} := 0$)

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = 2i\omega_{\mu\nu}k^\mu\partial_{k_\nu}a_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a_{\mathbf{k}} \tag{1}$$

Similarly, one can show that

$$[\omega_{\mu\nu}M^{\mu\nu}, a^\dagger_{\mathbf{k}}] = 2i\omega_{\mu\nu}k^\mu\partial_{k_\nu}a^\dagger_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a^\dagger_{\mathbf{k}} \tag{2}$$

Finally, we want to show that this generates an infinitesimal Lorentz transformation consistent with (a). From part (a), we have

$$U_\Lambda a_{\mathbf{k}}U^\dagger(\Lambda) = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a_{\Lambda\mathbf{k}}, \qquad U_\Lambda a^\dagger_{\mathbf{k}}U^\dagger(\Lambda) = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a^\dagger_{\Lambda\mathbf{k}}$$

We can expand these for infinitesimal transformations

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}, \qquad (\Lambda k)_i = k_i + \omega_i{}^\nu k_\nu, \qquad \omega_{\Lambda\mathbf{k}} = (\Lambda k)_0|_{k^0=\omega_{\mathbf{k}}} = \omega_{\mathbf{k}} - \omega_0{}^\nu k_\nu$$

Substituting these expressions into the ones above yield

$$a_{\mathbf{k}} + \frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = \left(1 - \frac{1}{2\omega_{\mathbf{k}}}\omega_0{}^\nu k_\nu\right)(a_{\mathbf{k}} + \omega_i{}^\nu k_\nu\partial_{k_i}a_{\mathbf{k}})$$
$$\frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = \left(-\frac{1}{2\omega_{\mathbf{k}}}\omega_0{}^\nu k_\nu + \omega_i{}^\nu k_\nu\partial_{k_i}\right)a_{\mathbf{k}} = -\frac{1}{2}\left(2\omega_{\mu\nu}k^\mu\partial_{k_\nu} + \omega_0{}^\nu\frac{k_i}{\omega_{\mathbf{k}}}\right)a_{\mathbf{k}}$$

Multiplying both sides by $-2i$ gives precisely (1). Replacing all instances of $a_{\mathbf{k}}$ with $a^\dagger_{\mathbf{k}}$ in this calculation gives (2), as desired. Therefore, $M^{\mu\nu}$ generates an infinitesimal Lorentz transformation acting on $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ via commutation.

(c) **Bonus.** Now consider a unitary operator generating finite Lorentz transformations,

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

where $\omega_{\mu\nu}$ are finite constants. Show that

$$U_\Lambda|0\rangle = |0\rangle$$

i.e. the vacuum is Lorentz invariant. Assuming the Lorentz transformations of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ in part (a), show that

$$U_\Lambda|k\rangle = |\Lambda k\rangle$$

Note that the expression for $M^{\mu\nu}$ in part (b) in terms of ladder operators is already normal ordered, with a lowering operator to the right in each term. Therefore, $M^{\mu\nu}|0\rangle = 0$ and

$$U_\Lambda|0\rangle = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)^n|0\rangle = 1|0\rangle = |0\rangle$$

where in the second equality, only the $n=0$ term does not vanish.

We can also act U_Λ on a momentum eigenstate $|k\rangle = \sqrt{2\omega_{\mathbf{k}}}a_{\mathbf{k}}^\dagger|0\rangle$:

$$U_\Lambda|k\rangle = \sqrt{2\omega_{\mathbf{k}}}U_\Lambda a_{\mathbf{k}}^\dagger U_\Lambda^\dagger U_\Lambda|0\rangle = \sqrt{2\omega_{\mathbf{k}}}\sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a_{\Lambda\mathbf{k}}|0\rangle = |\Lambda k\rangle$$

as desired. In the second equality we use the result from (a).

Question 2: Spatially Localized States of a Scalar Particle (25 points)
In a quantum field theory there is no natural way to define a position eigenvector $|\mathbf{x}\rangle$, as $\mathbf{x}$ is now simply a label, not an operator. Also, there is a fundamental conflict: a perfectly localized state in space is not a Lorentz covariant concept, as it picks out a reference frame (we cannot perfectly localize in time at the same time, as simultaneity is relative).

In this problem we will make these abstract statements concrete. Consider states of the form

$$|\mathbf{r}, t\rangle_f := \int d^3\mathbf{k} f(\mathbf{k})|k\rangle e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_\mathbf{k}t}$$

where $f(\mathbf{k})$ is a function to be determined, and $|k\rangle = \sqrt{2\omega_\mathbf{k}}a^\dagger_\mathbf{k}|0\rangle$ is the 1-particle state of momentum $\mathbf{k}$ and energy $\omega_\mathbf{k}$.
(a) Determine $f(\mathbf{k})$ by the condition of perfect localization,

$${}_f\langle \mathbf{r}_1, t|\mathbf{r}_2, t\rangle_f = \delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$$

We compute:

$$\begin{aligned}
{}_f\langle \mathbf{r}_1, t|\mathbf{r}_2, t\rangle_f &= \int d^3\mathbf{k}d^3\mathbf{k}' f(\mathbf{k})f^*(\mathbf{k}')\langle k'|k\rangle e^{-i(\mathbf{k}\cdot\mathbf{r}_1-\mathbf{k}'\cdot\mathbf{r}_2)+it(\omega_\mathbf{k}-\omega_{\mathbf{k}'})} \\
&= \int d^3\mathbf{k}d^3\mathbf{k}' f(\mathbf{k})f^*(\mathbf{k}') e^{-i(\mathbf{k}\cdot\mathbf{r}_1-\mathbf{k}'\cdot\mathbf{r}_2)+it(\omega_\mathbf{k}-\omega_{\mathbf{k}'})} 2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\
&= (2\pi)^3 \int d^3\mathbf{k}\ 2\omega_\mathbf{k}|f(\mathbf{k})|^2 e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}
\end{aligned}$$

We see that expression is equal to $\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) = \int đ^3k e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}$

$$|f(\mathbf{k})|^2 = \frac{1}{(2\pi)^6}\frac{1}{2\omega_\mathbf{k}}, \qquad f(\mathbf{k}) = \frac{1}{(2\pi)^3\sqrt{2\omega_\mathbf{k}}}$$

where in going from $|f|^2$ to f the phase is not fixed, but we set it to 1 for simplicity. Therefore, using the notation $r^\mu = (t, \mathbf{r})$ we have

$$|\mathbf{r}, t\rangle = \int đ^3k \frac{1}{\sqrt{2\omega_\mathbf{k}}}|k\rangle e^{-ik\cdot r} \tag{3}$$

(b) By acting the unitary operator U_Λ for a Lorentz transformation on the result in part (a), show that $|\mathbf{r}, t\rangle$ is not Lorentz invariant, i.e.

$$U_\Lambda|\mathbf{r}, t\rangle_f \neq |\Lambda\mathbf{r}, \Lambda t\rangle_f$$

Proof by contradiction. Suppose that we had $U_\Lambda|\mathbf{r}, t\rangle_f = |\Lambda\mathbf{r}, \Lambda t\rangle_f$. Then,

$${}_f\langle \Lambda\mathbf{r}_2, \Lambda t|\Lambda\mathbf{r}_1, \Lambda t\rangle_f = {}_f\langle \mathbf{r}_2, t|U^\dagger U|\mathbf{r}_1, t\rangle_f = {}_f\langle \mathbf{r}_2, t|\mathbf{r}_1, t\rangle_f$$

By part (a), the left-hand side is $\delta^{(3)}(\Lambda\mathbf{r}_1 - \Lambda\mathbf{r}_2)$, while the right-hand side is $\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$. One has:

$$\delta^{(3)}(\Lambda\mathbf{r}_1 - \Lambda\mathbf{r}_2) = \frac{\omega_\mathbf{k}}{\omega_{\Lambda\mathbf{k}}}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$$

so clearly the left and right-hand sides cannot be equal for generic Λ (a boost will change $\omega_\mathbf{k}$). This gives us our desired contradiction, therefore $|\mathbf{r}, t\rangle$ cannot be Lorentz-invariant.

(c) With $f(\mathbf{k})$ given by (a), consider the overlap $C(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2) = {}_f\langle \mathbf{r}_2,t_2|\mathbf{r}_1,t_1\rangle_f$. Evaluate $C(\mathbf{r},t)$ for a spacelike separation $|\mathbf{r}|>t$.
Suppose we interpret $|{}_f\langle \mathbf{r}_2,t_2|\mathbf{r}_1,t_1\rangle_f|^2$ as the probability for the particle originally at time $\mathbf{r}_1$ and time t_1 to transition to time t_2. Would the propagation be causal (confined to the forwards light-cone)?

Using (3), we immediately have

$$C(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2) = \int đ^3\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}e^{-i\omega_\mathbf{k}(t_2-t_1)}$$

This can be slickly evaluated by writing it as the derivative of a Lorentz-invariant integral:

$$C(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2) = 2i\partial_{t_2-t_1}\int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}e^{-i\omega_\mathbf{k}(t_2-t_1)}$$

The integral is now Lorentz-covariant, hence we can evaluate it any frame. We are interested in spacelike-separated points $(\mathbf{r}_1,t)$, $(\mathbf{r}_2,t)$, hence there exists a Lorentz transformation that makes these points simultaneous, i.e. $\Lambda t_1 = \Lambda t_2$. The integral can then be evaluated in spherical coordinates.

$$\begin{aligned}\int \frac{đ^3k}{2\omega_\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}e^{-i\omega_\mathbf{k}(t_2-t_1)} &= \int \frac{đ^3\Lambda\mathbf{k}}{2\omega_{\Lambda\mathbf{k}}} e^{i\Lambda\mathbf{k}\cdot(\Lambda\mathbf{r}_2-\Lambda\mathbf{r}_1)}e^{-i\omega_{\Lambda\mathbf{k}}(\Lambda t_2-\Lambda t_1)} = \int \frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}'}} e^{i\mathbf{k}'\cdot(\Lambda\mathbf{r}_2-\Lambda\mathbf{r}_1)} \\ &= \frac{1}{2(2\pi)^3}\int_0^\infty d|\mathbf{k}|\frac{|\mathbf{k}|^2}{\sqrt{|\mathbf{k}|^2+m^2}}\int_0^\pi d\phi\int_0^{2\pi} d\theta \sin\theta e^{ik|\mathbf{r}_1-\mathbf{r}_2|\cos\theta} \\ &= \frac{1}{2(2\pi)^2}\int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}}\frac{2\sin(k|\mathbf{r}_2-\mathbf{r}_1|)}{k|\mathbf{r}_2-\mathbf{r}_1|} \\ &= \frac{m}{4\pi^2|\mathbf{r}_2-\mathbf{r}_1|}K_1(m|\mathbf{r}_2-\mathbf{r}_1|)\end{aligned}$$

where $K_1(x)$ is the modified Bessel function of the second kind.

To compute $C(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2)$, we need to take the $\partial_{t_1-t_2}$ on this object. But this is a bit unclear: we have chosen a frame where $t_1=t_2$. To restore the time-dependence of this object, we use the fact that it is a Lorentz-invariant. This means that our integral can only depend on $|r_2-r_1| = \sqrt{-(t_2-t_1)^2+(\mathbf{r}_2-\mathbf{r}_1)^2}$ (using the notation $r=(\mathbf{r},t)$), which in the specified frame reduces to $|\mathbf{r}_1-\mathbf{r}_2|$. Therefore,

$$C(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2) = 2i\partial_{t_2-t_1}\left[\frac{m}{4\pi^2|r_2-r_1|}K_1(m|r_2-r_1|)\right] = \frac{im^2}{2\pi^2}\frac{t_2-t_1}{(r_1-r_2)^2}K_2(m|r_2-r_1|)$$

Interpreting $|{}_f\langle \mathbf{r}_2,t_2|\mathbf{r}_1,t_1\rangle_f|^2$ as a probability, we have:

$$|{}_f\langle \mathbf{r}_2,t_2|\mathbf{r}_1,t_1\rangle_f|^2 = \left|\frac{m^2}{2\pi^2}\frac{t_2-t_1}{(r_1-r_2)^2}K_2(m|r_2-r_1|)\right|^2$$

This is non-zero, meaning that the propagator is non-zero for generic spacelike separated points, i.e. propagation is non-causal.

(d) Compare your resulting state $|\mathbf{r},t\rangle_f$ in part (a) with the state

$$|x\rangle := \phi(x)|0\rangle$$

where $x=(\mathbf{x},t)$. Are the states $|x\rangle$ perfectly localized? Show that $|x\rangle$ is Lorentz covariant, i.e.

$$U_\Lambda|x\rangle = |\Lambda x\rangle$$

We compute

$$|x\rangle = \phi(x)|0\rangle = \int đ^3\mathbf{k}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}e^{-ik\cdot x}a_{\mathbf{k}}^\dagger|0\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}e^{-ik\cdot x}|k\rangle$$

Note that this last expression is written entirely of objects transforming simply under Lorentz transformations. The equal-time overlap is

$$\begin{aligned}
\langle x|x'\rangle &= \int đ^3\mathbf{k}đ^3\mathbf{k}'\frac{1}{2\omega_{\mathbf{k}}2\omega_{\mathbf{k}'}}e^{ik\cdot x-ik'\cdot x}\langle k'|k\rangle \\
&= \int đ^3\mathbf{k}đ^3\mathbf{k}'\frac{1}{2\omega_{\mathbf{k}}2\omega_{\mathbf{k}'}}e^{ik\cdot x-ik'\cdot x}2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \frac{m}{2\pi^2|\mathbf{x}-\mathbf{x}'|}K_1(m|\mathbf{x}-\mathbf{x}')
\end{aligned}$$

The integral in the second last equality has already been performed in part (c). The final expression is not equal to $\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, which would be the result if the states $|x\rangle$ were perfectly localized.

Lastly, we check Lorentz invariance.

$$U_\Lambda|x\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}e^{-ik\cdot x}U_\Lambda|k\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}e^{-ik\cdot x}|\Lambda k\rangle = \int \frac{đ^3\Lambda\mathbf{k}}{2\omega_{\Lambda\mathbf{k}}}e^{-i\Lambda k\cdot\Lambda x}|\Lambda k\rangle = |\Lambda x\rangle$$

(e) Consider a state

$$|\Psi\rangle = \int đ^3\mathbf{k}h(\mathbf{k})|k\rangle$$

with the corresponding 'wavefunction' defined as

$$\Psi(x) = \langle 0|\phi(x)|\Psi(x)\rangle$$

Find $h(\mathbf{k})$ so that at $t=0$, the single-particle wavefunction $\Psi(\mathbf{x})$ corresponding to $|\Psi\rangle$ is a Gaussian wave-packet centered around $\mathbf{x}_0$, with width a and momentum $\mathbf{p}$.

Given $|\Psi\rangle$ defined this way, at $t=0$ its wavefunction is given by

$$\Psi(\mathbf{x}) = \langle 0|\phi(x)|\Psi\rangle = \int đ^3\mathbf{k}h(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

This is just a 3D-Fourier transform. Therefore, choosing

$$h(\mathbf{k}) = (2\pi^{1/2}a)^{3/2}e^{-\frac{a^2}{2}(\mathbf{k}-\mathbf{p})^2}e^{-i\mathbf{k}\cdot\mathbf{x}_0} \qquad \Longrightarrow \qquad \Psi(\mathbf{x}) = \frac{1}{(\pi^{1/2}a)^{3/2}}e^{-\frac{1}{2a^2}(\mathbf{x}-\mathbf{x}_0)^2}e^{i\mathbf{p}\cdot\mathbf{x}}$$

which is a Gaussian centered around $\mathbf{x}_0$, with width a and momentum $\mathbf{p}$.

Question 3: Normal Ordering and Smeared Fields (25 points)
Consider the free real scalar field theory discussed in lecture.
(a) First show that

$$\langle 0|\phi(\mathbf{x},t)|0\rangle = 0$$

Evaluate the vacuum expectation value

$$\sigma^2 := \langle 0|\phi^2(\mathbf{x},t)|0\rangle$$

Express σ^2 as an integral over a single variable, and show that the integral is divergent. This result signifies that the vacuum is not empty! While the expectation value of ϕ is zero, the fluctuations of ϕ, as measured by σ^2, are non-zero, and in fact are infinitely large. This is a reflecton of that a QFT has an infinite number of degrees of freedom.

The expectation value of ϕ is

$$\langle 0|\phi(\mathbf{x},t)|0\rangle = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(\langle 0|a_{\mathbf{k}}|0\rangle e^{ik\cdot x} + \langle 0|a^\dagger_{\mathbf{k}}|0\rangle e^{-ik\cdot x}\right) = 0 + 0 = 0$$

Where we have used $a_{\mathbf{k}}|0\rangle = 0$, and $\langle 0|a^\dagger_{\mathbf{k}} = 0$.

The variance of ϕ has 4 sets of terms when we take the mode expansion, corresponding to combinations of creation/annihilation operators. The only set of terms that does not vanish is $\langle 0|a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}|0\rangle$, hence:

$$\begin{aligned}\sigma^2 = \langle 0|\phi(\mathbf{x},t)|0\rangle &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}|0\rangle e^{i(k-k')\cdot x} \\ &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(k-k')\cdot x} \\ &= \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{k^2+m^2} = \infty\end{aligned}$$

(b) The general philosophy of QFT is to regard operators like $\phi^2(\mathbf{x},t)$ as 'bad' operators. One then can introduce 'good' operators which do not suffer divergences, a procedure referred to as renormalization. One way to remove the divergence in part (a) is to introduce normal-ordered operators. The rule of normal ordering is whenever one has products of $a_{\mathbf{k}}$'s and $a^\dagger_{\mathbf{k}}$'s, to move all the $a_{\mathbf{k}}$'s to the right of the $a^\dagger_{\mathbf{k}}$'s. We denote the normal ordered version of an operator $\mathcal{O}$ by $:\mathcal{O}:$. For example,

$$:a_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}a_{\mathbf{k_3}}a^\dagger_{\mathbf{k_4}}:= a^\dagger_{\mathbf{k_2}}a^\dagger_{\mathbf{k_4}}a_{\mathbf{k_1}}a_{\mathbf{k_3}}$$

Express the normal-ordered operator $:\phi^2(\mathbf{x},t):$ in terms of $\mathbf{a_k}$ and $\mathbf{a}^\dagger_{\mathbf{k}}$. Show $\langle 0|\ :\phi^2(\mathbf{x},t):\ |0\rangle = 0$, and

$$\phi^2(\mathbf{x},t) =:\phi^2(\mathbf{x},t):\ +\sigma^2\mathbb{1}$$

The mode expansion for $\phi^2(\mathbf{x},t)$ is given by

$$\phi^2(\mathbf{x},t) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a_{\mathbf{k}}a_{\mathbf{k}}e^{i(k+k')\cdot x} + a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}}e^{-i(k+k')\cdot x} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}}e^{i(k-k')\cdot x} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}e^{-i(k-k')\cdot x}\right)$$

Of the 4 sets of terms composing the integrand, only the third is not normal ordered. Therefore,

$$:\phi^2(\mathbf{x},t):= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a_{\mathbf{k}}a_{\mathbf{k}}e^{i(k+k')\cdot x} + a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}}e^{-i(k+k')\cdot x} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}e^{i(k-k')\cdot x} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}e^{-i(k-k')\cdot x}\right)$$

Observe that each term in this expression has a lowering operator to the right, or a raising operator to the left. Therefore, when acting on the vaccum we have that $\langle 0|\,{:}\phi^2(\mathbf{x},t){:}\,|0\rangle = 0$.

Lastly we have seem that ${:}\phi^2(\mathbf{x},t){:}$ differs from the original $\phi^2(\mathbf{x},t)$ by just a single commutator:

$$\begin{aligned}\phi^2(\mathbf{x},t) - {:}\phi^2(\mathbf{x},t){:} &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}]e^{i(k-k')\cdot x} \\ &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(k-k')\cdot x} = \sigma\mathbb{1}\end{aligned}$$

where in the last line we quote the result from (a).

(c) Another way to introduce 'good' operators is to consider the 'smeared' field

$$\tilde{\phi}(\mathbf{x},t) := N\int d^3y\phi(\mathbf{y},t)e^{-\frac{1}{a^2}(\mathbf{x}-\mathbf{y})^2}, \qquad N = \frac{1}{\pi^{3/2}a^3}$$

The definition is motivated from the fact that the divergence in part (a) comes from having 2 ϕ's at the same spacetime point. Here we thus smear ϕ in a region of radius a. N is a normalization factor, chosen so that

$$\lim_{a\to 0}\tilde{\phi}(\mathbf{x},t) = \phi(\mathbf{x},t)$$

Show that

$$\langle 0|\tilde{\phi}(\mathbf{x},t)|0\rangle = 0$$

Now consider the fluctuations of $\tilde{\phi}$,

$$\tilde{\sigma}^2 := \langle 0|\tilde{\phi}^2(\mathbf{x},t)|0\rangle$$

Express $\tilde{\sigma}^2$ as an integral over a single variable, and show that it is finite.

We first compute the expectation, using the result from (a) that $\langle 0|\phi(\mathbf{y},t)|0\rangle = 0$.

$$\langle 0|\tilde{\phi}(\mathbf{x},t)|0\rangle = N\int d^3y e^{-\frac{1}{a^2}(\mathbf{x}-\mathbf{y})^2}\langle 0|\phi(\mathbf{y},t)|0\rangle = 0$$

Next, the variance.

$$\begin{aligned}\tilde{\sigma}^2 := \langle 0|\tilde{\phi}^2(\mathbf{x},t)|0\rangle &= N^2\int d^3\mathbf{y}d^3\mathbf{z}\frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a_\mathbf{k}a^\dagger_{\mathbf{k}'}|0\rangle e^{-(\mathbf{x}-\mathbf{y})^2/a^2}e^{-(\mathbf{x}-\mathbf{z})^2/a^2}e^{i\mathbf{k}\cdot\mathbf{y}-i\mathbf{k}'\cdot\mathbf{z}} \\ &= N^2\int d\!\!{}^-{}^3\mathbf{k}d^3\mathbf{y}d^3\mathbf{z}\frac{1}{2\omega_\mathbf{k}}e^{-(\mathbf{x}-\mathbf{y})^2/a^2}e^{-(\mathbf{x}-\mathbf{z})^2/a^2}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})} \\ &= \int\frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-\frac{1}{2}a^2\mathbf{k}^2} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{\sqrt{k^2+m^2}}e^{-\frac{1}{2}a^2k^2}\end{aligned}$$

This last integral can be evaluated in terms of hypergeometric or Bessel functions. Note to marker: student gets full credit if the integral is not evaluated. We find

$$\tilde{\sigma}^2 = \frac{1}{8a^2\pi^{3/2}}U\left(\frac{1}{2},0,\frac{a^2m^2}{2}\right) = \frac{m^2}{16\pi^2}e^{a^2m^2/4}\left(K_1(\tfrac{a^2m^2}{4}) - K_0(\tfrac{a^2m^2}{4})\right)$$

(d) Without evaluated the integral in part (c), show that in the limits of small and large a, the leading term in $\tilde{\sigma}^2$ may be written as

$$\tilde{\sigma}^2 \approx \alpha a^\gamma$$

Compute α and γ for each of these 2 limits. One should discover that at large a the average field approaches a classical variable, whereas at small a it is dominated by fluctuations.

We seek to compute the integral

$$\tilde{\sigma}^2 = \frac{1}{4\pi^2}\int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} e^{-\frac{1}{2}a^2k^2} = \frac{1}{4\pi^2 a^2}\int_0^\infty du \frac{u^2}{\sqrt{u^2+m^2a^2}} e^{-u^2/2}$$

First we expand for small a. The prefactor to the exponential can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{u\left(1+\frac{m^2a^2}{2u^2}+\cdots\right)} \approx u$$

Hence, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2 a^2}\int_0^\infty du u e^{-u^2/2} = \frac{1}{4\pi^2 a^2}$$

For large a, the prefactor to the exponential can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{ma\left(1+\frac{u^2}{2m^2a^2}+\cdots\right)} \approx \frac{u^2}{ma}$$

Hence, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2 m a^3}\int_0^\infty du u^2 e^{-u^2/2} = \frac{1}{\sqrt{32\pi^3} m a^3}$$

We see that $\tilde{\sigma}^2 \to \infty$ for small a, meaning that the field is dominated by fluctuations. On the other hand, $\tilde{\sigma}^2 \to 0$ for large a, meaning that the field behaves like a classical variable.

Question 4: Correlation Functions for a Complex Scalar (15 points)
Consider a theory of a complex scalar field ϕ which is invariant under a phase rotation of ϕ. The unitary operator U_α generating a phase rotation is

$$U_\alpha = e^{-i\alpha Q}, \qquad Q = i\int d^3x(\pi_{\phi^*}\phi^* - \pi_\phi\phi)$$

We also assume that the vacuum of the theory is invariant under the phase rotation,

$$U_\alpha|0\rangle = |0\rangle$$

The theory can be interacting. That is, in this problem, you should not use the mode expansion for ϕ discussed in lecture, which applies only to a free field theory.
(a) Show that

$$U_\alpha\phi(x)U_\alpha^\dagger = e^{i\alpha}\phi(x), \qquad U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi^*(x)$$

First consider the case where Q and $\phi(x)$ are evaluated at the same time t. Using Baker-Campbell-Hausdorff we compute

$$\begin{aligned}U_\alpha\phi(x)U_\alpha^\dagger = \phi(x) + \alpha\int d^3x'[-\phi(x')\pi_\phi(x'), \phi(x)] \\ + \frac{\alpha^2}{2!}\int d^3x'd^3x''[-\phi(x'')\pi_\phi(x''), [-\phi(x')\pi_\phi(x'), \phi(x)]] + \cdots\end{aligned}$$

Note that $[-\phi(x')\pi_\phi(x'), \phi(x)] = i\phi(x')\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, whose $\int d^3\mathbf{x}$-integral yields $i\phi(x)$. Therefore, doing this process n times for the n-th order term yields $i^n\phi(x)$, and

$$U_\alpha\phi(x)U_\alpha^\dagger = \left(1 + i\alpha + \cdots + \frac{(i\alpha)^n}{n!} + \cdots\right)\phi(x) = e^{i\alpha}\phi(x)$$

If instead, Q is computed at some time $t' = x'^0$ which is different from $t = x^0$, we can time-evolve $\phi(x)$ to the timeslice t', and use that $[H, Q] = 0$:

$$\begin{aligned}U_\alpha\phi(\mathbf{x},t)U_\alpha^\dagger &= e^{-i\alpha Q(t')}e^{iH(t-t')}\phi(\mathbf{x},t')e^{-iH(t-t')}e^{i\alpha Q(t')} \\ &= e^{iH(t-t')}e^{-i\alpha Q(t')}\phi(\mathbf{x},t')e^{i\alpha Q(t')}e^{-iH(t-t')} \\ &= e^{iH(t-t')}e^{i\alpha}\phi(\mathbf{x},t')e^{-iH(t-t')} = e^{i\alpha}\phi(\mathbf{x},t)\end{aligned}$$

where in the 3rd equality we quote the result we have just shown above.
An almost identical calculation yields

$$U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi(x)$$

(b) Show that

$$\langle 0|\phi(x)\phi(x')|0\rangle = 0$$

Using the result of part (a), along with the invariance of the vacuum, we have

$$\langle 0|\phi(x)\phi(x')|0\rangle = \langle U_\alpha 0|\phi(x)U_\alpha^\dagger U_\alpha\phi(x')U_\alpha^\dagger|0\rangle = e^{2i\alpha}\langle 0|\phi(x)\phi(x')|0\rangle$$

If we choose α not a multiple of 2π, then the equality is not satisfied unless $\langle 0|\phi(x)\phi(x')|0\rangle = 0$.

(c) Now consider a general n-point function of products of ϕ's and ϕ^*'s inserted at different spacetime points between the vacuum, i.e

$$\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle$$

Show that this vanishes whenever the numbers of ϕ's and ϕ^*'s are not the same.

Consider a general n-point function with M instances of ϕ and N instances of ϕ^*', with $n = M + N$. Similarly to (b), we have

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle &= \langle 0|U_\alpha\phi(x_1)U_\alpha^\dagger U_\alpha\cdots U_\alpha^\dagger U_\alpha\phi^*(x_i)U_\alpha^\dagger U_\alpha\cdots U_\alpha^\dagger U_\alpha\phi_(x_n)U_\alpha^\dagger|0\rangle\\ &= e^{i(M-N)\alpha}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle\end{aligned}$$

By picking again α arbitrary (e.g. irrational), we see that the equality is not satisfied for $M \neq N$ unless $\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle = 0$.
That is, the n-point function may only be non-vanishing is the numbers of ϕ's and ϕ^*'s are the same.

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 4 Solutions

March 7, 2023

Question 1: Properties of Wightman and Feynman Functions (20 points)

(a) For a free scalar field theory, using

$$U_\Lambda a_{\mathbf{k}} U_\Lambda^\dagger = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad U_\Lambda a_{\mathbf{k}}^\dagger U_\Lambda^\dagger = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}^\dagger$$

show that

$$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x)$$

We use the mode expansion for $\phi(x)$, and the transformation properties of the ladder operators:

$$\begin{aligned}
U_\Lambda \phi(x) U_\Lambda^\dagger &= U_\Lambda \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ik\cdot x} + a_{\mathbf{k}}^\dagger e^{ik\cdot x} \right) U_\Lambda^\dagger, \qquad k^0 = \omega_{\mathbf{k}} \\
&= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(\sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}} e^{-ik\cdot x} + \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}^\dagger e^{ik\cdot x} \right) \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} \sqrt{2\omega_{\Lambda\mathbf{k}}} \left(a_{\Lambda\mathbf{k}} e^{-ik\cdot x} + a_{\Lambda\mathbf{k}}^\dagger e^{ik\cdot x} \right) \\
&= \int \frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}}'} \sqrt{2\omega_{\mathbf{k}}'} \left(a_{\mathbf{k}'} e^{-ik'\cdot \Lambda x} + a_{\mathbf{k}'}^\dagger e^{ik'\cdot \Lambda x} \right) = \phi(\Lambda x)
\end{aligned}$$

where in the last line we make the change of variables $\mathbf{k}' = \Lambda\mathbf{k}$ (with $k'^0 = \omega_{\mathbf{k}'}$), and use that the measure $đ^3\mathbf{k}/2\omega_{\mathbf{k}}$ is Lorentz invariant.

(b) Now in the subsequent parts, let us consider a general interacting theory, where the mode expansion for ϕ given in lecture does not apply. But as far as the system is Lorentz and translation invariant,

$$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x), \qquad U_y \phi(x) U_y^\dagger = \phi(x+y)$$

are valid. Use this to prove that for the Wightman function $G_+(x.x') := \langle 0|\phi(x)\phi^\dagger(x')|0\rangle$, we have

$$G_+(x,x') = G_+(x-x')$$

Note that the same is true for the advanced, retarded, and Feynman two-point functions.

Using the translation-invariance of the vacuum, we have:

$$\begin{aligned}
G_+(x,x') = \langle 0|\phi(x)\phi^\dagger(x')|0\rangle &= \langle 0|U_y\phi(x)U_y^\dagger U_y \phi^\dagger(x') U_y^\dagger|0\rangle \\
&= \langle 0|\phi(x+y)\phi^\dagger(x'+y)|0\rangle = G_+(x+y, x'+y)
\end{aligned}$$

Since y is arbitrary, we can take $y = -x'$, which gives the desired result:

$$G_+(x,x') = G_+(x-x', 0) := G_+(x-x')$$

(c) Using $U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x)$ and the result of (b), show that one can write $G_+(x,x')$ as

$$G_+(x,x') = \theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)$$

where $G(y)$ is some function which satisfies

$$G(y) = G^*(y), \qquad \text{for } y > 0$$

We apply the same method in (b), except using Lorentz transformations instead of translations:

$$\begin{aligned} G_+(x,x') &= \langle 0|\phi(x)\phi^\dagger(x')|0\rangle = \langle 0|U_\Lambda \phi(x) U_\Lambda^\dagger U_\Lambda \phi^\dagger(x') U_\Lambda^\dagger|0\rangle \\ &= \langle 0|\phi(\Lambda x)\phi^\dagger(\Lambda x')|0\rangle = G_+(\Lambda(x-x')) \end{aligned}$$

Therefore, we have $G_+(x-x') = G_+(\Lambda(x-x'))$, i.e. our 2-point function is Lorentz invariant. For a given vector v^μ, there are 2 independent quantities we can build which are invariant under proper orthochronous Lorentz transformations: v^2 and $\theta(v^0)$. This follows immediately from definition. Lorentz transformations are (linear) transformations preserving a norm, and orthochronous restricts to those keeping the sign of v^0 fixed. Note that preserving parity gives no additional invariant quantity, as with 3D rotations. Therefore, we have the decomposition

$$G_+(x,x') = \theta(t-t')G_1((x-x')^2) + \theta(t'-t)G_2((x-x')^2)$$

Note further that G_1 and G_2 are not independent: complex conjugating this equation and observing $G_+(x,x') = G^*(x',x)$, we have $G_1(y) = G_2^*(y)$. Hence

$$G_+(x,x') = \theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)$$

Finally for $y = (x-x')^2 > 0$ one has $x - x'$ spacelike. Therefore, we can use an $\mathrm{SO}^+(3,1)$ transformation to go to a frame where $t' > t$, and a frame where $t > t'$. Since our expression is Lorentz-invariant, this demands the desired result,

$$G(y) = G^*(y)$$

(d) Now take ϕ to be real, and show that for the Feynman function G_F

$$G_F(x,x') = G((x-x')^2)$$

where the function G in the above equation is the same as that in part (c).

By definition, for ϕ real we have

$$G_F(x,x') = \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle = \theta(t-t')G_+(x,x') + \theta(t'-t)G_+(x',x)$$

Now we substitute our result from (c) for G_+.

$$\begin{aligned} G_F(x,x') = &\ \theta(t-t')\left(\theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)\right) \\ &+ \theta(t'-t)\left(\theta(t'-t)G((x-x')^2) + \theta(t-t')G^*((x-x')^2)\right) \\ = &\ \theta(t-t')G((x-x')^2) + \theta(t'-t)G((x-x')^2) = G((x-x')^2) \end{aligned}$$

where we use that $\theta(t-t')\theta(t'-t) = 0$, and $\theta(t-t') + \theta(t'-t) = 1$.

Question 2: Particle Production by an External Source (60 points)

Consider a free scalar field theory with external 'source' $J(x)$, whose Lagrangian density can be written

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi$$

where $\mathcal{L}_0$ is the Lagrangian density for a free scalar, and $J(x)$ is a fixed function. We assume it has the properties

$$J(t, \mathbf{x}) \to 0, \qquad t \to \pm\infty, \qquad |\mathbf{x}| \to \infty$$

and its Fourier transform

$$J(p) = \int d^4x e^{-ip\cdot x} J(x)$$

is analytic in the complex ω-plane.

Since $J(x)$ depends on time, the system does not have time translation symmetry. In particular, the vacuum at past infinity $|0, -\infty\rangle$ will be different from the one at future infinity $|0, +\infty\rangle$. Suppose we start with the vacuum state $|0, -\infty\rangle$ at $t = -\infty$. In the Heisenberg picture, the system remains in the same state $|0, -\infty\rangle$ at all times. At $t = +\infty$, the system is then not in the ground state (as $|0, -\infty\rangle \neq |0, +\infty\rangle$), and contains particle excitations. In other words, turning on a source $J(x)$ has produced particles. Below we will find the relation between $|0, -\infty\rangle$ and $|0, +\infty\rangle$, and calculate the probability of producing particles.

At $t = \mp\infty$, since $J = 0$ we have a free theory, and ϕ can be written

$$\phi(x) \to \phi_{\text{in/out}}(x) := \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left(a_{\text{in/out}}(\mathbf{k}) e^{ik\cdot x} + a^\dagger_{\text{in/out}}(\mathbf{k}) e^{-ik\cdot x} \right), \qquad t \to \mp\infty$$

where

$$[a_{\text{in/out}}(\mathbf{k}), a^\dagger_{\text{in/out}}(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

We then define the past/future vacuum as

$$a_{\text{in/out}}(\mathbf{k})|0, \mp\infty\rangle = 0, \qquad \langle 0, \mp\infty|0, \mp\infty\rangle = 1$$

Particles (at past/future infinity) can be defined by acting $a^\dagger_{\text{in/out}}$ on this vacuum. For example, and n-particle state at past/future infinity can be written as

$$|\mathbf{k}_1, \ldots, \mathbf{k}_n, \mp\infty\rangle = \prod_{i=1}^{n} \sqrt{2\omega_{\mathbf{k}_i}} a^\dagger_{\text{in/out}}(\mathbf{k}_i)|0, \mp\infty\rangle$$

with the normalization for a single particle state

$$\langle \mathbf{k}, \mp\infty|\mathbf{k}', \mp\infty\rangle = 2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Due to the presence of the external source $J(x)$, the past and future annihilation operators a_{in} and a_{out} are different. The field $\phi(t, \mathbf{x})$ with $-\infty < t < \infty$ interpolates between ϕ_{in} and ϕ_{out}.

(a) For general t, solve the classical equation of motion for this Lagrangian, and show that the solution can be written in a form

$$\phi(x) = \phi_0(x) + i \int d^4x' G(x - x') J(x') \tag{1}$$

where $\phi_0(x)$ is a solution of the homogeneous equation

$$(-\partial^2 + m^2)\phi_0(x) = 0$$

and $G(x - x')$ is a Green function satisfying

$$(-\partial^2 + m^2)G(x - x') = -i\delta^{(4)}(x - x')$$

We discussed various types of Greens functions. Which one should be used here?

The Euler-Lagrange equations pick up an extra term due from $J(x)\phi$ in the Lagrangian, and the equation of motion is given by

$$(\partial^2 - m^2)\phi = -J$$

We substitute the form for $\phi(x)$ given above into the left-hand side of equation of motion, and show it reproduces the source term.

$$\begin{aligned}(\partial^2 - m^2)\phi &= (\partial^2 - m^2)\phi_0(x) + (\partial^2 - m^2)i\int d^4x' G(x - x')J(x') \\ &= -i\int d^4x'(-\partial_x^2 + m^2)G(x - x')J(x') \\ &= -i\int d^4x'(-i)\delta^{(4)}(x - x')J(x') = -J(x)\end{aligned}$$

In the second line we use that ϕ_0 solves the homogeneous equation of motion. In the 3rd line we use the defining property of the Greens function as satisfying the inhomogeneous equation of motion with δ-source.

Here we should use the retarded Green's function, as causality requires that the field $\phi(t, \mathbf{x})$ at time t can only be influenced by the source at times $t' < t$.

(b) Now consider the quantum theory, and promote ϕ to a quantum operator, with (1) now an operator equation. Show that ϕ_0 in this equation should be given by ϕ_{in}.

We evaluate (1) as we take $t \to -\infty$, with $G = G_R$ the retarded Green's function. In this regime, $\theta(t - t') = 0$ for all t' on which $J(x)$ has support. Hence, $G_R(x - x') \propto \theta(t - t')$ also vanishes for these t'. We thus compute

$$\phi(x)|_{t\to-\infty} = \phi_0(x) + i\int d^4x' G(x - x')J(x') = \phi_0(x)$$

Finally, we know $\phi(x)|_{t\to-\infty} = \phi_{\text{in}}(x)$. Combining these equations we have $\phi_0(x) = \phi_{\text{in}}(x)$, as desired. Note that this equation only makes sense for $t \to -\infty$, since this is the domain of $\phi_{\text{in}}(x)$.

(c) Evaluate (1) at $t = +\infty$ to find the relation between $a_{\text{out}}(\mathbf{k})$ and $a_{\text{in}}(\mathbf{k})$.

Recall from class that the retarded Green's function is

$$G_R(x - x') = \theta(t - t')\int \frac{d\!\!^-{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}\right)$$

For $t \to +\infty$, for all points in the support of $J(t', \mathbf{x}')$ we have $\theta(t - t') = 1$. Using this and the result of (b), we compute

$$\begin{aligned}
\phi_{\text{out}}(x) &= \phi(x)|_{t\to\infty}(x) = \phi_{\text{in}}(x) + i\int d^4x' G(x-x')J(x') \\
&= \phi_{\text{in}}(x) + i\int d^4x' \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}\right) J(x') \\
&= \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\text{in}}(\mathbf{k})e^{ik\cdot x} + a_{\text{in}}^\dagger(\mathbf{k})e^{-ik\cdot x}\right) + i\int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot x}J(k) - e^{-ik\cdot x}J(-k)\right) \\
&= \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[\left(a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)e^{ik\cdot x} + \left(a_{\text{in}}^\dagger(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)e^{-ik\cdot x}\right]
\end{aligned}$$

Note that the 4-momenta are implicitly evaluated at $k = (\omega_{\mathbf{k}}, \mathbf{k})$, which is what we mean when we write $J(k) = J(\mathbf{k})$. In the last line, we also use that $J(x)$ is real for a real scalar, so its Fourier transform satisfies $J(-k) = J^*(k)$.

Finally, we can compare this to the mode expansion

$$\phi_{\text{out}}(x) := \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\text{out}}(\mathbf{k})e^{ik\cdot x} + a_{\text{out}}^\dagger(\mathbf{k})e^{-ik\cdot x}\right)$$

to relate the incoming and outgoing ladder operators,

$$a_{\text{out}} = a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}, \qquad a_{\text{out}}^\dagger = a_{\text{in}}^\dagger(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}$$

(d) Using (c), show that the expectation value λ for the total number of particles produced is given by

$$\lambda = \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}|J(k)|^2$$

The expectation value for the total number of particles produced is given by taking the expectation of the outgoing number operator with respect to our state. In the Heisenberg picture our state is constant, and is just $|0, -\infty\rangle$. Therefore,

$$\begin{aligned}
\lambda &= \langle 0, -\infty| \int d\!\!{}^{-}\!{}^3\mathbf{k}\, a_{\text{out}}^\dagger(\mathbf{k})a_{\text{out}}(\mathbf{k})|0, -\infty\rangle \\
&= \int d\!\!{}^{-}\!{}^3\mathbf{k}\langle 0, -\infty|\left(a_{\text{in}}^\dagger(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)\left(a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)|0, -\infty\rangle \\
&= \int d\!\!{}^{-}\!{}^3\mathbf{k}\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\langle 0, -\infty|0, -\infty\rangle = \int \frac{d\!\!{}^{-}\!{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}|J(\mathbf{k})|^2
\end{aligned}$$

(e) Show that we can write

$$a_{\text{out}}(\mathbf{k}) = S^\dagger a_{\text{in}}(\mathbf{k})S$$

with S a unitary operator given by

$$S = e^{iB}, \qquad B = \int d^4x J(x)\phi_{\text{in}}(x)$$

We compute this using Baker-Campbell-Hausdorff:

$$S^\dagger a_{\text{in}}(\mathbf{k})S = e^{-iB}a_{\text{in}}(\mathbf{k})e^{iB} = a_{\text{in}}(\mathbf{k}) - i[B, a_{\text{in}}(\mathbf{k})] - \frac{1}{2}[B,[B,a_{\text{in}}(\mathbf{k})]] + \cdots$$

This series terminates after the second term, since $[B, a_{\text{in}}(\mathbf{k})]$ is a c-number:

$$[B, a_{\text{in}}(\mathbf{k})] = \int d^4x J(x)[\phi_{\text{in}}(x), a_{\text{in}}(\mathbf{k})] = \int d^4x J(x) \int \frac{đ^3k'}{\sqrt{2\omega_{\mathbf{k}}}}[a^\dagger_{\text{in}}(\mathbf{k}'), a_{\text{in}}(\mathbf{k})]e^{-ik'\cdot x}$$
$$= -\frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \int d^4x J(x) e^{-ik\cdot x} = -\frac{1}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k})$$

Therefore,

$$S^\dagger a_{\text{in}}(\mathbf{k}) S = a_{\text{in}}(\mathbf{k}) + \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k}) = a_{\text{out}}(\mathbf{k})$$

(f) Use the result in part (e) to show that

$$S|0, +\infty\rangle = |0, -\infty\rangle, \qquad S|\mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty\rangle$$

This indicates that S is in fact the S-matrix operator of the system, i.e. for free theory states $|\alpha\rangle$, $|\beta\rangle$

$$S_{\beta\alpha} := \langle \beta, +\infty | \alpha, -\infty\rangle = \langle \beta, -\infty | S | \alpha, -\infty\rangle$$

To show that the state $S|0, +\infty\rangle$ is the incoming vacuum, we need to show that it is annihilated by all $a_{\text{in}}(\mathbf{k})$, and has the correct normalization. We compute

$$a_{\text{in}}(\mathbf{k}) S|0, +\infty\rangle = SS^\dagger a_{\text{in}}(\mathbf{k}) S|0, +\infty\rangle = S a_{\text{out}}(\mathbf{k})|0, +\infty\rangle = 0$$

Furthermore, S defined in part (e) is unitary (as both J and ϕ are real), so

$$\langle 0, +\infty | S^\dagger S | 0, +\infty\rangle = \langle 0, +\infty | 0, +\infty\rangle = 1$$

Therefore $S|0, +\infty\rangle$ can only differ from $|0, -\infty$ by a phase, which we can absorb into our definition of S. We have shown $S|0, +\infty\rangle = |0, -\infty\rangle$.

Now we show the desired equality for n-particle states:

$$\begin{aligned} S|\mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty\rangle &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} S a^\dagger_{\text{out}}(\mathbf{k}_1) \cdots a^\dagger_{\text{out}}(\mathbf{k}_n)|0, +\infty\rangle \\ &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} SS^\dagger a^\dagger_{\text{in}}(\mathbf{k}_1) S \cdots S^\dagger a^\dagger_{\text{in}}(\mathbf{k}_n) S|0, +\infty\rangle \\ &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} a^\dagger_{\text{in}}(\mathbf{k}_1) \cdots a^\dagger_{\text{in}}(\mathbf{k}_n)|0, -\infty\rangle \\ &= |\mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty\rangle \end{aligned}$$

In the 3rd line we use both $S|0, +\infty\rangle = |0, -\infty\rangle$ and the result in (e).

(g) In the following parts we will compute the probability of computing n particles. Before doing that, in this part we develop a technical tool to make that task easier. Show that we can write S as

$$S = e^{iB} = e^F e^G e^{-\lambda/2}$$

with λ as in (d), and

$$F = i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k}) a^\dagger_{\text{in}}(\mathbf{k}), \qquad G = i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} J(-\mathbf{k}) a_{\text{in}}(\mathbf{k})$$

Again we use the Baker-Campbell-Hausdorff formula, which in one of its forms states that

$$e^F e^G = \exp\left(F + G + \frac{1}{2}[F, G] + \frac{1}{12}[F, [F, G]] - \frac{1}{12}[G, [F, G]] + \cdots\right)$$

We first compute the commutator of F and G:

$$[F,G] = -\int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} J(\mathbf{k})J(-\mathbf{k}')[a_{\text{in}}^\dagger(\mathbf{k}), a_{\text{in}}(\mathbf{k}')]$$
$$= \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} J(\mathbf{k})J(-\mathbf{k}) = \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} |J(\mathbf{k})|^2 = \lambda$$

In the last line we use that $J(x)$ is real, so $J(-k) = J^*(k)$. This is a c-number, so all further commutators with operators vanish. The Baker-Campbell-Hausdorff formula reduces to

$$e^F e^G = e^{F+G+\lambda/2} \quad \Rightarrow \quad e^F e^G e^{-\lambda/2} = e^{F+G}$$

It remains to compute the right-hand side. We have

$$F + G = i\int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(J(\mathbf{k})a_{\text{in}}^\dagger(\mathbf{k}) + J(-\mathbf{k})a_{\text{in}}(\mathbf{k})\right)$$
$$= i\int d^4x J(x) \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(e^{-ik\cdot x}a_{\text{in}}^\dagger(\mathbf{k}) + e^{ik\cdot x}a_{\text{in}}(\mathbf{k})\right) = i\int d^4x J(x)\phi_{\text{in}}(x) = iB$$

Putting everything together, we have the desired result,

$$e^F e^G e^{-\lambda/2} = e^{iB} = S$$

(h) Use the results of (f) and (g) to find the vacuum to vacuum probability,

$$P_0 = |\langle 0, +\infty|0, -\infty\rangle|^2$$

P_0 is the probability of no particle production.

We compute

$$P_0 = |\langle 0, +\infty|0, -\infty\rangle|^2 = |\langle 0, -\infty|(S^\dagger)^\dagger|0, -\infty\rangle|^2 = e^{-\lambda}|\langle 0, -\infty|e^F e^G|0, -\infty\rangle|^2$$

Note that G is a linear combination of a_{in}'s, so $e^G|0,-\infty\rangle = e^0|0,-\infty\rangle = |\infty\rangle$. Similarly, F is a linear combination of $a_{\text{in}}^\dagger$'s, so $\langle 0,-\infty|e^F = \langle 0, -\infty|$. We thus find

$$P_0 = e^{-\lambda}|\langle 0, -\infty|0, -\infty\rangle|^2 = e^{-\lambda}$$

(i) Use the results of (f) and (g) to show that

$$\langle \mathbf{k}_1, \dots, \mathbf{k}_n, +\infty|0, -\infty\rangle = i^n J(\mathbf{k}_1)\cdots J(\mathbf{k}_n)e^{-\lambda/2}$$

We compute:

$$\langle \mathbf{k}_1, \dots, \mathbf{k}_n, +\infty|0, -\infty\rangle = \langle \mathbf{k}_1, \dots, \mathbf{k}_n, -\infty|S|0, -\infty\rangle = e^{-\lambda/2}\langle \mathbf{k}_1, \dots, \mathbf{k}_n, -\infty|e^F e^G|0, -\infty\rangle$$
$$= e^{-\lambda/2}\sqrt{2\omega_{\mathbf{k}_1}}\cdots\sqrt{2\omega_{\mathbf{k}_n}}\langle 0, -\infty|a_{\text{in}}(\mathbf{k}_1)\cdots a_{\text{in}}(\mathbf{k}_n)e^F|0, -\infty\rangle$$
$$= \frac{e^{-\lambda/2}}{n!}\sqrt{2\omega_{\mathbf{k}_1}}\cdots\sqrt{2\omega_{\mathbf{k}_n}}\langle 0, -\infty|a_{\text{in}}(\mathbf{k}_1)\cdots a_{\text{in}}(\mathbf{k}_n)F^n|0, -\infty\rangle$$

In line 1, we use from (f) that $S|\mathbf{k}_1, \dots, \mathbf{k}_n, +\infty\rangle = |\mathbf{k}_1, \dots, \mathbf{k}_n, -\infty\rangle$, and the decomposition of S in (g). In line 2, we use from (b) that $e^G|0, -\infty\rangle = |\infty\rangle$. In line 3, we Taylor expand e^F explicitly, and keep only the term F^n, as all other terms vanish (a term vanishes unless the number of creation operators equals the number of annihilation operators).

To finish the computation, we note that

$$\begin{aligned}
a_{\text{in}}(\mathbf{k}_n)F^n &= F^n a_{\text{in}}(\mathbf{k}_n) + [a_{\text{in}}(\mathbf{k}_n), F^n] \\
&= F^n a_{\text{in}}(\mathbf{k}_n) + i^n \int \frac{đ^3\mathbf{k}'_1}{\sqrt{2\omega_{\mathbf{k}'_1}}} \cdots \frac{đ^3\mathbf{k}'_n}{\sqrt{2\omega_{\mathbf{k}'_n}}} J(\mathbf{k}'_1)\cdots J(\mathbf{k}'_n)[a_{\text{in}}(\mathbf{k}_n), a_{\text{in}}(\mathbf{k}'_1)\cdots a_{\text{in}}(\mathbf{k}'_n)] \\
&= F^n a_{\text{in}}(\mathbf{k}_n) + n\frac{iJ(\mathbf{k}_n)}{\sqrt{2\omega_{\mathbf{k}_n}}} F^{n-1}
\end{aligned}$$

acting both sides on the state $|0,-\infty\rangle$, the first term on the right vanishes. We can use this result to simplify our previous equation:

$$\begin{aligned}
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty|0,-\infty\rangle &= \frac{e^{-\lambda/2}}{n!}\sqrt{2\omega_{\mathbf{k}_1}}\cdots\sqrt{2\omega_{\mathbf{k}_n}}\langle 0,-\infty|a_{\text{in}}(\mathbf{k}_1)\cdots a_{\text{in}}(\mathbf{k}_n)F^n|0,-\infty\rangle \\
&= \frac{e^{-\lambda/2}}{(n-1)!}\sqrt{2\omega_{\mathbf{k}_1}}\cdots\sqrt{2\omega_{\mathbf{k}_{n-1}}} iJ(\mathbf{k}_n)\langle 0,-\infty|a_{\text{in}}(\mathbf{k}_1)\cdots a_{\text{in}}(\mathbf{k}_{n-1})F^{n-1}|0,-\infty\rangle
\end{aligned}$$

By iterating this step n times in total, we obtain the desired result,

$$\begin{aligned}
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty|0,-\infty\rangle &= i^n J(\mathbf{k}_1)\cdots J(\mathbf{k}_n)e^{-\lambda/2}\langle 0,-\infty|0,-\infty\rangle \\
&= i^n J(\mathbf{k}_1)\cdots J(\mathbf{k}_n)e^{-\lambda/2}
\end{aligned}$$

(j) The probability dP of finding exactly n particles, with one particle each in the ranges $d^3\mathbf{k}_i$ around $\mathbf{k}_i$, $1 \le i \le n$, can be shown to be

$$dP = |\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty|0,-\infty\rangle|^2 \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}}$$

Use this formula to show that the probability of finding exactly n particles is

$$P_n = e^{-\lambda}\frac{\lambda^n}{n!}$$

This is a Poisson distribution with average particle number λ.

We compute

$$\begin{aligned}
P_n &= \frac{1}{n!}\int \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}} |\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty|0,-\infty\rangle|^2 = \frac{1}{n!}\int \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}} |J(\mathbf{k}_1)\cdots J(\mathbf{k}_n)|^2 e^{-\lambda} \\
&= \frac{1}{n!}e^{-\lambda}\left(\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}|J(\mathbf{k})|^2\right)^n = e^{-\lambda}\frac{\lambda^n}{n!}
\end{aligned}$$

Note that the integration over phase space contains a factor of $1/n!$ to avoid overcounting identical particles. In the second equality we use the result from (i).

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 5 Solutions

March 14, 2023

Question 1: A Useful Formula, and the Path Integral in Phase Space (20 points)

(a) Derive the following equations:

$$\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle = \int đp_i \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i)\right]$$
$$= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im\Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right]$$

We start by deriving the first equation:

$$\begin{aligned}
\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle &= \int dp_i \langle x_{i+1}| e^{-i\frac{\hat{p}^2}{2m}\Delta t} |p_i\rangle\langle p_i| e^{-i\Delta t V(\hat{x})} |x_i\rangle \\
&= \int dp_i e^{-i\frac{p_i^2}{2m}\Delta t} e^{-i\Delta t V(x_i)} \langle x_{i+1}|p_i\rangle\langle p_i|x_i\rangle \\
&= \int đp_i e^{-i\frac{p_i^2}{2m}\Delta t} e^{-i\Delta t V(x_i)} e^{ip_i x_{i+1}} e^{-ip_i x_i} \\
&= \int đp_i \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i)\right]
\end{aligned}$$

To get the second equation from this, we complete the square and compute the Gaussian integral, or equivalently use the identity $\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$. Hence,

$$\begin{aligned}
\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle &= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\Delta t}} \exp\left[-\frac{(x_{i+1} - x_i)^2}{2i\Delta t/m}\right] e^{-i\Delta t V(x_i)} \\
&= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im\Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right]
\end{aligned}$$

(b) Use the first equation in (a) to derive the following:

$$\langle x_a, t_a | x_b, t_b \rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t) \exp\left[i\int_{t'}^{t} dt(p\dot{x} - H)\right]$$

The integration in this expression should be understood as

$$\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t) = \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \int \frac{dx_1 dp_1}{2\pi} \cdots \int \frac{dx_{N-1}dp_{N-1}}{2\pi}$$

where we again divide the interval $[t_b, t_a]$ into N segments, with $t_0 = t_b$, $t_N = t_a$.

We denote the left-hand side of the equation in (a) by M_i. Now we do the same trick from class to derive the path-integral:

$$\begin{aligned}
\langle x_a, t_a | x_b, t_b \rangle &= \langle x_a | e^{-i\hat{H}(t_a - t_b)} | x_b \rangle = \langle x_a | (e^{-i\hat{H}\Delta t})^N | x_b \rangle \\
&= \int dx_1 \cdots dx_{N-1} \langle x_a | e^{-i\hat{H}\Delta t} | x_{N-1} \rangle \langle x_{N-1} | \cdots | x_1 \rangle \langle x_1 | e^{-i\hat{H}\Delta t} | x_b \rangle \\
&= \int dx_1 \cdots dx_{N-1} M_{N-1} \cdots M_0 \\
&= \int dx_1 \cdots dx_{N-1} đp_0 \cdots đp_{N-1} \prod_{i=0}^{N-1} \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + i p_i (x_{i+1} - x_i) \right] \\
&= \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \exp\left[i \sum_{i=0}^{N-1} \Delta t \left(p_i \frac{x_{i+1} - x_i}{\Delta t} - \frac{p_i^2}{2m} - V(x_i) \right) \right] \\
&= \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \exp\left[i \int_{t_b}^{t_a} dt (p\dot{x} - H) \right]
\end{aligned}$$

In the second line we insert the identity $N-1$ times, so that we can use the formula from (a) for each matrix element in the 4th line. In the 5th line the product of exponentials becomes a sum of exponents, which in the continuum case reduces to the integral in the last line.

Question 2: The Schrödinger Equation, Rederived (20 points)

The wavefunction $\psi(t,x)$ for a system at time t can be obtained from that at time t' by

$$\psi(t,x) = \int dx' K(x,t;x',t')\psi(x',t')$$

for the propagator given by

$$K(x,t;x',t') = \langle x,t|x',t'\rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)\exp\left[i\int_{t'}^{t} dt L(\dot{x},x)\right]$$

Show that $\psi(t,x)$ satisfies the Schrödinger equation,

$$i\partial_t\psi(t,x) = -\frac{1}{2m}\partial_x^2\psi(t,x) + V(x)\psi(t,x)$$

We therefore consider the wavefunction after an infinitesimal time step δt:

$$\psi(t+\delta t,x) = \int dy K(t+\delta t,x;t,y)\psi(t,y)$$

Equating both sides of this equation to order δ will lead to the Schrödinger equation.
The left hand side is simple:

$$\text{LHS} = \psi(t+\delta t,x) = \psi(t,x) + \delta t\partial_t\psi(t,x) + \mathcal{O}(\delta t^2)$$

To expand the right hand side, note for an infinitesimal δt that the propagator $K(t+\delta t,x;t,y)$ can be written as a single infinitesimal time step:

$$K(t+\delta t,x;t,y) = \left(\frac{m}{2\pi i\delta t}\right)^{1/2}\exp\left[i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)\right]$$

Therefore we expand:

$$\begin{aligned}
\text{RHS} &= \int dy K(t+\delta t,x;t,y)\psi(t,y)\\
&= \left(\frac{m}{2\pi i\delta t}\right)^{1/2}\int dy e^{i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)}\psi(t,y)\\
&= \left(\frac{m}{2\pi i}\right)^{1/2}\int du e^{i\left(\frac{m}{2}u^2 - \delta t V(x+u\sqrt{\delta t})\right)}\psi(t,x+u\sqrt{\delta t})\\
&= \left(\frac{m}{2\pi i}\right)^{1/2}\int du e^{i\frac{m}{2}u^2}(1-\delta t V(x))\left(\psi(t,x) + \sqrt{\delta t}\partial_x\psi(t,x)u + \frac{\delta t}{2}\partial^2 x\psi(t,u)u^2\right) + \mathcal{O}(\delta t^2)\\
&= \psi(t,x) + \delta t\left(-iV(x)\psi(t,x) + \frac{i}{2m}\partial_x^2\psi(t,x)\right) + \mathcal{O}(\delta t^2)
\end{aligned}$$

In the third line we have made the substitution $u\sqrt{\delta t} = y - x$, and in the 4th line we expand each term to order δt, neglecting higher order terms. In the subsequent line we perform the Gaussian integrals over u, noting that the order $(\delta t)^{1/2}$ term becauyse the integral is odd.

Setting the 2 sides equal, we have the Schrödinger equation as desired:

$$i\partial_t\psi(t,x) = \left(-\frac{1}{2m}\partial_x^2 + V(x)\right)\psi(t,x)$$

Question 3: The Free Particle (20 points)

For a free particle, i.e. $V(x) = 0$, perform explicitly the integrals over x_i, $i = 1, \ldots, N-1$ to show that

$$K(x_a, t_a; x_b, t_b) = \left(\frac{m}{2\pi i(t_a - t_b)}\right)^{1/2} \exp\left[\frac{im(x_a - x_b)^2}{2(t_a - t_b)}\right]$$

We evaluate the integrals in order, starting with the one over x_1. We denote $x_0 = x_b$ and $x_N = x_a$.

$$\begin{aligned}
K(x_a, t_a; x_b, t_b) &= \lim_{N\to\infty} \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_2 - x_1)^2 + (x_1 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \left(\frac{i\pi\Delta t}{m}\right)^{1/2} \int dx_2 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\Delta t}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{2}} \int dx_2 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\Delta t}\right)^{\frac{N-1}{2}} \frac{2}{\sqrt{2}} \left(\frac{i\pi\Delta t}{3m}\right)^{1/2} \int dx_3 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + \frac{1}{3}(x_3 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\Delta t}\right)^{\frac{N-2}{2}} \frac{1}{\sqrt{3}} \int dx_3 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_4 - x_3)^2 + \frac{1}{3}(x_3 - x_0)^2)} \\
&= \cdots = \lim_{N\to\infty} \left(\frac{m}{2\pi i N\Delta t}\right)^{1/2} e^{i\frac{m}{2N\Delta t}(x_a - x_b)^2} \\
&= \sqrt{\frac{m}{2\pi i N(t_a - t_b)}} e^{i\frac{m}{2}\frac{(x_a - x_b)^2}{t_a - t_b}}
\end{aligned}$$

Question 4: The Path Integral for a Free Particle, Revisited (20 points)

In this problem we evaluate the path integral for a free particle using a different method than Problem 3. For simplicity, take $x_a = x_b = 0, t_b = 0, t_a = T$. As discussed in class, K is a Gaussian path-integral of form:

$$K(0,T;0,0) = \int_{x(0)=0}^{x(T)=0} Dx(t) \exp\left[\frac{i}{2}\int dtdt' x(t)A(t,t')x(t')\right]$$

for some differential operator A.

(a) Write down the explicit expression for A.

The action for a free particle is given by

$$\begin{aligned} S[x(t)] &= \int dt \frac{m}{2}\partial_t x \partial_t x = -\frac{m}{2}\int dt x \partial_t^2 x = -\frac{m}{2}\int dtdt' x(t')\delta(t-t')\partial_t^2 x(t) \\ &= \frac{1}{2}\int dtdt' x(t')A(t',t)x(t), \qquad\qquad A(t',t) = -m\delta(t-t')\partial_t^2 \end{aligned}$$

Then, we write the propagator as

$$K(0,T;0,0) = \int_{x(0)=0}^{x(T)=0} Dx e^{iS} = \int_{x(0)=0}^{x(T)=0} Dx \exp\left[\frac{i}{2}\int_0^T dtdt' x(t')A(t',t)x(t)\right]$$

for the differential operator $A(t',t) = -m\delta(t-t)\partial_t^2$.

(b) Find all the eigenvalues of A. Show that the determinant of A can be written as

$$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2}$$

The eigenvectors of a second order derivative operator are precisely exponentials, $x(t) = f_\lambda(t) = e^{i\lambda t}$. We further need our eigenvectors to satisfy the boundary conditions $x(0) = x(T) = 0$, since the endpoints of our trajectory are fixed. Hence, a complete set of eigenvectors are given by sine functions, with momenta integer multiples of π/T. Without loss of generality we can restrict to $n > 0$.

$$f_{\lambda_n}(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{n\pi t}{T}\right)$$

Furthermore,

$$\int dt A(t',t) f_{\lambda_n}(t) = -m\int dt \delta(t-t')\sqrt{\frac{2}{T}}\partial_t^2 \sin\left(\frac{n\pi t}{T}\right) = m\frac{n^2\pi^2}{T^2}\sqrt{\frac{2}{T}}\sin\left(\frac{n\pi t}{T}\right) = \lambda_n f_{\lambda_n}(t)$$

for the eigenvalue $\lambda_n = m\frac{n^2\pi^2}{T^2}$. In this basis A is diagonal: $A_{mn} = m\frac{n^2\pi^2}{T^2}\delta_{nm}$. Finally, the determinant of an operator is obtained by multiplying all of its eigenvalues, giving us the desired expression:

$$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2}$$

(c) Since our propagator is Gaussian, it can be evaluated as

$$K(0,T;0,0) = \frac{C}{\sqrt{\det A}}$$

where C is some constant. By comparing this equation with the solution to Problem 3, show that the consistency of the 2 approaches requires

$$\frac{C}{\sqrt{\det A}} = \left(\frac{m}{2\pi i T}\right)^{1/2}$$

We substitute $x_a = x_b = 0, t_a = T, t_b = 0$ into the result in Problem 3:

$$K(0,T;0,0) = \left(\frac{m}{2\pi i T}\right)^{1/2} \exp\left[\frac{im0^2}{2T}\right] = \left(\frac{m}{2\pi i T}\right)^{1/2}$$

It therefore follows immediately from our work in (b) that

$$K(0,T;0,0) = \left(\frac{m}{2\pi i T}\right)^{1/2} = \frac{C}{\sqrt{\det A}} = \frac{CT}{\pi\sqrt{m}} \prod_{n=1}^{\infty} \frac{1}{n}$$

8.323 Problem Set 6 Solutions

March 21, 2023

Question 1: Particle Production by an External Source, Continued (10 points)

Consider again Problem 2 of Problem Set 4. Introduce

$$Z[J] = \int D\phi e^{i\int d^4x\mathcal{L}}, \qquad Z_0 = Z[J=0] = \int D\phi e^{i\int d^4x\mathcal{L}_0}$$

for the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi$$

We can show that, with an appropriate $i\epsilon$ prescription,

$$\langle 0, +\infty | 0. -\infty\rangle = \frac{Z[J]}{Z_0}$$

Use this to find the probability of no particle production

$$P_0 = |\langle 0, +\infty | 0, -\infty\rangle|^2$$

by directly evaluating the path integral. You should reproduce your answer in 2(h) of Problem Set 4.

We compute:

$$\begin{aligned}
Z[J] &= \int D\phi e^{i\int d^4x(\mathcal{L}_0 + J\phi)} \\
&= \int D\phi \exp\left[-\frac{i}{2}\int đ^4p \left(\phi^\dagger(p)(p^2+m^2-i\epsilon)\phi(p) - J^\dagger(p)\phi(p) - J(p)\phi^\dagger(p)\right)\right] \\
&= Z_0 \exp\left[\frac{i}{2}\int đ^4p \frac{|J(p)|^2}{p^2+m^2-i\epsilon}\right]
\end{aligned}$$

In the last line, we complete the square and perform a linear shift of ϕ, $\phi^\dagger$. Therefore,

$$\begin{aligned}
P_0 = \left|\frac{Z[J]}{Z[0]}\right|^2 &= \exp\left[i\int đ^4p|J(p)|^2 \ \mathrm{Im}\frac{1}{p^2+m^2-i\epsilon}\right] \\
&= \exp\left[-\pi\int đ^4p|J(p)|^2\delta(p^2+m^2)\right] \\
&= \exp\left[-\pi\int đ^4p|J(p)|^2\frac{1}{2\omega_{\mathbf{p}}}(\delta(p^0-\omega_{\mathbf{p}})+\delta(p^0+\omega_{\mathbf{p}}))\right] \\
&= \exp\left[-\pi\int \frac{đ^3\mathbf{p}}{2\pi}\left(|J(\omega_{\mathbf{p}},\mathbf{p})|^2 + |J(-\omega_{\mathbf{p}},\mathbf{p})|^2\right)\right] \\
&= \exp\left[-\int\frac{đ^3\mathbf{p}}{2\omega_{\mathbf{p}}}|J(p)|^2\right] = e^{-\lambda}
\end{aligned}$$

In the line 1, we use that $a - a^* = 2i\text{Im}(a)$. In line 2, we use the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{x - i\epsilon} = \text{PV}\left(\frac{1}{x}\right) + i\pi\delta(x)$$

where PV is the Cauchy principal value. Line 3 uses the identity from Problem Set 1 that

$$\delta(p^2 + m^2) = \frac{1}{2\omega_{\mathbf{p}}}(\delta(p^0 - \omega_{\mathbf{p}}) + \delta(p^0 + \omega_{\mathbf{p}}))$$

In line 4 we perform the p^0 integral. Finally, in line 5 we use the invariance of the $\int đ^3\mathbf{p}$ under $\mathbf{p} \to -\mathbf{p}$, along with $J(-p) = J^*(p)$ to show that the second term is equal to the first.

Question 2: Connected Diagrams (30 points)

Consider the $\lambda\phi^4$ theory discussed in lecture, with interaction Hamiltonian

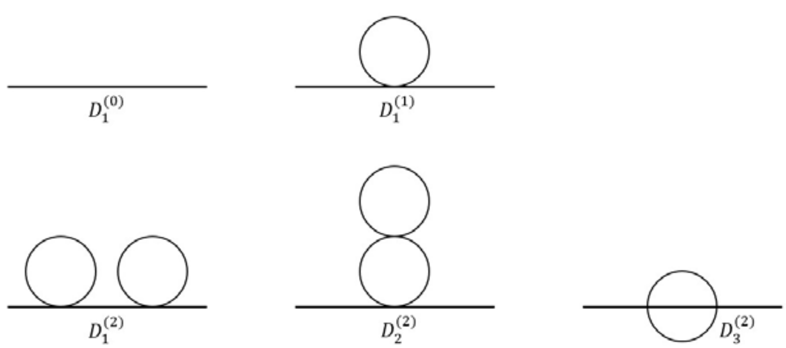
$$H_I = \frac{\lambda}{4!}\int d^3x\phi^4(x)$$

(a) List all connected diagrams for

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty} dtH_I}|0\rangle$$

to order $\mathcal{O}(\lambda^2)$, and give the symmetry factor for each diagram. For diagrams at orders $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\lambda^1)$, write down their expressions in both coordinate and momentum space.

Order	Diagram	S	Expressions
λ^0	$D_1^{(0)}$	1	$G_F^0(x_1-x_2)$ $(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}$
λ^1	$D_1^{(1)}$	2	$-\lambda\int d^4zG_F^0(x_1-z)G_F^0(x_2-z)G_F^0(0)$ $-i\lambda(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\int \bar{d}^4q\frac{-i}{q^2+m^2-i\epsilon}$
λ^2	$D_1^{(2)}$	2×2	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)G_F^0(0)^2$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\left(\int \bar{d}^4q\frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$D_2^{(2)}$	2×2	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)^2G_F^0(0)$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_2^2+m^2-i\epsilon}\int \bar{d}^4q\left(\frac{-i}{q^2+m^2-i\epsilon}\right)^2\int \bar{d}^4q\frac{-i}{q^2+m^2-i\epsilon}$
λ^2	$D_3^{(2)}$	$3!$	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)^3$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_2^2+m^2-i\epsilon}\int \bar{d}^4q_1\bar{d}^4q_2\frac{-i}{q_1^2+m^2-i\epsilon}\frac{-i}{q_2^2+m^2-i\epsilon}\frac{-i}{(q_1+q+q')^2+m^2-i\epsilon}$



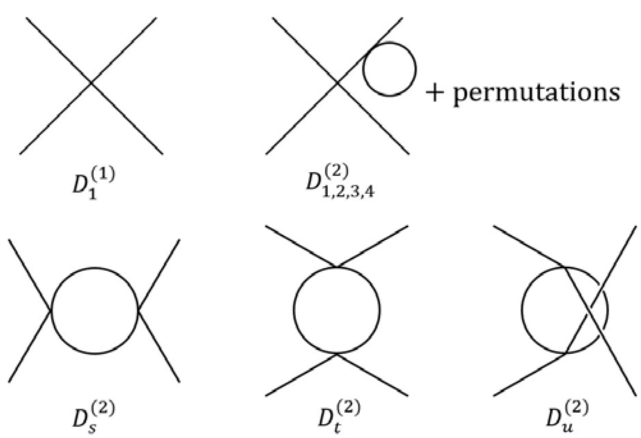
The symmetry factors for the order λ^2 diagrams are obtained as follows. $D_1^{(2)}$ receives a factor of 2 from each of the propagators starting and ending on the same point (x_1 and x_2). $D_2^{(2)}$ has one factor of 2 from the propagator starting and ending on the same point, and another factor of 2 from the 2 identical propagators connecting x_1 and x_2. $D_3^{(2)}$ has 3! from the 3 identical propagators connecting x_1 and x_2.

(b) List all connected diagrams of the four-point function

$$G_4(x_1, x_2, x_3, x_4) = \lambda\Omega|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle$$

to order $\mathcal{O}(\lambda^2)$, and choose 2 to write down their expressions in coordinate and momentum space. Here we take all momenta to be ingoing, and use the shorthand $G^0_F(x-y) =: G^0_{x,y}$.

Order	Diagram	S	Expressions
λ^1	$D_1^{(0)}$	1	$-i\lambda \int d^4z G^0_F(x_1-z)G^0_F(x_2-z)G^0_F(x_3-z)G^0_F(x_4-z)$
			$-i\lambda(2\pi)^4\delta^{(4)}(\sum_i p_i)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\frac{-i}{p_3^2+m^2-i\epsilon}\frac{-i}{p_4^2+m^2-i\epsilon}$
λ^2	$\sum_{i=1}^{4} D_i^{(2)}$	2	$(-i\lambda)^2\int d^4z_1d^4z_2G^0_{x_1,z_2}G^0_{z_1,z_2}G^0_{0,0}G^0_{x_2,z_1}G^0_{x_3,z_1}G^0_{x_4,z_1} + \sum_{i=2}^{4}(1\leftrightarrow i)$
			$-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\frac{-i}{p_1^2+m^2-i\epsilon}\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon} + \sum_{i=2}^{4}(1\leftrightarrow i)$
λ^2	$D_s^{(2)}$	2	$(-i\lambda)^2\int d^4z_1d^4z_2G^0_{x_1,z_1}G^0_{x_2,z_1}(G^0_{z_1,z_2})^2G^0_{x_3,z_2}G^0_{x_4,z_2}$
			$-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_2-q)^2+m^2-i\epsilon}$
λ^2	$D_t^{(2)}$	2	$(-i\lambda)^2\int d^4z_1d^4z_2G^0_{x_1,z_1}G^0_{x_3,z_1}(G^0_{z_1,z_2})^2G^0_{x_2,z_2}G^0_{x_4,z_2}$
			$-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_3-q)^2+m^2-i\epsilon}$
λ^2	$D_u^{(2)}$	2	$(-i\lambda)^2\int d^4z_1d^4z_2G^0_{x_1,z_1}G^0_{x_4,z_1}(G^0_{z_1,z_2})^2G^0_{x_2,z_2}G^0_{x_3,z_2}$
			$-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_4-q)^2+m^2-i\epsilon}$



For the $D_i^{(2)}$ diagrams, the symmetry factor of 2 comes from the propagator starting and ending on the same internal vertex. For the $D_{s,t,u}^{(2)}$ diagrams, it comes from the 2 identical propagators connecting x_1 and x_2.

Question 3: Vacuum Diagrams (30 points)
For a $\lambda\phi^4$ theory, consider the quantity

$$Z_0 = \langle 0|\mathrm{T}e^{-i\int_\infty^\infty dtH_I}|0\rangle$$

where the expectation is evaluated in the free theorem. We also assume the free theory vacuum $|0\rangle$ is properly normalized, i.e. $\langle 0|0\rangle = 1$.

(a) Consider

$$W_0 = \log Z_0$$

Show that W_0 can be written in a form

$$W_0 = \mathrm{cst} - i\epsilon VT$$

where cst is a constant independent of the spacetime volume, ϵ is the energy difference between the full and free theories, and VT is the total spacetime volume.

Method 1

We can write the path-integral as a ratio of matrix elements:

$$Z_0 = \langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle = \frac{\int D\phi e^{i(S_0+S_I)}}{\int D\phi e^{iS_0}} = \frac{\langle\phi = 0, \infty|\phi = 0, -\infty\rangle_\Omega}{\langle\phi = 0, \infty|\phi = 0, -\infty\rangle_0}$$

We first compute the numerator by inserting complete sets of eigenstates of H, at very early and late times. We further take $H \to H(1 - i\epsilon)$ to make the expression convergent.

$$\begin{aligned}\langle\phi = 0, \infty|\phi = 0, T/2\rangle_\Omega &= \lim_{T\to\infty}\sum_{n,m}\langle\phi = 0, \infty|n, T/2\rangle\langle n, T/2|m, -T/2\rangle\langle m, -T/2|\phi = 0, -T/2\rangle_\Omega \\ &= \lim_{T\to\infty}\sum_{n,m}\langle\phi = 0, \infty|n, T/2\rangle e^{-iE_m(1-i\epsilon)T}\delta_{n,m}\langle m, -T/2|\phi = 0, -T/2\rangle_\Omega\end{aligned}$$

For very large T, the dominant contribution to this sum is that with lowest E_m, i.e. vacuum E_Ω. Hence,

$$\langle\phi = 0, \infty|\phi = 0, T/2\rangle_\Omega = \lim_{T\to\infty}\Psi_\Omega[\phi = 0]\Psi^*_\Omega[\phi = 0]e^{-iE_\Omega T}$$

where $\Psi_\Omega[\phi = 0]$ measures the ground state overlap of the $\phi = 0$ state.

By the same procedure, we have

$$\langle\phi = 0, \infty|\phi = 0, T/2\rangle_0 = \lim_{T\to\infty}\Psi_0[\phi = 0]\Psi^*_0[\phi = 0]e^{-iE_0 T}$$

Putting both pieces together, and using that for a perturbation $E_\Omega - E_0 \ll 1$,

$$W_0 = \log Z_0 = \log\frac{|\Psi_\Omega[\phi = 0]|^2 e^{-iE_\Omega T}}{|\Psi_0[\phi = 0]|^2 e^{-iE_0 T}} \approx \mathrm{cst} - i(E_\Omega - E_0)T$$

This is of the desired form, where we identify $\epsilon = (E_\Omega - E_0)/V$. It is implied that $T \to \infty$.

Method 2

Alternatively, we can compute this directly by expanding in eigenstates of the full Hamiltonian:

$$\begin{aligned}Z_0 = \langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle &= \lim_{T\to\infty}\langle 0|e^{iH_0T}e^{-iHT}|0\rangle = \lim_{T\to\infty}e^{iE_0T}\langle 0|e^{-iHT}|0\rangle \\ &= \lim_{T\to\infty}e^{iE_0T}\sum_n\langle 0|e^{-iH(1-i\epsilon)T}|n\rangle\langle n|0\rangle = e^{i(E_0-E_\Omega)T}|\langle\Omega|0\rangle|^2\end{aligned}$$

As in Method 1, we take $H \to H(1-i\epsilon)$ to make the expression convergent. For very large T, the dominant contribution to this sum is that with lowest E_m, i.e. the vacuum E_Ω.
For a perturbation $E_\Omega - E_0 \ll 1$, so we have

$$W_0 = \log Z_0 \approx \log |\langle \Omega | 0 \rangle|^2 - i(E_\Omega - E_0)T$$

Again, we identify $\epsilon = (E_\Omega - E_0)/V$.

(b) The Feynman diagrams in the perturbative expansion of Z_0 have no external lines, and are often called vacuum diagrams/bubbles. We thus say that Z_0 is obtained by summing over vacuum diagrams. Show that W_0 is the sum of connected vacuum diagrams.

Let $\{V_i\}$ be the set of connected vacuum diagram contributions (including symmetry factors), and $\{\tilde{V}_I\rangle$ be the set of all vacuum diagram contributions.

Then, a general diagram $\tilde{V}_I$ consists of n_i^I V_i sub-diagrams, for each $V_i \in \{V_i\}$. In particular, we have explicitly

$$\tilde{V}_I = \frac{1}{S_I}\prod_i (V_i)^{n_i^I} = \prod_i \frac{1}{n_i^I!}(V_i)^{n_i^I}$$

Note that the expressions V_i already contain symmetry factors with associated with exchanging internal elements of subdiagrams. Therefore, the symmetry factor S_I above comes only from exchanging identical connected subdiagrams, of which there are n_i^I of type V_i. Hence $S_I = \prod_i n_i^I!$.

The full vacuum contribution comes from summing over all possible topologically distinct diagrams $\tilde{V}_I$. By the previous discussion, equivalently we may sum over all sets $\{n_i\}$:

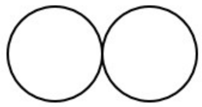
$$Z_0 = \sum_{\{n_i\}} \tilde{V}_I = \sum_{\{n_i\}} \prod_i \frac{1}{n_i!}(V_i)^{n_i} = \prod_i \sum_{n_i=0}^{\infty} \frac{1}{n_i!}(V_i)^{n_i} = \prod_i e^{V_i} = e^{\sum_i V_i}$$

Since $Z_0 = e^{W_0}$, we recover $W_0 = \sum_i V_i$ as desired, the sum of all connected vacuum diagrams.

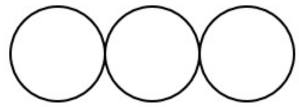
(c) Write down the expression for ϵ to order $\mathcal{O}(\lambda^2)$, in either coordinate or momentum space.

We first write down all the connected vacuum diagrams to order $\mathcal{O}(\lambda^2)$, using $\int d^4x = VT$.

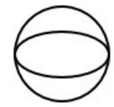
Order	Diagram	S	Expressions
λ^1	$V_1^{(1)}$	2^3	$-i\lambda VT G_F^0(0)^2$
			$-i\lambda\left(\int \text{đ}^4 q \frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$V_2^{(2)}$	2^4	$(-i\lambda)^2 VT G_F^0(0)^2 \int d^4z G_F^0(z)^2$
			$(-i\lambda)^2\left(\int \text{đ}^4 q \frac{-i}{q^2+m^2-i\epsilon}\right)^2 \int \text{đ}^4 q \left(\frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$V_3^{(2)}$	$2 \times 4!$	$(-i\lambda)^2 VT \int d^4z G_F^0(z)^4$
			$(-i\lambda)^2 \int \text{đ}^4 q_1 \text{đ}^4 q_2 \text{đ}^4 q_3 \frac{-i}{q_1^2+m^2-i\epsilon}\frac{-i}{q_2^2+m^2-i\epsilon}\frac{-i}{q_3^2+m^2-i\epsilon}\frac{-i}{(q_1+q_2+q_3)^2+m^2-i\epsilon}$



$V_1^{(1)}$



$V_2^{(2)}$



$V_2^{(2)}$

The symmetry factor of $V_1^{(1)}$ has 2^2 from propagators starting and ending on the same vertex, and another factor of 2 permuting the loops. The symmetry factor of $V_2^{(2)}$ has 2^2 from propagators starting and ending on the same vertex, a factor of 2 permuting the identical vertices, and another factor of 2 due to the 2 identical propagators connecting the internal vertices. The symmetry factor of $V_2^{(2)}$ has a factor of 2 permuting the identical vertices, 4! from the 4 identical propagators connecting the internal vertices.

Hence, we have

$$\begin{aligned}
\epsilon &= \frac{i}{VT}(V_1^{(1)} + V_2^{(2)} + V_3^{(2)}) + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda}{8}(G_F^0(0))^2 - \frac{i\lambda^2}{16}(G_F^0(0))^2 \int d^4x (G_F^0(x))^2 - \frac{i\lambda^2}{48} \int d^4x (G_F^0(x))^4 + \mathcal{O}(\lambda^3)
\end{aligned}$$

Question 4: General n-point Function (10 points)

Prove that in evaluating the n-point function $G_n(x_1, \ldots x_n)$, diagrams that contain factor(s) of vacuum diagrams all cancel. That is, G_n is obtained by summing over diagrams without any vacuum diagram factors. This stateement is true for any H_I, but it is enough to prove for the $\lambda\phi^4$ theory.

Method 1

We start with the expression

$$G_n(x_1, \ldots, x_n) = \frac{\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle}{\langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle}$$

The numerator can be expanded as:

$$\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^m\right\rangle$$

Observe that bubble diagrams must come from contractions between the L_I's themselves. We define $\langle\cdots\rangle_{\text{n.b.}}$ to be the contribution to a contraction from non-bubble diagrams. We can thus write the numerator as:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m-k}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle\\
&=\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\frac{1}{k!(m-k)!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m-k}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle\\
&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle\\
&=\sum_{k=0}^{\infty}\frac{1}{k!}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle\times\sum_{m=0}^{\infty}\frac{1}{m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m}\right\rangle_{\text{n.b.}}
\end{aligned}$$

In lines 2 and 3, all we have done is changing the summation indices (discrete change of variables). This allows us to factor out the vacuum contribution, which we do in line 4. The first factor is precisely the denominator of the Gell-Mann Low formula, thus

$$G_n(x_1, \ldots x_n) = \left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^m\right\rangle_{\text{n.b.}}$$

That is, the n-point function is obtained by summing over diagrams without any vacuum bubbles, as desired.

Method 2

Again, we start with the Gell-Mann Low formula. The idea is to factor the numerator into a sum of diagrams with no vacuum bubbles, times a factor containing all the vacuum-bubble dependence.

Let $\{V_i\}$ represent the connected vacuum diagram contributions (including symmetry factors), as in Problem 3(b). Further, let $\{D_i\}$ represent diagrams (including symmetry factors) contributing to the numerator in the Gell-Mann Low formula, that contain no vacuum bubbles.

For any diagram D_i, the sum of all diagrams contributing to the numerator which contain D_i as a subdiagram is precisely $D_i\prod_j V^j$. This follows from the argument in 3(a), and that all symmetry factors are accounted for–there are no additional symmetry factors between the D_i and V_j's. Furthermore, every contribution to the numerator must contain some D_i as a sub-diagram, therefore the numerator is

$$\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle = \sum_i \prod_j e^{V_j} D_i = Z_0 \sum_i D_i$$

where in the second equality we use from Problem 3(b) that $Z_0 = \prod_i e^{V_i}$. Therefore,

$$G_n(x_1,\ldots,x_n) = \frac{Z_0 \sum_i D_i}{Z_0} = \sum_i D_i$$

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 7 Solutions

April 4th, 2023

Question 1: Momentum Conservation (10 points)

Consider an interacting field theory of a real scalar ϕ. Assume that the theory is translation invariant. Introduce the Fourier transform of

$$G_n(x_1, \dots, x_n) = \langle \Omega \mathrm{T} \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$$
$$\tilde{G}_n(p_1, \dots, p_n) = \int d^4x_1 \cdots d^4x_n e^{-i\sum_i p_i \cdot x_i} G_n(x_1, \dots x_n)$$

Show that

$$\tilde{G}_n(p_1, \dots, p_n) \propto (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n)$$

The idea is to use translation invariance to remove the x_1 dependence in G_n, so taking its Fourier transform gives a δ-function when integrating over x_1.
We compute:

$$\begin{aligned}
\tilde{G}_n(p_1, \dots, p_n) &= \int d^4x_1 \cdots d^4x_n e^{-i\sum_{i=1}^n p_i \cdot x_i} G_n(x_1, x_2, \dots, x_n) \\
&= \int d^4x_1 d^4x_2 \cdots d^4x_n e^{-i\sum_{i=1}^n p_i \cdot x_i} G_n(0, x_2 - x_1, \dots, x_n - x_1) \\
&= \int d^4x_1 d^4x_2' \cdots d^4x_n' e^{-ix_1 \cdot p_1} e^{-i\sum_{i=2}^n (x_i' + x_1) \cdot p_i} G_n(0, x_2', \dots, x_n') \\
&= \int d^4x_2' \cdots d^4x_n' e^{-i\sum_{i=2}^n x_i' \cdot p_i} G(0, x_2', \dots, x_n') \int d^4x_1 e^{-ix_1 \cdot \sum_{i=1}^n p_i} \\
&= (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n) \tilde{G}'(p_2, \dots p_n)
\end{aligned}$$

In the second line we use translation invariance of the correlator. In the third line we change variables $x_i' = x_i - x_1$ for $i \geq 2$. In the last line we define $G'(x_2, \dots, x_n) := G(0, x_2, \dots, x_n)$ and $\tilde{G}'(p_2, \dots, p_n)$ its Fourier transform. We see that integrating over the x_1-dependence has given us the desired δ-function.

Question 2: Feynman Rules for a Complex Scalar Field (20 points)
For a complex scalar field particles and antiparticles are distinct, which we can think of as positively and negatively charged. The Feynman rules must distinguish them. We can make the distinction by including an arrow for each propagator indicating the flow of charge (note that this is separate from arrows indicating the flow of momentum, which are arbitrary). For a particle it is customary to point the arrow away from the external point in the initial state, and towards the external point in the final state. For an antiparticle, the direction is reversed. The charge arrows for propagators of the rest of a diagram then follow from charge conservation.

In order to avoid 2 kinds of arrows, we can simply align momentum arrows with the charge ones.
(a) Consider a complex scalar field theory

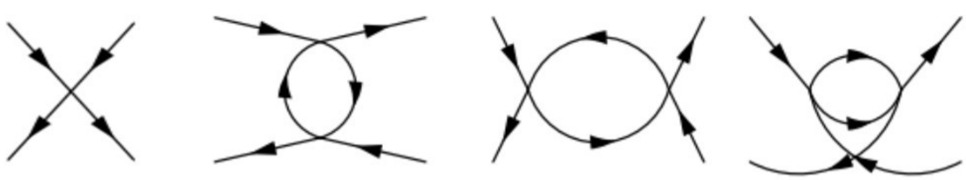
$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2$$

Write down the momentum space Feynman rules for this theory.
There are 2 Feynman rules: the propagator and a 4-point vertex. These correspond to the quadratic and 4th order terms in $\mathcal{L}$.

(b) Draw the connected diagrams for the scattering amplitude for the process

$$\phi + \phi^* \to \phi + \phi^*$$

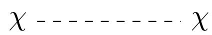
to order $\mathcal{O}(\lambda^2)$, where ϕ and ϕ^* denote particle and antiparticle.
We have 1 diagram at order λ. The rest are at order λ^2, corresponding to s, t, u-channel scattering. Note that we also have order λ^2 diagrams coming from corrections to the propagator, with loops on any of the 4 external legs, but these can be dropped since only amputated diagrams contribute to scattering amplitudes.



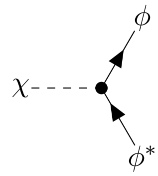
(c) Now suppose ϕ interacts with a real field χ via

$$\mathcal{L}_I = \lambda' \chi \partial^\mu \phi^* \partial_\mu \phi$$

That is, the full Lagrangian is the sum of the free (ϕ, ϕ^*) theory, the free χ theory, and $\mathcal{L}_I$. Suppose χ has mass M. Use a solid line for the ϕ propagator and a dashed line for the χ propagator. Write down the momentum space Feynman rules for the full theory.
We write down the new Feynman rules, for the χ propagator and the $\chi\phi^*\phi$ interaction: In the second



$$\frac{-i}{p^2 + M^2 - i\epsilon}$$



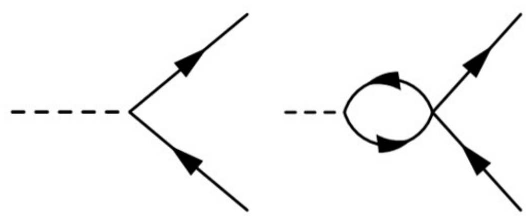
$$-i\lambda' p_1 \cdot p_2$$

diagram, the particles ϕ and ϕ^* have momentum p_1 and p_2, respectively. The Feynman rule comes from writing $i\mathcal{L}_I$ is momentum space, giving us the factor $i\lambda'(ip_1 \cdot p_2) = -i\lambda' p_1 \cdot p_2$.

(d) Suppose both $\lambda, \lambda' \sim \mathcal{O}(\epsilon)$, with ϵ a small parameter. Draw all the connected diagrams to order $\mathcal{O}(\epsilon^2)$ for the amplitude of the decay process

$$\chi \to \phi + \phi^*$$

We have 1 diagram at order ϵ given by the 3-point vertex. The rest at order ϵ^2 come from corrections to the external ϕ-propagators (which do not contribute to the decay process), and the 1-loop correction to the 3-point vertex (shown below).



Question 3: Higgs Production at LHC (10 points + 10 bonus)
The LHC collides protons with protons at very high energies. Each proton contains a number of quarks and gluons, so collisions of protons can also be considered as collisions of gluons. Here we consider a baby version of the Standard Model, which contains 3 types of scalar fields:

- Real gluon field g. Use a wavy line to denote its propagator.
- Complex quark field q. Use a solid line with an arrow to denote its propagator.
- Real Higgs field H. Use a dashed line to denote its propagator.

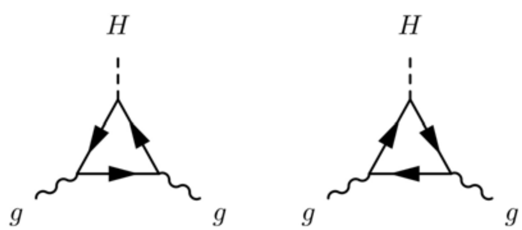
Suppose the interaction part of the theory is given by

$$\mathcal{L} = \lambda_1 g q^\dagger q + \lambda_2 H q^\dagger q$$

Note that there is no direct coupling between the gluon g and Higgs H. Assume that the couplings λ_1, λ_2 are small and of copmarable strength.
(a) The dominant channel of Higgs production at the LHC is gluon fusion, schematically written as

$$g + g \to H$$

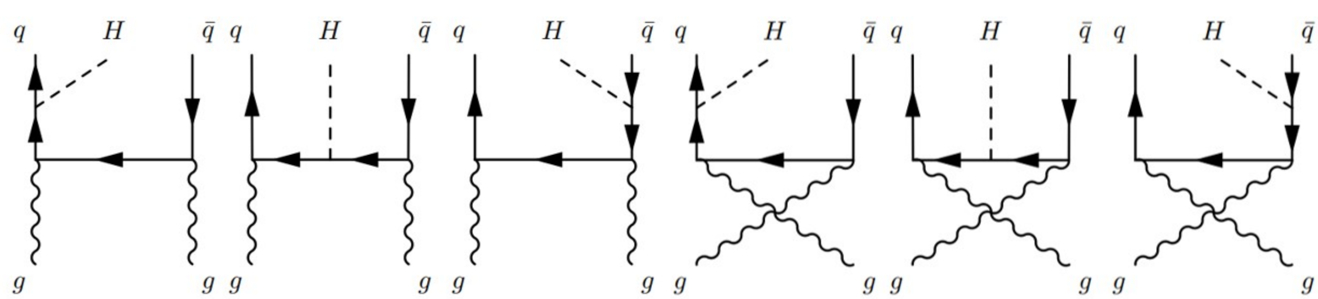
Draw the leading Feynman diagram(s) for this process.
The leading diagrams contribute at order $\lambda_1^2\lambda_2$, and are given by the 2 'triangle diagrams' below. These 2 diagrams give equal contributions in our case, but with realistic vector gluons and fermionic quarks, they do not.



(b) **Bonus.** Another channel for Higgs production is

$$g + g \to H + q + \bar{q}$$

where $\bar{q}$ denotes antiquark. Draw the leading Feynman diagram(s) for this process.
The leading diagrams contribute at order $\lambda_1^2\lambda_2$, and there are 6 of them given below.



Question 4: Properties of Gamma Matrices (25 points)

Without resorting to a particular representation, prove the following identities.

(a) $\text{Tr}\,\gamma^\mu = 0$

The canonical way to do this is define the matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$. Using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we immediately have $\{\gamma^\mu, \gamma^5\} = 0$ and $\gamma_5^2 = 1$.

Now, we compute

$$\text{Tr}\,\gamma^\mu = \text{Tr}\,\gamma^\mu\gamma^5\gamma^5 = -\text{Tr}\,\gamma^5\gamma^\mu\gamma^5 = -\text{Tr}\,\gamma^\mu\gamma^5\gamma^5 = -\text{Tr}\,\gamma^\mu$$

In the third equality we use that γ^5 anticommutes with γ^μ, and in the 4th line we use the cyclicity of the trace. Therefore, $\text{Tr}\,\gamma^\mu = 0$.

Note that the same argument goes through with $\gamma^5\gamma^5$ replaced by some $\gamma_\rho\gamma^\rho$ (no sum over ρ), with $\rho \neq \mu$.

(b) $\text{Tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$

We compute

$$\text{Tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\text{Tr}\,(\{\gamma^\nu, \gamma^\mu\}) = \frac{1}{2}\text{Tr}\,2\eta^{\mu\nu}\mathbb{1} = 4\eta^{\mu\nu}$$

In the second equality we use the cyclicity of the trace.

(c) $\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$

Again we use the γ^5 trick in (a):

$$\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = \text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5 = (-1)^3\text{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5 = -\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5 = -\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda$$

Therefore $\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$. Note that the same argument goes through with $\gamma^5\gamma^5$ replaced by some $\gamma_\rho\gamma^\rho$ (no sum over ρ), with $\rho \neq \mu, \nu, \lambda$

(d) $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q - 2i\Sigma^{\mu\nu}p_\mu q_\nu$, where we define $\not{p} = p_\mu\gamma^\nu$, and $\Sigma^{\mu\nu} = \dfrac{i}{4}[\gamma^\mu, \gamma^\nu]$

The first equality follows from contracting both sides of the anticommutation relation $\gamma^\mu\gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu$ with $p_\mu q_\nu$. To show the second equality, we use

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} - 2i\Sigma^{\mu\nu}$$

Contracting both sides by $p_\mu q_\nu$ gives the desired result.

(e) $\gamma^\mu\not{p}\gamma_\mu = -2\not{p}$

First note that

$$\gamma^\mu\gamma_\mu = \eta_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}\eta_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \eta_{\mu\nu}\eta^{\mu\nu}\mathbb{1} = 4\mathbb{1}$$

We thus compute

$$\gamma^\mu\gamma^\nu\gamma_\mu = -\gamma^\mu\gamma_\mu\gamma^\nu + 2\gamma^\mu\eta_\mu{}^\nu = -4\gamma^\nu + 2\gamma^\nu = -2\gamma^\nu$$

Contracting with p_ν gives the desired result.

Question 5: Conserved 'Probability' Current of the Driac Equation (15 points)
Starting from the Dirac equation

$$(\gamma^\mu \partial_\mu - m)\psi = 0$$

(a) Show that one can construct a current j^μ which is conserved,

$$\partial_\mu j^\mu = 0$$

Note that taking the adjoint of the Dirac equation gives

$$0 = \psi^\dagger((\gamma^\mu)^\dagger(-\overleftarrow{\partial_\mu}) - m) = \psi^\dagger(\overleftarrow{\partial_\mu}\gamma^0\gamma^0(\gamma^\mu)^\dagger) - m\gamma^0) = -\bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu + m)\gamma^0$$

In the first equality we use that the derivative $\partial_\mu = -ip_\mu$ is anti-Hermitian, since the momentum is Hermitian. In the second equality we use $(\gamma^0)^2 = -1$. In the third equality we use $(\gamma)^\mu = \gamma^0\gamma^\mu\gamma^0$. Therefore $\bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$.

Now consider the object $j^\mu := \bar{\psi}\gamma^\mu\psi$, where $\bar{\psi} := \psi^\dagger\gamma^0$. We compute

$$\partial_\mu j^\mu = \bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu)\psi + \bar{\psi}(\gamma^\mu\overrightarrow{\partial_\mu})\psi = -m\bar{\psi}\psi + m\bar{\psi}\psi = 0$$

(b) Show that the j^μ you constructed is real.
We take the conjugate:

$$(j^\mu)^* = (\bar{\psi}\gamma^\mu\psi)^* = \psi^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger\psi = \psi^\dagger(-(\gamma^0)^2)(\gamma^\mu)^\dagger(-\gamma^0)\psi = \bar{\psi}\gamma^0(\gamma^\mu)^\dagger\gamma^0\psi = \bar{\psi}\gamma^\mu\psi = j^\mu$$

Hence our current is real. In the 3rd equality we used that $(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0 = -\gamma^0$.

(c) Show that by choosing the overall sign of j^μ, the zeroth component of j^μ

$$\rho := j^0$$

can be made to be postive definite.
We compute

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = -\psi^\dagger\psi$$

This is negative definite. Thus by defining $j^\mu := -\bar{\psi}\gamma^\mu\psi$, we have $\rho = j^0 = \psi^\dagger\psi$ positive definite.

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 8 Solutions

April 11, 2023

Question 1: Proofs of Spinor Identities (21 points)

(a) From the definition

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}}$$

Prove the identity

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

Hint: use the identity

$$[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

By the Baker-Campbell-Hausdorff formula we have

$$e^A B e^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}L^n(A,B), \qquad L^n(A,B) := \underbrace{[A,\ldots,[A,[A,B]]\ldots]}_{n \text{ times}}$$

We wish to use this with $A = -\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}$ and $B = \gamma^\mu$. Following the hint, we use the identity

$$[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

We now compute

$$\begin{aligned}
L^n(A,B) &= L^{n-1}(A,[A,B]) = L^{n-1}\left(A, \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu\right) = \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu L^{n-1}(A,\gamma^\nu) \\
&= \left[\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\left[\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\nu{}_\lambda\right]L^{n-2}(A,\gamma^\lambda) = \cdots = \left[\left(\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^n\right]^\mu{}_\nu\gamma^\nu
\end{aligned}$$

In the last expression the power is taken in Lorentz space, i.e. $[(\omega\cdot\mathcal{J})^n]^\mu{}_\nu = (\omega\cdot\mathcal{J})^\mu{}_\alpha(\omega\cdot\mathcal{J})^\alpha{}_\beta\cdots(\omega\cdot\mathcal{J})^\lambda{}_\nu$. Substituting this into the BCH formula, we have

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^n\right]^\mu{}_\nu\gamma^\nu = \left(e^{+\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right)^\mu{}_\nu\gamma^\nu = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

(b) Prove the identity

$$S^\dagger = -\gamma^0 S^{-1}\gamma^0$$

From the definition, we start with

$$S(\Lambda)^\dagger = \exp\left(\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma\dagger}\right)$$

Now we use the conjugation of the gamma matrices, and $(\gamma^0)^2 = -1$.

$$\Sigma^{\mu\nu\dagger} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger = -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] = -\frac{i}{4}[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0] = -\frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = -\gamma^0\Sigma^{\mu\nu}\gamma^0$$

Substituting this in the exponential, we have

$$\begin{aligned} S(\Lambda)^\dagger &= \exp\left(-\frac{i}{2}\omega_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma}\gamma^0\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma}\gamma^0\right)^n \\ &= \gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^n(-1)^{n-1}\gamma^0 = -\gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^n\gamma^0 = -\gamma^0 S^{-1}(\Lambda)\gamma^0 \end{aligned}$$

In the third equality we use that $(\gamma^0)^2 = -1$ exactly $n-1$ times to pick up a factor of $(-1)^{n-1}$.

(c) From the Lorentz transformation of ψ, show that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform as a scalar and vector.
We know that under a Lorentz transformation,

$$\psi_\alpha(x) \to \psi'_\alpha(x) = S_\alpha{}^\beta(\Lambda)\psi_\beta(\Lambda^{-1}x) = S(\Lambda)\psi(\Lambda^{-1}x)$$

Using the result from (b), the Dirac conjugate transforms as

$$\bar{\psi}(x) \to \psi'^\dagger(x)\gamma^0 = \psi^\dagger(\Lambda^{-1}x)S(\Lambda)^\dagger\gamma^0 = \psi^\dagger(\Lambda^{-1}x)(-\gamma^0 S(\Lambda)^{-1}\gamma^0)\gamma^0 = \bar{\psi}(\Lambda^{-1}x)S(\Lambda)^{-1}$$

Now we can compute

$$\begin{aligned} \bar{\psi}\psi(x) &\to \bar{\psi}S(\Lambda)^{-1}S(\Lambda)\psi(\Lambda^{-1}x) = \bar{\psi}\psi(\Lambda^{-1}x) \\ \bar{\psi}\gamma^\mu\psi(x) &\to \bar{\psi}S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\psi(\Lambda^{-1}x) = \bar{\psi}S(\Lambda^{-1})\gamma^\mu S^{-1}(\Lambda^{-1})\psi(\Lambda^{-1}x) = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi(\Lambda^{-1}x) \end{aligned}$$

where in the last equality of the second line we use the result from (a). These are the transformation laws for a scalar and vector.

Question 2: More Spinor Identities (8 points)
Without using any explicit form uf u_s and v_s, show either one of

$$u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = 2E\delta_{rs}$$

In other words, show one of the following.

$$u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = -\frac{iE}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}), \qquad v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = \frac{iE}{m}\bar{v}_r(\mathbf{k})v_s(\mathbf{k})$$

The idea is to use the Dirac equation twice, once on $u^\dagger(\mathbf{k})$ (or $v^\dagger(\mathbf{k})$), and once on $u(\mathbf{k})$ (or $v(\mathbf{k})$). The Dirac equation, in all its forms, is given by

$$\begin{aligned} mu_s(\mathbf{k}) &= i\not{k}u_s(\mathbf{k}), \qquad & \bar{u}_s(\mathbf{k})m &= \bar{u}_s(\mathbf{k})i\not{k} \\ mv_s(\mathbf{k}) &= -i\not{k}v_s(\mathbf{k}), \qquad & \bar{v}_s(\mathbf{k})m &= -\bar{v}_s(\mathbf{k})i\not{k} \end{aligned}$$

We thus compute:

$$\begin{aligned} u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) &= -\frac{1}{2}\left(\bar{u}_r(\mathbf{k})\gamma^0 u_s(\mathbf{k}) + \bar{u}_r(\mathbf{k})\gamma^0 u_s(\mathbf{k})\right) = -\frac{i}{2m}\left(\bar{u}_r(\mathbf{k})\not{k}\gamma^0 u_s(\mathbf{k}) + \bar{u}_r(\mathbf{k})\gamma^0\not{k}u_s(\mathbf{k})\right) \\ &= -\frac{i}{2m}\bar{u}_r(\mathbf{k})k_\mu\{\gamma^\mu, \gamma^0\}u_s(\mathbf{k}) = \frac{ik_0}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = -\frac{iE}{m}\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs} \\ v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) &= +\frac{1}{2}\left(\bar{v}_r(\mathbf{k})\gamma^0 v_s(\mathbf{k}) + \bar{v}_r(\mathbf{k})\gamma^0 v_s(\mathbf{k})\right) = +\frac{iE}{m}\bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = 2E\delta_{rs} \end{aligned}$$

In the last step of both calculations we use the normalizations

$$\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = 2im\delta_{rs}, \qquad \bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = -2im\delta_{rs}$$

Question 3: Stress Tensor and Hamiltonian for the Dirac Theory. (21 points)

(a) The Dirac action is translationally invariant. Use the Noether procedure the construct the conserved currents $\Theta^{\mu\nu}$, i.e. the energy-momentum tensor.
The 'charge' density Θ^{00} for time translation is the energy density. Show that Θ^{00} indeed coincides with the Hamiltonian density $\mathcal{H}$ derived in class, i.e.

$$\Theta^{00} = \mathcal{H} = i\bar{\psi}(\gamma^i\partial_i - m)\psi$$

We begin with the Lagrangian density $\mathcal{L} = -i\bar{\psi}(\not{\partial} - m)\psi$.
The Noether current for a transformation is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)}\delta\Phi_A - \mathcal{F}^\mu, \qquad \delta\mathcal{L} = \partial_\mu \mathcal{F}^\mu$$

Under an spacetime transformation $x^\mu \to x^\mu + \epsilon^\mu$ the Lagrangian density transforms as $\delta\mathcal{L} = -\epsilon^\mu\partial_\mu\mathcal{L}$. The fields transform as $\delta\psi = -\epsilon^\mu\partial_\mu\psi$. Putting everything together, we have

$$j^\mu = -i\bar{\psi}\gamma^\mu(-\epsilon^\nu\partial_\nu\psi) + \epsilon^\mu\mathcal{L} = \epsilon^\nu(i\bar{\psi}\gamma^\mu\partial_\nu\psi - i\bar{\psi}\eta_{\nu\mu}(\not{\partial} - m)\psi) = i\epsilon_\nu\bar{\psi}(\gamma^{\mu\nu} - \eta^{\mu\nu}(\not{\partial} - m))\psi$$

We find that $j^\mu = \epsilon_\nu\Theta^{\mu\nu}$ for the stress-energy tensor

$$\Theta^{\mu\nu} = i\bar{\psi}(\gamma^\mu\partial^\nu - \eta^{\mu\nu}(\not{\partial} - m))\psi$$

In particular, the energy density is

$$\Theta^{00} = i\bar{\psi}(\gamma^0\partial^t + (\not{\partial} - m))\psi = i\bar{\psi}(-\gamma^0\partial_t + \gamma^0\partial_t + \gamma^i\partial_i - m)\psi = i\bar{\psi}(\gamma^i\partial_i - m)\psi = \mathcal{H}$$

(b) Show that using the Dirac equation, the Hamiltonian can be written as

$$H = i\int d^3x\psi^\dagger\partial_t\psi$$

Express H in terms of the operators $a^s_{\mathbf{k}}$, $a^{s\dagger}_{\mathbf{k}}$, $c^s_{\mathbf{k}}$, $c^{s\dagger}_{\mathbf{k}}$.

The Dirac equation tells us that

$$0 = i\bar{\psi}(\not{\partial} - m)\psi = i\bar{\psi}(\gamma^0\partial_0 + \gamma^i\partial_i - m)\psi \qquad \Longrightarrow \qquad i\bar{\psi}(\gamma^i\partial_i - m)\psi = -i\bar{\psi}\gamma^0\partial_0\psi = i\psi^\dagger\partial_t\psi$$

Therefore,

$$H = \int d^3x\Theta^{00} = \int d^3x i\bar{\psi}(\gamma^i\partial_i - m)\psi = i\int d^3x\psi^\dagger\partial_t\psi$$

Now we substitute the mode expansion:

$$\psi(x) = \int \frac{d\!\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[a^s_{\mathbf{k}}u_s(\mathbf{k})e^{ik\cdot x} + c^{s\dagger}_{\mathbf{k}}v_s(\mathbf{k})e^{-ik\cdot x}\right]$$

In terms of creation and annihilation operators, the Hamiltonian is thus

$$\begin{aligned}
H &= i\int d^3\mathbf{x}\frac{d\!\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\!\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left[a^{r\dagger}_{\mathbf{k}}u^\dagger_r(\mathbf{k})e^{-ik\cdot x} + c^r_{\mathbf{k}}v^\dagger_r(\mathbf{k})e^{ik\cdot x}\right](-ik'_0)\left[a^s_{\mathbf{k}'}u_s(\mathbf{k}')e^{ik'\cdot x} - c^{s\dagger}_{\mathbf{k}'}v_s(\mathbf{k}')e^{-ik'\cdot x}\right] \\
&= \int d^3\mathbf{x}\frac{d\!\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\!\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}k'_0\left[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}'}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}')e^{-i(k-k')\cdot x} - a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}'}u^\dagger_r(\mathbf{k})v_s(\mathbf{k}')e^{-i(k+k')\cdot x}\right. \\
&\qquad\qquad \left. + c^r_{\mathbf{k}}a^s_{\mathbf{k}'}v^\dagger_r(\mathbf{k})u_s(\mathbf{k}')e^{i(k+k')\cdot x} - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}'}v^\dagger_r(\mathbf{k})v_s(\mathbf{k}')e^{i(k-k')\cdot x}\right] \\
&= \int \frac{d\!\!\!{}^-{}^3\mathbf{k}}{2}\left[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}) - e^{2i\omega_{\mathbf{k}}t}a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{-\mathbf{k}}u^\dagger_r(\mathbf{k})v_s(-\mathbf{k}) + e^{-2i\omega_{\mathbf{k}}t}c^r_{\mathbf{k}}a^s_{-\mathbf{k}}v^\dagger_r(\mathbf{k})u_s(-\mathbf{k}) - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})v_s(\mathbf{k})\right] \\
&= \frac{1}{2}\int d\!\!\!{}^-{}^3\mathbf{k}\left[a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})u_s(\mathbf{k}) - c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})v_s(\mathbf{k})\right] \\
&= \int d\!\!\!{}^-{}^3\mathbf{k}\,\omega_{\mathbf{k}}\left[a^{s\dagger}_{\mathbf{k}}a^s_{\mathbf{k}} + c^{s\dagger}_{\mathbf{k}}c^s_{\mathbf{k}} - 2(2\pi)^3\delta^{(3)}(0)\right] = \int d\!\!\!{}^-{}^3\mathbf{k}\omega_{\mathbf{k}}(N_{\mathbf{k}} + \bar{N}_{\mathbf{k}}) + E_0
\end{aligned}$$

In the third equality we integrate over x to get a delta function in $\mathbf{k}$ and $\mathbf{k}'$, and subsequently do the integral over $\mathbf{k}'$. In the fourth equality we use the identities $u_r^\dagger(\mathbf{k})v_s(-\mathbf{k}) = v_r^\dagger(\mathbf{k})u_s(-\mathbf{k}) = 0$. In the fifth equality, we use $\{c^r_{\mathbf{k}}, c^{s\dagger}_{\mathbf{k}'}\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ to move the c-creation operator to the left, and the normalization $u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = v_r^\dagger(\mathbf{k})v_s(\mathbf{k}) = 2k^0\delta_{rs}$. In the final equality, we identify

$$N_{\mathbf{k}} = a^{s\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}, \qquad \bar{N}_{\mathbf{k}} = c^{s\dagger}_{\mathbf{k}}c^s_{\mathbf{k}}, \qquad E_0 = -\int d^3\mathbf{k}\ 2\omega_{\mathbf{k}}\delta^{(3)}(0)$$

(c) What is the vacuum energy density? Discuss the differences with that of a scalar.

The vacuum energy E_0^ψ is identified in (b). To get the vacuum energy density ε_0^ψ, we use that $\delta^{(0)} = \int d^3\mathbf{x}1$ and take the integrand:

$$\varepsilon_0^\psi = -\int d^3\mathbf{k}\ 2\omega_{\mathbf{k}} = -4\int d^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}$$

We can compare this to a complex scalar field:

$$\varepsilon_0^\phi = 2\int d^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}$$

A real scalar field has 2 degrees of freedom per momentum $\mathbf{k}$, each of which is a harmonic oscillator with negative vacuum energy $\omega_{\mathbf{k}}/2$. A Dirac field has 4 degrees of freedom per momentum $\mathbf{k}$ ($u_{1,2}$, $v_{1,2}$), each of which is a harmonic oscillator with positive vacuum energy $-\omega_{\mathbf{k}}/2$. In particular, if we have 2 complex scalars for each Dirac fermion, then the vacuum energy is exactly zero.

Question 4: Angular Momentum Operators (30 points)

The Dirac action is Lorentz invariant.

(a) Write down an infinitesimal Lorentz transformation for ψ.

We consider an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$.

A spinor and its conjugate transform as

$$\begin{aligned}
\delta\psi_\alpha(x) &= \psi'_\alpha(x) - \psi_\alpha(x) = S_\alpha{}^\beta(\Lambda)\psi_\beta(\Lambda^{-1}x) - \psi_\alpha(x)\\
&= \left(\delta_\alpha^\beta - \frac{i}{2}\omega_{\rho\sigma}(\Sigma^{\rho\sigma})_\alpha{}^\beta\right)\left(\psi_\beta(x) + \frac{i}{2}\omega_{\kappa\lambda}i(\eta^{\kappa\mu}\delta_\nu^\lambda - \eta^{\lambda\mu}\delta_\nu^\kappa)x_\nu\partial_\mu\psi_\beta(x)\right) - \psi_\alpha(x)\\
&= -\omega_{\rho\sigma}\left(\frac{i}{2}(\Sigma^{\rho\sigma})_\alpha{}^\beta + \delta_\alpha^\beta x^\sigma\partial^\rho\right)\psi_\beta(x)\\
\delta\bar\psi_\alpha(x) &= \bar\psi'_\alpha(x) - \bar\psi_\alpha(x) = \bar\psi_\beta(\Lambda^{-1}x)(S^{-1}(\Lambda))^\beta{}_\alpha - \bar\psi_\alpha(x)\\
&= \left(\bar\psi_\beta(x) + \frac{i}{2}\omega_{\kappa\lambda}i(\eta^{\kappa\mu}\delta_\nu^\lambda - \eta^{\lambda\mu}\delta_\nu^\kappa)\bar\psi_\beta(x)\overleftarrow{\partial_\mu}x_\nu\right)\left(\delta_\alpha^\beta + \frac{i}{2}\omega_{\rho\sigma}(\Sigma^{\rho\sigma})^\beta{}_\alpha\right) - \bar\psi_\alpha(x)\\
&= -\omega_{\rho\sigma}\bar\psi_\beta(x)\left(-\frac{i}{2}(\Sigma^{\rho\sigma})^\beta{}_\alpha + \delta_\alpha^\beta\overleftarrow{\partial^\rho}x^\sigma\right)
\end{aligned}$$

(b) Use the Noether procedure to construct the conserved charges $M^{\mu\nu}$ for with Lorentz transformations, and show that $M^{\mu\nu}$ can be written in terms of a 'spin' part and 'orbital angular momentum' part,

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$$

Show that the orbital part $L^{\mu\nu}$ has the same form as that of a scalar

$$L^{\mu\nu} = \int d^3x(x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu})$$

Where $\Theta^{\mu\nu}$ is the energy-momentum tensor from 3(a). Show that the spin part $S^{\mu\nu}$ can be written as

$$S^{\mu\nu} = -\int d^3x\psi^\dagger\Sigma^{\mu\nu}\psi$$

We start with $\mathcal{L} = -i\bar\psi(\partial\!\!\!/ - m)\psi$. Under the Lorentz transform in (a), $\mathcal{L}$ transforms as

$$\begin{aligned}
\delta\mathcal{L} &= -i(\delta\bar\psi(\partial\!\!\!/ - m)\psi + \bar\psi(\partial\!\!\!/ - m)\delta\psi)\\
&= i\omega_{\rho\sigma}\left[\bar\psi\left(-\frac{i}{2}\Sigma^{\rho\sigma} + \overleftarrow{\partial^\rho}x^\sigma\right)(\partial\!\!\!/ - m)\psi + \bar\psi(\partial\!\!\!/ - m)\left(\frac{i}{2}\Sigma^{\rho\sigma} + x^\sigma\partial^\rho\right)\psi\right]\\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} + i\omega_{\rho\sigma}\bar\psi\gamma^\sigma\partial^\rho\psi + \frac{1}{2}\omega_{\rho\sigma}\bar\psi(\Sigma^{\rho\sigma}\partial\!\!\!/ - \partial\!\!\!/\Sigma^{\rho\sigma})\psi\\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} + i\omega_{\rho\sigma}\bar\psi\gamma^\sigma\partial^\rho\psi - \frac{1}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\kappa{}_\lambda\bar\psi\gamma^\lambda\partial_\kappa\psi\\
&= -\omega_{\rho\sigma}x^\sigma\partial^\rho\mathcal{L} = -\omega_{\rho\sigma}\partial^\rho(x^\sigma\mathcal{L})
\end{aligned}$$

Note that in the third line, the second term comes from the product rule, in particular by $\partial\!\!\!/$ acting on the x^σ of the previous expression. In the fourth line we use that $[\Sigma^{\rho\sigma}, \gamma^\mu] = -(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$. In the last line we use the explicit form of $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$, and that $\omega_{\rho\sigma}$ is antisymmetric.

Therefore, the Noether current is:

$$
\begin{aligned}
j^\rho &= \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi_A)}\delta\Phi_A - \mathcal{F}^\rho = -i\bar{\psi}\gamma^\rho\delta\psi + \omega^{\rho\sigma}x_\sigma\mathcal{L} \\
&= \omega_{\mu\nu}\left(-\frac{1}{2}\bar{\psi}\gamma^\rho\Sigma^{\mu\nu}\psi + ix^\nu\bar{\psi}\gamma^\rho\partial^\mu\psi + \eta^{\mu\rho}x^\nu\mathcal{L}\right) \\
&= \omega_{\mu\nu}\left(-\frac{1}{2}\bar{\psi}\gamma^\rho\Sigma^{\mu\nu}\psi + \Theta^{\rho\mu}x^\nu\right)
\end{aligned}
$$

where we recall the energy-momentum tensor

$$
\Theta^{\mu\nu} = i\bar{\psi}(\gamma^\mu\partial^\nu - \eta^{\mu\nu}(\not{\partial} - m))\psi = i\bar{\psi}\gamma^\mu\partial^\nu\psi + \eta^{\mu\nu}\mathcal{L}
$$

We have an independent Noether current for each independent component of $\omega_{\mu\nu}$, of which there are 6. Therefore, we can define $j^\rho = -\frac{1}{2}\omega_{\mu\nu}J^{\rho\mu\nu}$, for the conserved currents

$$
J^{\rho\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^\rho\Sigma^{[\mu\nu]}\psi - \Theta^{\rho[\mu}x^{\nu]} = \bar{\psi}\gamma^\rho\Sigma^{\mu\nu}\psi + x^\mu\Theta^{\rho\nu} - x^\nu\Theta^{\rho\mu}
$$

The Noether charges are

$$
M^{\mu\nu} = \int d^3x J^{0\mu\nu} = \int d^3x\left[-\psi^\dagger\Sigma^{\mu\nu}\psi + x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu}\right] = S^{\mu\nu} + L^{\mu\nu}
$$

where we identify

$$
S^{\mu\nu} = -\int d^3x\psi^\dagger\Sigma^{\mu\nu}\psi, \qquad L^{\mu\nu} = \int d^3x(x^\mu\Theta^{0\nu} - x^\nu\Theta^{0\mu})
$$

Note that $S^{\mu\nu}$ and $L^{\mu\nu}$ are not separately conserved.

(c) Express the $S^{\mu\nu}$ in terms of the operators $a^s_{\mathbf{k}}$, $a^{s\dagger}_{\mathbf{k}}$, $c^s_{\mathbf{k}}$, $c^{s\dagger}_{\mathbf{k}}$.
We need the mode expansion

$$
\psi(x) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[a^s_{\mathbf{k}}u_s(\mathbf{k})e^{ik\cdot x} + c^{s\dagger}_{\mathbf{k}}v_s(\mathbf{k})e^{-ik\cdot x}\right]
$$

The process is straightforwards, very similar to 2(b). We substitute the mode expansion into $S^{\mu\nu}$, and expand to get 4 terms. We perform the integral over $\mathbf{x}$ to get a $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ or $\delta^{(3)}(\mathbf{k}+\mathbf{k}')$ for each term, and perform the $\mathbf{k}'$ integral to set $\mathbf{k}' = \mathbf{k}$ for the $u^\dagger\Sigma^{\mu\nu}u$ and $v^\dagger\Sigma^{\mu\nu}v$ terms, and set $\mathbf{k}' = -\mathbf{k}$ for the $u^\dagger\Sigma^{\mu\nu}v$ and $v^\dagger\Sigma^{\mu\nu}u$ terms. The result is

$$
\begin{aligned}
S^{\mu\nu} = -\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\Big[& a^{r\dagger}_{\mathbf{k}}a^s_{\mathbf{k}}u^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}u_s(\mathbf{k}) + e^{2i\omega_{\mathbf{k}}t}a^{r\dagger}_{\mathbf{k}}c^{s\dagger}_{-\mathbf{k}}u^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}v_s(-\mathbf{k}) \\
&+ e^{-2i\omega_{\mathbf{k}}t}c^r_{\mathbf{k}}a^s_{-\mathbf{k}}v^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}u_s(-\mathbf{k}) + c^r_{\mathbf{k}}c^{s\dagger}_{\mathbf{k}}v^\dagger_r(\mathbf{k})\Sigma^{\mu\nu}v_s(\mathbf{k})\Big]
\end{aligned}
$$

(d) From the expression in (c) for S^{ij}, keep only the time-independent part, denoted $\tilde{S}^{ij}$. Define

$$\mathbf{J}^2 = \frac{1}{2}\tilde{S}^{ij}\tilde{S}_{ij}$$

Show that the one-particle states constructed by acting $a_{\mathbf{k}}^{s\dagger}$ and $c_{\mathbf{k}}^{s\dagger}$ with $\mathbf{k} = 0$ (i.e. in the rest frame) on the vacuum are eigenstates of $\mathbf{J}^2$ with eigenvalues corresponding to that of a spin-$\frac{1}{2}$ particle.

We note that $S^{\mu\nu}$ has time-dependent terms. This is fine, since only the combination $M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$ is expected to be conserved. Thus in the analysis of angular momentum we may throw away these terms. We further normal order our operator by placing annihilation operators to the right, to get $\tilde{S}^{\mu\nu}$. The spatial components of this tensor are

$$\tilde{S}^{ij} = -\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} \left[a_{\mathbf{k}}^{r\dagger} a_{\mathbf{k}}^{s} u_r^\dagger(\mathbf{k})\Sigma^{\mu\nu} u_s(\mathbf{k}) - c_{\mathbf{k}}^{s\dagger} c_{\mathbf{k}}^{r} v_r^\dagger(\mathbf{k})\Sigma^{\mu\nu} v_s(\mathbf{k})\right]$$

Note that the second term picks up a -1 from normal ordering, because the operators are fermionic.

Now we compute the action of $\mathbf{J}^2 = \frac{1}{2}\tilde{S}^{ij}\tilde{S}_{ij}$ on the 1-particle state $a_{\mathbf{0}}^{t\dagger}|0\rangle$.

$$\begin{aligned}
\mathbf{J}^2 a_{\mathbf{0}}^{t\dagger}|0\rangle &= \frac{1}{2}\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}'}} \left[u_r^\dagger(\mathbf{k})\Sigma^{ij}u_s(\mathbf{k})\right]\left[u_{r'}^\dagger(\mathbf{k}')\Sigma_{ij}u_{s'}(\mathbf{k}')\right] a_{\mathbf{k}}^{r\dagger}a_{\mathbf{k}}^{s}a_{\mathbf{k}'}^{r'\dagger}a_{\mathbf{k}'}^{s'}a_{\mathbf{0}}^{t\dagger}|0\rangle \\
&= \frac{1}{2}\frac{1}{(2\omega_{\mathbf{0}})^2} u_r^\dagger(\mathbf{0})\Sigma^{ij}u_s(\mathbf{0})u_s^\dagger(\mathbf{0})\Sigma_{ij}u_t(\mathbf{0})a_{\mathbf{0}}^{r\dagger}|0\rangle \\
&= \frac{1}{8m^2}u_r^\dagger(\mathbf{0})\Sigma^{ij}\Sigma_{ij}[u_s(\mathbf{0})u_s^\dagger(\mathbf{0})]u_t(\mathbf{0})a_{\mathbf{0}}^{r\dagger}|0\rangle \\
&= \frac{1}{8m^2}u_r^\dagger(\mathbf{0})\Sigma^{ij}\Sigma_{ij}u_s(\mathbf{0})(2m\delta_{st})a_{\mathbf{0}}^{r\dagger}|0\rangle = \frac{1}{4m}u_r^\dagger(\mathbf{0})\Sigma^{ij}\Sigma_{ij}u_t(\mathbf{0})a_{\mathbf{0}}^{r\dagger}|0\rangle \\
&= \frac{1}{4m}\frac{3}{2}u_r^\dagger(\mathbf{0})u_t(\mathbf{0})a_{\mathbf{0}}^{r\dagger}|0\rangle = \frac{3}{4}a_{\mathbf{0}}^{t\dagger}|0\rangle
\end{aligned}$$

In the first line, note that any terms containing c's vanish, since the lowering operator anticommutes with $a^{t\dagger}$ to annihilate the vacuum. In the second line we use that

$$a_{\mathbf{k}}^{r\dagger}a_{\mathbf{k}}^{s}a_{\mathbf{k}'}^{r'\dagger}a_{\mathbf{k}'}^{s'}a_{\mathbf{0}}^{t\dagger}|0\rangle = \delta^{s't}(2\pi)^3\delta^{(3)}(\mathbf{k}')a_{\mathbf{k}}^{r\dagger}a_{\mathbf{k}}^{s}a_{\mathbf{k}'}^{r'\dagger}|0\rangle = \delta^{s't}\delta^{sr'}(2\pi)^6\delta^{(3)}(\mathbf{k}')\delta^{(3)}(\mathbf{k}-\mathbf{k}')a_{\mathbf{k}}^{r\dagger}|0\rangle$$

In the third line we use that $u_s(\mathbf{0})u_s^\dagger(\mathbf{0}) = i(i\not{k} + m)|_{\mathbf{k}=0} = m(i + \gamma^0)$. Further noting that $\{\gamma^0, \gamma^i\} = 0$, we see that this commutes with $\Sigma^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$. In the fourth line (and fifth line), we use the identity $u_r^\dagger(\mathbf{k})u_s(\mathbf{k}) = 2E\delta_{rs}$. Finally, in the fifth line we also use that

$$\begin{aligned}
\Sigma^{ij}\Sigma_{ij} &= -\frac{1}{16}[\gamma^i, \gamma^j][\gamma^i, \gamma^j] = -\frac{1}{8}(\gamma^i\gamma^j\gamma^i\gamma^j - \gamma^i\gamma^j\gamma^j\gamma^i) \\
&= -\frac{1}{8}(-(\gamma^i\gamma^i)^2 + 2\gamma^i\gamma^i - 3\gamma^i\gamma^i) = -\frac{1}{8}(-9 + 6 - 9) = \frac{3}{2}
\end{aligned}$$

Therefore, we see that the one-particle state $a_{\mathbf{0}}^{t\dagger}|0\rangle$ is an eigenstate of $\mathbf{J}^2$ with eigenvalue $s(s+1) = \frac{3}{4}$, as expected from a spin-1/2 particle.

The calculation is almost identical with the one-particle state $c_{\mathbf{0}}^{t\dagger}|0\rangle$: one merely replaces a's with c's and u's with v's. The one difference is in the third line, where instead we need to use the identity $v_s(\mathbf{0})v_s^\dagger(\mathbf{0}) = i(i\not{k} - m)|_{\mathbf{k}=0} = m(-i + \gamma^0)$. However, this still commutes with Σ^{ij}, so the same proof follows through.

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 9 Solutions

April 18, 2023

Question 1: Some Identities (10 points)
Define γ^5 as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Show it has the following properties: (a) $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = 1$
We compute:

$$\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)(\gamma^0\gamma^0\gamma^0)\\ &= -(-1)^4 i\gamma^0\gamma^3\gamma^2\gamma^1 = -i(-1)^3\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}$$

In the 2nd equality we use that $(\gamma^0)^2 = -1$. In the 4th equality we use the anticommutation properties of the gamma matrices.

Furthermore,

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3(-1)^2(-1)^1\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3 = -\eta^{00}\eta^{11}\eta^{22}\eta^{33} = -1$$

In the 3rd equality we use $(\gamma^\mu)^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$ (no sum over μ).

(b) $\{\gamma^5, \gamma^\mu\} = 0$ and $\mathrm{Tr}\gamma^5 = 0$
This is not difficult to compute for each $\mu = 0, 1, 2, 3$. Using the anticommutation relations, we have:

$$\begin{aligned}\{\gamma^5, \gamma^0\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^1\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 + \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)^2\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 + (-1)\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^2\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 + \gamma^2\gamma^0\gamma^1\gamma^2\gamma^3) = i((-1)\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 + (-1)^2\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3) = 0\\ \{\gamma^5, \gamma^3\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 + \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 + (-1)^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3) = 0\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathrm{Tr}\gamma^5 &= \mathrm{Tr}(\gamma^5\gamma^0\gamma_0) = \frac{1}{2}(\mathrm{Tr}(\gamma^5\gamma^0\gamma_0) + \mathrm{Tr}(\gamma_0\gamma^0\gamma^5))\\ &= \frac{1}{2}(\mathrm{Tr}(\gamma^5\gamma^0\gamma_0) + \mathrm{Tr}(\gamma^0\gamma^5\gamma_0)) = \frac{1}{2}\mathrm{Tr}(\{\gamma^5, \gamma^0\}\gamma_0\} = 0\end{aligned}$$

In the 3rd equality we use the cyclicity of the trace, and in the last equality we use the previous result that $\{\gamma^5, \gamma^0\} = 0$. Note that instead of γ^0 we could have chosen any γ^μ matrix.

Question 2: Feynman Propagator for Dirac Spinors (10 points)
Show that the Feynman Green function

$$D_F^{\alpha\beta}(x-y) = \langle 0|\mathrm{T}\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = i(\not{\partial}+m)_{\alpha\beta}G_F(x-y)$$

where G_F is the Feynman propagator for a free complex scalar of the same mass m.
We start by computing the Wightman function using the mode expansion of ψ.

$$\begin{aligned}
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a^r_\mathbf{k}a^{\dagger s}_{\mathbf{k}'}|0\rangle u^r_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}')e^{i(k\cdot x-k'\cdot y)} \\
&= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')u^r_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}')e^{i(k\cdot x-k'\cdot y)} = \int \frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}\Lambda^+_{\alpha\beta}(\mathbf{k})e^{ik\cdot(x-y)} \\
&= i(\not{\partial}+m)_{\alpha\beta}\int \frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{ik\cdot(x-y)} = i(\not{\partial}+m)_{\alpha\beta}G_+(x-y)
\end{aligned}$$

In the 3nd equality we use the identity

$$\sum_s u^s_\alpha(\mathbf{k})\bar{u}^s_\beta(\mathbf{k}) = \Lambda^+_{\alpha\beta}(\mathbf{k}) = i(i\not{k}+m)_{\alpha\beta}$$

In the last equality we identify $G_+(x-y) = \int \frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{ik\cdot(x-y)}$.

Similarly, we have:

$$\begin{aligned}
\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|c^s_\mathbf{k}c^{\dagger r}_{\mathbf{k}'}|0\rangle v^r_\alpha(\mathbf{k})\bar{v}^s_\beta(\mathbf{k}')e^{-i(k\cdot x-k'\cdot y)} \\
&= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{d\!\!{}^-{}^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')v^r_\alpha(\mathbf{k})\bar{v}^s_\beta(\mathbf{k}')e^{-i(k\cdot x-k'\cdot y)} = \int \frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}\Lambda^-_{\alpha\beta}(\mathbf{k})e^{-ik\cdot(x-y)} \\
&= -i(\not{\partial}+m)_{\alpha\beta}\int \frac{d\!\!{}^-{}^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-ik\cdot(x-y)} = -i(\not{\partial}+m)_{\alpha\beta}G_+(y-x)
\end{aligned}$$

Now we compute the Feynman Green's function:

$$\begin{aligned}
D_F^{\alpha\beta}(x-y) &= \theta(x^0-y^0)\langle 0|\psi^\alpha(x)\bar{\psi}^\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}^\beta(y)\psi^\alpha(x)|0\rangle \\
&= \theta(x^0-y^0)i(\not{\partial}_x+m)^{\alpha\beta}G_+(x-y) + \theta(y^0-x^0)i(\not{\partial}_x+m)^{\alpha\beta}G_+(y-x) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y) - i(\not{\partial}_x\theta(x^0-y^0))G_+(x-y) - i(\not{\partial}_x\theta(y^0-x^0))G_+(y-x) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y) - i\gamma^0(\delta(x^0-y^0)G_+(x-y) - \delta(x^0-y^0)G_+(y-x)) \\
&= i(\not{\partial}_x+m)^{\alpha\beta}G_F(x-y)
\end{aligned}$$

In the second last line we use that $\partial_{x^0}\theta(x^0-y^0) = -\partial_{x^0}\theta(y^0-x^0) = \delta(x^0-y^0)$. In the last line, we use that $G_+(0,\mathbf{x}-\mathbf{y}) = G_+(0,\mathbf{y}-\mathbf{x})$, which can easily be seen from the formula for G_+ above. This leaves us with the resired result.

Question 3: Chiral and Majorana Fermions. (50 points)
We consider the chiral representation, and write a Dirac spinor in terms of 2 chiral spinors ψ_L, ψ_R as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

(a) Show that under a rotation with parameters $\omega_{ij} = \epsilon_{ijk}\theta_k$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R(x)$$

A Dirac fermion transforms as

$$\psi'(x') = S(\Lambda)\psi(x) = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi(x)$$

In the chiral basis,

$$\Sigma^{ij} = \frac{i}{4}\left[\begin{pmatrix} 0 & i\bar{\sigma}^i \\ i\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\bar{\sigma}^j \\ i\sigma^j & 0 \end{pmatrix}\right] = \frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix}$$

Therefore for a rotation,

$$\begin{aligned} \psi'_L(x') &= e^{\frac{1}{8}\omega_{ij}[\sigma^i,\sigma^j]}\psi_L(x) = e^{\frac{1}{8}\epsilon_{ijk}\theta_k(2i\epsilon_{ijl}\sigma_l)}\psi_L(x) = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L(x) \\ \psi'_R(x') &= e^{\frac{1}{8}\omega_{ij}[\sigma^i,\sigma^j]}\psi_R(x) \qquad\qquad\qquad\qquad\quad = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R(x) \end{aligned}$$

(b) Show that under a boost with parameters $\omega_{0i} = \beta_i$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R(x)$$

Proceeding analogously, in the chiral basis,

$$\Sigma^{0i} = \frac{i}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\bar{\sigma}^i \\ i\sigma^i & 0 \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = -\Sigma^{i0}$$

Therefore for a boost,

$$\begin{aligned} \psi'_L(x') &= e^{-\frac{1}{4}(\omega_{0i}\sigma^i - \omega_{i0}\sigma^i)}\psi_L(x) = e^{-\frac{1}{2}\beta_i\sigma^i}\psi_L(x) = e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L(x) \\ \psi'_R(x') &= e^{+\frac{1}{4}(\omega_{0i}\sigma^i - \omega_{i0}\sigma^i)}\psi_L(x) = e^{+\frac{1}{2}\beta_i\sigma^i}\psi_R(x) = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R(x) \end{aligned}$$

(c) The Lagrangian density fo the Dirac theory contains a mass term of the form

$$\mathcal{L} = \cdots + im\bar{\psi}\psi = \cdots - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

Using the transformations in (a), (b), show that the mass term is Lorentz invariant, while a term of the form $m\psi_L^\dagger\psi_L$ or $m\psi_R^\dagger\psi_R$ is not.

We expand the Lagrangian into chiral spinors, in the chiral representation:

$$\begin{aligned} \mathcal{L} = -i\bar{\psi}(\not{\partial} - m)\psi &= -i(\psi_L^\dagger, \psi_R^\dagger)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} -m & i\bar{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \end{aligned}$$

Under a rotation, $(\psi_L, \psi_R) \to (e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_L, e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}\psi_R)$, and $(\psi_L^\dagger, \psi_R^\dagger) \to (\psi_L^\dagger e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}, \psi_R^\dagger e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}})$ by Hermiticity. The mass terms are both invariant. Under a boost, $(\psi_L, \psi_R) \to (e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L, e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R)$, and $(\psi_L^\dagger, \psi_R^\dagger) \to (\psi_L^\dagger e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}, \psi_R^\dagger e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}})$. The mass terms are again invariant, establishing full Lorentz invariance.
However, under a boost $\psi_L^\dagger\psi_L$ or $\psi_R^\dagger\psi_R$ are not invariant:

$$\psi_L^\dagger\psi_L \to \psi_L^\dagger e^{-\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_L, \qquad \psi_R^\dagger\psi_R \to \psi_R^\dagger e^{+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}}\psi_R$$

(d) The discussion in (c) may give the impression that it is not possible to write down a mass term with ψ_L or ψ_R alone. This is possible, with some more sophistication. For this purpose, first show that

$$\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^*$$

From this, show that $\sigma^2\psi_L^*$ transforms under Lorentz transformations in the same was as ψ_R.

We can check directly, either by explicit multiplication or using $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, that

$$\sigma^2\sigma^1\sigma^2 = -\sigma^1 = -(\sigma^1)^*, \qquad \sigma^2\sigma^3\sigma^2 = -\sigma^3 = -(\sigma^3)^*, \qquad \sigma^2\sigma^2\sigma^2 = \sigma^2 = -(\sigma^2)^*$$

Now consider the transformation properties of $\sigma^2\psi_L^*$.
Under rotations,

$$\sigma^2\psi_L'^*(x') = \sigma^2(e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}})^*\psi_L^*(x) = \sigma^2 e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*}\sigma^2\sigma^2\psi_L^*(x) = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*}\sigma^2\psi_L^*(x)$$

In the last equality we make use of the above identity. This can be made more precise by expanding the exponential as a power series, inserting $1 = \sigma^2\sigma^2$ between powers of $\boldsymbol{\sigma}$, using said identity, and resumming the series.
Under boosts,

$$\sigma^2\psi_L'^*(x') = \sigma^2(e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}})^*\psi_L^*(x) = \sigma^2 e^{-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^*}\sigma^2\sigma^2\psi_L^*(x) = e^{+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^*}\sigma^2\psi_L^*(x)$$

Therefore, $\sigma^2\psi_L^*$ transforms like ψ_R.

(e) From the observation in (d), construct a mass term using ψ_L alone, which is both Lorentz invariant and real. This is called the Majorana mass term. Show that it is identically zero if ψ_L consists of ordinary functions, while it is non-zero if ψ_L are anticommuting variables.

Consider the mass term $m\psi_L^\dagger\sigma^2\psi_L^*$. This is Lorentz invariant, as we showed in (c) that $\psi_L^\dagger\psi_R$ is invariant, and in (d) that $\sigma^2\psi_L^*$ transforms in the same way as ψ_R. To make it real, we add its Hermitian conjugate, which must also be Lorentz invariant. Therefore, consider

$$\mathcal{L}_{\chi m} = -\frac{1}{2}m\psi_L^\dagger\sigma^2\psi_L^* - \frac{1}{2}(m\psi_L^\dagger\sigma^2\psi_L^*)^* = -\frac{m}{2}(\psi_L^\dagger\sigma^2\psi_L^* + \psi_L^T\sigma^2\psi_L)$$

To see whether this vanishes, we write the second term in components:

$$\psi_L^T\sigma^2\psi_L = \psi_{La}(\sigma^2)^a{}_b\psi_L^b = (-1)^\epsilon\psi_L^b(\sigma^2)^a{}_b\psi_{La} = -(-1)^\epsilon\psi_L^b(\sigma^2)_b{}^a\psi_{La} = -(-1)^\epsilon\psi_L^T\sigma^2\psi_L$$

Here we introduce the notation $(-1)^\epsilon$, where $\epsilon = 0$ for regular functions, and $\epsilon = -1$ for Grassmann functions. When $\epsilon = 0$ we see that the term is the negative of itself, and vanishes. The same holds for its Hermitian conjugate, thus the Lagrangian vanishes identically. If the field is Grassmann, we get no such constraint.

(f) Now write down a Lorentz invariant full action using ψ_L alone, including both kinetic and mass terms. Write down equations of motion.

Consider now the Lagrangian

$$\mathcal{L} = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{m}{2}(\psi_L^\dagger \sigma^2 \psi_L^* + \psi_L^T \sigma^2 \psi_L)$$

Treating ψ_L and ψ_L^* as independent variables, the equations of motion are

$$i\sigma^\mu \partial_\mu \psi_L - m\sigma^2 \psi_L^* = 0$$
$$-i\partial_\mu \psi_L^\dagger \bar{\sigma}^\mu - m\psi_L^T \sigma^2 = 0$$

(g) Does the theory of part (f) have a conserved charge? Argue that such a theory can only describe neutral particles.

The theory has the usual conserved charges corresponding to spacetime and Poincaré symmetry. However, there are no internal symmetries, as the mass term is not invariant under a U(1) rotation $\psi \to e^{i\alpha}\psi$. Therefore, there is no U(1) conserved charge, and particle number is not conserved. This cannot describe charged particles, as it would violate charge conservation.

Question 4: Majorana Fermions (10 points)

In the Majorana representation, γ^μ are real, and thus ψ can be chosen to be real. Such a spinor is called a Majorana spinor. This has important physical consequences. Upon quantization, being real, a Majorana particle should not have an antiparticle (or equivalently, it is its own antiparticle).

We discussed in lecture that the concept of a Majorana spinor can be generalized to any representation of γ. If we can find a matrix B satisfying

$$B\gamma^\mu B^{-1} = (\gamma^\mu)^*$$

the Majorana condition is

$$\psi^* = B\psi$$

(a) In lecture we showed that in the chiral representation we can choose $B = \gamma^2$. Solve the Majorana condition in the chiral representation. Show that in this representation a Majorana spinor ψ can be expressed in terms of ψ_L alone.

For $B = \gamma^2$, the Majorana condition in the chiral representation is

$$B\psi = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} = \psi^*$$

This gives us $\psi_L^* = -i\sigma^2\psi_R$, and $\psi_R^* = i\sigma^2\psi_L$. Therefore $\psi_R = -i(\sigma^2)^*\psi_L^* = i\sigma^2\psi_L^*$, and we can write ψ in terms of purely ψ_L:

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix}$$

(b) Plug in the expression (in terms of ψ_L) for the Majorana spinor ψ from (a) into the Dirac action. Show it reduces to the action in part 3(f).

The Dirac Lagrangian becomes:

$$\begin{aligned}\mathcal{L} = -i\bar{\psi}(\partial\!\!\!/ - m)\psi &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_R^\dagger\bar{\sigma}^\mu\partial_\mu\psi_R - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \\ &= i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L + i\psi_L^T\sigma^2\bar{\sigma}^\mu\sigma^2\partial_\mu\psi_L^* - im(\psi_L^\dagger\sigma^2\psi_L^* - \psi_L^T\sigma^2\psi_L)\end{aligned}$$

To write this in a more familiar form we integrate the second term by parts and discarding the boundary contribution.

$$\begin{aligned}i\psi_L^T\sigma^2\bar{\sigma}^\mu\sigma^2\partial_\mu\psi_L^* = -i(\partial_\mu\psi_L^T)\sigma^2\bar{\sigma}^\mu\sigma^2\psi_L^* &= -i((\partial_\mu\psi_L^T)\sigma^2\bar{\sigma}^\mu\sigma^2\psi_L^*)^T = i\psi_L^\dagger(\sigma^2\bar{\sigma}^\mu\sigma^2)^T\partial_\mu\psi_L \\ &= i\psi_L^\dagger(\sigma^2\bar{\sigma}^\mu\sigma^2)^*\partial_\mu\psi_L = i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L\end{aligned}$$

In the third equality, we pick up a sign when we take the transpose because the fermionic fields anticommute. In the 4th equality we use the Hermiticity of the σ^μ's. In the last equality we use the relation from 3(d) that $(\sigma^2\bar{\sigma}^\mu\sigma^2)^* = \sigma^\mu$.

Therefore, the Lagrangian becomes:

$$\mathcal{L} = 2i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - im(\psi_L^\dagger\sigma^2\psi_L^* - \psi_L^T\sigma^2\psi_L)$$

This is identical to the Majorana action from problem 3(f) with the field redefinition $\psi_L \to \sqrt{2}e^{-i\pi/4}\psi_L$.

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 10 Solutions

April 25, 2023

Question 1: Chiral Symmetry (15 points)

Consider the Dirac action with $m = 0$. (a) Show that the action is invariant under transformations

$$\psi \to e^{i\alpha\gamma^5}\psi$$

The Dirac conjugate transforms as:

$$\bar{\psi}' = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = \psi^\dagger\gamma^0 e^{i\alpha\gamma_5} = \bar{\psi}e^{i\alpha\gamma_5}$$

where we use that $\{\gamma_5, \gamma^\mu\} = 0$. The second equality can be made more epxlicit by expanding the exponential, anticommuting the γ^0 through each term, and resummiong. Therefore, the massless Dirac Lagrangian transforms as

$$\mathcal{L}' = -i\bar{\psi}'\partial\!\!\!/\psi' = -i\bar{\psi}e^{i\alpha\gamma_5}\gamma^\mu e^{i\alpha\gamma^5}\psi\partial_\mu\psi = -i\bar{\psi}\gamma^\mu e^{-i\alpha\gamma_5}e^{i\alpha\gamma^5}\psi\partial_\mu\psi = -i\bar{\psi}\partial\!\!\!/\psi = \mathcal{L}$$

Therefore, the action is invariant.

(b) Construct the Noether current for the above symmetry.

Under an infinitesmial chiral rotation, $\delta\psi = i\alpha\gamma^5\psi$. Noting from (a) that the Lagrangian is invariant, the Noether current is thus:

$$j_5^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\gamma^5\psi$$

(c) Find how the mass term $m\bar{\psi}\psi$ transforms. Is it invariant?

The mass term transforms as:

$$m\bar{\psi}\psi \to m\bar{\psi}'\psi' = m\bar{\psi}e^{i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = e^{2i\alpha\gamma_5}m\bar{\psi}\psi$$

Since α is arbitrary, this is not left invariant under the chiral rotation.

Question 2: Quantizingf the Theory of Majorana Fermions (25 points)
Consider the theory of Majorana fermions discussed in Problem Set 9, written in terms of a 2-component complex spinor ψ_L

$$\mathcal{L}_L = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{m}{2}(\psi_L^T \sigma^2 \psi_L + \psi_L^\dagger \sigma^2 \psi_L^*)$$

where ψ^T denotes the transpose of ψ, and $\sigma^\mu = (1, \vec{\sigma})$.
(a) Write down the equal time canonical quantization relations.
For the Majorana Lagrangian above, the conjugate momentum to ψ_L is

$$\pi_L = i\psi_L^\dagger$$

Therefore, the canonical anticommutation relations are

$$\{\psi_{La}(t,\mathbf{x}), \psi_{Lb}(t,\mathbf{x}')\} = \{\psi_{La}^\dagger(t,\mathbf{x}), \psi_{Lb}^\dagger(t,\mathbf{x}')\} = 0, \qquad \{\psi_{La}(t,\mathbf{x}), \psi_{Lb}^\dagger(t,\mathbf{x}')\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

(b) Write down the equations of motion. In momentum space a general solution can be written as

$$\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}$$

Using this notation, write down a complete basis of solutions in the rest frame $\mathbf{p} = 0$.
The equations of motion are given by:

$$i\sigma^\mu \partial_\mu \psi_L - m\sigma^2 \psi_L^* = 0, \qquad -i\partial_\mu \psi_L^\dagger \sigma^\mu - m\psi_L^T \sigma^2 = 0$$

Using the ansatz $\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}$, we obtain the momentum space equation

$$[-p_\mu \sigma^\mu u_L(p) - m\sigma^2 v_L^*(p)]e^{ip\cdot x} + [p_\mu \sigma^\mu v_L(p) - m\sigma^2 u_L^*(p)]e^{-ip\cdot x} = 0$$

Both positive and negative frequency parts must vanish independently, which gives

$$p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0, \qquad p_\mu \sigma^\mu v_L(p) - m\sigma^2 u_L^*(p) = 0$$

Note that these are not independent: the first equation implies the second, as

$$p_\mu \sigma^\mu v_L = -\frac{1}{m}(p\cdot\sigma)\sigma_2(p\cdot\sigma)^* u_L^* = -\frac{1}{m}(p\cdot\sigma)(p\cdot\bar{\sigma})\sigma_2 u_L^* = -m\sigma_2 u_L^*$$

where in the first equality we use the complex conjugate form of $p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0$, in the second we use from Problem Set 9 that $\sigma_2 \sigma^* \sigma_2 = -\sigma_2$, and in the last equality we use that $(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2$.

Thus, without loss of generality, we look for solutions to $p_\mu \sigma^\mu u_L(p) + m\sigma^2 v_L^*(p) = 0$. In the rest frame, this becomes

$$u_L(\mathbf{0}) = \sigma_2 v_L(\mathbf{0})^* \quad\Longrightarrow\quad v_L(\mathbf{0}) = -\sigma_2 u_L^*(\mathbf{0})$$

This relates v_L to u_L. We still have the freedom to specify the polarization $u_L(\mathbf{0})$. First, consider the normalized eigenvectors $\xi_{\{s\in\pm\}}$ of σ^3. That is, $\sigma^3 \xi_\pm = \pm 1$, and $\xi_s^\dagger \xi_r = \delta_{rs}$. We let this be our solution space of $u_L(\mathbf{0})$, and set:

$$u_{Ls}(\mathbf{0}) = \sqrt{m}\xi_s, \quad v_{Ls} = -\sqrt{m}\sigma^2 \xi_s, \qquad \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are normalized so that $u_{Lr}^\dagger(\mathbf{0})u_{Lr}(\mathbf{0}) = m\delta_{rs}$.

(c) Verify the following expresssions give a complete basis of solutions for general p:

$$u_s(p) = \sqrt{-p\cdot\bar{\sigma}}\xi_s, \qquad v_s(p) = -\sqrt{-p\cdot\bar{\sigma}}\sigma^2\xi_s$$

where ξ_s, $s = \pm$ are respectively eigenvectors of σ^3 with eigenvalues ± 1.
By the discussion in (b), to check these spinors are solutions of the equations of motion, it is sufficient to check that $p\cdot\sigma u_L(p) + m\sigma^2 v_L^*(p) = 0$. We compute:

$$\begin{aligned}
p\cdot\sigma u_L(p) + m\sigma^2 v_L^*(p) &= (p\cdot\sigma)(-p\cdot\bar{\sigma})^{1/2}\xi_s + m\sigma^2(-\sqrt{-p\cdot\bar{\sigma}}\sigma^2\xi_s)^* \\
&= -(-p\cdot\sigma)^{1/2}(-p\cdot\sigma)^{1/2}(-p\cdot\bar{\sigma})^{1/2}\xi_s + m\sigma^2(\sqrt{-p\cdot\bar{\sigma}})^*\sigma^2\xi_s \\
&= -m\sqrt{-p\cdot\sigma}\xi_s + m(\sqrt{-p\cdot\sigma})\xi_s = 0
\end{aligned}$$

In the second line we use $p\cdot\sigma = -(-p\cdot\sigma)^{1/2}(-p\cdot\sigma)^{1/2}$. In the third line, we use that $\sqrt{-p\cdot\sigma}\sqrt{-p\cdot\bar{\sigma}} = m$. We also use that $\sigma^2(\sqrt{-p\cdot\bar{\sigma}})^*\sigma^2 = \sqrt{-p\cdot\sigma}$: this can be showed by expanding the square root as a Taylor series, and repeatedly applying the identity $\sigma_2\sigma^*\sigma_2 = -\sigma_2$ from Problem Set 9.

We show that $u_s(p)$ give a complete basis of solutions for any p. Solutions only exist on the mass-shell, so without loss of generality consider any fixed $p = (\omega_{\mathbf{p}}, \mathbf{p})$. It follows from the discussion in (b) that the solution space is 2-dimensional. Furthermore,

$$u_+^\dagger(\mathbf{p})u_-(-\mathbf{p}) = \xi_+\sqrt{-p\cdot\bar{\sigma}}^\dagger\sqrt{-p\cdot\sigma}\xi_- = \xi_+\sqrt{-p\cdot\bar{\sigma}}\sqrt{-p\cdot\sigma}\xi_- = m\xi_+\xi_- = 0$$

Therefore, the $u_\pm(\mathbf{p})$ are always orthogonal. Since neither vanish for any $\mathbf{p}$, they are always linearly independent, and thus span the solution space for given momentum.

(d) Write down the mode expansion for the quantum operator ψ_L
The mode expansion is given by

$$\begin{aligned}
\psi_L(x) &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}^{(s)}u_{Ls}(\mathbf{k})e^{ik\cdot x} + a_{\mathbf{k}}^{\dagger(s)}v_{Ls}(\mathbf{k})e^{-ik\cdot x}\right) \\
\psi_L^\dagger(x) &= \int \frac{d\!\!{}^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}^{\dagger(s)}u_{Ls}^\dagger(\mathbf{k})e^{-ik\cdot x} + a_{\mathbf{k}}^{(s)}v_{Ls}^\dagger(\mathbf{k})e^{ik\cdot x}\right)
\end{aligned}$$

where the relation between $u_L(\mathbf{p})$ and $v_L(\mathbf{p})$ is given in part (b).

(e) Define the vacuum and construct the single particle states, with proper normalization. Discuss the differences between the particles in this theory and those of the Dirac theory.
The vacuum is defined by the state annihilated by all the $a_{\mathbf{k}}^{(s)}$'s, i.e. $a_{\mathbf{k}}^{(s)}|0\rangle = 0$.
The single particle states are defined as usual (with Lorentz invariant inner product):

$$|k, s\rangle = \sqrt{2\omega_{\mathbf{k}}}a_{\mathbf{k}}^{\dagger(s)}|0\rangle$$

Note that the theory for Majoranas (as constrasted with the Dirac theory) has only particles with no antiparticles. Equivalently, each particle is its own antiparticle.

Question 3: Gaussian Integrals for Grassmann Variables (16 points)
Show the following identities:

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} = \det A$$
$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \theta_k \theta_l^* e^{-\theta_i^* A_{ij} \theta_j} = (A^{-1})_{kl} \det A$$

We first expand out the exponential as a Taylor series:

$$e^{-\theta_i^* A_{ij} \theta_j} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \prod_{k=1}^{n} A_{i_k j_k} \theta_{i_k}^* \theta_{j_k} = \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} A_{i_k j_k} \theta_{j_k} \theta_{i_k}^*$$

Note that any term of order $k > N$ will vanish: there must be at least 2 of some θ_k, thus the term is zero because $\theta_k^2 = 0$. Furthermore, any term with $k < N$ vanishes when taking the $2N$ integrals, because $\int d\theta \, 1 = 0$. Therefore only the Nth order term survives:

$$\begin{aligned}
\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} &= \frac{1}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) (\theta_{j_1} \theta_{i_1}^*) \cdots (\theta_{j_N} \theta_{i_N}^*) \\
&= \frac{(-1)^{N(N-1)/2}}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) (\theta_{j_1} \cdots \theta_{j_N})(\theta_{i_1}^* \cdots \theta_{i_N}^*) \\
&= \frac{(-1)^{N(N-1)/2}}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \epsilon_{j_1 \cdots j_N} \epsilon_{i_1 \cdots i_N} (\theta_1 \cdots \theta_N)(\theta_1^* \cdots \theta_N^*) \\
&= \frac{1}{N!} A_{i_1 j_1} \cdots A_{i_N j_N} \epsilon_{j_1 \cdots j_N} \epsilon_{i_1 \cdots i_N} \int d\theta_N^* d\theta_N \cdots d\theta_1^* d\theta_1 (\theta_1 \theta_1^*) \cdots (\theta_N \theta_N^*) \\
&= \frac{1}{N!} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} A_{i_1 j_1} \cdots A_{i_N j_N} = \det A
\end{aligned}$$

In line 2, we have moved all of the θ's to the left and θ^*'s to the right, picking up a total factor of $(-1)^{(N-1)+(N-2)+\cdots+1} = (-1)^{N(N-1)/2}$. In line 3 we use that $\theta_{i_1} \cdots \theta_{i_N} = \epsilon_{i_1 \cdots i_N} \theta_1 \cdots \theta_N$, and likewise for the θ^*'s. In line 4 we change the order of the Grassmann variables again, picking up another factor of $(-1)^{N(N-1)/2}$ cancelling the factor from before. Note also that it doesn't matter what order we place the $d\theta_i^* d\theta_i$'s, since pairs of Grassmann variables always commute with each other. In line 5 we do the Grassmann integrals successively. Finally we recognize the Leibniz formula for the determinant.

To obtain the second equation, we differentiate both sides of the first equation by A_{lk}. On the left hand side we get

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} (-\theta_l^* \theta_k) = \int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} \theta_k \theta_l^*$$

On the right-hand side we can Jacobi's formula, $\delta \det A = \det A \; \mathrm{Tr}(A^{-1} \delta A) = (A^{-1})_{kl} \det A$, for δA zero everywhere except the kl-th entry. Therefore,

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij} \theta_j} \theta_k \theta_l^* = (A^{-1})_{kl} \det A$$

Question 4: Yukawa Theory (24 points)
Consider the Yukawa theory discussed in lecture,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = i\bar{\psi}(\partial\!\!\!/ - m)\psi = g\phi\bar{\psi}\psi$$

Denote the propagator of a ϕ particle by a dashed line, and that of ψ by a solid line (with arrow). We will call p the particle excitation of ψ, and $\bar{p}$ the antiparticle excitation of ψ.
(a) Consider the process

$$\bar{p} + \bar{p} \to \bar{p} + \bar{p}$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1) (p_2, s_2) and (p_1', s_1') (p_2', s_2') respectively.

We identify the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$. There is an t-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\begin{aligned}\mathcal{M} &= ig^2\left[\frac{\bar{v}_{s_1}(p_1)v_{s_1'}(p_1')\bar{v}_{s_2}(p_2)v_{s_2'}(p_2')}{(p_1 - p_1')^2 + m_\phi^2 - i\epsilon} - \frac{\bar{v}_{s_1}(p_1)v_{s_2'}(p_2')\bar{v}_{s_2}(p_2)v_{s_1'}(p_1')}{(p_1 - p_2')^2 + m_\phi^2 - i\epsilon}\right]\\ &= -ig^2\left[\frac{\bar{v}_{s_1}(p_1)v_{s_1'}(p_1')\bar{v}_{s_2}(p_2)v_{s_2'}(p_2')}{t - m_\phi^2 + i\epsilon} - \frac{\bar{v}_{s_1}(p_1)v_{s_2'}(p_2')\bar{v}_{s_2}(p_2)v_{s_1'}(p_1')}{u - m_\phi^2 + i\epsilon}\right]\end{aligned}$$

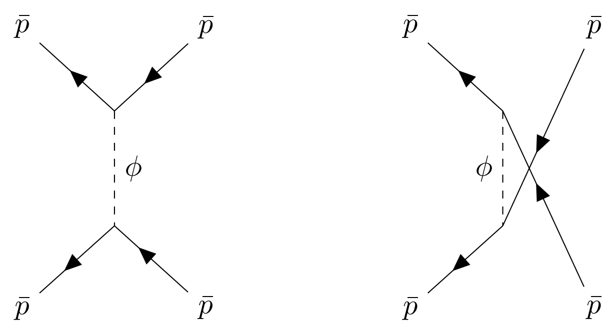


Figure 1: $\bar{p}\bar{p} \to \bar{p}\bar{p}$ scattering

(b) Consider the process

$$p + \bar{p} \to \phi + \phi$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1) (p_2, s_2) and p_1', p_2'.

There is an t-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\mathcal{M} = g^2\left[\frac{\bar{v}_{s_2}(p_2)(m_p + i(p\!\!\!/_1 - p\!\!\!/_1'))u_{s_1}(p_1)}{t - m_p^2 + i\epsilon} + \frac{\bar{v}_{s_2}(p_2)(m_p + i(p\!\!\!/_1 - p\!\!\!/_2'))u_{s_1}(p_1)}{u - m_p^2 + i\epsilon}\right]$$

We have made use of the expression for the fermion propagator, $-\dfrac{1}{ik\!\!\!/ - m + i\epsilon} = \dfrac{m + ik\!\!\!/}{k^2 + m^2 - i\epsilon}$.

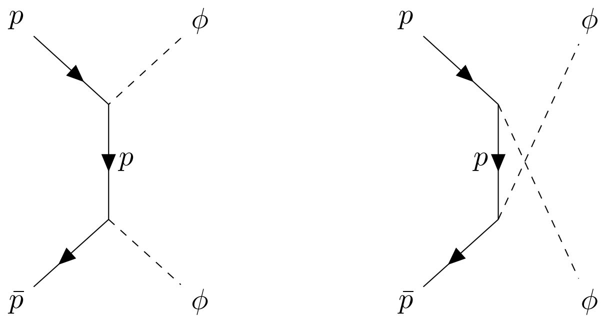


Figure 2: $p\bar{p} \to \phi\phi$ scattering

(c) Consider the process

$$p + \phi \to p + \phi$$

Draw the lowest order Feynman diagrams, and write down the corresponding scattering amplitude. Take the initial and final states to have momenta and polarizations (p_1, s_1), p_2 and (p_1', s_1'), p_2'.

There is an s-channel and a u-channel diagram. Using the Feynman rules, the amplitude is:

$$\mathcal{M} = g^2 \left[\frac{\bar{u}_{s_1'}(p_1')(m_p + i(\not{p}_1 + \not{p}_2))u_{s_1}(p_1)}{s - m_p^2 + i\epsilon} + \frac{\bar{u}_{s_1'}(p_1')(m_p + i(\not{p}_1 - \not{p}_2'))u_{s_1}(p_1)}{u - m_p^2 + i\epsilon} \right]$$

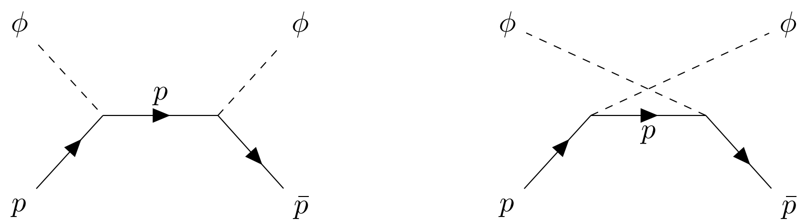


Figure 3: $p\phi \to p\phi$ scattering

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 11 Solutions

May 2, 2023

Question 1: Quantization of Maxwell Theory in the Lorentz Gauge (50 points)
In the Lorentz gauge we consider the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2$$

where ξ is an arbitrary real parameter (and different ξ's give equivalent theories). It is convenient to take $\xi = 1$, in which case

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$$

The complete set of solutions to the operator equations are

$$A_\mu(x) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \sum_{\alpha=0}^{3} \left[\mathcal{E}_\mu^{(\alpha)} a_\mathbf{k}^{(\alpha)} e^{ik\cdot x} + \mathcal{E}_\mu^{(\alpha)*} a_\mathbf{k}^{(\alpha)\dagger} e^{-ik\cdot x}\right]$$

where $\omega_\mathbf{k} = |\mathbf{k}|$, and $k^\mu = (|\mathbf{k}|, \mathbf{k})$. The polarization vectors $\mathcal{E}_\mu^{(\alpha)}$ are defined by

$$\mathcal{E}_\mu^{(0)} = (1, \mathbf{0}), \qquad \mathcal{E}_\mu^{(3)} = (0, \mathbf{k}/|\mathbf{k}|), \qquad \mathcal{E}_\mu^{(1,2)} = (0, \boldsymbol{\epsilon}_{1,2}), \qquad \boldsymbol{\epsilon}_{1,2}\cdot\mathbf{k} = 0$$

where $\boldsymbol{\epsilon}_{1,2}$ are orthogonal unit-norm spatial vectors.

With the canonical commutation relations

$$[A_\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{x}')] = i\delta_\mu^\nu \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \qquad \pi^\mu = \partial_0 A^\mu$$

we find that

$$[a_\mathbf{k}^{(\alpha)}, a_{\mathbf{k}'}^{(\beta)\dagger}] = \eta^{\alpha\beta}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

with all other commutators vanishing. We define the vacuum as

$$a_\mathbf{k}^{(\alpha)}|0\rangle = 0, \qquad \forall \alpha, \mathbf{k}$$

and the 'big' Hilbert space is defined as

$$\mathcal{H}_{\text{big}} = \{|\psi\rangle \text{ obtained by acting } a_\mathbf{k}^{(\alpha)\dagger} \text{ on } |0\rangle\}$$

As discussed in lecture, $\mathcal{H}_{\text{big}}$ contains states with negative norms, and thus unphysical states. Indeed $\mathcal{H}_{\text{big}}$ follows from the gauge-fixed Lagrangian, which by itself is not Maxwell theory.

To obtain the Maxwell theory, we still need to impose $\partial_\mu A^\mu = 0$ and get rid of the residual gauge freedom. We will see in this problem that by imposing $\partial_\mu A^\mu = 0$ we get rid of the negative-norm unphysical states in $\mathcal{H}_{\text{big}}$. But this is not enough: we will observe that some states possess zero norm, which can be attributed to the presence of residual gauge freedom. By eliminating these null states, we obtain the physical Hilbert space, which contains only 2 transverse massless degrees of freedom rather than 4.

The enforcement of $\partial_\mu A^\mu$ at the quantum level is subtle. Remember that in the Coulomb gauge, classically we apply the gauge condition $\nabla_i A_i = 0$ as part of the equations of motion, which becomes an operator equation at the quantum level. In the Lorentz gauge, classically we only need to impose 'boundary conditions' to ensure that the equation $\partial^2(\partial_\mu A^\mu) = 0$ has the trivial solution $\partial_\mu A^\mu = 0$. This implies that, at the quantum level, we cannot enforce $\partial_\mu A^\mu = 0$ as an operator equation. Instead, we will have to do something weaker, imposing a variant of it as a condition on the physical states.

(a) Calculate

$$[A_0(t, \mathbf{x}), \partial_\mu A^\mu(t, \mathbf{x}')]$$

and from your result explain why we cannot impose $\partial_\mu A^\mu = 0$ as an operator equation.

In the mode expansion for A_0, only the $\alpha = 0$ components survive, since $\mathcal{E}^{(0)}$ is the only polarization with a time-like component.

$$A_0 = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left[a_\mathbf{k}^{(0)} e^{ik\cdot x} + a_\mathbf{k}^{(0)\dagger} e^{-ik\cdot x} \right]$$

Next, we compute $\partial_\mu A^\mu$ in the mode expansion.

$$\begin{aligned}
\partial_\mu A^\mu &= i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \sum_{\alpha=0}^{3} \left[(k \cdot \mathcal{E}^{(\alpha)}) a_\mathbf{k}^{(\alpha)} e^{ik\cdot x} - (k \cdot \mathcal{E}^{(\alpha)*}) a_\mathbf{k}^{(\alpha)\dagger} e^{-ik\cdot x} \right] \\
&= i \int đ^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{2}} \left[(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)}) e^{ik\cdot x} - (a_\mathbf{k}^{(0)\dagger} + a_\mathbf{k}^{(3)\dagger}) e^{-ik\cdot x} \right]
\end{aligned}$$

where we use $k \cdot \mathcal{E}^{(0)} = \omega_\mathbf{k}$, $k \cdot \mathcal{E}^{(1,2)} = 0$, $k \cdot \mathcal{E}^{(3)} = \omega_\mathbf{k}$ by using the explicit form of the polarizations. Therefore,

$$\begin{aligned}
[A_0(t, \mathbf{x}), \partial_\mu A^\mu(t, \mathbf{x}')] &= \frac{i}{2} \int \frac{đ^3\mathbf{k} đ^3\mathbf{k}'}{\sqrt{\omega_\mathbf{k}/\omega_{\mathbf{k}'}}} \left[a_\mathbf{k}^{(0)} e^{ik\cdot x} + a_\mathbf{k}^{(0)\dagger} e^{-ik\cdot x}, (a_{\mathbf{k}'}^{(0)} + a_{\mathbf{k}'}^{(3)}) e^{ik'\cdot x'} - (a_{\mathbf{k}'}^{(0)\dagger} + a_{\mathbf{k}'}^{(3)\dagger}) e^{-ik'\cdot x'} \right] \\
&= -\frac{i}{2} \int \frac{đ^3\mathbf{k} đ^3\mathbf{k}'}{\sqrt{\omega_\mathbf{k}/\omega_{\mathbf{k}'}}} \left([a_\mathbf{k}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}] e^{i(k\cdot x - k'\cdot x')} + [a_{\mathbf{k}'}^{(0)}, a_\mathbf{k}^{(0)\dagger}] e^{i(-k\cdot x + k'\cdot x')} \right) \\
&= -i \int đ^3\mathbf{k} đ^3\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_\mathbf{k}}} (-(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')) e^{i(k\dot{x} - k'\cdot x')} \\
&= i \int đ^3\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')} = i(2\pi)^3 \delta^3(\mathbf{x} - \mathbf{x}')
\end{aligned}$$

In the 3rd equality, we used that the two terms in line 2 are equal, upon change of variables $\mathbf{k} \leftrightarrow \mathbf{k}'$.

We see that the commutator is non-vanishing. This prevents us from imposing $\partial_\mu A^\mu = 0$ as an operator equation, which would imply the commutator vanishes identically.

(b) We also cannot impose that 'physical states' satisfy

$$\partial_\mu A^\mu |\psi\rangle = 0$$

as $\partial_\mu A^\mu |0\rangle \neq 0$, and we want to keep the vacuum as physical. So, to define physical states we need a weaker condition. It turns out the condition eliminating all negative norm states while keeping $|0\rangle$ is

$$\partial^\mu A_\mu^{(-)} |\psi\rangle = 0$$

where $A_\mu^{(-)}$ denotes the annihilation part of A_μ. We now impose that the set of physical states $\mathcal{H}_{\text{small}} \subseteq \mathcal{H}_{\text{big}}$ satisfies this equation. Show that there is no negative-norm state in $\mathcal{H}_{\text{small}}$.

We show that any negative-norm state $|\psi\rangle \in \mathcal{H}_{\text{big}}$ does not satisfy the defining condition $\partial^\mu A_\mu^{(-)}|\psi\rangle = 0$, and thus does not belong to $\mathcal{H}_{\text{small}}$.

From the calculation in part (a), we have:

$$\partial^\mu A_\mu^{(-)} = i\int đ^3\mathbf{k}\sqrt{\frac{\omega_\mathbf{k}}{2}}(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)})e^{ik\cdot x}$$

The negative norm states in $\mathcal{H}_{\text{big}}$ come from applying instances of the raising operator $a_\mathbf{p}^{(0)\dagger}$. We prove the desired result for the negative norm states $|\psi\rangle = a_\mathbf{p}^{(0)\dagger}|\chi\rangle$, where $|\chi\rangle$ is a positive norm state. (Note: this is a sufficient, but not necessary condition for negative norm state, as a negative norm state can be obtained from a linear combination of negative, null, and positive norm states. Taking these into account is more difficult, and is more or less equivalent to doing the rest of the problem. We therefore defer this to part (g)). We compute

$$\begin{aligned}
\partial^\mu A_\mu^{(-)}|\psi\rangle &= i\int đ^3\mathbf{k}\sqrt{\frac{\omega_\mathbf{k}}{2}}(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)})e^{ik\cdot x}a_\mathbf{p}^{(0)\dagger}|\chi\rangle \\
&= -i\sqrt{\frac{\omega_\mathbf{p}}{2}}e^{ip\cdot x}|\chi\rangle + i\int đ^3\mathbf{k}\sqrt{\frac{\omega_\mathbf{k}}{2}}e^{ik\cdot x}a_\mathbf{p}^{(0)\dagger}a_\mathbf{k}^{(3)}|\chi\rangle
\end{aligned}$$

This cannot vanish, as the first term is non-zero, and the second term is orthogonal to the first.

(c) Show that

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0, \qquad \forall|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}$$

That is, $\partial_\mu A^\mu$ has zero matrix element among states in $\mathcal{H}_{\text{small}}$.

Note from the mode expansion in (b) that

$$\partial_\mu A^\mu = i\int đ^3\mathbf{k}\sqrt{\frac{\omega_\mathbf{k}}{2}}\left[(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)})e^{ik\cdot x} - (a_\mathbf{k}^{(0)\dagger} + a_\mathbf{k}^{(3)\dagger})e^{-ik\cdot x}\right] = \partial^\mu A_\mu^{(-)} - \partial^\mu A_\mu^{(-)\dagger}$$

Therefore, for $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}$ we have

$$\langle\psi'|\partial^\mu A_\mu|\psi\rangle = \langle\psi'|\partial^\mu A_\mu^{(-)}\psi\rangle - \langle\partial^\mu A_\mu^{(-)}\psi'|\psi\rangle = 0 - 0 = 0$$

(d) Introduce

$$b_\mathbf{k}^{(\pm)} = \frac{1}{\sqrt{2}}(a_\mathbf{k}^{(3)} \pm a_\mathbf{k}^{(0)}), \qquad b_\mathbf{k}^{(\pm)\dagger} = \frac{1}{\sqrt{2}}(a_\mathbf{k}^{(3)\dagger} \pm a_\mathbf{k}^{(0)\dagger})$$

Show that the physical state condition can be written as

$$b_\mathbf{k}^{(+)}|\psi\rangle = 0$$

From the mode expansion in (b), the physical state condition is

$$\partial^\mu A_\mu^{(-)}|\psi\rangle = i\int đ^3\mathbf{k}\sqrt{\frac{\omega_\mathbf{k}}{2}}e^{ik\cdot x}(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)})|\psi\rangle = i\int đ^3\mathbf{k}\sqrt{\omega_\mathbf{k}}e^{ik\cdot x}b_\mathbf{k}^{(+)}|\psi\rangle = 0$$

Since the states $b_\mathbf{k}^{(+)}|\psi\rangle$, $b_{\mathbf{k}'}^{(+)}|\psi\rangle$ are orthogonal for $\mathbf{k} \neq \mathbf{k}'$, for the above equation to hold we must have $b_\mathbf{k}^{(+)}|\psi\rangle = 0$ for all $\mathbf{k}$.

(e) We are not yet done, as $\mathcal{H}_{\text{small}}$ still contains zero-norm states. To see this, work out the commutators

$$[b_{\mathbf{k}}^{(+)}, b_{\mathbf{k}'}^{(+)\dagger}], \qquad [b_{\mathbf{k}}^{(-)}, b_{\mathbf{k}'}^{(-)\dagger}], \qquad [b_{\mathbf{k}}^{(+)}, b_{\mathbf{k}'}^{(-)\dagger}], \qquad [b_{\mathbf{k}}^{(-)}, b_{\mathbf{k}'}^{(+)\dagger}]$$

and show that $\mathcal{H}_{\text{small}}$ can also be described as

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } a_{\mathbf{k}}^{(1)\dagger}, a_{\mathbf{k}}^{(2)\dagger}, b_{\mathbf{k}}^{(+)\dagger} \text{ on } |0\rangle\}$$

In other words, a physical state can have no $b_{\mathbf{k}}^{(-)\dagger}$ excitations.

The operators have the commutation relations

$$[b_{\mathbf{k}}^{(\pm)}, b_{\mathbf{k}'}^{(\pm)\dagger}] = \frac{1}{2}([a_{\mathbf{k}}^{(3)}, a_{\mathbf{k}'}^{(3)\dagger}] + [a_{\mathbf{k}}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}]) = 0$$
$$[b_{\mathbf{k}}^{(\pm)}, b_{\mathbf{k}'}^{(\mp)\dagger}] = \frac{1}{2}([a_{\mathbf{k}}^{(3)}, a_{\mathbf{k}'}^{(3)\dagger}] - [a_{\mathbf{k}}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}]) = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$$

From (d), the condition for $|\psi\rangle \in \mathcal{H}_{\text{small}}$ is

$$b_{\mathbf{k}}^{(+)}|\psi\rangle = 0, \ \forall \mathbf{k}$$

Suppose $|\psi\rangle$ is generated by applying some number of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$, $b_{\mathbf{k}}^{(+)\dagger}$ to the vacuum. By the commutation relations above, to compute $b_{\mathbf{k}}^{(+)}|\psi\rangle$ we may commute the instance of $b_{\mathbf{k}}^{(+)}$ to the very right, where it annihilates the vacuum to give 0.

Now suppose there are any instances of $b_{\mathbf{k}'}^{(-)\dagger}$ in generating $|\psi\rangle$ by applying creation operators to vacuum. Due to the commutation relations these serve as obstructions to commuting $b_{\mathbf{k}}^{(+)}$ to the very right, and doing so one picks up non-zero terms proportional to $\delta^3(\mathbf{k}-\mathbf{k}')$, which cannot cancel.

Combining these results, one finds the desired result that

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } a_{\mathbf{k}}^{(1)\dagger}, a_{\mathbf{k}}^{(2)\dagger}, b_{\mathbf{k}}^{(+)\dagger} \text{ on } |0\rangle\}$$

(f) Show that any state $|\psi\rangle \in \mathcal{H}_{\text{small}}$ with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations has zero norm, and its overlap with any state in $\mathcal{H}_{\text{small}}$ is zero. Such states are called null states, and cannot have any physical significance.

Any state in $|\psi\rangle \in \mathcal{H}_{\text{small}}$ can be produced by acting some combination of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$, $b_{\mathbf{k}}^{(+)\dagger}$ on $|0\rangle$. Taking the Hermitian conjugate, any state $\langle\psi'| \in \mathcal{H}_{\text{small}}$ can be produced by acting some $a_{\mathbf{k}}^{(1)}$, $a_{\mathbf{k}}^{(2)}$, $b_{\mathbf{k}}^{(+)}$ on $\langle 0|$.

Suppose now we take $\langle\psi'|\psi\rangle$, where $|\psi\rangle$ has at least one instance of $b_{\mathbf{k}}^{(+)}$. Since there are no instances of $b_{\mathbf{k}}^{(-)\dagger}$ operators, the $b_{\mathbf{k}}^{(+)}$ commutes with all the raising/lowering operators in the product. One can thus commute it all the way to the left, where it annihilates $\langle 0|$. Therefore, $\langle\psi'|\psi\rangle = 0$. This includes the case where $|\psi'\rangle = |\psi\rangle$, so in particular a state with a $b_{\mathbf{k}}^{(+)\dagger}$ excitation has zero norm.

The same proof holds if $|\psi\rangle$ is a linear combination of states, each given by acting creation operators on the vacuum, where at least one has a $b_{\mathbf{k}}^{(+)\dagger}$ excitation.

(g) Show that any state in $\mathcal{H}_{\text{small}}$ with non-zero norm must have the form

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle$$

where $|\psi_T\rangle$ contains only excitations of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$ (i.e. transverse components), and $|\chi\rangle$ a null state. $|\psi\rangle$ should be physically equivalent to $|\psi_T\rangle$, as they differ only by a null state. We can then forget about the null states, and define

$$\mathcal{H}_{\text{phys}} = \{|\psi_T\rangle\}$$

$\mathcal{H}_{\text{phys}}$ contains only positive-norm states, and is identical to that obtained in the Coulomb gauge.
We showed in (f) that any state with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations is a null state. Since we can expand an arbitrary state as a sum of n-particle states using creation and annihilation operators, we can write any state in $|\psi\rangle \in \mathcal{H}_{\text{small}}$ as

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle$$

where we put all terms with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations into $|\chi\rangle$, and all terms without $b_{\mathbf{k}}^{(+)\dagger}$ excitations into $|\psi_T\rangle$. By definition, $|\chi\rangle$ is a null state, and $|\psi_T\rangle$ contains only $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$ excitations.

In particular, one finds $\langle\psi|\psi\rangle = \langle\psi_T|\psi_T\rangle > 0$, which proves (b) in full generality.

(h) Let us call excitations of $b_{\mathbf{k}}^{(+)\dagger}$ null photons. To understand their physical interpretation, consider the 'wave-function' $\chi_\mu(x)$ if the single null photon state

$$\chi_\mu(x) = \langle 0|A_\mu(x)|\mathbf{k},+\rangle, \qquad |\mathbf{k},+\rangle = \sqrt{2\omega_{\mathbf{k}}}\,b_{\mathbf{k}}^{(+)\dagger}|0\rangle$$

Note that the above definition of wavefunction $\chi_\mu(x)$ is the straightforward generalization to vector fields of our previous discussion for scalar fields.
Show that $\chi_\mu(x)$ can be written as

$$\chi_\mu(x) = \partial_\mu\lambda(x)$$

where $\lambda(x)$ is some function which satisfies the equation for a massless scalar

$$\partial_\mu\partial^\mu\lambda = 0$$

This shows that a null photon can be interpreted as a guage transformation from the vacuum. To see this, recall that the Lorentz gauge $\partial_\mu A_\mu = 0$ leaves residual gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad \partial_\mu\partial^\mu\lambda = 0$$

Thus, $\chi_\mu(x)$ can be considered a residual gauge transformation from $A_\mu = 0$.
Using the mode expansion for $A_\mu(x)$, we have

$$\begin{aligned}\chi_\mu(x) = \langle 0|A_\mu(x)|\mathbf{k},+\rangle &= \int d\!\!{}^-{}^3\mathbf{p}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{p}}}}e^{ip\cdot x}\epsilon_\mu^{(\alpha)}\langle 0|a_{\mathbf{p}}^{(\alpha)}b_{\mathbf{k}}^{(+)\dagger}|0\rangle \\ &= \frac{1}{\sqrt{2}}e^{ik\cdot x}(-\epsilon_\mu^{(0)} + \epsilon_\mu^{(3)}) = \frac{1}{\sqrt{2}\omega_{\mathbf{k}}}e^{ik\cdot x}k_\mu = -\frac{i}{\sqrt{2}\omega_{\mathbf{k}}}\partial_\mu(e^{ik\cdot x})\end{aligned}$$

In the second line we used the explicit forms of $\epsilon_\mu^{(\alpha)}$, and that $k_\mu = (-\omega_{\mathbf{k}}, \mathbf{k})$. Therefore, $\chi_\mu(x) = \partial_\mu\lambda(x)$ with $\lambda(x) = -\frac{i}{\sqrt{2}\omega_{\mathbf{k}}}e^{ik\cdot x}$. Since $k^2 = 0$, we immediately have $\partial^2\lambda = 0$.

(i) Show that the conclusion of part (h) holds for any wavepacket of a null photon,

$$|f\rangle = \int \mathrm{d}^3\mathbf{k} f(\mathbf{k})|\mathbf{k},+\rangle$$

More generally, for $|f\rangle = \int d^3\mathbf{k} f(\mathbf{k})|\mathbf{k},+\rangle$, we compute

$$\langle 0|A_\mu(x)|\mathbf{k},+\rangle = \int \frac{d^3\mathbf{k}}{\sqrt{2}\omega_{\mathbf{k}}}e^{ik\cdot x}f(\mathbf{k})k_\mu = \partial_\mu\tilde{f}(x)$$

where $\tilde{f}(x) := -i\int \frac{d^3\mathbf{k}}{\sqrt{2}\omega_{\mathbf{k}}}f(\mathbf{k})e^{ik\cdot x}$. We again have $k_\mu = (-\omega_{\mathbf{k}}, \mathbf{k})$ and $k^2 = 0$, so all the results from (h) carry over.

Question 2: Casimir Effect in 1 Dimension (30 points)

Until now, we have disregarded the vacuum's zero-point energy as an unobservable (infinite) shift in the energy scale. However, as Casimir demonstrated in 1948, differences in vacuum zero point energies are observable. This phenomenon is known as the Casimir effect. A simplest example is a small attractive force between 2 close parallel conducting plates, due to quantum vacuum fluctuations of the EM field. The force is caused by a change in vacuum energy of the EM field which results from the boundary conditions imposed by the plates.

In this problem, we will explore the Casimir effect. For technical simplicity, we consider a toy version. As we have seen, after quantization the EM field has the same number of degrees of freedom as 2 massless scalar fields. Thus, consider a free, massless, real scalar field ϕ in 1 spatial dimension,

$$S = -\frac{1}{2}\int \mathrm{d}x\mathrm{d}t\partial_\mu\phi\partial^\mu\phi$$

The vacuum of the system has infinite zero point energy. Denote it by E_0. Now imagine we put 2 'plates' at $x = 0$ and $x = a$ such that ϕ is required to vanish at the location of the plates,

$$\phi(x = 0, t) = \phi(x = a, t) = 0$$

Adding plates which impose additional boundary conditions on ϕ disturbs the vacuum, and results in a different zero-point energy $E(a)$.

Even though E_0 and $E(a)$ are both infinite, their different turns out to be finite, and physically meaningful. In fact, the difference

$$U(a) = E(a) - E_0$$

can be considered as the potential energy between the 2 plates. Changing a modifies the potential energy, resulting in a force between the plates which can be measured experimentally.

In this problem we compute $U(a)$. Taking the difference between infinities is a highly dangerous thing to do, and in principle one can get any answer, so we will need to be careful.

Both E_0 and $E(a)$ have 2 sources of infinities, one from the infinite volume, the other from there being infinite local degrees of freedom. It is convenient to separate them by putting the system in a box with finite size $L \gg a$. We will take $L \to \infty$ at the end. More explicitly, we require ϕ to satisfy a periodic boundary condition corresponding to a circle of size L,

$$\phi(x, t) = \phi(x + L, t)$$

(a) In the vacuum (i.e. before putting plates), write down the mode expansion for ϕ and calculate its zero-point energy E_0. Your answer should have the form

$$E_0 = \frac{1}{2}\sum_n \omega_n$$

where ω_n is the energy for each mode. Specify both ω_n and the range of summation.

We are interested in a real scalar field with periodicity L, i.e. $\phi(x + L, t) = \phi(x, t)$. We write ϕ in terms of creation and annihilation operators, this time restricting to modes apropriate for the circle.

$$\phi(x, t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2L\omega_n}}(a_n e^{-i\omega_n t + ik_n x} + a_n^\dagger e^{i\omega_n t - ik_n x}), \qquad k_n = \frac{2\pi n}{L}, \quad \omega_n = \frac{2\pi|n|}{L}$$

Substituting this expansion into the Hamiltonian, and using the commutator $[a_m, a_n^\dagger] = \delta_{m,n}$, one has

$$H = \sum_{n=-\infty}^{\infty} \omega_n N_n + \frac{1}{2} \sum_{n=\infty}^{\infty} \omega_n$$

We find that

$$E_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{2\pi |n|}{L} = \frac{1}{2} \sum_{n \geq 1} \frac{4\pi n}{L}$$

(b) Adding the plates separates the system into 2 segments, one with size a, and one with size $L - a$. In both segments one has Dirichlet boundary conditions at either end. Therefore it is enough to work out one of them. Find the mode expansion for ϕ in the region $[0, a]$ and zero point $\varepsilon(a)$. Again your answer should have the form

$$\varepsilon(a) = \frac{1}{2} \sum_n \tilde{\omega}_n$$

Specify both $\tilde{\omega}_n$ and the range of summation. The total zero-point energy of the system in the presence of the plates is thus

$$E(a) = \varepsilon(a) + \varepsilon(L - a)$$

We proceed as before, but now write ϕ in terms of modes appropriate for Dirichlet boundary conditions on the interval $[0, a]$:

$$\phi(x, t) = \sum_{n=1}^{\infty} \frac{\sin \tilde{\omega} x}{\sqrt{a \tilde{\omega}_n}} (a_n e^{-i\tilde{\omega}_n t} + a_n^\dagger e^{i\tilde{\omega}_n t}), \qquad \tilde{\omega}_n = \frac{\pi n}{a}$$

Substituting this expansion into the Hamiltonian, and using the commutator $[a_m, a_n^\dagger] = \delta_{m,n}$, one has

$$H = \sum_{n=1}^{\infty} \tilde{\omega}_n N_n + \frac{1}{2} \sum_{n=\infty}^{\infty} \tilde{\omega}_n$$

We find that

$$\varepsilon(a) = \frac{1}{2} \sum_{n \geq 1} \frac{\pi n}{a}$$

(c) Both sums in (a) and (b) are divergent. There is not much sense in taking their difference. To do this, we will first make them finite by introducing a 'UV-cutoff' Λ, and change the sums to

$$E_0 = \frac{1}{2} \sum_n \omega_n e^{-\omega_n / \Lambda}, \qquad \epsilon(a) = \frac{1}{2} \sum_n \tilde{\omega}_n e^{-\tilde{\omega}_n / \Lambda}$$

Evaluate these expressions with finite Λ.

Using the identity

$$\sum_{n=1}^{\infty} n e^{-an} = \frac{e^a}{(e^a - 1)^2} = \frac{1}{4 \sinh^2(a/2)}$$

We find

$$E_0 = \frac{1}{2}\sum_{n\geq 1}\frac{4\pi n}{L}e^{-\frac{2\pi n}{L\Lambda}} = \frac{\pi}{2L\sinh^2(\pi/L\Lambda)}$$
$$\varepsilon(a) = \frac{1}{2}\sum_{n\geq 1}\frac{\pi n}{a}e^{-\frac{\pi n}{a\Lambda}} = \frac{\pi}{8a\sinh^2(\pi/2a\Lambda)}$$

(d) Expand the answers from part (c) in the limit $\Lambda \to \infty$. You will find they become divergent. Keep terms which are divergent and finite, but throw out all terms which go to zero in the limit.

We obtain

$$E_0 = \frac{\pi}{2L\sinh^2(\pi/L\Lambda)} \sim \frac{L\Lambda^2}{2\pi} - \frac{\pi}{6L} + \mathcal{O}(\Lambda^{-2})$$
$$\varepsilon(a) = \frac{\pi}{8a\sinh^2(\pi/2a\Lambda)} \sim \frac{a\Lambda^2}{2\pi} = \frac{\pi}{24a} + \mathcal{O}(\Lambda^{-2})$$

(e) From your answers in part (d), find

$$U(a) = E(a) - E_0$$

You should find $U(a)$ is finite in the limit $\Lambda \to \infty$. Now take the limit $L \to \infty$ in $U(a)$, and find the force between the plates.

Combining the above terms, we have

$$U(a) = \varepsilon(a) + \varepsilon(L-a) - E_0 = \frac{\pi}{6L} - \frac{\pi}{24a} - \frac{\pi}{24(L-a)}$$

Taking $L \to \infty$ gives

$$U(a) = -\frac{\pi}{24a}, \qquad F(a) = -\partial_a U(a) = -\frac{\pi}{24a^2}$$

8.323 Relativistic Quantum Field Theory I
Spring 2023

8.323 Problem Set 12 Solutions

May 14, 2023

Question 1: Decay of a Scalar Particle (20 points)

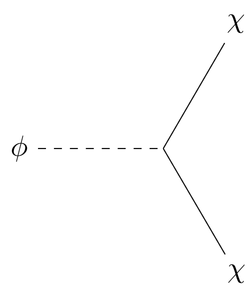
Consider a theory with 2 scalar fields ϕ and χ,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g\phi\chi^2$$

Assume $M > 2m$ and the coupling g is smnall. Calculate the decay rate Γ of ϕ-particles to the lowest order in g.

We can read off the Feynman rules from the Lagrangian. They are $\frac{-i}{p^2+M^2-i\epsilon}$ for a ϕ propagator, $\frac{-i}{p^2+m^2-i\epsilon}$ for a χ propagator, and ig for the 3-point vertex.

At leading order, the decay $\phi \to \chi\chi$ is given by a single diagram. Using the Feynman rules, the amplitude is simply $\mathcal{M} = ig$.



Now we substitute this into the formula for differential decay rate in the rest frame of ϕ:

$$\begin{aligned} d\Gamma &= \frac{1}{2M}|\mathcal{M}|^2(2\pi)^4\delta^4(k_1+k_2-k_\phi)\frac{\not{d}^3\mathbf{k}_1}{2\omega_{\mathbf{k}_1}}\frac{\not{d}^3\mathbf{k}_2}{2\omega_{\mathbf{k}_2}} \\ &= \frac{g^2}{2M}\frac{1}{4\omega_{\mathbf{k}}^2}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^2}\delta(2\omega_{\mathbf{k}} - k_\phi^0) = \frac{g^2}{2M}\frac{\sqrt{E^2-m^2}}{32\pi^2 E}\mathrm{d}E\mathrm{d}\Omega_2\delta(E - M/2) \end{aligned}$$

Performing the integrals, and multiplying by 1/2 for identical final state particles, we have

$$\Gamma = \frac{g^2}{32\pi M}\sqrt{1 - \frac{4m^2}{M^2}}$$

Question 2: Cross-Sections (20 points)

In this problem we consider a toy model of the process in which an electron-positron collision produces a quark-antiquark final state through an intermediate photon: $e^+e^- \to \gamma \to q\bar{q}$. The role of the electron and positron is played by a massless scalar field ψ. The photon is represented by a massless scalar field ϕ. Finally, quarks are represented by a massive scalar field χ. The relevant Lagrangian density is

$$\mathcal{L}_L = -\frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g'\phi\psi^2 + \frac{1}{2}g\phi\chi^2$$

Assume that couplings g, g' are small, and of comparable magnitude.

(a) Consider the process

$$\psi\psi \to \chi\chi$$

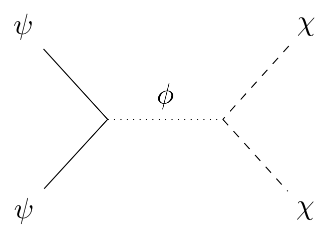
Compute the total cross-section to lowest order in couplings. Express your answer in terms of s, t, u.

The Mandelstam variables for $2 \to 2$ scattering are given by

$$s = -(p_1+p_2)^2, \qquad t = -(p_1-p_3)^2, \qquad u = -(p_1-p_4)^2, \qquad s+t+u = \sum_{i=1}^{4} m_i^2$$

The Feynman rules are $\frac{-i}{p^2-i\epsilon}$ for a ϕ propagator, $\frac{-i}{p^2-i\epsilon}$ for a ψ propagator, $\frac{-i}{p^2+m^2-i\epsilon}$ for a χ propagator, ig for the $\phi\chi^2$ interaction, and ig' for the $\phi\psi^2$ interaction.

At leading order, the process $\psi\psi \to \chi\chi$ is given by a single s-channel diagram.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = (ig')\frac{-i}{(p_1+p_2)^2 - i\epsilon}(ig) = igg'\frac{1}{s}$$

To compute the cross-section, we first work out kinematics. We can take the momentum to be

$$\begin{aligned} k_1 &= (E, 0, 0, E), & k_2 &= (E, 0, 0, -E) \\ k_3 &= (E, 0, k'\sin\theta, k'\cos\theta), & k_3 &= (E, 0, -k'\sin\theta, -k'\cos\theta), \qquad k'^2 := E^2 - m^2 \end{aligned}$$

In terms of the Mandelstam variables, this gives

$$s = 4E^2, \qquad t = -E^2 - k'^2 + 2Ek'\cos\theta, \qquad u = -E^2 - k'^2 - 2Ek'\cos\theta$$

The differential scattering amplitude is

$$d\sigma = \frac{1}{64\pi^2 s}\frac{k'}{E}|\mathcal{M}|^2 \mathrm{d}\Omega_2 = \frac{g^2 g'^2}{64\pi^2 s}\frac{k'}{E}\frac{1}{s^2}\mathrm{d}\Omega_2$$

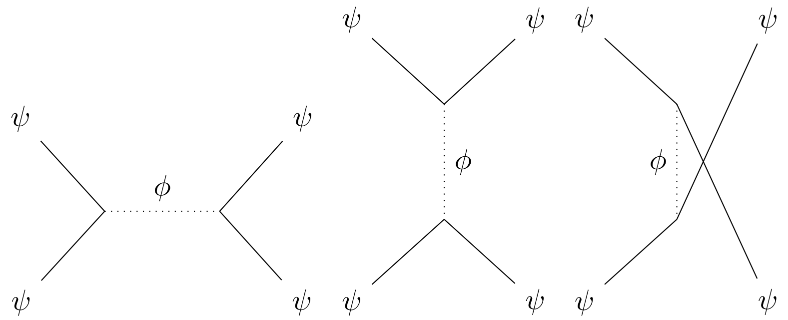
Multiplying by 1/2 for identical final state particles and performing the angular integration, the total scattering amplitude is

$$\sigma = \frac{1}{2}\int \mathrm{d}\sigma = \frac{(gg')^2}{32\pi s^3}\sqrt{1 - \frac{4m^2}{s}}$$

(b) Consider the process

$$\psi\psi \to \psi\psi$$

Compute the differential cross-section to lowest order. Express your answer in terms of s, t, u.
At leading order, the process $\psi\psi \to \psi\psi$ is given by s,t,u-channel diagrams.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = (ig)\left[\frac{-i}{(p_1+p_2)^2 - i\epsilon} + \frac{-i}{(p_1-p_3)^2 - i\epsilon} + \frac{-i}{(p_1-p_4)^2 - i\epsilon}\right](ig) = -ig^2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Proceeding in the same way as (a) with the same kinematics, the differential cross section is

$$\mathrm{d}\sigma = \frac{g^4}{64\pi^2 s}\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)^2 \mathrm{d}\Omega_2 = \frac{g^4}{64\pi^2 s}\left(1 - \frac{4}{\sin^2\theta}\right)^2 \mathrm{d}\Omega_2$$

Note: in principle, we would multiply by 1/2 for identical particles and perform the integral for the total scattering cross section, but the θ-integral is divergent at $\theta = 0$ and $\theta = \pi$. These are known as collinear singularities, associated with the final state particles moving along the same axis. It is related to subtleties in defining asymptotic states for massless particles, such as how to distinguish between multiple collinear massless particles with a single massless particle of the same total energy.

Question 3: Rutherford Scattering (30+10 points) The cross-section for scattering of an electron by the Coulombfield of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$\mathcal{H}_I = \int \mathrm{d}^3 x e\bar{\psi}\gamma^\mu\psi A_\mu$$

where $\psi(x)$ is the usual quantized Dirac field.

(a) Show the T-matrix element for electron scattering off a localized classical potential, is to lowest order

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p)$$

where $\tilde{A}_\mu(q)$ is the usual 4D Fourier transform of $A_\mu(x)$.

We compute S-matrix elements:

$$\begin{aligned}
{}_{\text{out}}\langle p'|p\rangle_{\text{in}} = \langle p'|e^{-i\int \mathrm{d}^4 x H_I}|p\rangle &= \langle p'|p\rangle - ie\int \mathrm{d}^4 x A_\mu(x)\langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle + \mathcal{O}(e^2)\\
&= \langle p'|p\rangle - ie\int \mathrm{d}^4 x A_\mu(x)\bar{u}(p')\gamma^\mu u(p)e^{i(p-p')\cdot x} + \mathcal{O}(e^2)\\
&= \langle p'|p\rangle - ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p-p') + \mathcal{O}(e^2)
\end{aligned}$$

where in the second line we used the mode expansion for $\psi(x)$ and $\bar{\psi}(x)$.

Comparing this to the definition of the transfer matrix

$${}_{\text{out}}\langle p'|p\rangle_{\text{in}} = \langle p'|p\rangle + \langle p'|iT|p\rangle$$

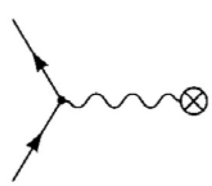
we find that to lowest order in e,

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p)$$

(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p'|iT|p\rangle := i\mathcal{M}(2\pi)\delta(E_f - E_i)$$

where E_i and E_f are the initial and final energies of the particle. We adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu\tilde{A}_\mu(\mathbf{q}),$$

where $\tilde{A}_\mu(\mathbf{q})$ is the 3D Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross-section for scattering off a time-independent, localized potential is

$$d\sigma = \frac{1}{|\mathbf{v}_i|}\frac{1}{2E_i}\frac{d^3\mathbf{p}_f}{2E_f}|\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)|^2(2\pi)\delta(E_f - E_i)$$

where $\mathbf{v}_i$ is the particle's initial velocity. Integrate over $|\mathbf{p}_f|$ to find a simple expression for $d\sigma/d\Omega$.

We compute the cross section by putting our system in a large box of dimensions $T \times V$. Spatial momenta are discretized to multiples of $2\pi/L$, therefore the density of states is $V/(2\pi)^3$. There is 1 initial state excitation and 1 final state excitation, so we compute:

$$d\sigma = \frac{\text{scattering probability } dP}{\text{incident flux}} = \frac{1}{T|\mathbf{v}_i|/V} \frac{|\langle p_f|S|p_i\rangle|^2}{\langle p_f|p_f\rangle\langle p_i|p_i\rangle} \prod_{\text{final states}} \mathrm{d}^3\mathbf{p}_f \frac{V}{(2\pi)^3}$$
$$= \frac{V}{T|\mathbf{v}_i|} \frac{(|\mathcal{M}|2\pi\delta(E_f - E_i))^2}{2E_i V} \frac{\mathrm{d}^3\mathbf{p}_f}{2E_f V} \frac{V}{(2\pi)^3} = \frac{1}{|\mathbf{v}_i|} \frac{d^3\mathbf{p}_f}{2E_i 2E_f} |\mathcal{M}|^2 2\pi\delta(E_f - E_i)$$

where in the 3rd line we used that $(2\pi\delta(E_f - E_i))^2 = 2\pi\delta(E_f - E_i)2\pi\delta(0) = T2\pi\delta(E_f - E_i)$. All factors of T and V have cancelled out, as expected.

Finally, we integrate over $|\mathbf{p}_f|$ to obtain the differential cross section:

$$\frac{d\sigma}{d\Omega} = \int \frac{p^2\mathrm{d}p}{(2\pi)^3} \frac{1}{|\mathbf{v}_i|} \frac{1}{4E_i E_f} |\mathcal{M}|^2 2\pi\delta(E_f - E_i) = \int \frac{p^2\mathrm{d}p}{(2\pi)^3} \frac{1}{|\mathbf{v}_i|} \frac{1}{4E_i^2} |\mathcal{M}|^2 \frac{E_i}{|\mathbf{p}_i|} 2\pi\delta(p - |\mathbf{p}_i|)$$
$$= \frac{1}{4(2\pi)^2} \frac{|\mathbf{p}_i|}{E_i|\mathbf{v}_i|} |\mathcal{M}|^2 = \frac{|\mathcal{M}|^2}{16\pi^2}$$

(c) Specialize to the case of electron scattering from a Coulomb potential, $A^0 = Ze/4\pi r$. Working in the non-relativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}$$

Refer to (d).

(d) **Bonus.** Repeat the calculation of (c) in the relativistic regime. After averaging over the spin of the initial state and summing over the spin of the final state, show that the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2 \sin^2(\theta/2)} \left(1 - \beta^2 \sin^4 \frac{\theta}{2}\right)$$

where $\alpha = e^2/4\pi$ the fine structure constant, $\mathbf{p}$ the electron momentum, and β its velocity.

Using the Feynman rules, the amplitude is

$$\mathcal{M} = e\bar{u}(p')\gamma^\mu \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})u(p)$$

Therefore, the spin-summed/averaged amplitude squared is

$$\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^2}{2} \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})\tilde{A}_\nu(\mathbf{p}' - \mathbf{p}) \sum_{s,s'} \bar{u}_s(p')\gamma^\mu u_{s'}(p)\bar{u}_{s'}(p)\gamma^\nu u_s(p')$$
$$= \frac{e^2}{2} \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})\tilde{A}_\nu(\mathbf{p}' - \mathbf{p}) \mathrm{Tr}[\gamma^\mu(\not{p} + m)\gamma^\nu(\not{p}' + m)]$$
$$= 2e^2[2(p \cdot \tilde{A}(\mathbf{p}' - \mathbf{p}))(p' \cdot \tilde{A}(\mathbf{p}' - \mathbf{p})) + (m^2 + p \cdot p')\tilde{A}^2(\mathbf{p}' - \mathbf{p})]$$

Now we specialize to a Coulomb potential, $A^0 = \frac{Ze}{4\pi r}$, $A^i = 0$. The Fourier transform of $A^0(x)$ is divergent owing to the $1/r$ behavior, but we can compute it by adding a 'photon mass regulator' e^{-mr}, and take $m \to 0$ in the end.

$$A^0(\mathbf{k}) = \lim_{m\to 0} \int \mathrm{d}^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-mr} \frac{Ze}{4\pi r} = \frac{Ze}{2} \lim_{m\to 0} \int \mathrm{d}r\mathrm{d}\theta\, r\sin\theta e^{-mr} e^{-i|\mathbf{k}|r\cos\theta}$$
$$= \lim_{m\to 0} \frac{Ze}{|\mathbf{k}|^2 + m^2} = \frac{Ze}{|\mathbf{k}|^2}$$

Therefore, using that $E_f = E_i$ for a time-independent potential, we have

$$\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = 2e^2(m^2 + EE' + \mathbf{p}\cdot\mathbf{p}')\frac{Z^2e^2}{|\mathbf{p}' - \mathbf{p}|^4} = \frac{Z^2e^4(1-\beta^2\sin^2(\theta/2))}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}$$

Using the result from (b), the differential cross section is thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\frac{1}{2}\sum_{\text{spin}}|\mathcal{M}|^2 = \frac{Z^2\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

In the non-relativistic limit only the first term contributes, which is the result desired in (c).

Question 4: Electron-Muon Scattering (20+20 points)

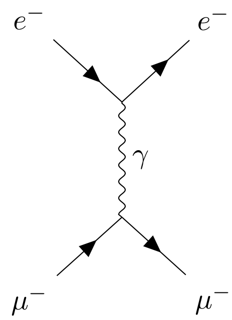
Consider the scattering process

$$e^- + \mu^- \to e^- + \mu^-$$

The electron and muon have masses m_e and m_μ, respectively.

(a) Calculate the scattering amplitude $\mathcal{M}$.

At leading order, the process $\psi\psi \to \chi\chi$ is given by a single t-channel diagram.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = -ie^2 \frac{1}{t + i\epsilon} [\bar{u}_{s_1'}(p_1')\gamma^\mu u_{s_1}(p_1)][\bar{u}_{s_2'}(p_2')\gamma_\mu u_{s_2}(p_2)]$$

(b) Calculate $\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2$, averaging over the spins of the initial state, and summing over the spins of the final state. Express your answer in terms of s,t,u.

Squaring the amplitude, summing over final spins, and averaging over initial spins, we have

$$\begin{aligned}
\frac{1}{4}\sum_{s_1,s_2,s_1',s_2'} |\mathcal{M}|^2 &= \frac{e^4}{4t^2} \sum_{s_1,s_2,s_1',s_2'} [\bar{u}_{s_1'}(p_1')\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_1)\gamma^\mu u_{s_1'}(p_1')][\bar{u}_{s_2'}(p_2')\gamma_\mu u_{s_2}(p_2)\bar{u}_{s_2}(p_2)\gamma_\mu u_{s_2'}(p_2')] \\
&= \frac{e^4}{t^2}\mathrm{Tr}[\gamma^\mu \Lambda_+(p_1)\gamma^\nu \Lambda_+(p_1')]\mathrm{Tr}[\gamma_\mu \Lambda_+(p_2)\gamma_\nu \Lambda_+(p_2')] \\
&= \frac{4e^4}{t^2}[(m_e^2 + p_1\cdot p_1')\eta^{\mu\nu} - p_1^\mu p_1'^\nu - p_1^\nu p_1'^\mu][(m_\mu^2 + p_2\cdot p_2')\eta_{\mu\nu} - p_{2\mu}p_{2\nu}' - p_{2\nu}p_{2\mu}'] \\
&= \frac{8e^4}{t^2}(p_1\cdot p_2 p_1'\cdot p_2' + p_1\cdot p_2' p_1'\cdot p_2 + m_e^2 p_2\cdot p_2' + m_\mu^2 p_1\cdot p_1' + 2m_e^2 m_\mu^2) \\
&= \frac{8e^4}{t^2}\left(\frac{s^2}{4} + \frac{u^2}{4} - (m_e^2 + m_\mu^2)(s+u) + \frac{3}{2}(m_e^2+m_\mu^2)^2\right)
\end{aligned}$$

In the second line we use the identity $\sum_s u_s(p)\bar{u}_s(p) = \Lambda_+(p) = i(i\not{p} + m)$. In the third line, we use the gamma trace identities

$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0, \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

in order to simplify

$$\mathrm{Tr}[\gamma^\mu(i\not{p} + m)\gamma^\nu(i\not{q} + m)] = \eta^{\mu\nu}(4m^2 + p\cdot q) - 4p^\mu q^\nu - 4p^\nu q^\mu$$

In the last line, we use the Mandelstam identities

$$s = -(p_1+p_2)^2 = m_e^2 + m_\mu^2 - 2p_1\cdot p_2, \qquad u = -(p_1 - p_2')^2 = m_e^2 + m_\mu^2 + 2p_1\cdot p_2'$$

(c) **Bonus.** Calculate the differential cross-section in the center of mass frame.
We first work out kinematics. In the center of mass frame, we can take the momenta to be

$$p_1 = (E_1, 0, 0, p), \qquad p_2 = (E_2, 0, 0, -p)$$
$$p_1' = (E_1, 0, p\sin\theta, p\cos\theta), \qquad p_2' = (E_2, 0, -p\sin\theta, -p\cos\theta), \qquad p'^2 := E^2 - m^2$$

In terms of Mandelstam variables,

$$\begin{aligned} s &= -(p_1 + p_2)^2 = E_{\text{cm}}^2 = (E_1 + E_2)^2 \\ t &= -(p_1 - p_1')^2 = 2p^2(\cos\theta - 1) \\ u &= -(p_1 - p_2')^2 = (E_2 - E_1)^2 - 2p^2(\cos\theta + 1) \end{aligned}$$

In the center of mass frame, the spin-summed/averaged differential cross-section is

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} &= \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\frac{1}{4}\sum |\mathcal{M}|^2 \\ &= \frac{e^4}{8\pi^2 s t^2}\left(\frac{s^2}{4} + \frac{u^2}{4} - (m_e^2 + m_\mu^2)(s+u) + \frac{3}{2}(m_e^2 + m_\mu^2)^2\right)^2 \end{aligned}$$

(d) **Bonus.** Obtain the differential cross-section in the muon rest frame. Show that in the limit $m_\mu \to \infty$, one recovers the cross section of Rutherford scattering.
In the muon rest frame, the kinematics becomes

$$p_1 = (E_1, 0, 0, p), \qquad p_2 = (m_\mu, 0, 0, 0)$$
$$p_1' = (E_1', 0, p'\sin\theta, p'\cos\theta), \qquad p_2' = (E_1 - E_1' + m_\mu, 0, -p'\sin\theta, p - p'\cos\theta)$$

The Lorentz-invariant differential cross section is given by

$$d\sigma = \frac{1}{4E_A E_B |v_A - v_B|}\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 d\Pi_{\text{LIPS}}$$

The spin-summed/averaged amplitude is given in (b). In the muon rest frame,

$$E_1 E_2 |v_1 - v_2| = |E_1 p_2^z - E_2 p_1^z| = m_\mu p$$

The final state phase space factor is

$$d\Pi_{\text{LIPS}} = (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')\frac{d^3\mathbf{p}_1'}{2E_1'}\frac{d^3\mathbf{p}_2'}{2E_2'} = \frac{1}{(2\pi)^2}\frac{1}{4E_1'E_2'}\delta(E_1 + m_\mu - E_1' - E_2')p'^2 \mathrm{d}p' \mathrm{d}\Omega_2$$

for $E_1'^2 = m_e^2 + p'^2$, and $E_2'^2 = m_\mu^2 + |\mathbf{p}_1' - \mathbf{p}_1|^2 = m_\mu^2 + p'^2 + p^2 - 2pp'\cos\theta$.

Now we take the limit as $m_\mu^2 \to \infty$. We have $E_2' \approx m_\mu$,therefore the delta function imposes $E_1 \approx E_1'$, thus $p \approx p'$. The final state phase factor simplifies to

$$d\Pi_{\text{LIPS}} \approx \frac{1}{16\pi^2}\frac{p}{m_\mu}\mathrm{d}\Omega_2$$

The leading m_μ dependence from the amplitude is

$$\begin{aligned} \frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 &= \frac{8e^4}{t^2}(p_1 \cdot p_2 p_1' \cdot p_2' + p_1 \cdot p_2' p_1' \cdot p_2 + m_e^2 p_2 \cdot p_2' + m_\mu^2 p_1 \cdot p_1' + 2m_e^2 m_\mu^2) \\ &\approx \frac{e^4 m_\mu^2}{2p^4 \sin^4(\theta/2)}(2E_1^2 - m_e^2 - E_1^2 + p^2\cos\theta + 2m_e^2) \\ &= \frac{e^4 m_\mu^2}{2p^4\sin^4(\theta/2)}(2E_1^2 - 2p^2\sin^2(\theta/2)) = \frac{e^4 m_\mu^2}{\beta p^2 \sin^4(\theta/2)}(1 - \beta^2\sin^2(\theta/2)), \qquad \beta = p/E_1 \end{aligned}$$

Putting everything together, the differential cross section in the muon rest frame for $m_\mu^2 \to \infty$ is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4m_\mu p}\frac{1}{16\pi^2}\frac{p}{m_\mu}\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 \approx \frac{\alpha^2}{4p^2\beta^2\sin^4(\theta/2)}(1-\beta^2\sinh^2(\theta/2))$$

This matches the result from Problem 3.

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Spring 2023