Quantum Field Theory I (8.323) Spring 2023
Assignment 1

Feb. 6, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Sec. 2.1 and 2.2
- Weinberg vol 1 Chap. 1

Review of Special Relativity: Lorentz transformations

- We use the notation

$$x^\mu = (x^0, x^i) = (t, x^1, x^2, x^3) = (t, \vec{x}) \ . \tag{1}$$

When used in the argument of a function we often simply write x^μ as x, e.g.

$$\phi(x^\mu) \equiv \phi(x) \ . \tag{2}$$

The four momentum is written as

$$p^\mu = (p^0, p^i) = (E, \vec{p}) \tag{3}$$

and the derivative

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right) = (\partial_t, \nabla) \ . \tag{4}$$

- We use the mostly-plus form of the Minkowski metric, i.e.

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta_{\mu\nu} \tag{5}$$

where $\mathrm{diag}(\cdots)$ denotes a diagonal matrix with diagonal entries given by $\cdots$. $\eta_{\mu\nu}$ is the inverse of $\eta^{\mu\nu}$, as

$$\eta_{\mu\lambda}\eta^{\lambda\nu} = \delta_\mu{}^\nu \tag{6}$$

where $\delta_\mu{}^\nu$ is the Kronecker delta symbol.

- Note

$$x_\mu = \eta_{\mu\nu} x^\nu = (-t, \vec{x}), \qquad p_\mu = (-p^0, p^i) = (-E, \vec{p}) \ . \tag{7}$$

- We will also use the notation

$$x^2 \equiv x^\mu x_\mu = -t^2 + \vec{x}^2, \tag{8}$$

$$p^2 \equiv p_\mu p^\mu = -E^2 + \vec{p}^2 \tag{9}$$

and

$$p \cdot x \equiv p_\mu x^\mu = p^\mu x_\mu = -Et + \vec{x} \cdot \vec{p}\ . \tag{10}$$

- A Lorentz transformation acts on x^μ and p^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{11}$$

where the matrix $\Lambda^\mu{}_\nu$ satisfies the relation

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta^{\rho\lambda} = \eta^{\mu\nu} \tag{12}$$

or in a matrix notation

$$\Lambda \eta \Lambda^t = \eta \tag{13}$$

where the superscript t denotes transpose. We can raise and lower the indices of Λ by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and equation (13) can also be written as

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\lambda \eta_{\rho\lambda} = \eta_{\mu\nu}\ . \tag{14}$$

- Under a Lorentz transformation (11), a scalar field transforms as

$$\phi(x) \to \phi'(x') = \phi(x)\ ; \tag{15}$$

a vector field transforms as

$$A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)\ ; \tag{16}$$

a second rank tensor field transforms as

$$T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho T_{\lambda\rho}(x) \tag{17}$$

and so on.

- Infinitesimal Lorentz transformations take the form

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu \tag{18}$$

where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \qquad \omega_\mu{}^\nu = \eta^{\nu\lambda}\omega_{\mu\lambda} \tag{19}$$

are infinitesimal numbers.

Problem Set 1

1. **Review: Quantum harmonic oscillator in the Heisenberg picture (25 points)**

   Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω

   $$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2) \ . \tag{20}$$

   In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

   $$[\hat{x}(t), \hat{p}(t)] = i \ . \tag{21}$$

   Here and below we set $\hbar = 1$.

   (a) Obtain the Heisenberg evolution equations for $\hat{x}(t)$ and $\hat{p}(t)$.

   (b) Suppose the initial conditions at $t = 0$ are given by

   $$\hat{x}(0) = \hat{x}, \qquad \hat{p}(0) = \hat{p} \tag{22}$$

   find $\hat{x}(t)$ and $\hat{p}(t)$.

   (c) It is convenient to introduce operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ defined by

   $$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}(\hat{a}(t) + \hat{a}^\dagger(t)), \qquad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}(\hat{a}(t) - \hat{a}^\dagger(t)) \ . \tag{23}$$

   Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy equal-time commutation relation

   $$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \ . \tag{24}$$

   (d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (f) Suppose the initial conditions at $t = 0$ are given by

   $$\hat{a}(0) = \hat{a}, \qquad \hat{a}^\dagger(0) = \hat{a}^\dagger \tag{25}$$

   find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (g) Express $\hat{x}(t), \hat{p}(t)$ and the Hamiltonian H in terms of $\hat{a}$ and $\hat{a}^\dagger$.

2. **Review: Lorentz transformations (15 points)**

(a) Prove that the four-dimensional δ-function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \tag{26}$$

is Lorentz invariant, i.e

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p}) \tag{27}$$

where $\tilde{p}^\mu$ is a Lorentz transformation of p.

(b) Show that

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \tag{28}$$

is Lorentz invariant, i.e.

$$= \omega_1' \delta^{(3)}(\vec{k}_1' - \vec{k}_2') \ . \tag{29}$$

$\vec{k}_1$ and $\vec{k}_2$ are respectively the spatial part of four-vectors $k_1^\mu = (\omega_1, \vec{k}_1)$ and $k_2^\mu = (\omega_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 = k_2^2 = -m^2 \ . \tag{30}$$

$k_1'^\mu = (\omega_1', \vec{k}_1')$ and $k_2'^\mu = (\omega_2', \vec{k}_2')$ are related to k_1^μ, k_2^μ by a same Lorentz transformation.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$ prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k), \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \tag{31}$$

is Lorentz invariant in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(\tilde{k}) \tag{32}$$

where $\tilde{k}^\mu = \Lambda^\mu{}_\nu k^\nu$ is a Lorentz transformation of k^μ.

3. **A complex scalar field (20 points)**

Consider the field theory of a complex value scalar field $\phi(x)$ with action

$$S = \int d^4x \left[-\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2) \right], \qquad |\phi|^2 = \phi\phi^* \ . \tag{33}$$

One could either consider the real and imaginary parts of ϕ, or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.

(a) Check the action (33) is Lorentz invariant (see (15)) and find the equations of motion.

(b) Find the canonical conjugate momenta for ϕ and ϕ^*, and the Hamiltonian H for (33).

(c) The action (33) is invariant under transformation

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^* \tag{34}$$

for arbitrary constant α. When α is small, i.e. for an infinitesimal transformation, (34) become

$$\delta\phi = i\alpha\phi, \qquad \delta\phi^* = -i\alpha\phi^* \tag{35}$$

Use Noether theorem to find the corresponding conserved current j^μ and conserved charge Q.

(d) Use equations of motion of part (a) to verify directly that j^μ is indeed conserved.

4. **The energy-momentum tensor for the complex scalar field theory (20 points)**

In this problem we work out the energy-momentum tensor of the complex scalar theory (33).

(a) Under a spacetime translation

$$x^\mu \to x'^\mu = x^\mu + a^\mu \tag{36}$$

a scalar field transforms as

$$\phi'(x') = \phi(x) \ . \tag{37}$$

Show that the action (33) is invariant under transformation $\phi(x) \to \phi'(x)$.

(b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether theorem to find the corresponding conserved currents $T^{\mu\nu}$.

(c) The conserved charge for a time translation

$$H = \int d^3x \, T^{00} \tag{38}$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x \, T^{0i} \tag{39}$$

should be identified with the total momentum. Thus $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explicit expressions for H and P^i. Compare H obtained here with the Hamiltonian of problem 3(b).

(d) Use equations of motion of problem 3(a) to verify directly that $T^{\mu\nu}$ is indeed conserved.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 2

Feb. 14, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 2
- Weinberg vol 1 Chap. 1

Notes:

1. **Conventions on Fourier transform and the Dirac delta function**

   - Fourier transform of $\phi(\vec{x}, t)$ is defined as

   $$\tilde{\phi}(\vec{k}, \omega) = \int dt d^3\vec{x}\, e^{i\omega t - i\vec{k}\cdot\vec{x}}\, \phi(\vec{x}, t) \tag{1}$$

   with the inverse transform given by

   $$\phi(\vec{x}, t) = \int \frac{d\omega}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\omega t + i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{k}, \omega)\ . \tag{2}$$

   We will often suppress the tilde on $\tilde{\phi}(\vec{k}, \omega)$ and simply write it as $\phi(\vec{k}, \omega)$, distinguishing it from $\phi(\vec{x}, t)$ by their arguments.

   - Note

   $$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k), \tag{3}$$

   and it higher dimensional generalizations

   $$\int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}) \tag{4}$$

2. **Lorentz transformations**

- A Lorentz transformation acts on x^μ and p^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{5}$$

  where the matrix $\Lambda^\mu{}_\nu$ satisfies the relation

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta^{\rho\lambda} = \eta^{\mu\nu} \tag{6}$$

  or in a matrix notation

$$\Lambda \eta \Lambda^t = \eta \tag{7}$$

  where the superscript t denotes transpose. We can raise and lower the indices of Λ by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and equation (7) can also be written as

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\lambda \eta_{\rho\lambda} = \eta_{\mu\nu} \ . \tag{8}$$

- Under a Lorentz transformation (5), a scalar field transforms as

$$\phi(x) \to \phi'(x') = \phi(x) \ ; \tag{9}$$

  a vector field transforms as

$$A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) \ ; \tag{10}$$

  a second rank tensor field transforms as

$$T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho T_{\lambda\rho}(x) \tag{11}$$

  and so on.

- Infinitesimal Lorentz transformations take the form

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu \tag{12}$$

  where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \qquad \omega_\mu{}^\nu = \eta^{\nu\lambda}\omega_{\mu\lambda} \tag{13}$$

  are infinitesimal numbers.

3. All single-particle states used below follow relativistic normalization, i.e.

$$\boxed{|k\rangle = \sqrt{2\omega_{\vec{k}}}\, a^\dagger_{\vec{k}} |0\rangle} \ . \tag{14}$$

Problem Set 2

1. **Problem with relativistic quantum mechanics (20 points)**

   The Schrodinger equation for a free non-relativistic particle is

$$i\partial_t\psi(\vec{x},t) = -\frac{1}{2m}\nabla^2\psi(\vec{x},t) \ . \tag{15}$$

   The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

$$\partial_t^2\psi(\vec{x},t) - \nabla^2\psi(\vec{x},t) + m^2\psi(\vec{x},t) = 0 \ . \tag{16}$$

   We emphasize that in both (15) and (16), $\psi(\vec{x},t)$ is interpreted as a wave function for dynamical variable $\vec{x}(t)$ rather than a dynamical field.

   (a) As a reminder, derive from (15) the continuity equation for the probability

$$\partial_t\rho + \nabla\cdot\vec{J} = 0, \tag{17}$$

   where

$$\rho = |\psi|^2, \qquad \vec{J} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \ . \tag{18}$$

   (b) Suppose $\psi(\vec{x},t)$ has the plane wave form, i.e.

$$\psi(\vec{x},t) \propto e^{i\vec{k}\cdot\vec{x}} \tag{19}$$

   for some real vector $\vec{k}$, find the solutions to (16).

   (c) Show that the Klein-Gordon equation also leads to a continuity equation (17) with now ρ and $\vec{J}$ given by

$$\rho = \frac{i}{2m}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right), \qquad \vec{J} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \ . \tag{20}$$

   (d) Argue that ρ in (20) cannot be interpreted as probability density.

2. **Commutation relations of annihilation and creation operators (20 points)**

   For the real scalar field theory discussed in lecture, i.e.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{21}$$

   we showed that the time evolution of quantum operator $\phi(\vec{x},t)$ is given by

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}u_{\vec{k}}(\vec{x},t) + a_{\vec{k}}^\dagger u_{\vec{k}}^*(\vec{x},t)\right) \tag{22}$$

where

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}, \qquad u_{\vec{k}}(\vec{x}, t) = e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \ . \tag{23}$$

We use $\pi(\vec{x}, t)$ to denote the momentum density conjugate to ϕ. The canonical commutation relations among ϕ and π are

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0 = [\pi(\vec{x}, t), \pi(\vec{x}', t)], \quad [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^{(3)}(\vec{x} - \vec{x}') \ . \tag{24}$$

(a) Show that it is enough to impose (24) at $t = 0$. In other words, once we impose them at $t = 0$, then the relations at general t are automatically satisfied.

Note: This statement in fact applies not only to $V(\phi) = \frac{1}{2}m^2\phi^2$, *but any potential* $V(\phi)$.

(b) Express $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ in terms of $\phi(\vec{k})$ and $\pi(\vec{k})$, where $\phi(\vec{k})$ and $\pi(\vec{k})$ are Fourier transforms of $\phi(\vec{x}, t = 0)$ and $\pi(\vec{x}, t = 0)$, i.e.

$$\phi(\vec{k}) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \phi(\vec{x}, t = 0) \tag{25}$$

and similarly for π.

(c) Using the expressions you derived in part (b) to deduce the commutations relations

$$[a_{\vec{k}}, a_{\vec{k}'}], \qquad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}], \qquad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] \tag{26}$$

from the commutation relations (24) at $t = 0$.

3. **Expressing Noether charges in terms of creation and annihilation operators (20 points)**

In pset 1 you obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory (21).

(a) Express the Hamiltonian H of (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(b) Express the conserved charges P^i for spatial translations for (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(c) Starting with

$$\phi(0, 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} + a^\dagger_{\vec{k}} \right) \tag{27}$$

show that under the action of translation operators

$$\phi(\vec{x}, t) = e^{iHt - iP^i x^i} \phi(0, 0) e^{-iHt + iP^i x^i} \ . \tag{28}$$

Note: This problem becomes trivial if you recall the following formula for a harmonic oscillator

$$e^{i\alpha N} a e^{-i\alpha N} = e^{-i\alpha} a, \qquad N = a^\dagger a \tag{29}$$

and α is a constant.

4. **Noether charges for Lorentz symmetries of the real scalar field theory (20 points + 10 bonus points)**

   In this problem we work out the conserved current corresponding to Lorentz symmetries of (21).

   (a) Consider an infinitesimal Lorentz transformation (12)–(13). Show that (12) satisfies (6) to first order in $\omega_{\mu\nu}$, so does give a Lorentz transformation.

   (b) Write down how ϕ transforms under an infinitesimal Lorentz transformation (see (9)) and show that the conserved Noether current for this transformation can be written as

   $$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} \tag{30}$$

   where $T^{\mu\nu}$ is the conserved energy-momentum tensor which we have already derived in pset 1.

   Note: this part does not involve complicated calculations. If you find yourself in a massive calculation, pause, and try to find a simpler approach.

   (c) Use the conservation of the energy-momentum tensor to verify that the current (30) is indeed conserved, i.e.

   $$\partial_\mu J^{\mu\lambda\nu} = 0 \ . \tag{31}$$

   This problem is complete if you finish the above parts. The part below is an instructive exercise, but is calculation heavy. It is given as a bonus problem (10 extra points) for those of you who would like to have more fun.

   (d) Consider the conserved charges associated with $J^{\mu\lambda\nu}$

   $$M^{\lambda\nu} = \int d^3x \, J^{0\lambda\nu} \tag{32}$$

   Express the conserved charges $M^{\mu\nu}$ for Lorentz symmetries for (21) in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 3

Feb. 21, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 2
- Weinberg vol 1 Chap. 1

Notes: useful formulae

1. All single-particle states used below follow relativistic normalization, i.e.
$$\boxed{|k\rangle = \sqrt{2\omega_{\vec{k}}}\, a^\dagger_{\vec{k}}|0\rangle}\ . \tag{1}$$
2. The unitary quantum operator generating a spacetime translation y^μ is written as
$$U_y = e^{-iP^\mu y_\mu} = e^{iHy^0 - iP^i y^i}, \qquad P^\mu = (H, P^i)\ . \tag{2}$$
It should be understood that the vacuum energy is already subtracted from H. U_y acts on creation and annihilation operators as
$$U_y a_{\vec{k}} U_y^\dagger = e^{-i\omega_{\vec{k}} y^0 + ik^i y_i} a_{\vec{k}} = e^{ik\cdot y} a_{\vec{k}}, \qquad U_y a^\dagger_{\vec{k}} U^\dagger_y = e^{-ik\cdot y} a^\dagger_{\vec{k}}\ . \tag{3}$$
3. The quantum operator generating a Lorentz transformation is
$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{4}$$
where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are finite constants.
4. For a complex scalar field we also have the unitary operator which generates a phase rotation
$$U_\alpha = e^{-i\alpha Q}, \tag{5}$$
where
$$Q = i\int d^3x\ [\pi_{\phi^*}\phi^* - \pi_\phi \phi]\ . \tag{6}$$

1. **Lorentz transformations for operators and states (15 points)**

   (a) From the commutation relation of creation and annihilation operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$

   $$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \tag{7}$$

   and the result of problem 2(b) of pset 1 argue that $a_{\vec{k}}, a^\dagger_{\vec{k}}$ should transform under a Lorentz transformation as

   $$a_{\vec{k}} \quad \to \quad \tilde{a}_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a_{\Lambda\vec{k}} \tag{8}$$

   $$a^\dagger_{\vec{k}} \quad \to \quad \tilde{a}^\dagger_{\vec{k}} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a^\dagger_{\Lambda\vec{k}} \ . \tag{9}$$

   (b) Using the expression of $M_{\mu\nu}$ of problem 4(c) of Pset 2 to compute

   $$\frac{1}{2}[\omega_{\mu\nu} M^{\mu\nu}, a_{\vec{k}}], \quad \frac{1}{2}[\omega_{\mu\nu} M^{\mu\nu}, a^\dagger_{\vec{k}}] \tag{10}$$

   where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal constant. Show that this indeed generates an infinitesimal Lorentz transformation in $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ which are consistent with (8)–(9).

   (c) Now consider unitary operator generating finite Lorentz transformations

   $$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} \tag{11}$$

   where $\omega_{\mu\nu}$ are finite constants. Show that

   $$U_\Lambda |0\rangle = |0\rangle \tag{12}$$

   i.e. the vacuum is Lorentz invariant. One can derive equations (8)–(9) by explicitly evaluating $U_\Lambda a_{\vec{k}} U^\dagger_\Lambda$ and $U_\Lambda a^\dagger_{\vec{k}} U^\dagger_\Lambda$. I will NOT ask you to show this. Assuming (8)–(9) show that

   $$U_\Lambda |k\rangle = |\Lambda k\rangle \ . \tag{13}$$

   [**Note:** this part shows that *the vacuum is Lorentz invariant.*]

2. **Spatially localized states of a scalar particle (25 points)**

   In a quantum field theory there is no natural way to define position eigenvector $|\vec{x}\rangle$ as $\vec{x}$ is now simply a label, not an operator. Also there is a fundamental conflict: a perfectly localized state in space is not a Lorentz covariant concept as it picks out a reference frame (we cannot perfectly localize in time at the same time).

In the problem we will make these abstract statements concrete.

Consider states of the form

$$|\vec{r}, t\rangle_f \equiv \int d^3\vec{k}\, f(\vec{k})\, |k\rangle\, e^{-i\vec{k}\cdot\vec{r}+i\omega_{\vec{k}}t} , \tag{14}$$

where $f(\vec{k})$ is a function to be determined and $|k\rangle$ is the one-particle state (1) of momentum $\vec{k}$ and energy $\omega_{\vec{k}}$.

(a) Determine $f(\vec{k})$ by the condition of perfect localization

$$_f\langle \vec{r}_1, t|\vec{r}_2, t\rangle_f = \delta^{(3)}(\vec{r}_1 - \vec{r}_2) . \tag{15}$$

(b) By acting unitary operator U_Λ for a Lorentz transformation (as defined in (4)) on your result in part (a) show that $|\vec{r}, t\rangle$ is *not* Lorentz covariant, i.e.

$$U_\Lambda|\vec{r}, t\rangle_f \neq |\Lambda\vec{r}, \Lambda t\rangle_f \tag{16}$$

where $\Lambda\vec{r}$ and Λt denote Lorentz transformation of $\vec{r}, t$.

(c) With $f(\vec{k})$ given by (a), consider the overlap $C(\vec{r}_1-\vec{r}_2, t_1-t_2) = {}_f\langle \vec{r}_2, t_2|\vec{r}_1, t_1\rangle_f$. Evaluate $C(\vec{r}, t)$ for a spacelike separation: $|\vec{r}| > t$.

Suppose we interpret $|\vec{r}_2, t_2|\vec{r}_1, t_1\rangle_f|^2$ as the probability for the particle originally at $\vec{r}_1$ at time t_1 to transition to $\vec{r}_2$ at time t_2. Would the propagation be causal (*i.e.* confined to the forward light-cone)?

Note: The above parts suggest that it is *not* sensible to interpret the state determined from part (a) describing a particle localized at $\vec{r}$ at time t.

(d) Compare your resulting state $|\vec{r}, t\rangle_f$ in part (a) with the state

$$|x\rangle \equiv \phi(x)|0\rangle \tag{17}$$

where $x = (\vec{x}, t)$. Are the states $|x\rangle$ perfectly localized? Show that $|x\rangle$ is Lorentz covariant, i.e.

$$U_\Lambda|x\rangle = |\Lambda x\rangle . \tag{18}$$

(e) Consider a state

$$|\Psi\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\, h(\vec{k})|k\rangle \tag{19}$$

with the corresponding "wave function" defined as

$$\Psi(x) = \langle 0|\phi(x)|\Psi\rangle . \tag{20}$$

Find $h(\vec{k})$ so that at $t = 0$, the single-particle wave function $\Psi(\vec{x})$ (as defined in lecture) corresponding to $|\Psi\rangle$ is a Gaussian wave packet center around $\vec{x}_0$ with width a, and momentum $\vec{p}$.

3. **Normal ordering and smeared fields (25 points)**

Consider the free real scalar field theory discussed in lectures.

(a) First show that

$$\langle 0|\phi(\vec{x},t)|0\rangle = 0 \ . \tag{21}$$

Evaluate the vacuum expectation value

$$\sigma^2 \equiv \left\langle 0|\phi^2(\vec{x},t)|0\right\rangle \ . \tag{22}$$

Express σ^2 as an integral over a single variable and show that the integral is divergent.

This result tells you that the vacuum is not empty! While the expectation value of ϕ is zero, the fluctuations of ϕ, as measured by σ^2, are nonzero, and in fact are infinitely large. This is a reflection of that a QFT has an infinite number of degrees of freedom.

(b) The general philosophy of QFT is to regard operators like $\phi^2(\vec{x},t)$ as "bad" operators. One then introduces "good" operators which do not suffer divergences, a procedure often referred to as renormalization. One way to remove the divergence in part (a) is to introduce normal ordered operators. The rule of normal ordering is that whenever you see products of $a_{\vec{k}}$'s and $a^\dagger_{\vec{k}}$'s move all the $a_{\vec{k}}$'s to the right of $a^\dagger_{\vec{k}}$'s. We denote the normal ordered version of an operator $\mathcal{O}$ by $:\mathcal{O}:$. For example,

$$:a_{\vec{k}_1} a^\dagger_{\vec{k}_2} a_{\vec{k}_3} a^\dagger_{\vec{k}_4}: = a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_4} a_{\vec{k}_1} a_{\vec{k}_3} \ . \tag{23}$$

Express normal ordered operator $:\phi^2(\vec{x},t):$ in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. Show that $\langle 0| :\phi^2(\vec{x},t): |0\rangle = 0$ and

$$\phi^2(\vec{x},t) = :\phi^2(\vec{x},t): + \sigma^2 \mathbf{1} \tag{24}$$

where $\mathbf{1}$ is the identity operator.

(c) Another way to introduce "good" operators is to consider the "smeared" field

$$\tilde{\phi}(\vec{x},t) \equiv N \int d^3y\, \phi(\vec{y},t)\, e^{-\frac{(\vec{x}-\vec{y})^2}{a^2}}, \quad N = \frac{1}{\pi^{\frac{3}{2}} a^3} \ . \tag{25}$$

The definition is motivated from the fact that the divergence in part (a) comes from having two ϕ's at the same spacetime point. In (25) we thus "smear" (or average) ϕ in a region of radius a. N is a normalization factor, chosen so that

$$\lim_{a\to 0}\tilde{\phi}(\vec{x},t) = \phi(\vec{x},t)\ . \tag{26}$$

Show that

$$\left\langle 0|\tilde{\phi}(\vec{x},t)|0\right\rangle = 0\ . \tag{27}$$

Now consider the fluctuation of $\tilde{\phi}$

$$\tilde{\sigma}^2 \equiv \left\langle 0\left|\tilde{\phi}^2(\vec{x},t)\right|0\right\rangle\ . \tag{28}$$

Express $\tilde{\sigma}^2$ as an integral over a single variable and show that it is finite.

(d) Without evaluating the integral in part (c), show that in the limits of small a and large a, the leading term in $\tilde{\sigma}^2$ may be written as

$$\tilde{\sigma}^2 \approx \alpha a^\gamma \tag{29}$$

and calculate α and γ for each of these two limits. You should discover that at large a the average field approaches a classical variable whereas at small a it is dominated by fluctuations.

4. **Correlation functions for a complex scalar (15 points)**

Consider a theory of a complex scalar field ϕ which is invariant under a phase rotation of ϕ. The unitary operator U_α generating a phase rotation is written in (5). We also assume that the vacuum of the theory is invariant under the phase rotation,[1]

$$U_\alpha|0\rangle = |0\rangle\ . \tag{30}$$

The theory can be interacting. That is, in this problem, you should *not* use the mode expansion for ϕ discussed in lecture which applies only to a free field theory.

(a) Show that

$$U_\alpha\phi(x)U_\alpha^\dagger = e^{i\alpha}\phi(x), \qquad U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi^*(x)\ . \tag{31}$$

(b) Show that

$$\langle 0|\phi(x)\phi(x')|0\rangle = 0\ . \tag{32}$$

[1]Note that it is possible that the Lagrangian is invariant under phase rotations, but the vacuum is not. The phenomenon is normally referred to as "spontaneous symmetry breaking," a subject of QFT III. This happens, e.g. in superluids and superconductors, whose many fascinating properties arise from spontaneous symmetry breaking.

(c) Now consider a general n-point function of products of ϕ's and $\phi^\dagger$'s inserted at different spacetime points between the vacuum, i.e.

$$\left\langle 0|\phi(x_1)\cdots\phi^\dagger(x_i)\cdots\phi_n(x_n)|0\right\rangle \ . \tag{33}$$

Show that the above quantity vanishes whenever the numbers of ϕ's and $\phi^\dagger$'s are not the same.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 4

Feb. 28, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.4
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. Here we give the definitions of various functions for a complex scalar field ϕ:

$$\text{Wightman function } G_+(x,x') = \langle 0|\phi(x)\phi^\dagger(x')|0\rangle \tag{1}$$
$$\text{retarded function } G_R(x,x') = \theta(t-t')\Delta(x,x') \tag{2}$$
$$\text{advanced function } G_A(x,x') = -\theta(t'-t)\Delta(x,x') \tag{3}$$

where

$$\Delta(x,x') = \langle 0|[\phi(x),\phi^\dagger(x')]|0\rangle \tag{4}$$

and the Feynman function

$$\begin{aligned} G_F(x,x') &= \langle 0|T\phi(x)\phi^\dagger(x')|0\rangle \\ &= \theta(t-t')\langle 0|\phi(x)\phi^\dagger(x')|0\rangle + \theta(t'-t)\langle 0|\phi^\dagger(x')\phi(x)|0\rangle\ . \end{aligned} \tag{5}$$

Note that these definitions may differ from various books by an overall factor i or minus sign.

2. The unitary quantum operator generating a spacetime translation y^μ is written as

$$U_y = e^{-iP^\mu y_\mu} = e^{iHy^0 - iP^i y^i}, \qquad P^\mu = (H, P^i)\ . \tag{6}$$

It should be understood that the vacuum energy is already subtracted from H. U_y acts on creation and annihilation operators as

$$U_y a_{\vec{k}} U_y^\dagger = e^{-i\omega_{\vec{k}} y^0 + ik^i y_i} a_{\vec{k}} = e^{ik\cdot y} a_{\vec{k}}, \qquad U_y a_{\vec{k}}^\dagger U_y^\dagger = e^{-ik\cdot y} a_{\vec{k}}^\dagger\ . \tag{7}$$

3. The quantum operator generating a Lorentz transformation is

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{8}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are finite constants. It acts on creation and annihilation operators as

$$U_\Lambda a_{\vec{k}} U_\Lambda^\dagger \quad = \quad = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a_{\Lambda\vec{k}} \tag{9}$$

$$U_\Lambda a_{\vec{k}}^\dagger U_\Lambda^\dagger \quad = \quad = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a_{\Lambda\vec{k}}^\dagger\ . \tag{10}$$

Problem Set 4

1. **Properties of Wightman and Feynman functions (20 points)**

   (a) For a free scalar field theory, using (9) and (10), show that

   $$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x)\ . \tag{11}$$

   (b) Now in all the subsequent parts, let us consider a general interacting theory, where the mode expansion given in lecture for ϕ does *not* apply. But as far as the system is translational and Lorentz invariant, both (11) and (below U_y is given by (6))

   $$U_y \phi(x) U_y^\dagger = \phi(x+y) \tag{12}$$

   are valid[1]. Using (12) to prove that for Wightman function introduced in (1)

   $$G_+(x,x') = G_+(x-x')\ . \tag{13}$$

   You need not to show it for other functions, but should realize the same is true for all two-point functions defined at the beginning of this pset.

   (c) Using (11) and the result of part (b) show that one can write $G_+(x,x')$ as

   $$G_+(x,x') = \theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2) \tag{14}$$

   where $G(y)$ is some function which satisfies

   $$G(y) = G^*(y), \qquad \text{for } y > 0\ . \tag{15}$$

[1] One can prove this by using the expressions of H, P^i and $M^{\mu\nu}$ in terms of ϕ and π_ϕ and canonical commutation relations between ϕ and π_ϕ.

(d) Now take ϕ to be real and show that for Feynman function G_F

$$G_F(x, x') = G((x - x')^2) \tag{16}$$

where function G in the above equation is the same as that in (14).

2. **Particle production by an external source (60 points)**

This problem is developed from Peskin and Schroeder's example on p. 32 and prob. 4.1. It is the simplest example in which the S-matrix can be calculated exactly.

Consider a free scalar field theory with an external "source" $J(x)$, whose Lagrangian density can be written as

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi \tag{17}$$

where $\mathcal{L}_0$ is the Lagrangian density for a free scalar and $J(x)$ is a fixed function. We assume that it has the the following properties

$$J(t, \vec{x}) \to 0, \qquad t \to \pm\infty, \quad |\vec{x}| \to \infty \tag{18}$$

and its Fourier transform

$$J(p) = \int d^4x \; e^{-ip\cdot x} J(x) \tag{19}$$

is analytic in the complex ω plane.

Since $J(x)$ depends on time, the system does not have time translation symmetry. In particular, the vacuum at past infinity $|0, -\infty\rangle$ will be *different* from the one at future infinity $|0, +\infty\rangle$. Suppose we start with the vacuum state $|0, -\infty\rangle$ at $t = -\infty$, in the Heisenberg picture, the system remains in the same state $|0, -\infty\rangle$ at all times. At $t = +\infty$, the system is then *not* in the ground state (as $|0, -\infty\rangle \neq |0, +\infty\rangle$), and contains particle excitations. In other words, turning on a source $J(x)$ has produced particles. Below we will find the relation between $|0, -\infty\rangle$ and $|0, +\infty\rangle$, and calculate the probability for producing particles.

At $t = -\infty$, since $J = 0$, we have a free theory, and ϕ can be written as

$$\phi(x) = \phi_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[a_{in}(\vec{k})e^{ik\cdot x} + a_{in}^\dagger(\vec{k})e^{-ik\cdot x}\right], \qquad t \to -\infty \tag{20}$$

where

$$[a_{in}(\vec{k}), a_{in}^\dagger(\vec{k}')] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') \; . \tag{21}$$

We can then define the past vacuum as

$$a_{in}(\vec{k})|0, -\infty\rangle = 0, \qquad \langle 0, -\infty|0, -\infty\rangle = 1 \tag{22}$$

and particles (in the past infinity) can be defined by acting $a_{in}^\dagger$ on this vacuum. For example an n-particle state in the past infinity can be written as

$$|\vec{k}_1, \cdots \vec{k}_n, -\infty\rangle = \sqrt{2\omega_{\vec{k}_1}} \cdots \sqrt{2\omega_{\vec{k}_n}} a_{in}^\dagger(\vec{k}_1) \cdots a_{in}^\dagger(\vec{k}_n)|0, -\infty\rangle \tag{23}$$

with normalization for a single-particle state

$$\left\langle \vec{k}, -\infty|\vec{k}', -\infty \right\rangle = 2\omega_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') \ . \tag{24}$$

Similarly, at $t = +\infty$, since $J = 0$, we again have a free theory, and ϕ can be written as

$$\phi(x) = \phi_{out} \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{out}(\vec{k})e^{ik\cdot x} + a_{out}^\dagger(\vec{k})e^{-ik\cdot x} \right], \qquad t \to +\infty \ . \tag{25}$$

with the future vacuum defined as

$$a_{out}(\vec{k})|0, +\infty\rangle = 0, \qquad \langle 0, +\infty|0, +\infty\rangle = 1 \tag{26}$$

and an n-particle state written as

$$|\vec{k}_1, \cdots \vec{k}_n, +\infty\rangle = \sqrt{2\omega_{\vec{k}_1}} \cdots \sqrt{2\omega_{\vec{k}_n}} a_{out}^\dagger(\vec{k}_1) \cdots a_{out}^\dagger(\vec{k}_n)|0, +\infty\rangle \ . \tag{27}$$

Due to the presence of the external source $J(x)$, the past and future annihilation operators a_{in} and a_{out} are different. $\phi(t, \vec{x})$ with $-\infty < t < +\infty$ interpolates between (20) and (25).

We consider that the system starts in the vacuum, i.e.

$$|\Psi\rangle = |0, -\infty\rangle \ . \tag{28}$$

(a) For general t, solve the classical equation of motion for (17) and show that the solution can be written in a form

$$\phi(x) = \phi_0(x) + i\int d^4x\, G(x - x')J(x') \tag{29}$$

where $\phi_0(x)$ is a solution of the homogeneous equation

$$(-\partial^2 + m^2)\phi_0(x) = 0 \tag{30}$$

and $G(x - x')$ is a Green function satisfying

$$(-\partial^2 + m^2)G(x - x') = -i\delta^{(4)}(x - x') \ . \tag{31}$$

We have discussed various types of Green functions, the retarded, advanced, and Feynman Green functions. Which one we should use here?

(b) Now consider the quantum theory and promote ϕ to a quantum operator, with (29) now an operator equation. Show that ϕ_0 in (29) should be given by ϕ_{in} introduced in (20).

(c) Evaluate (29) at $t = +\infty$ to find the relation between $a_{out}(\vec{k})$ and $a_{in}(\vec{k})$.

(d) Using (c) show that the expectation value λ for the total number of particles produced is given by

$$\lambda = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} |J(k)|^2 \ . \tag{32}$$

(e) Show that we can write

$$a_{out}(\vec{k}) = S^\dagger a_{in}(\vec{k}) S, \tag{33}$$

with S a unitary operator given by

$$S \equiv e^{iB}, \qquad B = \int d^4x \, J(x)\phi_{in}(x) \ . \tag{34}$$

(f) Use (33) to show that

$$S|0, +\infty\rangle = |0, -\infty\rangle \tag{35}$$

and

$$S|\vec{k}_1, \cdots \vec{k}_n, +\infty\rangle = |\vec{k}_1, \cdots \vec{k}_n, -\infty\rangle \ . \tag{36}$$

Note: The results of this part indicate that S is in fact the S-matrix operator of the system, i.e. for any free theory state $|\alpha\rangle$ and $|\beta\rangle$,

$$S_{\beta\alpha} \equiv \langle \beta, +\infty | \alpha, -\infty\rangle = \langle \beta, -\infty | S | \alpha, -\infty\rangle \ . \tag{37}$$

(g) In the following parts we will compute the probability of producing n particles. Before doing that, in this part we develop a technical tool to make that task easier. Show that we can write S as

$$S = e^{iB} = e^F e^G e^{-\frac{1}{2}\lambda} \tag{38}$$

with

$$F = i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} J(k) a^\dagger_{in}(k), \quad G = i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} J(-k) a_{in}(k) \tag{39}$$

and λ was introduced in (32).

(h) Use (35) and (38) to find the vacuum to vacuum probability , i.e.

$$P_0 = |\langle 0, +\infty | 0, -\infty\rangle|^2 \ . \tag{40}$$

P_0 is the probability of no particle production.

(i) Use (36) and (38) to show that

$$\left\langle \vec{k}_1, \cdots, \vec{k}_n, +\infty | 0, -\infty \right\rangle = i^n J(k_1) \cdots J(k_n) e^{-\frac{1}{2}\lambda} . \tag{41}$$

(j) The probability dP of finding exactly n particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, $\cdots$, one particle in the range $d^3\vec{k}_n$ around $\vec{k}_n$ can be shown to be given by

$$dP = \left| \left\langle \vec{k}_1, \cdots, \vec{k}_n, +\infty | 0, -\infty \right\rangle \right|^2 \prod_{i=1}^{n} \left(\frac{d^3\vec{k}_i}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_i}} \right) . \tag{42}$$

Use (42) to show that the probability of finding exactly n particles is

$$P_n = e^{-\lambda} \frac{\lambda^n}{n!} . \tag{43}$$

This is Poisson distribution with average particle number λ.

[*Note:* I am not asking you to derive (42). The equation of course looks quite intuitive and many of you may be able to guess it. We will discuss its precise derivation a bit later.]

Remarks on problem 2:

1. While the setup of this problem is very simple, the notion that in a time-dependent situation the past and future vacua are inequivalent (and the resulting particle production) is very general and plays very important roles in cosmology and black holes. In particular, the famous Hawking radiation from black holes is also a consequence of this.

2. We notice that only the on-shell part of $J(k)$ (i.e. with $k^0 = \omega_k$) contributes to the particle production. This is just field theory analog of the resonance effect for a forced harmonic oscillator.

3. Suppose we turn on the source J very slowly and very slowly turn it off, from the adiabatic theorem of quantum mechanics, we expect the system to remain in the vacuum throughout, i.e. there is no particle production. Indeed to see this in our discussion, in the adiabatic limit $J(\omega, \vec{k})$ will be nonzero only for ω close to zero and there is no on-shell component.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 5

Mar. 7, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.4
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. Path integral is a subject covered by graduate quantum mechanics class. In lectures I tried to give a self-contained exposition of the essential ideas, but have not time to be comprehensive. If you have not done path integral before you should consult standard quantum mechanics textbooks on this subject. A very nice, and detailed discussion of the formalism with many examples and applications is the classic book by *Feynman and Hibbs "Quantum Mechanics and Path Integrals."*

   The problems in this pset are aimed to remind you of the subject (or familiarize it if you have not done it before).

2. For a non-relativistic particle moving in one-dimension with a Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{1}$$

   we have, in the path integral formalism,

$$\begin{aligned} & K(x_a, t_a; x_b, t_b) = \langle x_a, t_a | x_b, t_b \rangle & (2) \\ = \; & \lim_{N \to \infty} \left(\frac{m}{2\pi i \Delta t} \right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1} \; \exp\left[i\Delta t \sum_{i=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 - V(x_i) \right) \right] & \\ \equiv \; & \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) \; \exp\left[i \int_{t'}^{t} dt \, L(\dot{x}, x) \right] & (3) \end{aligned}$$

   where $L = \frac{1}{2} m \dot{x}^2 - V(x)$ is the Lagrangian and $N\Delta t = t_a - t_b$.

Problem Set 5

1. **A useful formula and path integral in phase space (20 points)**

   (a) Derive equations (4) and (5) below.

$$
\begin{aligned}
&\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle \\
&= \int \frac{dp_i}{2\pi} \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + i p_i (x_{i+1} - x_i)\right] \qquad (4) \\
&= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{i m \Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right] \qquad (5)
\end{aligned}
$$

   Equation (5) was used in lecture to derive (3).

   (b) Using (4) to derive an alternative expression of (2):

$$
\langle x_a, t_a | x_b, t_b \rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \, \exp\left[i \int_{t'}^{t} dt \, (p\dot{x} - H)\right] . \qquad (6)
$$

   The integration $\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t)$ in the above expression should be understood as

$$
\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \equiv \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \int \frac{dx_1 dp_1}{2\pi} \cdots \int \frac{dx_{N-1} dp_{N-1}}{2\pi} \qquad (7)
$$

   where we again divide the interval $[t', t]$ into N segments with $t_0 = t', t_N = t$.

2. **Schrodinger equation rederived (20 points)**

   Using the propagator introduced in (3), the wave function $\psi(x, t)$ for a system at time t can be obtained from that at time t' by

$$
\psi(t, x) = \int dx' \, K(x, t; x', t') \, \psi(x', t') \, . \qquad (8)
$$

   Show that $\psi(t, x)$ satisfies the Schrodinger equation

$$
i\partial_t \psi(t, x) = -\frac{1}{2m} \partial_x^2 \psi(t, x) + V(x) \psi(t, x) \, . \qquad (9)
$$

   [Hint:] We can write the wave function at time $t + \delta t$ from that at time t by:

$$
\psi(t + \delta t, x) = \int dy \, K(t + \delta t, x; t, y) \, \psi(t, y) \, . \qquad (10)
$$

With δt small, $K(t+\delta t, x; t, y)$ can be written as a single infinitesimal step of (3), i.e.

$$K(t+\delta t, x; t, y) = \left(\frac{m}{2\pi i \delta t}\right)^{\frac{1}{2}} \exp\left[i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)\right] . \tag{11}$$

Proving (9) then amounts to a careful analysis of the $\delta t \to 0$ limit of the difference between (10) and $\psi(t,x)$.

3. **Free particle (20 points)**

   For a free particle, i.e. $V(x) = 0$, perform explicitly the integrals over $x_i, i = 1, \cdots N-1$ in (3) and show that

   $$K(x_a, t_a; x_b, t_b) = \left(\frac{m}{2\pi i(t_a - t_b)}\right)^{\frac{1}{2}} \exp\left[\frac{im(x_a - x_b)^2}{2(t_a - t_b)}\right] . \tag{12}$$

   [**Hint:**] First do the integral for x_1 and try to find a pattern.

4. **Path integral for a free particle revisited (20 points)**

   In this problem we evaluate the path integral for a free particle using a different method from Prob. 3. For simplicity we take $x_a = x_b = 0$ and $t_b = 0$ and $t_a = T$. As discussed in lecture K is a Gaussian path integral of the form

   $$K(0,T;0,0) = \int_{x(0)=0}^{x(T)=0} Dx(t) \exp\left[\frac{i}{2}\int dtdt'\, x(t)A(t,t')x(t')\right] \tag{13}$$

   for some differential operator A.

   (a) Write down the explicit expression for A.

   (b) Find all the eigenvalues of A. Show that the determinant of A can be written as

   $$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2} . \tag{14}$$

   (c) Since (13) is Gaussian, it can be evaluated as

   $$K(0,T;0,0) = \frac{C}{\sqrt{\det A}} \tag{15}$$

   where C is some constant. By comparing the above equation with (12) show that consistency of the two approaches requires

   $$\frac{C}{\sqrt{\det A}} = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} \tag{16}$$

i.e.

$$C \prod_{n=1}^{\infty} \frac{T}{\sqrt{m}\pi} \frac{1}{n} = \left(\frac{m}{2\pi i T} \right)^{\frac{1}{2}} \ . \tag{17}$$

Note: While the above equation looks rather bizarre, it can in fact be shown explicitly by discretizing the path integral as in the second line of (3), carefully calculating C, and then taking $N \to \infty$ (after discretizing, A becomes a finite matrix).

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 6

Mar. 14, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.6
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. We showed that for a quantum field theory of real scalar ϕ, n-point time-ordered correlation functions can be expressed in terms of path integrals as

$$\begin{aligned} G_n(x_1, \cdots, x_n) &= \langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle \\ &= \frac{\int D\phi \, e^{iS[\phi]} \phi(x_1) \cdots \phi(x_n)}{\int D\phi \, e^{iS[\phi]}} \end{aligned} \tag{1}$$

where the path integrals of both upstairs and downstairs are defined by integrating over all configurations of $\phi(t, \vec{x})$ satisfying

$$\lim_{t \to \pm\infty} \phi(t, \vec{x}) = 0, \quad \lim_{|\vec{x}| \to \infty} \phi(t, \vec{x}) = 0 \ . \tag{2}$$

2. In perturbation theory, it is convenient to express G_n in terms of free theory quantities as

$$G_n(x_1, \cdots x_n) = \frac{\left\langle 0 \middle| T\phi(x_1)\phi(x_2) \cdots \phi(x_n) e^{-i\int_{-\infty}^{\infty} H_I dt} \middle| 0 \right\rangle}{\left\langle 0 \middle| T e^{-i\int_{-\infty}^{\infty} H_I dt} \middle| 0 \right\rangle} \tag{3}$$

Note that

$$\int d^4x \, \mathcal{L}_I = -\int dt \, H_I \tag{4}$$

and we will use both these two forms.

3. When we write expressions like

$$\left\langle 0 \left| T\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty} H_I dt} \right| 0 \right\rangle \tag{5}$$

the time ordering includes both $\phi(x_1)\phi(x_2)$ and $e^{-i\int_{-\infty}^{\infty} H_I dt}$. One might be puzzled by what we mean by time ordering of $e^{-i\int_{-\infty}^{\infty} H_I dt}$ as there is a time integral in the exponential. One should understand $e^{-i\int_{-\infty}^{\infty} H_I dt}$ as a power series and then do time ordering inside the integrals. More explicitly,

$$Te^{-i\int_{-\infty}^{\infty} H_I dt} \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \cdots dt_n \, T\left(H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right) \ . \tag{6}$$

4. Peskin and Schroeder's notion of connected diagrams, as discussed on p.97-p.98, is confusing and non-standard, as it includes diagrams with disconnected parts! To make things even more confusing, they use different notions of connected diagrams in Sec. 4.4 and Sec. 4.6. The definition of Sec. 4.6 (see p.114) is the same as that in lecture.

   Below by **connected diagrams** we mean diagrams with no disconnected parts. For example, among diagrams in eq (4.58) on p.99 of Peskin and Schroeder, only the first diagram in the second line and the diagrams in the third line are connected.

5. Some of you may have observed that some expressions for correlation functions I wrote down in lectures are divergent. You will also find divergences in various expressions you will write down in this pset. As is the case for the energy density of the free scalar field theory we already encountered, such divergences are ubiquitous in quantum field theories and reflect that a QFT has an infinite number of *local* degrees of freedom. We then have to understand two issues:

   (a) How to make sense of physical observables which are seemingly divergent.

   (b) Why we can do perturbation theory if the terms in perturbative series are divergent.

   The first issue is addressed by the program of renormalization, which is a mathematical procedure for extracting unambiguous, finite answers out of apparently divergent expressions.

   The renormalization program, developed in late 1940's and early 1950's, was hugely successful in getting sensible answers which compared well with experiments. It, however, does not address the second issue.

   The second issue was only understood after the development of Wilson's renormalization group program in late 1960's and early 1970's.

Both these issues will be discussed in QFT2. In QFT1, we will only concern ourselves with the leading term in perturbative expansions of scattering amplitudes, which as we will see are not divergent.

Problem Set 6

1. **Particle production by an external source: continued (10 points)**

   Consider again Prob. 2 of Pset 4. Introduce

$$Z[J] = \int D\phi \, e^{i\int d^4x\, \mathcal{L}}, \quad Z_0 = Z[J=0] = \int D\phi \, e^{i\int d^4x\, \mathcal{L}_0} \tag{7}$$

   where $\mathcal{L}$ is given by equation (17) of pset 4, which we copy here for your convenience

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi \ . \tag{8}$$

   Using similar tricks as we introduced in lecture to derive (1), you can show that

$$\langle 0, +\infty | 0, -\infty \rangle = \frac{Z[J]}{Z_0} \tag{9}$$

   with an appropriate $i\epsilon$ prescription. I will not ask you to derive the above equation, but you should think about yourself how it can be derived.

   Using (9) to find the probability of no particle production

$$P_0 = |\langle 0, +\infty | 0, -\infty \rangle|^2 \ . \tag{10}$$

   by directly evaluating the path integrals (7). You should verify that it reproduces the answer you obtained in part 2(h) of pset 4.

2. **Connected diagrams (30 points)**

   Below the notion of connected diagrams follows that defined in item 4 of Notes at the beginning.

   Consider the $\lambda\phi^4$ theory discussed in lecture, i.e.

$$H_I = \frac{\lambda}{4!}\int d^3x \, \phi^4(x) \ . \tag{11}$$

(a) List all *connected* diagrams for

$$\left\langle 0|T\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty} H_I dt}|0\right\rangle \tag{12}$$

to order $O(\lambda^2)$, and give the symmetry factor for each diagram.

For diagrams at $O(\lambda^0)$ and $O(\lambda)$ orders write down their expressions in *both* coordinate and momentum space.

At order $O(\lambda^2)$ choose one diagram to write down its coordinate and momentum space expressions.

You do not need to evaluate the integrals.

(b) List all *connected* diagrams of the four-point function

$$G_4(x_1, x_2, x_3, x_4) = \langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle \tag{13}$$

to order $O(\lambda^2)$, and choose *two of them* to write down their respective expressions in *both* coordinate and momentum space. You do not need to do the integrals.

3. **Vacuum diagrams (30 points)**

For $\lambda\phi^4$ theory (11), consider the quantity

$$Z_0 = \left\langle 0\middle|Te^{-i\int_{-\infty}^{\infty} H_I dt}\middle|0\right\rangle \tag{14}$$

where the expectation value is evaluated in the free theory. We also assume that the free theory vacuum $|0\rangle$ is properly normalized, i.e. $\langle 0|0\rangle = 1$.

(a) Consider

$$W_0 = \log Z_0 \ . \tag{15}$$

Show that W_0 can be written in a form

$$W_0 = C - i\varepsilon VT \tag{16}$$

where C is some constant independent of the spacetime volume, ε is the energy density difference between the full and free theories, and VT is the total *spacetime* volume. Note $T = t - t'$ where $t \to \infty$ and $t' \to -\infty$ are respectively upper and lower limits of the path integrals.

(b) The Feynman diagrams in perturbative expansion of Z_0 have no external lines, which are often called vacuum diagrams or vacuum bubbles. We thus say that Z_0 is obtained by summing over vacuum diagrams. Show that W_0 is the sum of *connected* vacuum diagrams.

Note that the statement in fact applies to all theories.

Hint: consult discussion of Peskin and Schroeder on p.96-98. The discussion of Peskin and Schroeder is not great, but hopefully you can get the key idea and write down your own proof.

(c) Write down the expression for ε to order $O(\lambda^2)$. You can write it either in coordinate or momentum space. It is enough to write it formally as integrations of Feynman propagators. You do not need to evaluate the integrals.

Note: the expression is divergent!

4. **General n-point function (10 points)**

Prove that in evaluating n-point function (3), diagrams that contain factor(s) of vacuum diagrams all cancel. That is, G_n is obtained by summing all diagrams without any vacuum diagram factors. In lecture we saw a specific example of this cancellation when considering two-point function at order $O(\lambda)$. Here you will see that the cancellation is not an accident.

The statement is in fact is true for any H_I. But it is enough if you could prove it for the $\lambda\phi^4$ theory.

Hint: Peskin and Schroeder gives a proof for the two-point function on p.96-98, which can be easily generalized to n-point functions. The discussion of Peskin and Schroeder is not great, but hopefully you can get the key idea and write down your own proof.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 7

Mar. 21, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes:

- Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{1}$$

- The Dirac equation has the form

$$(\gamma^\mu \partial_\mu - m)\psi = 0 \ . \tag{2}$$

- A possible choice for gamma matrices is

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \ . \tag{3}$$

Problem Set 7

1. **Momentum conservation (10 points)**

   Consider an interacting field theory of real scalar ϕ. Assume that the theory is translation invariant. Introduce the Fourier transform of

$$G_n(x_1, \cdots x_n) = \langle \Omega | T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \tag{4}$$

   as

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) = \int d^4x_1 \cdots d^4x_n \, e^{-i\sum_{i=1}^n p_i\cdot x_i} G_n(x_1, \cdots x_n) \ . \tag{5}$$

   Show that

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) \propto (2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots p_n) \ . \tag{6}$$

   Note: Recall $G_n(p_1, p_2, \cdots, p_n)$ discussed in lecture is defined as

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) \equiv (2\pi)^4\delta^{(4)}(p_1 + p_2 + \cdots p_n) G_n(p_1, p_2, \cdots, p_n) \ . \tag{7}$$

2. **Feynman rules for a complex scalar field (20 points)**

   For a complex scalar field, particle and antiparticle are distinct, which we can think of as positively and negatively charged. The Feynman rules must distinguish them. We can make the distinction by including an arrow for each propagator indicating the flow of charge. Note that this is different from the arrows for momentum flows which we sometimes draw. The directions of "momentum" arrows are completely arbitrary, while those of the "charge" arrows in a diagram are not and should reflect charge conservation. *For a particle it is customary to point the arrow away from the external point in the initial state, and toward the external point in the final state.* For an antiparticle, the direction is reversed. The charge arrows for propagators of the rest of a diagram then follow from charge conservation.

   In order to avoid two kinds of arrows, we can simply align momentum arrows with the charge ones.

   (a) Consider a complex scalar field theory

   $$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 \ . \tag{8}$$

   Write down the momentum space Feynman rules for this theory.

   (b) Draw the *connected* diagrams for the *scattering amplitude* of the process

   $$\phi + \bar{\phi} \to \phi + \bar{\phi} \tag{9}$$

   to order $O(\lambda^2)$, where ϕ and $\bar{\phi}$ denote particle and antiparticle respectively. You only need to draw the diagrams.

   (c) Now suppose ϕ interacts with a real field χ via

   $$\mathcal{L}_I = \lambda' \chi \partial^\mu \phi^* \partial_\mu \phi \ . \tag{10}$$

   i.e. the full Lagrangian is the sum of (8) and (10) plus a free theory part for χ. Suppose χ has mass M. Use a solid line for the ϕ propagator and a dashed line for the χ propagator. Write down the momentum space Feynman rules for the full theory.

   (d) Suppose both $\lambda, \lambda' \sim O(\epsilon)$ with ϵ a small parameter. Draw all the *connected* diagrams for the amplitude of the decay process

   $$\chi \to \phi + \bar{\phi} \tag{11}$$

   to order $O(\epsilon^2)$.

3. **Higgs production at LHC (10 points + 10 bonus points)**

   At LHC people collides a proton with a proton at very high energies. Each proton contains a number of quarks and gluons. So collisions of protons can also be considered as collisions of gluons. Here we consider a baby version of the Standard Model which contains three types of scalar fields:

   - real gluon field g. Use a wavy line to denote its propagator.
   - *complex* quark field q (quark and antiquark are different). Use a solid line to denote its propagator. Remember the arrow!
   - real Higgs field H. Use a dashed line to denote its propagator.

   Suppose the interaction part of the theory is given by

$$\mathcal{L}_I = \lambda_1 g q^\dagger q + \lambda_2 H q^\dagger q \ . \tag{12}$$

   Note that there is no direct coupling between gluon g and Higgs H. You can assume that couplings λ_1, λ_2 are small and of comparable strength.

   (a) The dominant channel for Higgs production at LHC is the so-called gluon fusion process, which can be schematically written as

$$g + g \to H \tag{13}$$

   Draw the leading Feynman diagram(s) for (13).

   (b) **Bonus problem (10 pts):** Another channel for Higgs production is

$$g + g \to H + q + \bar{q} \tag{14}$$

   where $\bar{q}$ denotes antiquark. Draw the leading Feynman diagram(s) for such a process.

4. **Properties of gamma matrices (25 points)**

   Without resorting to a particular representation, prove the following identities

   (a) $\mathrm{Tr}\,\gamma^\mu = 0$.

   (b) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$.

   (c) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda) = 0$.

   (d) $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q - 2i\Sigma^{\mu\nu} p_\mu q_\nu$.

   (e) $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$

   where we have defined

$$\not{p} \equiv p_\mu \gamma^\mu, \quad \Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ . \tag{15}$$

5. **Conserved "probability" current of the Dirac equation (15 points)**

Starting from the Dirac equation (2),

(a) show that one can construct a current j^μ which is conserved

$$\partial_\mu j^\mu = 0 \ . \tag{16}$$

(b) Show that j^μ you constructed is real.

(c) Show that by choosing the overall sign of j^μ, the zeroth component of j^μ

$$\rho \equiv j^0 \tag{17}$$

can be made to be positive definite.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 8

Apr. 4, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes: conventions and some useful formulae

1. Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

and

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form

$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$

and the action is given by

$$S = -i\int d^4x\, \bar{\psi}(\not\partial - m)\psi \ . \tag{4}$$

3. A spinor ψ transforms under a Lorentz transformation Λ as

$$\psi'_\alpha(x') = S_\alpha{}^\beta(\Lambda)\psi_\beta(x), \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5}$$

with

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\lambda\rho}\Sigma^{\lambda\rho}}, \tag{6}$$

and

$$\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu = i\left(\eta^{\lambda\mu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\lambda{}_\nu\right), \qquad \Sigma^{\lambda\rho} = \frac{i}{4}[\gamma^\lambda, \gamma^\rho] \ . \tag{7}$$

4. Note the relations

$$[\Sigma^{\lambda\rho}, \gamma^\mu] = -\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu\gamma^\nu, \tag{8}$$

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu, \tag{9}$$

$$S^\dagger = -\gamma^0 S^{-1}\gamma^0 \tag{10}$$

5. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

6. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as

$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{11}$$

   $u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal

$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{12}$$

7. With normalization (11), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{13}$$

   and the orthogonal relations (12) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{14}$$

   These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (11).

8. With normalization (11), one can also show that

$$\Lambda_+(\vec{k}) \;=\; \sum_{s=1,2} u_s(\vec{k}) \otimes \bar{u}_s(\vec{k}) = i(i\not{k} + m), \tag{15}$$

$$\Lambda_-(\vec{k}) \;=\; \sum_{s=1,2} v_s(\vec{k}) \otimes \bar{v}_s(\vec{k}) = -i(-i\not{k} + m) \ . \tag{16}$$

9. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}^{(s)} u_s(\vec{k})e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k})e^{-ik\cdot x} \right] \ . \tag{17}$$

   where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \tag{18}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{19}$$

Problem Set 8

1. **Some proofs (21 points)**

   (a) From the definitions of (6), prove (9).

   Hint: use (8).

   Note: in lecture we derived the infinitesimal version of (9). *Here you need to prove the finite version.*

   (b) Prove (10).

   (c) From Lorentz transformation of ψ, show that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform under respectively as a scalar and a vector.

2. **Some identities (8 points)**

   Without using any explicit form of u_s and v_s, show: one of the relations in (13). In other words, show one of the following

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = -\frac{iE}{m}\bar{u}^r(\vec{k})u_s(\vec{k}), \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = \frac{iE}{m}\bar{v}^r(\vec{k})v_s(\vec{k}) \ . \tag{20}$$

3. **Stress tensor and the Hamiltonian for the Dirac theory (21 points)**

   (a) The Dirac action is translationally invariant. Use the Noether procedure to construct the corresponding conserved currents $\Theta^{\mu\nu}$, i.e. the energy-momentum tensor.

   The "charge" density Θ^{00} for time translation is the energy density. Show that your Θ^{00} indeed coincides with the Hamiltonian density $\mathcal{H}$ we derived in lecture, i.e.

$$\Theta^{00} = \mathcal{H} = i\bar{\psi}(\gamma^i\partial_i - m)\psi \ . \tag{21}$$

   (b) Show that using the Dirac equations the Hamiltonian can be written as

$$H = i\int d^3x\, \psi^\dagger\partial_t\psi \tag{22}$$

   and express H in terms of $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ introduced in (17).

   (c) What is the vacuum energy density? (You can leave the answer as an integral). Discuss the differences with that for a scalar.

4. **Angular momentum operators (30 points)**

   The Dirac action (4) is Lorentz invariant.

   (a) Write down an infinitesimal Lorentz transformation for ψ.

(b) Use the Noether procedure to construct the conserved *charges* $M^{\mu\nu}$ associated with Lorentz transformations, and show that $M^{\mu\nu}$ can be written as a sum of a "spin" part $S^{\mu\nu}$ and an "orbital angular momentum" part $L^{\mu\nu}$, i.e.

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu} \ . \tag{23}$$

Show that the orbital part $L^{\mu\nu}$ has the same form as that of a scalar, i.e.

$$L^{\mu\nu} = \int d^3x \left(x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu} \right) \tag{24}$$

where $\Theta^{\mu\nu}$ is the energy-momentum tensor you obtained in 2(a). Show that the spin part $S^{\mu\nu}$ can be written as

$$S^{\mu\nu} = - \int d^3x \, \psi^\dagger \Sigma^{\mu\nu} \psi \ . \tag{25}$$

(c) Express the $S^{\mu\nu}$ in terms of $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$.

(d) From the expression you obtained from part (c) for S^{ij}, keep only the time-independent part, which we will denote as $\tilde{S}^{ij}$. Define

$$\vec{J}^2 \equiv \frac{1}{2} \tilde{S}^{ij} \tilde{S}_{ij} \ . \tag{26}$$

Show that the one-particle states constructed by acting $(a_{\vec{k}}^{(s)})^\dagger$ and $(c_{\vec{k}}^{(s)})^\dagger$ with $\vec{k} = 0$ (i.e. in the rest frame) on the vacuum are eigenstates of $\vec{J}^2$ with eigenvalues corresponding to that of a spin-$\frac{1}{2}$ particle.

Note: the reason that we need not worry about the time-dependent part of S^{ij} is that the part will be canceled by a contribution from L^{ij}. M^{ij} is conserved: so the time dependent parts of S^{ij} and L^{ij} have to cancel each other.

Note: the restriction to the rest frame is important here as S_{ij} is not covariant under a Lorentz transformation. A covariant version of it is called the Pauli-Lubanski pseudovector, which we will not go into here.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 9

Apr. 11, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes: conventions and some useful formulae

1. Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

and

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form

$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$

and the action is given by

$$S = -i\int d^4x\, \bar{\psi}(\not{\partial} - m)\psi \ . \tag{4}$$

3. A spinor ψ transforms under a Lorentz transformation Λ as

$$\psi'_\alpha(x') = S_\alpha{}^\beta(\Lambda)\psi_\beta(x), \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5}$$

with

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\lambda\rho}\Sigma^{\lambda\rho}}, \tag{6}$$

and

$$\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu = i\left(\eta^{\lambda\mu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\lambda{}_\nu\right), \qquad \Sigma^{\lambda\rho} = \frac{i}{4}[\gamma^\lambda, \gamma^\rho] \ . \tag{7}$$

Also note

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu \ . \tag{8}$$

4. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

5. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as

$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{9}$$

$u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal

$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{10}$$

6. With normalization (9), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{11}$$

and the orthogonal relations (10) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{12}$$

These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (9).

7. With normalization (9), one can also show that

$$\Lambda_+(\vec{k}) = \sum_{s=1,2} u_s(\vec{k}) \otimes \bar{u}_s(\vec{k}) = i(i\not{k} + m), \tag{13}$$

$$\Lambda_-(\vec{k}) = \sum_{s=1,2} v_s(\vec{k}) \otimes \bar{v}_s(\vec{k}) = -i(-i\not{k} + m) \ . \tag{14}$$

8. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}^{(s)} u_s(\vec{k})e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k})e^{-ik\cdot x} \right] \ . \tag{15}$$

where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \tag{16}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{17}$$

9. In the chiral representation, the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \ . \tag{18}$$

10. In this course we will not be able to talk about applications of Majorana fermions, which have played important roles in models for neutrinos. During the last decade, it has also received wide interests in condensed matter physics and quantum computing as a possible avenue for topological quantum computing. Ettore Majorana came up with the idea of Majorana fermions in 1937, at the age of 31. Less than one year later he mysteriously disappeared; he boarded a ship from Palermo to Naples, but never got off it.

    Fermi once said:

    "There are several categories of scientists in the world; those of second or third rank do their best but never get very far. Then there is the first rank, those who make important discoveries, fundamental to scientific progress. But then there are the geniuses, like Galilei and Newton. Majorana was one of these."

    You can read more about Majorana at:

    http://www.ccsem.infn.it/em/EM_genius_and_mystery.pdf.

Problem Set 9

1. **Some identities (10 points)**

   Define γ^5 as
   $$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{19}$$
   Show that it has the following properties:

   (a) $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = 1$.

   (b) $\{\gamma^5, \gamma^\mu\} = 0$ and $\mathrm{Tr}\,\gamma^5 = 0$.

2. **Feynman propagator for Dirac spinors (10 points)**

   Show that the Feynman Green function
   $$D_F^{\alpha\beta}(x-y) \equiv \left\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\right\rangle = i(\partial\!\!\!/ + m)_{\alpha\beta} G_F(x-y) \tag{20}$$
   where G_F is the Feynman propagator for a free complex scalar of the same mass m.

3. **Chiral and Majorana fermions (50 points)**

   In this problem we consider the chiral representation (18), and write a Dirac spinor ψ in terms of two chiral spinors ψ_L and ψ_R as
   $$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ . \tag{21}$$

(a) Show that under a rotation with parameters $\omega_{ij} = \epsilon_{ijk}\theta_k$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\psi_R(x) \ . \tag{22}$$

(b) Show that under a boost with parameters $\omega_{0i} = \beta_i$, $\psi_{L,R}$ transform as

$$\psi'_L(x') = e^{-\frac{1}{2}\vec{\beta}\cdot\vec{\sigma}}\psi_L(x), \qquad \psi'_R(x') = e^{\frac{1}{2}\vec{\beta}\cdot\vec{\sigma}}\psi_R(x) \ . \tag{23}$$

(c) The Lagrangian density for the Dirac theory contains a mass term of the form

$$\mathcal{L} = \cdots + im\bar{\psi}\psi = \cdots + im(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \ . \tag{24}$$

Using the transformations of parts (a) and (b) show that the above mass term is Lorentz invariant, while a term of the form

$$m\psi_L^\dagger\psi_L \quad \text{or} \quad m\psi_R^\dagger\psi_R \tag{25}$$

is not.

(d) The discussion in part (c) might give the impression that it is not possible to write down a mass term with ψ_L or ψ_R alone. In fact, it is possible, with a bit more sophistication. For this purpose, first show that

$$\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^* \ . \tag{26}$$

From the above equation show that $\sigma^2\psi_L^*$ transforms under Lorentz transformation in the same way as ψ_R.

(e) From the observation of part (d), construct a mass term using ψ_L alone, which is both Lorentz invariant and real. (This mass term is called the Majorana mass term for reasons which will be clear in problem 4.)

Show that the mass term is *identically* zero if ψ_L consists of ordinary functions, while it is nonzero if ψ_L are anti-commuting variables.

(f) Now write down a Lorentz invariant full action using ψ_L alone which includes both kinetic and mass terms. Write down equations of motion.

(g) Does the theory of part (f) has a conserved charge? Argue such a theory can only describe neutral particles (thus cannot be a theory of electron).

4. **Majorana fermions (10 points)**

In the Majorana representation, γ^μ are real and thus ψ can be chosen to be real. Such a spinor is call a Majorana spinor. This has important physical consequences. Upon quantization, being real, a Majorana particle should not have an anti-particle (or equivalently it is its own anti-particle).

We discussed in lecture that the concept of a Majorana spinor can be generalized to any representation of γ, if we could find a matrix B satisfying

$$B\gamma^{\mu}B^{-1} = \gamma^{\mu *}, \tag{27}$$

and the Majorana condition is

$$\psi^{*} = B\psi \ . \tag{28}$$

(a) In lecture we showed that in the chiral representation (18) we can choose $B = \gamma^2$. Solve the Majorana condition (28) in the chiral representation. Show that in this representation a Majorana spinor ψ can be expressed in terms of ψ_L alone.

(b) Plug in the expression (in terms of ψ_L) for Majorana ψ you obtained in part (a) into the Dirac action. Show that it reduces to the action you found in part 3(f)! (That is why the mass term you found in part 3(e) is called the Majorana mass term.)

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 10

Apr. 18, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3
- Peskin & Schroeder Chap. 9.5
- Peskin & Schroeder Chap. 4.7

Notes: conventions and some useful formulae

1. Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

   and

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form

$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$

   and the action is given by

$$S = -i\int d^4x\, \bar{\psi}(\not{\partial} - m)\psi \ . \tag{4}$$

3. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

4. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as

$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{5}$$

   $u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal

$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{6}$$

5. With normalization (5), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{7}$$

and the orthogonal relations (6) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{8}$$

These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (5).

6. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}^{(s)}u_s(\vec{k})e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k})e^{-ik\cdot x}\right] \ . \tag{9}$$

where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'), \tag{10}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{11}$$

Problem Set 10

1. **Chiral symmetry (15 points)**

   Consider the Dirac action with $m = 0$.

   (a) Show that the action is invariant under transformations

   $$\psi \to e^{i\alpha\gamma^5}\psi \ . \tag{12}$$

   (b) Construct the Noether current for the above symmetry.

   (c) Find how the mass term $m\bar{\psi}\psi$ transforms under (12). Is it invariant?

2. **Quantizing the theory of Majorana fermions (25 points)**

   Consider the theory of Majorana fermions discussed in Pset 9, written in terms of a two-component complex spinor ψ_L

   $$\mathcal{L}_L = i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - \frac{m}{2}(\psi_L^T\sigma^2\psi_L + \psi_L^\dagger\sigma^2\psi_L^*). \tag{13}$$

   where ψ^T denotes the transpose of ψ, and $\sigma^\mu = (1, \vec{\sigma})$.

(a) Write down the equal time canonical quantization relations.

(b) Write down the classical equations of motion. In momentum space a general solution can be written as

$$\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x} . \tag{14}$$

Using the above notations, write down a complete basis of solutions in the rest frame (i.e. $\vec{p} = 0$).

(c) Verify the following expressions give a complete basis of solutions for general p (below $\bar{\sigma} = (1, -\vec{\sigma})$)

$$\begin{aligned} u_s(p) &= \sqrt{-p\cdot\bar{\sigma}}\zeta_s & (15) \\ v_s(p) &= -\sqrt{-p\cdot\bar{\sigma}}\sigma^2\zeta_s & (16) \end{aligned}$$

where $\zeta_s, s = \pm$ are respectively eigenvectors of σ^3 with eigenvalues $\pm$.

(d) Write down the mode expansion for quantum operator ψ_L.

(e) Define the vacuum and construct the single-particle states (properly normalized). Discuss the differences between the particles in this theory and those of the Dirac theory.

3. **Gaussian integrals for Grassman variables (16 points)**

Show that

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i)\ e^{-\theta_i^* A_{ij}\theta_j} = \det A \tag{17}$$

and

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i)\ \theta_k\theta_l^* e^{-\theta_i^* A_{ij}\theta_j} = \det A (A^{-1})_{kl}\ . \tag{18}$$

4. **Yukawa theory (24 points)**

Consider the Yukawa theory discussed in lecture

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - i\bar{\psi}(\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\psi\ . \tag{19}$$

Denote the propagator of a ϕ particle by a dashed line and that of ψ by a solid line (with arrow). We will call p the particle excitation of ψ and $\bar{p}$ and the anti-particle excitation of ψ.

(a) Consider the process

$$\bar{p} + \bar{p} \to \bar{p} + \bar{p} \tag{20}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), (p_2, s_2)$ and $(p_1', s_1'), (p_2', s_2')$ respectively.

(b) Consider the process

$$p + \bar{p} \to \phi + \phi \tag{21}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), (p_2, s_2)$ and p_1', p_2' respectively.

(c) Consider the process

$$p + \phi \to p + \phi \tag{22}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), p_2$ and $(p_1', s_1'), p_2'$ respectively.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 11

Apr. 25, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 9.4

Notes:

Problem Set 11

1. **Quantization of Maxwell in the Lorentz gauge: null states and residual gauge transformations (50 points)**

   In the Lorentz gauge we consider the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2 \tag{1}$$

where ξ is an arbitrary real parameter (and different ξ's give equivalent theories). It is convenient to take $\xi = 1$, in which case

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \ . \tag{2}$$

The complete set of solutions to operator equations following from (2)

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\sum_{\alpha=0}^{3}\left[\mathcal{E}_\mu^{(\alpha)}a_{\vec{k}}^{(\alpha)}e^{ik\cdot x} + \mathcal{E}_\mu^{(\alpha)*}(a_{\vec{k}}^{(\alpha)})^\dagger e^{-ik\cdot x}\right] \tag{3}$$

where $\omega_{\vec{k}} = |\vec{k}|$ and $k^\mu = (|\vec{k}|, \vec{k})$. $\mathcal{E}_\mu^{(\alpha)}$ are polarization vectors defined by

$$\mathcal{E}_\mu^{(0)} = (1, \vec{0}), \quad \mathcal{E}_\mu^{(3)} = \left(0, \frac{\vec{k}}{|\vec{k}|}\right), \quad \mathcal{E}_\mu^{(1,2)} = (0, \vec{\epsilon}_{1,2}), \quad \vec{\epsilon}_{1,2}\cdot\vec{k} = 0 \tag{4}$$

where $\vec{\epsilon}_{1,2}$ are orthogonal unit-norm spatial vectors.

With canonical commutation relations

$$[A_\mu(t,\vec{x}), \pi^\nu(t'\vec{x}')] = i\delta_\mu^\nu \delta^{(3)}(\vec{x}-\vec{x}'), \quad \pi^\mu = \partial_0 A^\mu \tag{5}$$

we find that

$$[a^{(\alpha)}_{\vec{k}}, (a^{(\beta)}_{\vec{k}'})^\dagger] = \eta^{\alpha\beta}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') \tag{6}$$

with all other commutators vanishing. We define the vacuum as

$$a^{(\alpha)}_{\vec{k}}|0\rangle = 0, \quad \forall \alpha, \vec{k} \tag{7}$$

and the "big" Hilbert space is defined as

$$\mathcal{H}_{\rm big} = \{|\psi\rangle \text{ obtained by acting } (a^{(\alpha)}_{\vec{k}})^\dagger \text{ on } |0\rangle\} \ . \tag{8}$$

As discussed in lecture, $\mathcal{H}_{\rm big}$ contains states with negative norms, and thus unphysical states. Indeed $\mathcal{H}_{\rm big}$ follows from (2), which by itself is not the Maxwell theory.

To obtain the Maxwell theory we still need to impose $\partial_\mu A^\mu = 0$ and get rid of the residual gauge freedom. In this problem we will guide you to do this. We will see that by imposing $\partial_\mu A^\mu = 0$ we get rid of the negative-norm unphysical states in $\mathcal{H}_{\rm big}$. But this is not enough. We will observe that some states possess zero norm, which can be attributed to the presence of residual gauge freedom. By eliminating these null states, we obtain the physical Hilbert space, which contains only two transverse massless degrees of freedom rather than four.

The enforcement of $\partial_\mu A^\mu = 0$ at the quantum level is subtle. Remember that in the Coulomb gauge, classically we apply the gauge condition $\nabla \cdot \vec{A} = 0$ as part of the equations of motion, which becomes an operator equation at the quantum level. In the Lorentz gauge, classically we only need to impose "boundary conditions" to ensure that the equation $\partial^2(\partial_\mu A^\mu) = 0$ has the trivial solution $\partial_\mu A^\mu = 0$. This implies that, at the quantum level, we cannot enforce $\partial_\mu A^\mu = 0$ as an operator equation (as you will check yourself below). Instead, we will have to do something weaker, imposing a variant of it as a condition on the physical states.

(a) Calculate

$$[A_0(t,\vec{x}), \partial_\mu A^\mu(t.\vec{x}')] \tag{9}$$

and from your result explain why we cannot impose $\partial_\mu A^\mu = 0$ as an operator equation.

(b) We also cannot impose that "physical states" satisfy

$$\partial_\mu A^\mu |\psi\rangle = 0 \tag{10}$$

as $\partial_\mu A^\mu|0\rangle \neq 0$. We do want to keep $|0\rangle$ to be a physical state. So to define physical states we need a weaker condition. It turns out the condition that eliminates all negative norm states while keeping the vacuum $|0\rangle$ is

$$\partial^\mu A_\mu^{(-)}|\psi\rangle = 0 \tag{11}$$

where $A_\mu^{(-)}$ denotes the annihilation part of A_μ. We now impose that physical states should satisfy (11) and denote the set of physical states in $\mathcal{H}_{\text{big}}$ to be $\mathcal{H}_{\text{small}}$.

Show that there is no negative-norm state in $\mathcal{H}_{\text{small}}$.

(c) Show that

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0, \quad \forall|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}, \tag{12}$$

that is, $\partial_\mu A^\mu$ has zero matrix element among states in $\mathcal{H}_{\text{small}}$.

(d) Introduce

$$b_{\vec{k}}^{(\pm)} = \frac{1}{\sqrt{2}}\left(a_{\vec{k}}^3 \pm a_{\vec{k}}^0\right), \qquad (b_{\vec{k}}^{(\pm)})^\dagger = \frac{1}{\sqrt{2}}\left(a_{\vec{k}}^{3\dagger} \pm a_{\vec{k}}^{0\dagger}\right) . \tag{13}$$

Show that the physical state condition (11) can be written as

$$b_{\vec{k}}^{(+)}|\psi\rangle = 0 . \tag{14}$$

(e) We are not done yet as $\mathcal{H}_{\text{small}}$ still contain zero-norm states. To see this, first work out the commutation relations

$$[b_{\vec{k}}^+, (b_{\vec{k}'}^+)^\dagger], \qquad [b_{\vec{k}}^-, (b_{\vec{k}'}^-)^\dagger], \qquad [b_{\vec{k}}^+, (b_{\vec{k}'}^-)^\dagger], \qquad [b_{\vec{k}}^-, (b_{\vec{k}'}^+)^\dagger] . \tag{15}$$

and show that $\mathcal{H}_{\text{small}}$ can also be described as

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } (a_{\vec{k}}^1)^\dagger, (a_{\vec{k}}^2)^\dagger, (b_{\vec{k}}^+)^\dagger \text{ on } |0\rangle\} . \tag{16}$$

In other words, a physical state can have no $(b_{\vec{k}}^-)^\dagger$ excitations.

(f) Show that any state $|\psi\rangle$ in $\mathcal{H}_{\text{small}}$ with nonzero $(b_{\vec{k}}^+)^\dagger$ excitations has zero norm and its overlap with any state in $\mathcal{H}_{\text{small}}$ is zero. Such states (which are called null states) clearly cannot have any physical significance.

(g) Show that any state with nonzero norm then must have the form

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle \tag{17}$$

where $|\psi_T\rangle$ contains only excitations of $(a_{\vec{k}}^1)^\dagger, (a_{\vec{k}}^2)^\dagger$ (i.e. transverse components) and $|\chi\rangle$ is a null state.

$|\psi\rangle$ should be physically equivalent to $|\psi_T\rangle$ as they differ only by a null state.

We can then forget about the null states and define

$$\mathcal{H}_{\text{phys}} = \{|\psi_T\rangle\} \ . \tag{18}$$

$\mathcal{H}_{\text{phys}}$ contains only positive-norm states and is identical to that obtained in the Coulomb gauge.

This concludes the discussion of quantization in the Lorentz gauge. For the rest the problem we explore a bit further the nature of null states and their physical interpretation.

(h) Let us call excitations of $(b^+_{\vec{k}})^\dagger$ null photons. To understand the physical interpretation of a null photon, let us consider the "wave function" $\chi_\mu(x)$ of the single null photon state

$$\chi_\mu(x) = \Big\langle 0\big|A_\mu(x)\big|\vec{k},+\Big\rangle, \qquad |\vec{k},+\rangle = \sqrt{2\omega_{\vec{k}}}(b^+_{\vec{k}})^\dagger|0\rangle \ . \tag{19}$$

Note that the above definition of wave function $\chi_\mu(x)$ is the straightforward generalization to vector field of our previous discussion for a scalar field. Show that $\chi_\mu(x)$ can be written as

$$\chi_\mu(x) = \partial_\mu\lambda(x) \tag{20}$$

where $\lambda(x)$ is some function which satisfies the equation for a massless scalar

$$\partial_\mu\partial^\mu\lambda = 0 \ . \tag{21}$$

This shows that a null photon can be interpreted as a gauge transformation from the vacuum.

[Recall that the Lorentz gauge $\partial_\mu A_\mu = 0$ leaves residual gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad \partial_\mu\partial^\mu\lambda = 0 \ . \tag{22}$$

Thus $\chi_\mu(x)$ can be considered as a residual gauge transformation from $A_\mu = 0$.]

(i) Show that the conclusion of part (h) holds for any wave packet of a null photon

$$|f\rangle = \int d^3\vec{k}\, f(\vec{k})|\vec{k},+\rangle, \tag{23}$$

i.e. the wave function for $|f\rangle$ again has the form of a residual gauge transformation from $A_\mu = 0$.

2. **Casimir effect in one dimension (30 points)**

Until now, we have disregarded the vacuum's zero-point energy as an unobservable (infinite) shift in the energy scale. However, as Casimir demonstrated in

1948, *differences* in vacuum zero-point energies are, however, observable. This phenomenon is known as the Casimir effect. A simplest example of the Casimir effect is a small attractive force between two close parallel conducting plates, due to quantum vacuum fluctuations of the electromagnetic field. The force is caused by change in vacuum energy of the electromagnetic field that results from the boundary conditions imposed by the plates.

In this problem we will explore the Casimir effect. For technical simplicity, we will consider a toy version of the effect, which captures all the essential physics. As we have learned, after quantization the electromagnetic field has the same number of degrees of freedom as two massless scalar fields. So we will consider a massless scalar field instead of the Maxwell theory. Also instead of three spatial dimensions we consider one spatial dimension.

Consider a free, *massless*, real scalar field ϕ in one spatial dimension, i.e.

$$S = -\frac{1}{2}\int dxdt\, \partial_\mu \phi \partial^\mu \phi \ . \tag{24}$$

The vacuum of the system has an infinite zero-point energy. Let us denote it by E_0. Now imagine we put two "plates" at $x = 0$ and $x = a$ such that ϕ is required to vanish at the location of the plates,[1] i.e.

$$\phi(x = 0, t) = \phi(x = a, t) = 0 \ . \tag{25}$$

Adding plates which imposes additional boundary conditions on ϕ disturbs the vacuum, and results in a different zero-point energy $E(a)$.

Even though both E_0 and $E(a)$ are infinite, their difference turns out to be finite and physically meaningful. In fact the difference

$$U(a) = E(a) - E_0 \tag{26}$$

can be considered as the potential energy between the two plates. Changing a modifies the potential energy and results in a force between the plates which can be measured experimentally!

In this problem I will guide you to calculate $U(a)$. As you learned in calculus, taking the differences between infinities is a highly dangerous thing to do. One can in principle get any answer. So we will need to be very careful.

Both E_0 and E_a have two sources of infinities, one from the infinite volume, the other from there are infinite number of *local* degrees of freedom. It is convenient to separate these two infinities by putting the system in a box with finite size $L \gg a$. We will take L to infinity at the end of the calculation. More explicitly, we require ϕ to satisfy a periodic boundary condition

$$\phi(x, t) = \phi(x + L, t) \tag{27}$$

[1] In real-life Casimir effects, the plates are simply conducting plates.

i.e. putting the system on a circle of size L.

(a) In the vacuum (i.e. before putting the plates), write down the mode expansion for ϕ and calculate its zero-point energy E_0. Your answer should have the form

$$E_0 = \frac{1}{2}\sum_n \omega_n \tag{28}$$

where ω_n is the energy for each mode. You should specify both ω_n and the range of summation.

(b) Adding the plates separates the system into two segments, one has size a, the other has size $L-a$. In both segments one has Dirichlet boundary conditions (25) at two ends. (Remember the system is on a circle.) So it is enough to work out the story for one of them. Find the mode expansion for ϕ in the region $[0,a]$ and zero-point energy $\varepsilon(a)$. Again your answer should have the form

$$\varepsilon(a) = \frac{1}{2}\sum_n \tilde{\omega}_n \tag{29}$$

and you should specify both $\tilde{\omega}_n$ and the range of summation. The total zero-point energy of the system in the presence of the plates is thus

$$E(a) = \varepsilon(a) + \varepsilon(L-a)\ . \tag{30}$$

(c) Both sums (28) and (30) are divergent. There is not much sense in taking the difference between them. To take the difference we will first make them finite. We will do this by introducing a "UV cutoff" Λ, i.e. change the sums to[2]

$$E_0 = \frac{1}{2}\sum_n \omega_n e^{-\frac{\omega_n}{\Lambda}} \tag{31}$$

and

$$\varepsilon(a) = \frac{1}{2}\sum_n \tilde{\omega}_n e^{-\frac{\tilde{\omega}_n}{\Lambda}}\ . \tag{32}$$

Clearly taking the limit $\Lambda \to \infty$ *inside* the sum we recover (28) and (29). Now evaluate (31) and (32) with a finite Λ.

(d) Expand the answers you get in part (c) in the limit $\Lambda \to \infty$. You will find they become divergent. Keep terms which are divergent and finite, but throw away all terms which go to zero in the limit (i.e. throw away all terms with a negative power of Λ).

[2]Note that by introducing Λ, we suppress the contributions from "high" energy modes with $\omega_n, \tilde{\omega}_n \gg \Lambda$.

(e) From your answers in part (d), find

$$U(a) = E(a) - E_0 \ . \tag{33}$$

You should find $U(a)$ is finite in the limit $\Lambda \to \infty$. Now take the limit $L \to \infty$ in $U(a)$, and find the force between the plates in the $L \to \infty$ limit.

8.323 Relativistic Quantum Field Theory I
Spring 2023

Quantum Field Theory I (8.323) Spring 2023
Assignment 12

May. 2, 2023

- Please remember to put **your name** at the top of your paper.
- **This is the last pset of the semester, with total points 90+30 (30 are bonus points).**

Readings

- Peskin & Schroeder Chap. 4.5 – 4.6
- Peskin & Schroeder Chap. 4.7–4.8
- Peskin & Schroeder Chap. 5

Notes:

1. For discussion of S-matrix and cross section, you can also read Weinberg Vol1. Sec. 3.1, 3.2 and 3.4. Another good source is Sredicki Vol1. chap. 11, which I partially followed in my discussion.

2. Decay rate: consider the decay process

$$p_1 \to k_1 + k_2 + \cdots + k_n \ . \tag{1}$$

Then the differential decay rate of decaying into n final particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, and etc., is given by

$$d\Gamma = \frac{1}{2E_1}|M|^2 d\mu \tag{2}$$

where E_1 is the energy of the decaying particle, M is the corresponding scattering amplitude, and $d\mu$ is the Lorentz invariant differential measure

$$d\mu = (2\pi)^4 \delta^{(4)}\left(p_\alpha - p_\beta\right) \prod_{i=1}^{n} \frac{d^3\vec{k}_i}{(2\pi)^3} \frac{1}{2k_i^0} \tag{3}$$

where p_α and p_β denote the total momentum of the initial and final state respectively.

The full decay rate for this process is given by

$$\Gamma = \frac{1}{\Lambda} \int d\Gamma \tag{4}$$

where the integration is over all momenta. $1/\Lambda$ is a numerical factor which is 1 if the final particles are all distinct. If there are a subset of m *identical* particles in the final state, then $\Lambda = m!$ as the integrations over all momenta over-count the phase space.

3. Cross section: consider the scattering process

$$p_1 + p_2 \to k_1 + k_2 + \cdots + k_n \ . \tag{5}$$

Then the differential cross section of finding n final particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, and etc., is given by

$$d\sigma = \frac{1}{4\Sigma}|M|^2 d\mu, \qquad \Sigma = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{6}$$

where M is the corresponding scattering amplitude, and $d\mu$ is again given by (3). m_1, m_2 are the masses of p_1 and p_2 respectively. The total cross section is given by

$$\sigma_{\text{tot}} = \frac{1}{\Lambda} \int d\sigma \tag{7}$$

where as in (4) $1/\Lambda$ is a numerical factor which accounts for the over-counting of phase space if there are identical particles in the final state.

4. It is often convenient to introduce

$$s = -(p_1 + p_2)^2 \tag{8}$$

which is the invariant mass square for the whole system.

5. It is often convenient to consider the center of mass frame where

$$\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}_{cm} \ . \tag{9}$$

The magnitude $|\vec{p}_{CM}|$ can be expressed in terms of s by solving the equation

$$\sqrt{s} = \sqrt{\vec{p}_{cm}^2 + m_1^2} + \sqrt{\vec{p}_{cm}^2 + m_2^2} \ . \tag{10}$$

Equation (6) can also be written as

$$d\sigma = \frac{|M|^2}{4|\vec{p}_{cm}|\sqrt{s}} d\mu \ . \tag{11}$$

6. Two-to-two scattering: for two-to-two scattering

$$p_1 + p_2 \to k_1 + k_2 \tag{12}$$

in addition to (8) it is convenient to introduce

$$t \equiv -(p_1 - k_1)^2, \qquad u = -(p_1 - k_2)^2 \ . \tag{13}$$

Note that

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \tag{14}$$

where m_1' and m_2' are the masses of k_1 and k_2 respectively. In the center of mass frame, from momentum conservation, the final spatial momenta satisfy

$$\vec{k}_1 = -\vec{k}_2 \equiv \vec{k}_{cm} \tag{15}$$

where $|\vec{k}_{cm}|$ can again be expressed in terms of s.

The differential cross section can be further simplified and written as

$$d\sigma = \frac{|M|^2}{64\pi^2 s} \frac{|\vec{k}_{cm}|}{|\vec{p}_{cm}|} d\Omega_{cm} \tag{16}$$

where $d\Omega_{cm}$ denotes the angular part of $d^3\vec{k}_{cm}$.

Problem Set 12

1. **Decay of a scalar particle (20 points)**

   Consider a theory with two scalar fields ϕ and χ

   $$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}M^2\phi^2 - \tfrac{1}{2}(\partial_\mu \chi)^2 - \tfrac{1}{2}m^2\chi^2 + \tfrac{1}{2}\, g\phi\chi^2 \, . \tag{17}$$

   Assume $M > 2m$ and the coupling constant g is small. Calculate the decay rate Γ of ϕ-particles to the lowest order in g.

2. **Cross-sections (20 points)**

   In this problem we consider a toy model of the process in which an electron-positron collision produces a quark-antiquark final state through an intermediate photon: $e^+e^- \to \gamma \to q\,\bar{q}$. The role of the electron and that of the positron is played by a massless scalar field ψ. The photon is represented by a massless scalar field ϕ. Finally, the quarks are represented by a massive scalar field χ. The relevant Lagrangian density is

   $$\mathcal{L} = -\frac{1}{2}(\partial\psi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g'\phi\psi^2 + \frac{1}{2}g\phi\chi^2 \, . \tag{18}$$

   Assume that couplings g and g' are small and of comparable magnitude.

(a) Consider the process:

$$\psi\,\psi \to \chi\,\chi\ . \tag{19}$$

Compute the total cross section to lowest order in couplings. Express your answer in terms of s, t, u variables.

(b) Consider the process:

$$\psi\,\psi \to \psi\,\psi\ . \tag{20}$$

Compute the differential cross section to lowest order in couplings. Express your answer in terms of s, t, u variables .

3. **Rutherford scattering (30 + 10 points)**

(a)–(c) Peskin & Schroeder: Prob. 4.4.

The following part is counted as bonus (10 points)

(d) Repeat the calculation of part (c) in the relativistic regime. After averaging over the spin of the initial state and summing over the spin of the final state, show that the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\vec{p}|^2\beta^2\sin^2\frac{\theta}{2}}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right) \tag{21}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant, $\vec{p}$ is the spatial momentum of the electron, and β is its velocity.

4. **Electron-Muon scattering (20+20 points)**

Consider the scattering process

$$e^- + \mu^- \to e^- + \mu^-\ . \tag{22}$$

Electron and muon have mass m_e and m_μ respectively.

(a) Calculate the scattering amplitude $\mathcal{M}$.

(b) Calculate

$$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 \tag{23}$$

where $\sum_{\text{spin}}$ denotes average over the spins of the initial state and sum over the spins of the final state. Express your answer in terms of s, t, u variables defined in (8) and (13).

The following two parts are bonus parts (20 points)

(c) Calculate the differential cross section in the center of mass frame.

(d) Obtain the differential cross section in the muon rest frame. Show that in the limit $m_\mu \to \infty$, one recovers the cross section of Rutherford scattering, equation (21).

[Hint: you can consult Sec. 5.4 of Peskin and Schroeder which contains all the key results. But you need to work out some of the intermediate steps yourself.]

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