

The Standard Model

The Guideline

Our purpose in theoretical physics is not to describe the world as we find it, but to explain – in terms of a few fundamental principles – why the world is the way it is.

Steven Weinberg

1. Plan and Introduction

These lecture notes cover the Standard Model (SM) —the SM is constructed from first principles.

We begin by demonstrating that essentially only particles with spin/helicity 0, $1/2$, 1, $3/2$, 2 can describe matter and interactions, using spacetime symmetries, soft theorems, gauge redundancies, Ward identities, and perturbative unitarity.

The remaining freedom lies in the choice of the Yang-Mills gauge group and matter representations.

Effective field theories (EFTs) are a central theme throughout the course, with the 4-Fermi interactions and chiral perturbation theory serving as key examples.

Both gravity and the SM itself are treated as EFTs, specifically as the SMEFT (Standard Model Effective Field Theory).

Key phenomenological aspects of the SM are covered, including the Higgs mechanism, Yukawa couplings, the CKM matrix, the GIM mechanism, neutrino oscillations, running couplings, and asymptotic freedom.

The discussion of anomalies and their non-trivial cancellations in the SM is detailed.

Simple examples of calculations, such as scattering amplitudes and decay rates, are provided and I will mention a range of wild extensions as we proceed.

1.1 INTRODUCTION

The Standard Model is one of the greatest scientific achievements of all time.

It consistently describes all known fundamental particles and their interactions with the exception of gravity that is still properly described at low energies.

In this sense we can now explain any fundamental physical phenomenon at the smallest distances that can be probed experimentally with spectacular success.

We may therefore claim that, so far, the Standard Model is the fundamental theory of Nature.

The Standard Model is the most successful application of quantum field theory when it comes to experimental verification.

It is the final conclusion of many decades of intense research both on the theoretical and experimental side.

Its structure was finally completed after the celebrated discovery of the Higgs particle in 2012.

Over the decades since its ingredients were combined, thousands of measurements have been made at energies $E \leq 1 \text{ TeV}$, all consistent with the Standard Model.

The Standard Model describes the physics of the building blocks of all visible matter: spin 1/2 quarks and leptons interacting via three fundamental forces, each mediated by spin 1 particles known as gauge bosons.

Electrically charged particles feel the electromagnetic force by exchanging photons as described by Quantum Electrodynamics (QED).

The electromagnetic interactions are of long range due to the fact that photons are massless.

In contrast, the short range weak force is responsible for certain radioactive decays such as the neutron β -decay and plays a crucial role in the thermonuclear interactions within stars.

The mediators of this interaction are the massive W and Z bosons.

Their large mass is responsible for the short range of the interaction.

The strong force binds quarks into nucleons (protons and neutrons) and indirectly nucleons into nuclei; the carriers of the strong force are appropriately called the gluons.

Leptons, such as electrons and neutrinos do not feel the strong force.

Particles made out of quarks are called hadrons which can be either baryons (made up of 3 quarks) and mesons (a quark-antiquark pair).

The aim of these lectures is threefold.

First, to let you appreciate all the twists and turns that drove scientists in the past century to discover and develop the Standard Model.

A historical perspective is important to appreciate the magnitude of the achievements, but also the surprises and human drama that came with the development of new ideas in particle physics.

Most importantly, it demonstrates that research is not a straight line of well developed arguments as usually presented in textbooks and lecture notes, but rather a windy road with occasionally unforeseen twists and turns before a proper understanding emerges.

The second goal is for you to internalize that, despite the Standard Model being only one in an infinite number of possible field theories, its structure is extraordinary rigid and compelling.

Following general principles within our basic theories, Special Relativity¹ and Quantum Mechanics, we take a constructive approach arguing that it is essentially unavoidable that elementary particles are determined by unitary representations of the Poincaré group limiting their nature to only a handful of possibilities, namely spins 0, 1/2, 1, 3/2, 2 out of an infinite number of possible spins.

Further, we will see why gauge invariance defining the three non-gravitational interactions is only a redundancy needed to properly describe the interactions.

Lastly, even though Field Theory is the basic formalism to describe interactions, the fundamental objects are actually the particles themselves, whereas fields are a necessary tool to describe local interactions among particles.

The third target of these lectures is for you to get familiar with the physical details of the Standard Model and be able to reproduce some of the key calculations and results that led to its successful completion.

Throughout the lectures emphasis will be given to basic principles and potential loopholes that may be important to guide us towards the unknown physics beyond the Standard Model.

That is, we provide crucial methods to build new theories (or models) of nature – a skill that is vital for any theoretical physicist.

We follow the guideline as in the quote of Steven Weinberg earlier with the key word being to explain rather than describe.

We emphasize the explanatory power of the Standard Model towards all the experiments, but also towards some of the approximate or accidental symmetries such as isospin, flavor, baryon number, etc.

Just like the Standard Model provides a UV description of Effective Field Theories (EFTs) such as the Fermi theory of weak interactions and the pion dynamics of Yukawa, it should itself only be regarded as an EFT once gravity or other UV physics is included.

The lectures only assume a basic knowledge of group theory(as we have done) and a first exposure to quantum field theory although an effort is made to be as self-contained as possible(a second exposure).

Most concepts we covered in QFT will be repeated and presented in new ways and I hope your understanding level will thereby skyrocket!

Subjects such as path integrals, quantization of non-abelian gauge theories and renormalization group are only briefly summarized since we just finished a set of lectures about that stuff.

The presentation aims at preparing you to think about the fundamental ideas underlying the Standard Model.

1.2 Summary and Motivation

Let us start with a first overall glimpse at the Standard Model just to introduce the concepts that will be developed during the subsequent lectures

1.2.1 A Brief Introduction to the Standard Model

Definition:

The Standard Model (SM) is a construction (or model or theory) that describes all the known elementary particles and their interactions in terms of relativistic quantum field theories.

Ingredients:

- (1) *Spacetime*. The spacetime is (3 + 1)-dimensional Minkowski spacetime with (global) symmetry group

$$G_{\text{gl}} = \mathbb{R}^{3,1} \rtimes \text{O}(3, 1) \tag{1.2.1}$$

that is the semi-direct product of spacetime translations and the Lorentz group, corresponding to the Poincaré group of special relativity.

- (2) *Matter*. The particle content can be classified by the spin s (or helicity), i.e., there are the Higgs H with $s = 0$ as well as 3 families of quarks and leptons with $s = 1/2$, see also table 1.1.

Name	Label	$SU(3)_C, SU(2)_L, U(1)_Y$	Spin/Helicity
Quarks	$Q_L^i = (u_L^i, d_L^i)$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})$	$\frac{1}{2}$
	u_R^i	$(\bar{\mathbf{3}}, \mathbf{1}, \frac{2}{3})$	$\frac{1}{2}$
	d_R^i	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{1}{3})$	$\frac{1}{2}$
Leptons	$L_L^i = (\nu_L^i, e_L^i)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$\frac{1}{2}$
	e_R^i	$(\mathbf{1}, \mathbf{1}, -1)$	$\frac{1}{2}$
	ν_R^{i*}	$(\mathbf{1}, \mathbf{1}, 0)$	$\frac{1}{2}$
Higgs	H	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})$	0
Gluons	g_α	$(\mathbf{8}, \mathbf{1}, 0)$	1
W/Z -Bosons	$W^\pm, Z^0$	$(\mathbf{1}, \mathbf{3}, 0)$	1
Photon	γ	$(\mathbf{1}, \mathbf{1}, 0)$	1
Graviton*	$h_{\mu\nu}$	$(\mathbf{1}, \mathbf{1}, 0)$	2

Table 1.1: Particle content of the Standard Model and the corresponding group representations. The right-handed neutrino and the graviton are included here for completeness with the understanding that the couplings of the graviton to all other particles can be studied as long as the energies are small enough in terms of an effective QFT.

- (3) *Interactions.* The interactions are given by 3 gauge interactions with associated gauge bosons of spin $s = 1$ and, in general, also gravitational interactions transmitted by a particle of spin $s = 2$ known as the graviton.

The gauge forces are encoded in the gauge (or local) symmetry group

$$G_{\text{SM}} = \underbrace{\text{SU}(3)_C}_{\text{color}} \times \underbrace{\text{SU}(2)_L}_{\text{left}} \times \underbrace{\text{U}(1)_Y}_{\text{hypercharge}} \quad (1.2.2)$$

strong
electroweak

where the subindex C refers to color with $\text{SU}(3)_C$ determining the strong interactions.

The strong force binds quarks into nucleons and nucleons into nuclei; the carrier of the strong force is appropriately called the gluon.

The L in $\text{SU}(2)_L$ refers to left-handed in the sense that weak interactions only act on left-handed particles.

Finally Y refers to hypercharge.

The gauge group in (1.2.2) is broken by the Higgs boson through a non-zero vacuum expectation value, to a subgroup, namely

$$G_{\text{SM}} \xrightarrow{\text{SSB}} \underbrace{\text{SU}(3)_C}_{\text{QCD}} \times \underbrace{\text{U}(1)_{EM}}_{\text{QED}} . \quad (1.2.3)$$

In the process referred to as spontaneous symmetry breaking (SSB), the corresponding gauge bosons of the broken group ($W^\pm$ and Z^0) receive a mass, but there is also a remaining massless boson corresponding to the unbroken $U(1)_{\text{EM}}$ with EM standing for electromagnetic.

This is the familiar photon of the electromagnetic interactions.

To reiterate, after the breaking of the symmetry, only the gluons and photons remain massless.

Photons are free to move but gluons together with quarks are confined within the particles of strong interactions such as protons and neutrons.

The representations of the particles involved are summarized in table 1.1.

We work in a particular representation where $\alpha \in U(1)_Y$ acts on $\psi \in C$ in such a way that $\psi \rightarrow \alpha^{6iy}\psi$, i.e., weak hypercharges appear in integer multiples of $1/6$.

Keep in mind that different definitions are commonly used in the literature!

- (4) *Three families.* For the quarks and leptons, there are 3 distinct families coming with the same copies of the representation:

$$\text{Leptons: } \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \text{Quarks: } \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} .$$

Only the first family (with electron, its neutrino and up and down quarks) are enough to make the matter we know.

The second (muon, its neutrino, charm and strange quarks) and third (tau-lepton, its neutrino, top and bottom quarks) are more massive and the corresponding particles are unstable having the particles of the first family as end results of their decay.

It is remarkable that these simple ingredients are enough to account for the structure of the Universe as we know it including every single experience and measurement we make.

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Some comments:

- Chirality. Since the right-handed quarks u_R and d_R transform under the trivial representation of $SU(2)_L$ in table 1.1, that is to say they are $SU(2)_L$ -singlets, they do not feel the $SU(2)_L$ -interactions.

This is why we call the Standard Model a chiral gauge theory.

Physically, this implies that left- and right-handed fermions feel certain gauge interactions differently, i.e., they couple non-democratically to the mediators (gauge fields) of a given force.

Mathematically speaking, left- and right-handed fermions transform in different $SU(2)_L$ -representations.

Thus, the weak interaction is not parity invariant under exchange of left- and right-handed particles.

- *Charge quantization.* The electric charge is defined as

$$Q = T_3 + Y \tag{1.2.4}$$

where T_3 is the third generator of $SU(2)_L$ which is the diagonal matrix with entries $(1/2, -1/2)$ and Y the hypercharge under $U(1)_Y$.

For instance to compute the electric charge of the left-handed electron we observe that its hypercharge is $Y = -1/2$ and its value of T_3 is $-1/2$ giving $Q = -1$.

For the right-handed electron the corresponding T_3 value is zero because it is a singlet and then charge and hypercharge are the same (here +1 for the positron).

Computing the electric charges of quarks give multiples of $1/3$ instead of integers as we are familiar for electrons and protons.

- *Consistency conditions.* We observe that the assignment of these numbers such as hypercharge and the different representations of the Standard Model particles is not arbitrary.

For instance it is easy to verify the following conditions for the hypercharge

$$\sum_{Left} Y - \sum_{Right} Y = \sum_{Left} Y^3 - \sum_{Right} Y^3 = 0. \quad (1.2.5)$$

Also the total number of particles transforming as a $\mathbf{3}$ of $SU(3)_C$ equals the number of $\bar{\mathbf{3}}$ ($\#\mathbf{3} = \#\bar{\mathbf{3}}$) and the total number of $SU(2)_L$ doublets ($\#2$) is even.

Any modification of these numbers would render the theory mathematically inconsistent.

This will be crucial in ensuring anomaly cancellation within the Standard Model.

If the Standard Model was not chiral, these conditions would be trivially satisfied.

It is the chiral structure of the Standard Model that makes it subject to potential inconsistencies and therefore makes it more interesting when they are satisfied.

- *Coleman-Mandula theorem.* The total symmetry of the Standard Model is given by a direct product between a spacetime and an internal symmetry

$$\text{spacetime} \otimes \text{internal gauge} = (\mathbb{R}^{3,1} \rtimes \text{SO}(3, 1)) \otimes (\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y) . \quad (1.2.6)$$

The Coleman-Mandula theorem states that this structure is the most general⁶ for the full symmetry group, i.e., a direct product of the Poincaré group and an internal (gauge) group.

- *Gravity as an effective field theory.* We can treat gravity only as what is called an effective QFT (EFT) by working with energies E well below the *Planck scale*:

$$E \ll M_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{18} \text{GeV} . \quad (1.2.7)$$

At energies $E \sim M_P$, quantum effects of gravity become important and the EFT has to be replaced by a more fundamental theory that is ultra-violet complete.

But for energies well below $E \ll M_P$ working with quantum aspects of gravity as an EFT are predictable and reliable.

We will have to say more about the role of gravitational interactions within the Standard Model later.

- *Accidental symmetries.* There are accidental symmetries known as Baryon number B and Lepton number L .

That is, the total number of baryons, such as the neutron and proton, and the total number of leptons such as the electron and neutrino are conserved in every interaction.

- *Approximate symmetries.* The three families of quarks and leptons in which the members of each family behave the same as the other families except that the particles are heavier for each generation (e.g., the muon is like a heavier copy of the electron, the top quark of the up quark, etc.) implies that there are approximate symmetries known as flavor symmetries such as $SU(3)_f$ known as the eightfold way, see Fig. 1.2.

This flavor $SU(3)_f$ should not be confused with the color $SU(3)_c$ which is the symmetry describing the strong interactions.

- *Phases of the Standard Model.* The Standard Model is relatively simple, although not the simplest model we can imagine.

Actually, it is rich enough to illustrate the 3 main phases of gauge theories: The Coulomb phase for $U(1)_{EM}$ meaning that the corresponding gauge boson, the photon, moves freely; the confining phase for $SU(3)_c$ meaning that the interactions are so strong that the corresponding gauge bosons, the gluons, and the quarks are confined within hadrons; and the Higgs phase for the weak interactions in $SU(2)_L \times U(1)_Y$ meaning that the corresponding force is short range since the gauge bosons W, Z are heavy after symmetry breaking.

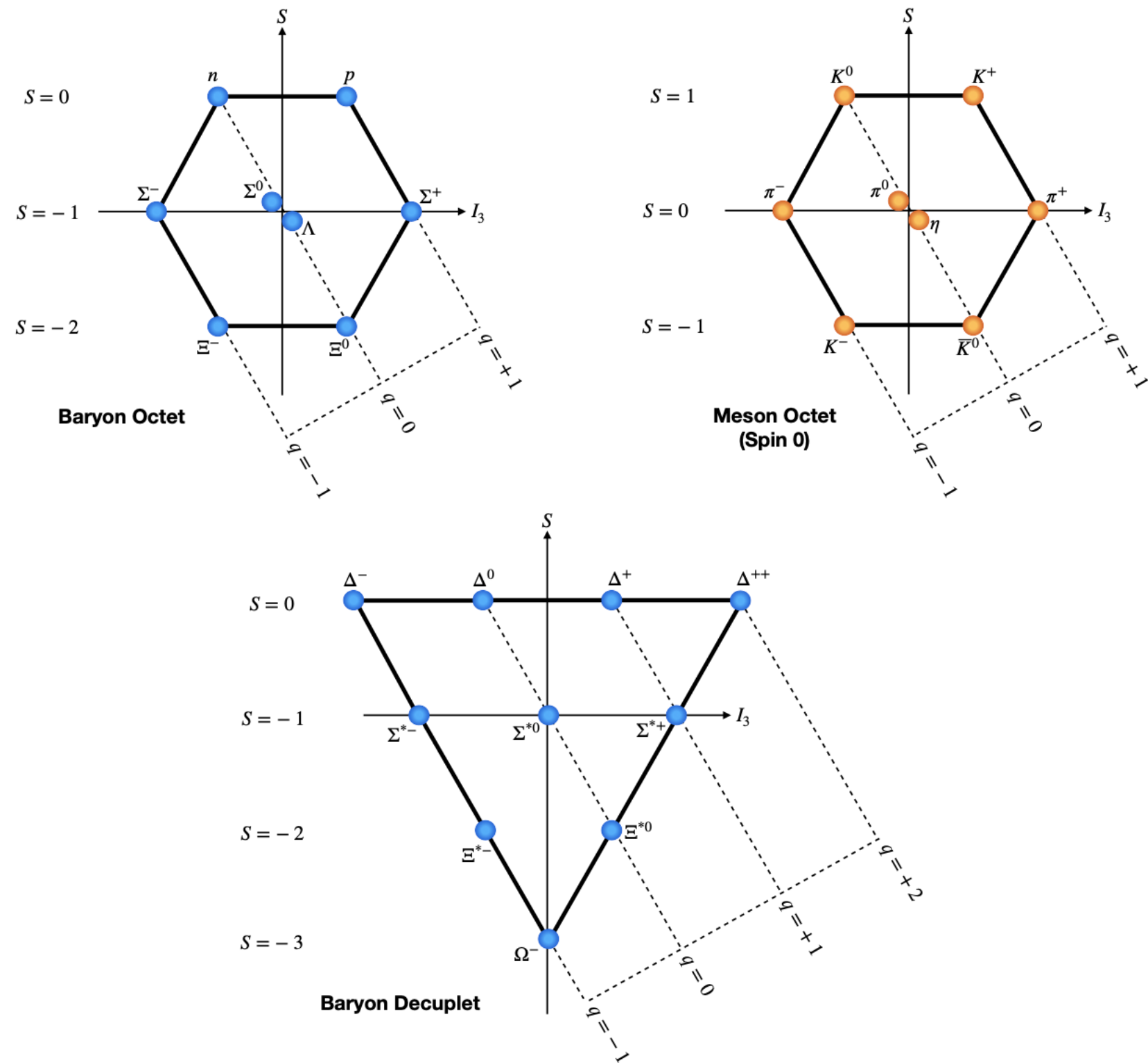


Figure 1.2: The *eightfold way*. The weight lattice of hadrons in their respective $SU(3)_f$ representations with the baryon octet (top left) corresponding to **8** of $SU(3)_f$, pseudoscalar meson octet (top right) again corresponding to **8** of $SU(3)_f$ and the baryon decuplet (bottom) corresponding to **10** of $SU(3)_f$. Here, S denotes 'strangeness' and I_3 isospin. The prediction of the Ω^- particle and its subsequent discovery lead strong credibility to this approximate symmetry.

Why do I like the SM so much?

As you now see.....

- It is fundamental.
- Robustness.
- It is true!
- It is the best test of validity of QFT.
- Cosmology.
- It is incomplete!

So let us begin our journey.

2. SPACETIME SYMMETRIES

Importance

If it were not for these symmetries, the work of science would have to be redone in every new laboratory and in every passing moment.

Steven Weinberg

In this part, we review basic techniques for constructing suitable representations of the Poincaré group – the symmetry group of (Minkowski) spacetime – which defines Special Relativity and is relevant for constructing the Standard Model.

We begin with a general discussion of the Poincaré algebra and its properties before introducing spinor representations of the Lorentz group $SO(3, 1)$.

Subsequently, we will detail Wigner's classification of irreducible representations of the Poincaré group which define for us elementary particles.

We discuss in detail how these particles have to transform under discrete spacetime transformations which allows us to count all of the relevant polarization states.

Among others, we will find that massless particles with helicity ≥ 0 have always just two degrees of freedom which will severely constrain the field theories to be studied in the reminder of these lectures.

2.1 Poincaré symmetry and spinors

For the vast majority of this lecture, we will be interested in describing relativistic processes involving particles moving in $(3 + 1)$ -dimensional Minkowski space $\mathbb{R}^{3,1}$.

Any theory describing such phenomena must respect the symmetry inherited from the spacetime geometry – heuristically, physical processes should not depend on the observer’s initial frame.

This is the *principle of relativity* stating that the laws of physics are the same in all viable frames of reference.

For particle physics in $\mathbb{R}^{3,1}$, the spacetime symmetry group in question is the *Poincaré group* $\mathcal{P}(3, 1)$.

The Poincaré group corresponds to the basic symmetries of special relativity, it acts on the Minkowski spacetime coordinates x^μ via

$$x^\mu \mapsto x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} x^\nu + \underbrace{a^\mu}_{\text{translation}}, \quad \mu, \nu = 0, 1, 2, 3. \quad (2.1.1)$$

It is sometimes convenient to use a shorthand notation $\{\cdot|\cdot\}$ for such a transformation where

$$x'^\mu = \{\Lambda|a\} x^\mu \equiv \Lambda^\mu{}_\nu x^\nu + a^\mu. \quad (2.1.2)$$

Formally, the Poincaré group $\mathcal{P}(3, 1)$ or $\text{ISO}(3, 1)$ is the isometry group of Minkowski spacetime.

It is a semidirect product of spacetime translations and the transformations corresponding to the Lorentz group of special relativity

$$\mathcal{P}(3, 1) = \mathbb{R}^{3,1} \rtimes \mathrm{O}(3, 1) , \quad (2.1.3)$$

which is just a fancy way of saying that $\mathcal{P}$ leaves $\mathbb{R}^{3,1}$ invariant and every Poincaré transformation can be decomposed into the product of the form

$$\{\Lambda|a\} = \{\mathbb{1}_{4\times 4}|a\}\{\Lambda|0\} . \quad (2.1.4)$$

The Lorentz transformations belong to the orthogonal $\mathrm{O}(3, 1)$ group that leaves the metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \quad (2.1.5)$$

in the line element

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \quad (2.1.6)$$

invariant, i.e.,

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad \text{or} \quad \Lambda^T \eta \Lambda = \eta . \quad (2.1.7)$$

From this equation we can easily see that

$$\det \Lambda = \pm 1 \quad (2.1.8)$$

and taking the 00 component

$$\left(\Lambda^0_0\right)^2 - \left(\Lambda^1_0\right)^2 - \left(\Lambda^2_0\right)^2 - \left(\Lambda^3_0\right)^2 = 1 \quad \Longrightarrow \quad |\Lambda^0_0| \geq 1. \quad (2.1.9)$$

Therefore the Lorentz group has 4 disconnected components according to the signs of $\det \Lambda$ and Λ^0_0 .

We will mostly discuss those transformations Λ connected to the identity, i.e., the *proper orthochronous group* $\text{SO}(3, 1)^\uparrow$ for which $\det \Lambda = 1$ (proper) and $\Lambda^0_0 \geq 1$ (orthochronous).

All $\text{O}(3, 1)$ transformations can be obtained by combining the $\text{SO}(3, 1)^\uparrow$ transformations with:

$$\{\mathbb{1}, \Lambda_P, \Lambda_T, \Lambda_{PT}\} \quad (2.1.10)$$

where

- $\mathbb{1}$ is the identity matrix,
- $\Lambda_P = \text{diag}(1, -1, -1, -1)$ is the *parity* transformation,
- $\Lambda_T = \text{diag}(-1, 1, 1, 1)$ is *time reversal*, and
- $\Lambda_{PT} = \Lambda_P \times \Lambda_T$ is combined parity and time reversal.

These four elements form a group known as *Klein's four-group*.

From now on we will concentrate on those transformations connected to the identity and drop the arrow on $\text{SO}(3, 1)$ for simplifying the notation.

For the same reason, we work from now on with the proper orthochronous Poincaré group and simply write $\mathcal{P}(3, 1) \equiv \mathcal{P}_+^\uparrow = \mathbb{R}^{3,1} \rtimes \text{SO}(3, 1)^\uparrow$.

2.1.1 The Poincaré Algebra

Let us consider infinitesimal Poincaré transformations $\{\Lambda|a\} \in \mathcal{P}(3, 1)$ for which

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu; \quad a^\mu = \epsilon^\mu \quad \omega^\mu{}_\nu, \epsilon^\mu \ll 1. \quad (2.1.11)$$

Plugging this back in (2.1.7), we find

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2) \stackrel{!}{=} \eta_{\rho\sigma}. \quad (2.1.12)$$

In mathematics, $\stackrel{!}{=}$ denotes a useful, important, but unexpected equality.

To linear order, we deduce that

$$\omega_{\sigma\rho} = -\omega_{\rho\sigma} \quad (2.1.13)$$

is an anti-symmetric tensor which has 6 free parameters.

Together with the 4 translations e^μ , a general Poincaré transformations must have 10 parameters.

This is the dimensionality of the Poincaré group.

In order to determine the algebra we can exponentiate the group elements.

We will do this considering the action of operators acting on the Hilbert space $\mathcal{H}$ relevant in quantum mechanics.

A Poincaré transformation will be represented by a unitary operator $U(\Lambda, a)$ acting on the Hilbert space vectors $|\Psi\rangle \in \mathcal{H}$.

$$|\Psi\rangle \rightarrow U(\Lambda, a)|\Psi\rangle \quad U = U^\dagger. \quad (2.1.14)$$

Near the identity, we can expand to linear order

$$U(1 + \omega, \epsilon) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_\mu P^\mu \quad (2.1.15)$$

where $M^{\mu\nu} = -M^{\nu\mu}$ and P^μ are the generators of the group.

Since U is unitary, both $M^{\mu\nu}$ and P^μ are Hermitian.

As usual in group theory, the above can be used to determine the algebra satisfied by the generators.

First, since translations commute, their generators also commute with each other

$$[P^\mu, P^\nu] = 0. \quad (2.1.16)$$

Let us now consider the commutator $[P^\sigma, M^{\mu\nu}]$ by analyzing how P transforms under Lorentz transformations.

For this we can consider the dual role of P^μ .

On the one hand, it is a vector that should transform (to leading order in $\omega_{\mu\nu}$) as

$$\begin{aligned}
 P^\sigma &\rightarrow \Lambda^\sigma{}_\rho P^\rho \\
 &= (\delta^\sigma{}_\rho + \omega^\sigma{}_\rho) P^\rho \\
 &= P^\sigma + \frac{1}{2} (\omega_{\alpha\rho} - \omega_{\rho\alpha}) \eta^{\sigma\alpha} P^\rho \\
 &= P^\sigma + \frac{1}{2} \omega_{\alpha\rho} (\eta^{\sigma\alpha} P^\rho - \eta^{\sigma\rho} P^\alpha)
 \end{aligned} \tag{2.1.17}$$

On the other hand, P^σ as an operator transforms as

$$\begin{aligned}
 P^\sigma &\rightarrow U^\dagger P^\sigma U \\
 &= \left(\mathbb{1} + \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \right) P^\sigma \left(\mathbb{1} - \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \right) \\
 &= P^\sigma - \frac{i}{2} \omega_{\mu\nu} (P^\sigma M^{\mu\nu} - M^{\mu\nu} P^\sigma)
 \end{aligned} \tag{2.1.18}$$

Comparing both expressions we find the commutator $[P^\sigma, M^{\mu\nu}]$:

$$[P^\sigma, M^{\mu\nu}] = -i (P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \tag{2.1.19}$$

A similar argument can be used for the commutators of $M^{\mu\nu}$.

Therefore the generators of the Poincaré group are $M^{\mu\nu}$ and P^σ with algebra:

Poincaré algebra

$$[P^\mu, P^\nu] = 0 \quad (2.1.20)$$

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \quad (2.1.21)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma} \eta^{\nu\rho} + M^{\nu\rho} \eta^{\mu\sigma} - M^{\mu\rho} \eta^{\nu\sigma} - M^{\nu\sigma} \eta^{\mu\rho}) \quad (2.1.22)$$

As an example, a 4-dimensional matrix representation for the $M^{\mu\nu}$ is

$$(M^{\rho\sigma})^\mu{}_\nu = -i(\eta^{\mu\sigma} \delta^\rho{}_\nu - \eta^{\rho\mu} \delta^\sigma{}_\nu) . \quad (2.1.23)$$

Also, the definition of the operators

$$(M^{\rho\sigma})^\mu{}_\nu = i(x^\mu \partial_\nu - x_\nu \partial^\mu) \quad (2.1.24)$$

is a representation of the Lorentz generators acting on the space of functions.

Similarly, the operator

$$P^\mu = i \partial^\mu \quad (2.1.25)$$

is the generator of translations in the representation defined by (2.1.24).

It is easy to verify that these operators satisfy the Poincaré algebra.

2.1.2 Properties of the Poincaré group

Let us summarize the basic properties of the Poincaré group.

Conservation laws

Recall that $P^0 = H$ corresponds to the Hamiltonian and thus

$$[P^0, P^\mu] = 0 \tag{2.1.26}$$

implies *conservation of energy and momentum*, whereas

$$[P^0, M^{ij}] = 0 \quad i, j = 1, 2, 3 \tag{2.1.27}$$

amounts to *conservation of angular momentum*.

There is no conservation law associated to the M^{0i} generators since they do not commute with P^0 .

Correspondence of $SO(3, 1)$ and $SU(2) \otimes SU(2)$

There is a correspondence between the algebras of $SO(3, 1)$ and $SU(2) \otimes SU(2)$, namely

$$SO(3, 1) \leftrightarrow SU(2) \otimes SU(2) . \tag{2.1.28}$$

This means that the representations of the $SO(3, 1)$ algebra can be determined by those of $SU(2) \oplus SU(2)$.

This works as follows.

The generators J_i of rotations and K_i of Lorentz boosts can be defined as

$$J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} , \quad K_i = M_{0i} , \quad (2.1.29)$$

therefore, using the Poincaré algebra we can easily derive the commutation relations

$$\left[J_i, J_j \right] = i\epsilon_{ijk} J_k , \quad \left[J_i, K_j \right] = i\epsilon_{ijk} K_k , \quad \left[K_i, K_j \right] = -i\epsilon_{ijk} J_k . \quad (2.1.30)$$

In order to identify the different representations of the Lorentz group, it is instructive to consider the linear combinations

$$A_i = \frac{1}{2} \left(J_i + iK_i \right) , \quad B_i = \frac{1}{2} \left(J_i - iK_i \right) \quad (2.1.31)$$

which are neither Hermitian nor anti-Hermitian if J_i and K_i are Hermitian.

They satisfy SU(2) commutation relations,

$$\left[A_i , A_j \right] = i\epsilon_{ijk} A_k , \quad \left[B_i , B_j \right] = i\epsilon_{ijk} B_k , \quad \left[A_i , B_j \right] = 0 . \quad (2.1.32)$$

These are two independent copies of the SU(2) algebra, but keeping in mind that the operators A_i and B_i are not Hermitian.

In contrast, the combination $\mathbf{J} = \mathbf{A} + \mathbf{B}$ is Hermitian and corresponds to the physical spin.

We can then interpret $\mathbf{J}$ as the physical spin which itself generates an SU(2) group.

Recall that irreducible representations $\bar{R}_{\Lambda_j}$ of SU(2) are labelled by $j = 0, 1/2, \dots$ with j determined from the eigenvalues of the *quadratic Casimir Operator* which is nothing but the “total angular momentum” $J^2 = J_1^2 + J_2^2 + J_3^2$.

It commutes with the three generators J_i and satisfies

$$J^2 |j\rangle = j(j+1) |j\rangle \quad j = 0, \frac{1}{2}, 1, \dots \quad (2.1.33)$$

Hence, we can use those eigenvalues to label the irreducible representations of SU(2).

In the case of SO(3, 1), we denote representations of SU(2) $\otimes$ SU(2) as

$$(A, B), \quad \text{with } A, B = 0, \frac{1}{2}, 1, \dots \quad (2.1.34)$$

Since $\mathbf{J} = \mathbf{A} + \mathbf{B}$, we can see using the standard addition of angular momenta that the representation (A, B) corresponds to spins $j = |A - B| \oplus (|A - B| + 1) \cdots \oplus (A + B)$.

Under parity P with $x^0 \mapsto x^0$ and $\mathbf{x} \rightarrow -\mathbf{x}$, we have

$$J_i \mapsto J_i, \quad K_i \mapsto -K_i \implies A_i \leftrightarrow B_i. \quad (2.1.35)$$

Therefore, A and B are interchanged under parity transformation

$$(A, B) \xleftrightarrow{P} (B, A) . \quad (2.1.36)$$

We then call A the *left-handed* and B the *right-handed* component of (A, B).

Below, we will use this notion to label representations of SO(3,1) and to define their handedness (or chirality).

Universal cover of SO(3, 1)

There is a homomorphism (not an isomorphism)

$$\text{SO}(3, 1) \cong \text{SL}(2, \mathbb{C}) , \quad (2.1.37)$$

where SL(2, C) is the group of 2 x 2 complex matrices with unit determinant.

To see this, take a 4 vector X and a corresponding (2 x 2)-matrix $\tilde{x}$,

$$X = x_\mu e^\mu = (x_0, x_1, x_2, x_3) , \quad \tilde{x} = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - \text{i}x_2 \\ x_1 + \text{i}x_2 & x_0 - x_3 \end{pmatrix} , \quad (2.1.38)$$

where σ^μ is the 4 vector of Pauli matrices

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -\text{i} \\ \text{i} & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} . \quad (2.1.39)$$

Transformations $X \mapsto \Lambda X$ under $SO(3, 1)$ leave the modulus square

$$|X|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad (2.1.40)$$

invariant, whereas the action of $SL(2, \mathbb{C})$ mapping

$$\tilde{x} \mapsto N \tilde{x} N^\dagger \quad (2.1.41)$$

with $N \in SL(2, \mathbb{C})$ preserves the determinant

$$\det \tilde{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2. \quad (2.1.42)$$

This equals $|X|^2$.

The map between $SL(2, \mathbb{C})$ and $SO(3, 1)$ is 2-1, since $N = \pm \mathbb{1}_2$ both correspond to $\Lambda = 1_4$, but $SL(2, \mathbb{C})$ has the advantage of being simply connected, that means that $SL(2, \mathbb{C})$ is the *Universal Covering Group* of the Lorentz group.

This is important since it is the simply connected group manifold that is continuously connected to the identity operator.

Furthermore, since the map between $SL(2, \mathbb{C})$ and $SO(3, 1)$ is 2-1, a rotation by an angle θ in $SO(3, 1)$ is mapped to the matrix $\text{diag}(e^{i\theta/2}, e^{-i\theta/2})$ in $SL(2, \mathbb{C})$.

This in turn implies that it is only rotations by $\theta = 4\pi$ (and not 2π) that give the identity in $SL(2, \mathbb{C})$.

This will turn out to be a crucial observation in order to describe particles of half-integer spin as we will see later.

Let us briefly see why the manifold of $SL(2, \mathbb{C})$ is simply connected.

By the polar decomposition of matrices, an $SL(2, \mathbb{C})$ matrix N can be written as $N = e^H U$ where e^H is a positive Hermitian matrix and U is unitary.

Since e^H has positive determinant then $\det N = 1$ implies $\text{Tr } H = 0$ and $\det U = 1$.

Therefore H and U can be written as

$$H = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \quad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix} \quad (2.1.43)$$

with a, b, c real parameters and x, y, z, w constrained by $x^2 + y^2 + z^2 + w^2 = 1$ therefore the manifold of $SL(2, \mathbb{C})$ is $\mathbb{R}^3 \times S^3$ which are simply connected, whereas the manifold for $SO(3, 1)$ is $\mathbb{R}^3 \simeq S^3/\mathbb{Z}_2$ which is doubly connected.

Being the covering group, it is $SL(2, \mathbb{C})$ that is reached by exponentiating the algebra and therefore we are let to consider the representations of $SL(2, \mathbb{C})$ that we discuss next.

2.2 Spinor representations of the Lorentz group

Above, we already established that $SL(2, \mathbb{C})$ is the universal covering group of the Lorentz group $SO(3, 1)$.

This should ring a bell: in quantum mechanics, we learned that $SU(2)$ is the double cover of $SO(3)$ and therefore it is representations of $SU(2)$ that are the ones to be considered since $SU(2)$ is simply connected and $SO(3)$ is not.

We observe a similar phenomenon for the Lorentz group in the sense that representation theory of $SL(2, \mathbb{C})$ is the relevant one to study.

Concentrating only on representations of $SO(3, 1)$ we would miss the fundamental representations which are the spinor representations which we define next.

2.2.1 Representations and invariant tensors of $SL(2, \mathbb{C})$

To begin, we define the basic representations of $SL(2, \mathbb{C})$. Let $N \in SL(2, \mathbb{C})$, then we have

- The fundamental representation ψ_β transforming as

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta, \quad \alpha, \beta = 1, 2 \quad (2.2.1)$$

The elements of this representation ψ_α are called **left-handed Weyl spinors**.

- The conjugate fundamental representation $\bar{\chi}_{\dot{\beta}}$ transforming as

$$\bar{\chi}'_{\dot{\alpha}} = N_{\dot{\alpha}}^*{}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}}, \quad \dot{\alpha}, \dot{\beta} = 1, 2 \quad (2.2.2)$$

Here $\bar{\chi}_{\dot{\beta}}$ are called **right-handed Weyl spinors**.

- The contravariant representations ψ^β and $\bar{\chi}^{\dot{\beta}}$

$$\psi'^\alpha = \psi^\beta (N^{-1})_\beta{}^\alpha, \quad \bar{\chi}'^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}} (N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}} \quad (2.2.3)$$

as the dual representations of the two above.

The fundamental and conjugate representations are the basic representations of $SL(2, \mathbb{C})$ and the Lorentz group, giving then the importance to spinors as the basic objects of special relativity, a fact that could be missed by not realizing the connection of the Lorentz group and $SL(2, \mathbb{C})$.

We will show now that the contravariant representations are however not independent by explicitly showing how indices can be raised (or lowered) using specific tensors.

To see this, we consider three different ways to raise and lower indices.

- The metric tensor $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$ is invariant under $SO(3, 1)$ and is therefore used to lower and raise spacetime indices.
- The analogue for $SL(2, \mathbb{C})$ is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}, \quad (2.2.4)$$

since it is invariant under $SL(2, \mathbb{C})$ transformations

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} (N^{-1})_\rho{}^\alpha (N^{-1})_\sigma{}^\beta = \epsilon^{\alpha\beta} \cdot \det N^{-1} = \epsilon^{\alpha\beta}. \quad (2.2.5)$$

That is why $\epsilon^{\rho\sigma}$ is used to raise and lower indices

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta , \quad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}} , \quad (2.2.6)$$

so contravariant representations are not independent.

- To handle mixed $\text{SO}(3, 1)$ - and $\text{SL}(2, \mathbb{C})$ -indices, recall that the transformed components x_μ should look the same, whether we transform the vector X via $\text{SO}(3, 1)$ or the matrix $\tilde{x} = x_\mu \sigma^\mu$

$$(x_\mu \sigma^\mu)_{\alpha\dot{\alpha}} \mapsto N_\alpha{}^\beta (x_\nu \sigma^\nu)_{\beta\dot{\gamma}} N_{\dot{\alpha}}^*{}^{\dot{\gamma}} = (\Lambda^{-1})_\mu{}^\nu x_\nu \sigma^\mu , \quad (2.2.7)$$

so the right transformation rule is

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = N_\alpha{}^\beta (\sigma^\nu)_{\beta\dot{\gamma}} (\Lambda)^\mu{}_\nu N_{\dot{\alpha}}^*{}^{\dot{\gamma}} . \quad (2.2.8)$$

This may be interpreted by saying that the Pauli matrices are invariant under a combined $\text{SO}(3, 1)$ transformation on its spacetime index with a $\text{SL}(2, \mathbb{C})$ on its matrix indices. Similar relations hold for the quantity

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}} = (\mathbb{1}, -\boldsymbol{\sigma}) . \quad (2.2.9)$$

Note that this is the definition of $\bar{\sigma}$ and no other connection with σ such as complex conjugation should be assumed despite the notation. Note in particular the chosen location of the dotted and undotted indices which differ between σ^μ and $\bar{\sigma}^\mu$. The order is conventional and keeps track of how the corresponding quantity transforms under $\text{SL}(2, \mathbb{C})$. Both σ and $\bar{\sigma}$ will play an important role next. In fact, we can already deduce that the *Clifford algebra*

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} \mathbb{1}_2 \quad (2.2.10)$$

appears naturally in our analysis which will give rise to *Dirac spinors* further below.

2.2.2 Generators of $SL(2, \mathbb{C})$ and Weyl spinors

Let us define tensors $\sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}$ as antisymmetrized products of σ matrices

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &:= \frac{i}{4} \left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu \right)_\alpha{}^\beta \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &:= \frac{i}{4} \left(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{aligned}$$

which satisfy the Lorentz algebra

$$\left[\sigma^{\mu\nu}, \sigma^{\lambda\rho} \right] = i \left(\eta^{\mu\rho} \sigma^{\nu\lambda} + \eta^{\nu\lambda} \sigma^{\mu\rho} - \eta^{\mu\lambda} \sigma^{\nu\rho} - \eta^{\nu\rho} \sigma^{\mu\lambda} \right) \quad (2.2.11)$$

and similarly for $\bar{\sigma}^{\mu\nu}$.

Then $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ correspond to the generators of the Lorentz algebra in the spinor representation.

Under a finite Lorentz transformation with parameters $\omega_{\mu\nu}$, Weyl spinors transform as follows

$$\begin{aligned}\psi_\alpha &\mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta \psi_\beta && \text{(left-handed)} \\ \bar{\chi}^{\dot{\alpha}} &\mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}} && \text{(right-handed)}\end{aligned}$$

Now consider the spins with respect to the SU(2)s spanned by the A_i and B_i :

$$\begin{aligned}J_i &= \frac{1}{2}\sigma_i, & K_i &= -\frac{i}{2}\sigma_i &\implies \psi_\alpha : (A, B) &= \left(\frac{1}{2}, 0\right), & \text{left-handed} \\ J_i &= \frac{1}{2}\sigma_i, & K_i &= +\frac{i}{2}\sigma_i &\implies \bar{\chi}^{\dot{\alpha}} : (A, B) &= \left(0, \frac{1}{2}\right), & \text{right-handed}.\end{aligned}$$

Recall the the Pauli matrices correspond to generators of the rotation group in the $j = 1/2$ representation since

$$\sum_i \left(\frac{\sigma_i}{2}\right)^2 = \frac{3}{4} = j(j+1) \quad \text{for} \quad j = \frac{1}{2}. \quad (2.2.12)$$

The expressions above also justify the aforementioned term left- and right-handed components for A, B .

The difference and independence between the left- and right-handed representations of $SL(2, \mathbb{C})$ indicates that there is no reason to assume that parity is a fundamental symmetry and will be the reason behind the fact that the Standard Model is chiral in the sense that left- and right-handed representations are not the same.

The belief that physicists before Yang and Lee had assuming parity should be an inherent symmetry of Nature is not justified and it is not surprising then that the laws of Nature are not invariant under parity as we will see later.

The concept of chirality is ubiquitous not only in the Standard Model, but also more generally in various areas of modern physics (and biology).

Some useful identities concerning the σ^μ and $\sigma^{\mu\nu}$ can be found in various sources.

For now, let us just mention the identities

$$\sigma^{\mu\nu} = \frac{1}{2i} \epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} , \quad \bar{\sigma}^{\mu\nu} = -\frac{1}{2i} \epsilon^{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma} , \quad (2.2.13)$$

known as *self duality* and *anti-self duality* respectively.

They are important because naively $\sigma^{\mu\nu}$ being antisymmetric seems to have $4 \times 3/2$ components, but the self duality conditions reduces this by half.

We then need the two sets of generators $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ to complete the 6 independent generators of $SL(2, \mathbb{C})$.

There are reference books illustrating many of the calculations for 2-component spinors.

Products of Weyl spinors

We define the product of two Weyl spinors as

$$\chi\psi = \chi^\alpha \psi_\alpha = -\chi_\alpha \psi^\alpha, \quad \bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, \quad (2.2.14)$$

particularly,

$$\psi\psi = \psi^\alpha \psi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \psi_\alpha = \psi_2 \psi_1 - \psi_1 \psi_2. \quad (2.2.15)$$

Choose the ψ_α to be *anticommuting Grassmann numbers*, $\psi_1 \psi_2 = -\psi_2 \psi_1$, so $\psi\psi = 2\psi_2 \psi_1$.

From the definitions

$$\psi_\alpha^\dagger = \bar{\psi}_{\dot{\alpha}}, \quad \bar{\psi}^{\dot{\alpha}} = \psi_\beta^* (\sigma^0)^{\beta\dot{\alpha}} \quad (2.2.16)$$

it follows that

$$(\chi\psi)^\dagger = \bar{\chi}\bar{\psi}, \quad (\psi \sigma^\mu \bar{\chi})^\dagger = \chi \sigma^\mu \bar{\psi} \quad (2.2.17)$$

which justifies the $\nearrow$ contraction of dotted indices in contrast to the $\searrow$ contraction of undotted ones.

In general we can generate all higher dimensional representations of the Lorentz group by products of the fundamental representation $(1/2, 0)$ and its conjugate $(0, 1/2)$.

The computation of tensor products

$$\left(\frac{r}{2}, \frac{s}{2}\right) = \left(\frac{1}{2}, 0\right)^{\otimes r} \otimes \left(0, \frac{1}{2}\right)^{\otimes s} \quad (2.2.18)$$

can be reduced to successive application of the elementary SU(2) rule (for $j \neq 0$)

$$\left(\frac{j}{2}\right) \otimes \left(\frac{1}{2}\right) = \left(\frac{j-1}{2}\right) \oplus \left(\frac{j+1}{2}\right) . \quad (2.2.19)$$

Let us give two examples for tensoring Lorentz representations:

- $\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

Bispinors with different chiralities can be expanded in terms of the $\sigma_{\alpha\dot{\alpha}}^{\mu}$. Actually, the σ matrices form a complete orthonormal set of 2×2 matrices with respect to the trace $\text{Tr}\{\sigma^{\mu}\bar{\sigma}^{\nu}\} = 2\eta^{\mu\nu}$:

$$\psi_{\alpha} \bar{\chi}_{\dot{\alpha}} = \frac{1}{2} (\psi \sigma_{\mu} \bar{\chi}) \sigma_{\alpha\dot{\alpha}}^{\mu} \quad (2.2.20)$$

Hence, two spinor degrees of freedom with opposite chirality give rise to a Lorentz vector $\psi\sigma_{\mu}\bar{\chi}$.

- $\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)$

Alike bispinors require a different set of matrices to expand, $\epsilon_{\alpha\beta}$ and $(\sigma^{\mu\nu})_{\alpha}{}^{\gamma}\epsilon_{\gamma\beta} =: (\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}$. The former represents the unique antisymmetric 2×2 matrix, the latter

provides the symmetric ones. Note that the (anti-)self duality reduces the number of linearly independent $\sigma^{\mu\nu}$'s (over $\mathbb{C}$) from 6 to 3:

$$\psi_\alpha \chi_\beta = \frac{1}{2} \epsilon_{\alpha\beta} (\psi\chi) + \frac{1}{2} \left(\sigma^{\mu\nu} \epsilon^T \right)_{\alpha\beta} (\psi \sigma_{\mu\nu} \chi) \quad (2.2.21)$$

The product of spinors with alike chiralities decomposes into two Lorentz irreducibles, a scalar $\psi\chi$ and a self-dual antisymmetric rank two tensor $\psi \sigma_{\mu\nu} \chi$. The counting of independent components of $\sigma^{\mu\nu}$ from its self-duality property precisely provides the right number of three components for the $(1, 0)$ representation. Similarly, there is an anti-self dual tensor $\bar{\chi} \bar{\sigma}^{\mu\nu} \bar{\psi}$ in $(0, 1)$.

2.2.3 Dirac and Majorana spinors

Here, we give the dictionary connecting the ideas of Weyl spinors with the more standard Dirac theory in $D = 4$ dimensions.

Dirac spinor

A Dirac spinor Ψ_D is defined to be the direct sum of two Weyl spinors $\psi, \bar{\chi}$ of opposite chirality. It therefore falls into a reducible representation of the Lorentz group,

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \left(0, \frac{1}{2} \right) \oplus \left(\frac{1}{2}, 0 \right) . \quad (2.2.22)$$

The Dirac analogue of the Weyl spinors' sigma matrices are the 4 x 4 gamma matrices γ^μ subject to the *Clifford algebra*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1} . \quad (2.2.23)$$

Due to the reducibility, the generators of the Lorentz group take block diagonal form

$$\Sigma^{\mu\nu} = \frac{i}{4} \gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \quad (2.2.24)$$

and naturally obey the same algebra like the irreducible blocks $\sigma^{\mu\nu}$, $\bar{\sigma}^{\mu\nu}$

$$[\Sigma^{\mu\nu}, \Sigma^{\lambda\rho}] = i \left(\eta^{\mu\rho} \Sigma^{\nu\lambda} + \eta^{\nu\lambda} \Sigma^{\mu\rho} - \eta^{\mu\lambda} \Sigma^{\nu\rho} - \eta^{\nu\rho} \Sigma^{\mu\lambda} \right) . \quad (2.2.25)$$

To disentangle the two inequivalent Weyl representations, one defines the chiral matrix γ^5 as

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} , \quad (2.2.26)$$

such that the $\psi(\chi)$ components of a Dirac spinors have eigenvalues (chirality) -1 (+1) under γ^5

$$\gamma^5 \Psi_D = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} . \quad (2.2.27)$$

Hence, one can define projection operators P_L, P_R ,

$$P_L = \frac{1}{2} (\mathbb{1} - \gamma^5) , \quad P_R = \frac{1}{2} (\mathbb{1} + \gamma^5) , \quad (2.2.28)$$

eliminating one part of definite chirality, i.e.,

$$P_L \Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} , \quad P_R \Psi_D = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (2.2.29)$$

The fact that Lorentz generators preserve chirality can also be seen from $\{\gamma^5, \gamma^\mu\} = 0$ implying $[\gamma^5, \Sigma^{\mu\nu}] = 0$.

Finally, define the *Dirac conjugate* $\bar{\Psi}_D$ and *charge conjugate spinor* Ψ_D^C

$$\bar{\Psi}_D = (\chi^\alpha, \bar{\psi}_{\dot{\alpha}}) = \Psi_D^\dagger \gamma^0 , \quad \Psi_D^C = C \bar{\Psi}_D^T = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} , \quad (2.2.30)$$

where C denotes the *charge conjugation matrix*

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} . \quad (2.2.31)$$

There is a third basic type of spinors called *Majorana spinors* Ψ_M which have the property $\psi_\alpha = \chi_\alpha$,

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = \Psi_M^C, \quad (2.2.32)$$

which are neutral under charge conjugation.

A general Dirac spinor (and its charge conjugate) can be decomposed in terms of Majorana spinors as

$$\Psi_D = \Psi_{M1} + i\Psi_{M2}, \quad \Psi_D^C = \Psi_{M1} - i\Psi_{M2}. \quad (2.2.33)$$

Note that there can be no spinors in 4 dimensions which are both Majorana and Weyl.

This is a dimension dependent property.

It can be shown that in dimensions $2 \bmod 8$ it is possible to have spinors which are both Majorana and Weyl.

2.3 Unitary Representations of the Poincaré group

We now will combine the 2 fundamental theories of special relativity and quantum mechanics to find the unitary representations of the Poincaré group on quantum states.

As usually unitarity is required in order to have invariant observables (such as matrix elements).

Being non-compact, the Poincaré group does not have finite dimensional unitary representations.

Recap: the rotation group in Quantum Mechanics

Before we consider the Poincaré group, let us reiterate some facts about unitary representations of the rotation group $SU(2)$ in quantum mechanics.

Recall that the rotation group $SU(2)$ has generators $\{J_i : i = 1, 2, 3\}$ satisfying the algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k . \quad (2.3.1)$$

Next, we define the Casimir operator

$$J^2 = \sum_{i=1}^3 J_i^2 . \quad (2.3.2)$$

In general, for a given group G , the Casimir operators are operators that commute with all the generators.

They are important because Schur's Lemma guarantees that they are proportional to the identity within a given representation and therefore their eigenvalues can be used to label the representations.

For compact Lie groups, the number of Casimir operators equals the rank of the group.

In the present case of $SU(2)$, the Casimir operator J^2 commutes indeed with all the J_i ,

$$[J^2, J_i] = 0 \quad \forall i = 1, 2, 3 , \quad (2.3.3)$$

and labels irreducible representations by eigenvalues $j(j+1)$ of J^2 , that is,

$$J^2 |j; \lambda\rangle = j(j+1) |j; \lambda\rangle . \quad (2.3.4)$$

Within these representations, λ parametrizes the degeneracy of states in the same representation obtained by acting with the ladder operators on the highest weight state.

Thus, we can make a choice and diagonalize the states with respect to J_3 with eigenvalues $\lambda = j_3$ so that

$$J_3 |j; j_3\rangle = j_3 |j; j_3\rangle , \quad j_3 = -j, -j+1, \dots, j-1, j . \quad (2.3.5)$$

Hence, the corresponding states are labelled like $|j; j_3\rangle$.

These are in fact unitary representations and, since $SU(2)$ is compact, they are finite-dimensional.

The latter will cease to be true for the Poincaré group.

The Poincaré group

Now, let us consider the Poincaré group.

The takeaway message from our recap about labelling irreducible representations of $SU(2)$ is that we simply need to find the corresponding Casimir operators.

In this case there are two Casimir operators.

The first one corresponds to the square of the momenta: $C_1 = P^\mu P_\mu$ which can be easily checked that it commutes with all the generators $P^\mu, M^{\mu\nu}$.

The second one involves the *Pauli-Ljubanski vector* W_μ ,

$$W_\mu := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \quad (2.3.6)$$

where $\epsilon_{0123} = -\epsilon^{0123} = -1$.

This operator satisfies the following commutation relations

$$[W_\mu, P_\nu] = 0, \quad (2.3.7)$$

$$[W_\mu, M_{\rho\sigma}] = i (\eta_{\mu\rho} W_\sigma - \eta_{\mu\sigma} W_\rho), \quad (2.3.8)$$

$$[W_\mu, W_\nu] = -i \epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma. \quad (2.3.9)$$

From these commutations relations we can check that a second Casimir corresponds to $C_2 = W^\mu W_\mu$.

Notice that at this level the Pauli-Ljubanski vector only provides a short way to express the second Casimir.

Even though W_μ has standard commutation relations with the generators of the Poincaré group $M_{\mu\nu}$, P_μ stating that it transforms as a vector under Lorentz transformations and commutes with P_μ (invariant under translations), the commutator $[W_\mu, W_\nu] \sim \epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma$ implies that the W_μ 's by themselves are not generators of any algebra since the right hand side is quadratic and not linear in the corresponding operators.

Summarizing, one can show that the Casimir operators for the Poincaré group are given by

Casimir operators of the Poincaré group

$$C_1 = P^\mu P_\mu, \quad C_2 = W^\mu W_\mu. \quad (2.3.10)$$

It is easy to verify that

$$[C_{1,2}, P^\mu] = [C_{1,2}, M^{\mu\nu}] = 0. \quad (2.3.11)$$

Poincaré multiplets are therefore labelled $|m, \omega; \lambda_i\rangle$ so that

$$C_1 |m, \omega; \lambda_i\rangle = m^2 |m, \omega; \lambda_i\rangle, \quad C_2 |m, \omega; \lambda_i\rangle = f(m, \omega) |m, \omega; \lambda_i\rangle, \quad (2.3.12)$$

that is, m^2 are the eigenvalues of C_1 and $f(m, \omega)$ the ones of C_2 .

We have to work a little harder to determine the labels ω and the exact expression for the $f(m, \omega)$.

As above, states within those irreducible representations carry extra labels λ_i corresponding to all operators that can be diagonalized simultaneously (such as J_3 for $SU(2)$).

One of the λ_i corresponds to the eigenvalue p^μ of the generator P^μ as a label.

To find more labels, take the eigenvalue p^μ of P^μ as given and look for all elements of the Lorentz group that commute with P^μ .

This defines the *Little* or *Stability group* which we denote as $L(p_\mu)$.

Note that within a multiplet, at fixed momentum, the operator P^μ can be replaced by its eigenvalue p^μ and then the Pauli-Ljubanski vector can be seen as the combination of the generators of the Lorentz group that commutes with the momentum operator and its commutation relations determine the algebra of the Little group (now the right hand side of $[W_\mu, W_\nu]$ is a linear combination of the W_ρ 's since the $P^\mu = p^\mu$'s are just numbers within the multiplet).

Our ultimate goal is to obtain unitary irreducible representations of the Poincaré group.

This can be achieved using the arguments above that can be summarized as the following theorem.

Theorem 2.3.1 Let p^μ be some fixed 4-vector.

a) On the orbit (all 4-vectors q^μ for which there exists $\Lambda \in SO(3,1)$ so that $\Lambda^\mu{}_\nu q^\nu = p^\mu$.) $O(p^\mu)$, the independent components of W^μ form a Lie algebra $L(L(p^\mu))$ of the Little group $L(p^\mu)$.

b) For every *unitary* irreducible representation of $L(p^\mu)$, there exists an induced representation of the Poincaré group $P(3, 1)$.

c) The unitary irreducible representations of $P(3, 1)$ are characterized by eigenvalues of the Casimirs P^2 and W^2 .

We will now consider the different representations determined by fixing the momenta for different values and signs of $P^\mu P_\mu$.

- $P^\mu P_\mu = m^2 > 0$ (**Massive particles**).

Valid choices of eigenvectors include $p^\mu = (m, 0, 0, 0)$ which have rotations as their little group since $p^i = 0, i = 1, 2, 3$, i.e., $L(p^\mu) = \text{SO}(3)$.

Due to the completely antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ in the definition of W_μ , it follow

$$W_0 = 0, \quad W_i = -m J_i \quad \Longrightarrow \quad C_2 = m^2 J^2. \quad (2.3.13)$$

Thus, we have

$$C_1 |m, \omega; \lambda_i\rangle = m^2 |m, \omega; \lambda_i\rangle, \quad C_2 |m, \omega; \lambda_i\rangle = m^2 j(j+1) |m, \omega; \lambda_i\rangle, \quad (2.3.14)$$

This identifies ω with j , while the remaining labels are specified as $\lambda_i \in \{p^\mu, j_3\}$.

Note that once the p^μ are fixed within a representation the components of the Pauli-Ljubanski vector do satisfy an algebra (since within one representation we can replace P^μ by p^μ and in this case the algebra is essentially the same as the rotation group since $W_i \propto J_i$).

This algebra defines the Little group that has the well known finite dimensional representations.

Hence, every particle with nonzero mass is an irreducible representation of the Poincaré group with labels $|m, j; p^\mu, j_3\rangle$.

This defines a one-particle state and, in particular, an elementary particle of mass m and spin j .

It is important to emphasize that the existence of these quantum states corresponding to elementary particles is a general consequence of the two basic theories, quantum mechanics and special relativity.

We may then define elementary particles as unitary irreducible representations of the Poincaré group.

This is a remarkable result since it is a way to mathematically define the basic building blocks of nature.

- $P^\mu P_\mu = 0$ (**Massless particles**).

The simplest realization is selecting the origin $p^\mu = (0, 0, 0, 0)$ which is Lorentz invariant.

Even though this seems like a trivial case, it corresponds to a state with no particles, the vacuum state $|0\rangle$.

We will see the importance of this state later on.

In order to have non-trivial representations corresponding to particle states, we can take the momentum of the form $p^\mu = (E, 0, 0, E)$ which implies

$$(W_0, W_1, W_2, W_3) = E (J_3, -J_1 + K_2, -J_2 - K_1, -J_3) \quad (2.3.15)$$

$$[W_1, W_2] = 0, \quad [W_3, W_1] = -iE W_2, \quad [W_3, W_2] = iE W_1. \quad (2.3.16)$$

These commutation relations are those for the Euclidean group in two dimensions (translations generated by W_1, W_2 and rotations generated by W_3 , acting on an abstract two dimensional space) and again define the Little group for massless particles.

This group, contrary to the massive case, has infinite dimensional unitary representations known as *continuous spin representations*.

A simple way to see this is to realize that W_1 and W_2 commute with each other and can be simultaneously diagonalized with eigenvalues w_1, w_2 .

If $w_1, w_2 \neq 0$, then

$$W^\mu W_\mu = -(w_1^2 + w_2^2) := -\rho^2 \quad \Rightarrow \quad w_1 = \rho \cos \theta, \quad w_2 = \rho \sin \theta. \quad (2.3.17)$$

Hence, the representation can be labelled as $|0, \rho; p^\mu, \theta\rangle$.

Therefore, the existence of these representations would imply particles with an extra continuous label on top of the momenta p^μ .

Since particles with these extra continuous labels have not been seen in nature, in order to proceed we concentrate only on the finite dimensional representations.

This is not entirely satisfactory since, contrary to the massive case in which we extracted the most general implications of special relativity and quantum mechanics without any further assumptions, here we have to make an ad-hoc restriction to concentrate only on finite dimensional representations.

Restricting to finite dimensional representations, $SO(2)$ is the relevant subgroup of the Little group generated by W_3 as w_1, w_2 vanish.

In that case, $W^\mu = \lambda P^\mu$ and states are labelled as $|0,0, p^\mu, \lambda\rangle := |p^\mu, \lambda\rangle$, where λ is called helicity and corresponds to the component of angular momentum in the direction of motion of the particle.

Since we have seen that it is only rotations by 4π and not 2π that leave the physics invariant, we should expect

$$e^{2\pi i \lambda} |p^\mu, \lambda\rangle = \pm |p^\mu, \lambda\rangle \quad (2.3.18)$$

which requires λ to be integer or half integer $\lambda = 0, 1/2, 1, \dots$

Notice that contrary to the massive case in which the integer or half-integer nature of spin was dictated by group theory, i.e., the representations of $SU(2)$, for the massless case we need to use a topological argument related to the simply connected nature of $SL(2, \mathbb{C})$.

We will see that essentially all the particles of the Standard Model will come from these massless representations of the Poincaré group, e.g., $\lambda = 0$ (Higgs), $\lambda = 1/2$ (quarks, leptons), $\lambda = 1$ ($\gamma, W^\pm, Z^0, g$) and $\lambda = 2$ (graviton).

We will see that parity transforms states of helicity $\lambda, |p^\mu, \lambda\rangle$ to $|p^\mu, -\lambda\rangle$ and therefore, if parity is conserved, states such as the photon and graviton have two degrees of freedom corresponding to $\lambda = \pm 1, \lambda = \pm 2$ respectively.

- $P^\mu P_\mu = -m^2 < 0$ (**tachyons**).

A typical momentum can be $p^\mu = (0, m, 0, 0)$.

This would correspond to a particle moving in a space-like trajectory (moving faster than light).

In particular, it would contradict causality.

In some cases, these particles appear in physical theories when instead of expanding around a minimum of the energy we expand around a maximum and their presence would only indicate that we are expanding around the 'wrong' vacuum.

Once a minimum is identified and the expansion is done around the minimum of the energy the particle would correspond to a normal massive particle as described above.

We encounter such a situation further below when discussing the electroweak phase transition in Part 6.

All in all, we deduce that the states for massless and massive particles are finite dimensional representations of $SO(3)$ (massive) or $SO(2)$ (massless).

It is important to emphasize that the existence of the aforementioned quantum states corresponding to elementary particles is a general consequence of the two basic theories, quantum mechanics and special relativity.

We may then define elementary particles as unitary irreducible representations of the Poincaré group.

This is a remarkable result since it is a way to mathematically define the basic building blocks of nature.

Next we use these results to introduce quantum fields in an attempt to build up an off-shell framework to describe particle interactions.

But before we get there, we have to briefly discuss the effects of discrete spacetime transformations which, as we will see throughout these lectures, will also have important consequences for the Standard Model.

2.4 Discrete spacetime transformations

So far we have only considered the representations of the proper orthochronous Lorentz group.

Let us now consider the disconnected components of the Lorentz group and consider the action of parity and time reversal that, as we saw before, together with the identity and their product define the Klein group.

In the following, we denote operators representing a general Poincaré transformation $\{\Lambda|a\} \in \mathcal{P}(3, 1)$ as $U(\Lambda, a)$.

The transformation matrices for parity P and time reversal T can be written

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.4.1)$$

As operators acting on the Hilbert space of quantum states, we denote them as

$$\mathcal{P} = U(\Lambda_P, 0), \quad \mathcal{T} = U(\Lambda_T, 0). \quad (2.4.1)$$

For a general Poincaré transformation $\{\Lambda|a\} \in \mathcal{P}(3, 1)$, we have

$$\mathcal{P}^{-1}U\mathcal{P} = U(\Lambda_P^{-1}\Lambda\Lambda_P, \Lambda_P a), \quad \mathcal{T}^{-1}U\mathcal{T} = U(\Lambda_T^{-1}\Lambda\Lambda_T, \Lambda_T a). \quad (2.4.1)$$

Expanding as before around the identity using

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad a^\mu = \epsilon^\mu, \quad \omega^\mu{}_\nu, \quad \epsilon^\mu \ll 1$$

$$U(\Lambda, a) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_\mu P^\mu, \quad (2.4.4)$$

and recalling that $M^{\mu\nu}$ transforms as a tensor under Lorentz transformations, i.e.,

$$\frac{i\omega_{\rho\sigma}}{2}\mathcal{P}^{-1}M^{\rho\sigma}\mathcal{P} = \frac{i\omega_{\rho\sigma}}{2}\Lambda_P^\rho{}_\mu\Lambda_P^\sigma{}_\nu M^{\mu\nu}, \quad (2.4.5)$$

we compute

$$\mathcal{P}^{-1}J_i\mathcal{P} = J_i, \quad \mathcal{P}^{-1}K_i\mathcal{P} = -K_i, \quad (2.4.6)$$

$$\mathcal{P}^{-1}P_i\mathcal{P} = -P_i, \quad \mathcal{P}^{-1}P^0\mathcal{P} = P^0. \quad (2.4.7)$$

This is as expected since under parity we expect that the 0-th component of the vector P^μ ($P^0 = E$) is invariant whereas the spatial components change sign.

Also the angular momentum J_i should be invariant (being an axial vector as in classical mechanics $\mathbf{J} = \mathbf{r} \times \mathbf{p}$).

It is straightforward to see that the parity operator is indeed unitary.

However, if we follow the same procedure to obtain the transformations under time reversal we encounter a problem.

If the time reversal operator is also unitary it would map $\mathcal{T}^{-1}P^0\mathcal{T} = -P^0$.

That means it would change positive energies to negative energies that seems unphysical.

To interpret this result, we have the following theorem due to Wigner:

Theorem 2.4.1 (Wigner). Transformations on a Hilbert space preserving probabilities are either

- unitary and linear, i.e.,

$$\langle U\Phi | U\Psi \rangle = \langle \Phi | \Psi \rangle, \quad U(\alpha|\Phi\rangle + \beta|\Psi\rangle) = \alpha U|\Phi\rangle + \beta U|\Psi\rangle \quad (2.4.8)$$

- or anti-unitary and anti-linear, that is,

$$\langle U\Phi | U\Psi \rangle = \langle \Phi | \Psi \rangle^*, \quad U(\alpha|\Phi\rangle + \beta|\Psi\rangle) = \alpha^* U|\Phi\rangle + \beta^* U|\Psi\rangle \quad (2.4.9)$$

Now, in order to preserve positive energies, $\mathcal{T}$ must be an anti-unitary and anti-linear operator

$$\mathcal{T}(i|\Phi\rangle) = -i\mathcal{T}|\Phi\rangle, \quad (2.4.10)$$

so that

$$\mathcal{T}^{-1}J_i\mathcal{T} = -J_i, \quad \mathcal{T}^{-1}K_i\mathcal{T} = K_i, \quad (2.4.11)$$

$$\mathcal{T}^{-1}P_i\mathcal{T} = -P_i, \quad \mathcal{T}^{-1}P^0\mathcal{T} = P^0. \quad (2.4.12)$$

Then, the time reversal operator $\mathcal{T}$ maps positive energies to positive energies.

Let us come back to our one-particle states defined in section 2.3 and understand their transformation behavior under parity and time reversal.

One can show that:

- For massive particles, we have the transformation properties

$$|m, j; p^\mu, j_3\rangle \begin{cases} \xrightarrow{\mathcal{P}} & \eta_P |m, j; p'^\mu, j_3\rangle , \\ \xrightarrow{\mathcal{T}} & \eta_T (-1)^{j-j_3} |m, j; p'^\mu, -j_3\rangle \end{cases} \quad (2.4.13)$$

with $|\eta_P| = |\eta_T| = 1$ and p'^μ is the result of the corresponding transformation acting on p^μ (under parity the spatial components change sign, under time reversal only the time component change sign, etc.).

- For massless particles, one finds

$$|p^\mu, \lambda\rangle \begin{cases} \xrightarrow{\mathcal{P}} & \eta'_P e^{\mp i\pi\lambda} |p'^\mu, -\lambda\rangle , \\ \xrightarrow{\mathcal{T}} & \eta'_T e^{\pm i\pi\lambda} |p'^\mu, \lambda\rangle . \end{cases} \quad (2.4.14)$$

Note that for $\lambda \neq 0$ there has to exist the opposite helicity states.

In particular,

- the photon and the graviton have $\lambda = \pm 1, \lambda = \pm 2$ respectively which means that for each of them the $\pm\lambda$ represent two states of the same particle, since both gravitation and electromagnetism are invariant under parity (both graviton and photon are their own antiparticle).
- if the neutrino were massless, then $\lambda = \pm 1/2$ may have a different interpretation, for instance $\lambda = +1/2$ could be identified with the neutrino and $\lambda = -1/2$ with the antineutrino, since the weak interactions are not invariant under parity.

Even though, as we will see, the neutrinos are expected to have a mass, it is still an open question if the neutrinos are or are not their own anti-particles.

We conclude this section with some comments that will be very relevant in the next chapters:

- Massive particles of spin $j = 1$ have $2j + 1 = 3$ polarisation states, namely $j_3 = -1, 0, 1$. In contrast, massless particles of helicity $\lambda = 1$ have only 2 polarisation states with $\lambda = \pm 1$.
- Massive particles of spin $j = 2$ have $2j + 1 = 5$ polarisation states, namely $j_3 = -2, -1, 0, 1, 2$. In contrast, massless particles of helicity $\lambda = 2$ have **still** only 2 polarisation states with $\lambda = \pm 2$.

3. FROM PARTICLES TO FIELDS

Particles vs Fields

If it turned out that some physical system could not be described by a quantum field theory, it would be a sensation; if it turned out that the system did not obey the rules of quantum mechanics and relativity, it would be a cataclysm.

Steven Weinberg

In the previous part we stressed that only assuming two fundamental theories of nature, special relativity and quantum mechanics, the fundamental physical entities are the elementary particles labeled by the quantum number specified by the representations of the Poincaré group $|m, j; p^\mu, j_3\rangle$ and $|p^\mu, \lambda\rangle$ describing massive and massless particles.

This part is devoted to the study of interactions among these particles.

The general requirements of Poincaré invariance, locality and unitarity will let us to introduce *fields* as “functions” of spacetime which are operators made out of creation and annihilation operators that create and destroy the corresponding particles.

We emphasize that fields are only a tool to describe interactions among the particles.

Their introduction includes more conditions than just the assumptions of special relativity and quantum mechanics.

However, fields are the key objects to describe interactions among particles and their use goes beyond the study of interactions among particles.

They are key ingredients in any interacting theory which requires local interactions such as condensed matter systems.

So their use is across different disciplines.

They are often presented as the basic objects of high energy physics with the particles appearing as their excitations.

The two descriptions are somehow manifestations of the wave-particle duality of quantum mechanics.

Our emphasis on particles rather than fields as the fundamental objects resides on the fact that it may be possible that some of the ingredients assumed in the introduction of fields may be overcome in future descriptions of nature beyond the Standard Model.

The spirit of this class is not only to introduce the basic tools to describe the Standard Model, but also to identify the key ingredients and assumptions that may help in shaping formulations beyond the Standard Model.

However, for the rest of the class, we will use the powerful tool of field theory.

3.1 Particle interactions and fields

In the preceding section, we learned about the concept of one-particle states.

Here, our aim is to describe interactions among many particles.

Putting together Poincaré invariance with the extra assumptions of unitarity and locality will lead us to superpositions of the aforementioned one-particle states corresponding to fields.

These objects allow us to develop a formalism known as Quantum Field Theory (QFT) that is suitable to describe local interactions among particle states.

3.1.1 Many particle states

Let us begin by trying to understand how we can describe relativistic processes in a quantum mechanical theory.

Clearly, $E = mc^2$ tells us that mass and energy are on equal footing which further implies that particles can be annihilated into energy.

To describe such processes, we aim at combining, as we said several times before, Lorentz invariance of special relativity with the notion of quantum mechanics and add further conditions such as locality to describe interactions.

To this end, we initially need to introduce the space of multi-particle states.

The Hilbert space $\mathcal{H}$ of all particle states can be decomposed as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \quad (3.1.1)$$

where

- $\mathcal{H}_0$ encodes 0-particle states, i.e., the vacuum $|0\rangle$
- $\mathcal{H}_1$ includes 1-particle states, e.g., those generated by the creation operator $a^\dagger(p, \lambda)$ from the vacuum state,

$$|p^\mu, \lambda\rangle = a^\dagger(p, \lambda) |0\rangle . \quad (3.1.2)$$

- $\mathcal{H}_2$ includes 2-particle states, e.g.,

$$|p_1^\mu, \lambda_1; p_2^\mu, \lambda_2\rangle = a^\dagger(p_2, \lambda_2) |p_1^\mu, \lambda_1\rangle = \pm |p_2^\mu, \lambda_2; p_1^\mu, \lambda_1\rangle . \quad (3.1.3)$$

Here, the $+$ sign refers to integer spin/helicity states (**bosons**), whereas the $-$ sign to half-integer spin/helicity states (**fermions**).

- ...

Note that this is one of the most important - signs in science since it is the origin of the Pauli exclusion principle that implies that nuclei, atoms, molecules and therefore all matter, have a non-trivial structure.

As usual the creation and annihilation operators satisfy

$$\begin{aligned} [a(p, \lambda), a(p', \lambda')]_{\pm} &= [a^\dagger(p, \lambda), a^\dagger(p', \lambda')]_{\pm} = 0 , \\ [a(p, \lambda), a^\dagger(p', \lambda')]_{\pm} &\propto \delta(p - p') \delta_{\lambda\lambda'} \end{aligned} \quad (3.1.4)$$

for bosons $[\cdot, \cdot]_- = [\cdot, \cdot]$ which are the standard commutators, whereas for fermions we change commutators for anti-commutators which may be written as $[\cdot, \cdot]_+ = \{\cdot, \cdot\}$.

In this way, we construct the full Hilbert space of particle states that we would like to describe with a dedicated quantum theory.

3.1.2 Interactions and Fields

Interactions among many particle states are determined by computing the S-matrix.

Typically, we can think of scattering processes as starting from an initial state $|\alpha^{\text{in}}\rangle$ at $t = -\infty$ and ending up with an out-state $|\beta^{\text{out}}\rangle$ at $t = +\infty$, see Fig. 3.1.

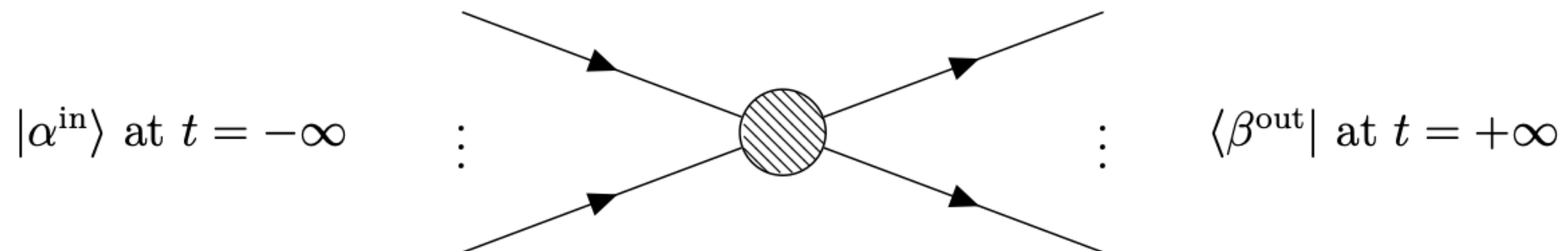


Figure 3.1: A cartoon representation of a scattering process.

In between, particles interact in a complicated way by, e.g., colliding with each other or splitting up in a bunch of new particles.

We define the S-matrix as

$$S_{\beta\alpha} = \langle \beta^{\text{out}} | \alpha^{\text{in}} \rangle = \delta_{\beta\alpha} + (2\pi)^4 \delta(p_\beta - p_\alpha) \mathcal{M}_{\beta\alpha} . \quad (3.1.5)$$

The first term stands from the trivial event of no interactions at all.

So the interesting physics is encoded in the Matrix Elements $\mathcal{M}_{\beta\alpha}$.

The standard questions we can ask for particle interactions are:

a) **Decay Rates**. The probability of decay of one particle to several particles.

This is the simplest case in which the $|\alpha^{\text{in}}\rangle$ state is one single particle.

The S-matrix reduces to the probability of decay of the original particle to its daughter states, see Fig. 3.2.

It is usually represented as $\Gamma(\alpha^{\text{in}} \rightarrow \beta^{\text{out}})$.

The decay rate per unit of phase space volume of the final states can be explicitly computed via

$$d\Gamma = \frac{d\Pi_{\text{LIPS}}}{2E_\alpha} |\mathcal{M}_{\beta\alpha}|^2 \quad (3.1.6)$$

where $d\Pi_{\text{LIPS}}$ stands for Lorentz invariant phase space volume

$$d\Pi_{\text{LIPS}} \equiv (2\pi)^4 \delta^4(p_\alpha^\mu - \sum p) \quad (3.1.7)$$

The important point for us is that Γ is determined by $|\mathcal{M}_{\beta\alpha}|^2$ integrated and summed over all final momentum and spin states, see Appendix A at the end of this section for details.

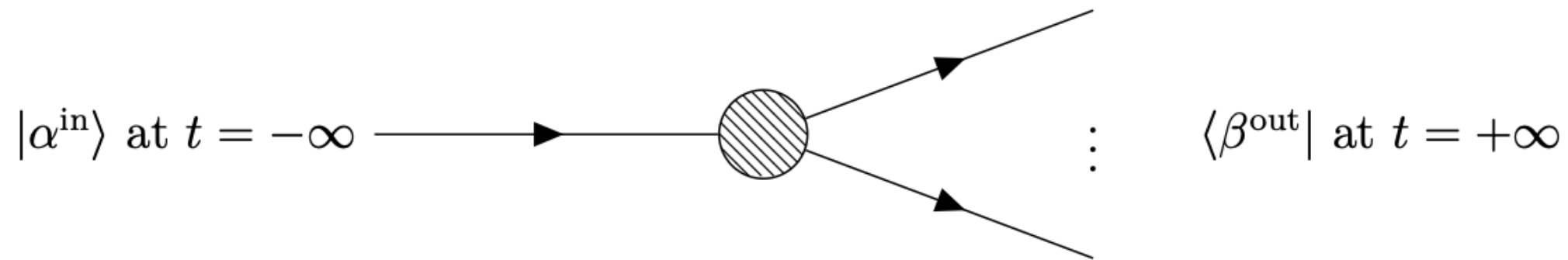


Figure 3.2: A cartoon representation of a decay process.

b) Cross Sections. As we know from Classical Mechanics, cross sections are the quantities that measure how often a scattering process between several particles happens.

Cross sections are usually labelled as σ and are also proportional to $|\mathcal{M}_{\beta\alpha}|^2$, see Appendix A at the end of this section for details.

Therefore our goal is to find a way to determine $|\mathcal{M}_{\beta\alpha}|^2$ given a configuration of initial and final states interacting in a particular way.

In order to determine the matrix elements $|\mathcal{M}_{\beta\alpha}|^2$, in general, we require the following conditions on interactions:

a) **Poincaré invariance of amplitudes.** The S-matrix is invariant under Poincaré transformations.

b) **Unitarity:** Probabilities add up to 1 and are preserved under time evolution by $U = e^{-iHt}$.

For the S-matrix, unitarity implies

$$\int d\beta S_{\beta\gamma}^* S_{\beta\alpha} = \int d\beta \langle \gamma^{\text{in}} | \beta^{\text{out}} \rangle \langle \beta^{\text{out}} | \alpha^{\text{in}} \rangle = \langle \gamma^{\text{in}} | \alpha^{\text{in}} \rangle = \delta(\alpha - \gamma). \quad (3.1.8)$$

or as an operator equation

$$S^\dagger S = 1. \quad (3.1.9)$$

c) **Stability:** This is the requirement that the energy should be bounded from below and there is a state of minimum energy, the vacuum $|0\rangle$.

d) **Locality** (cluster decomposition): Observables $\mathcal{O}_1, \mathcal{O}_2$ commute at space-like distances, i.e.,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0, \quad (3.1.10)$$

which is usually referred to as *microcausality*.

As a consequence of the last requirement of locality, we deduce that the *Hamiltonian* H is determined by a local function, the *Hamiltonian density* $\mathcal{H}(\mathbf{x}, t)$, which is defined at each space point.

More precisely, we define the Hamiltonian as

$$H = \int \mathcal{H}(\mathbf{x}, t) d^3x \quad (3.1.11)$$

corresponding to the sum of energies at each point in space.

Similarly, the *Lagrangian* L is obtained from the *Lagrangian density* $\mathcal{L}(\mathbf{x}, t)$ via

$$L = \int \mathcal{L}(\mathbf{x}, t) d^3x \quad (3.1.12)$$

with associated action

$$S = \int \mathcal{L}(\mathbf{x}, t) d^4x. \quad (3.1.13)$$

This locality requirement is crucial to determine interactions and is known as *cluster decomposition* which means that experiments performed at large enough distances do not affect each other.

Now, we arrive at a conundrum: $\mathcal{H}$ and $\mathcal{L}$ are operators in position space, but particle states as derived above are defined in momentum space.

The way out is pretty obvious: we need to apply Fourier transformations to describe the corresponding states in terms of “functions” of spacetime coordinates – objects that we call **fields**.

For any particle of given momentum and spin/helicity, we define a field Φ_α as

$$\Phi_\alpha(x^\mu) = A_\alpha(x^\mu) + B_\alpha^*(x^\mu) \quad (3.1.14)$$

in terms of

$$A_\alpha(x^\mu) = \sum_\lambda \int dp u_\alpha(p, \lambda) a(p, \lambda) e^{ip_\mu x^\mu}, \quad (3.1.15)$$

$$B_\alpha^*(x^\mu) = \sum_\lambda \int dp v_\alpha(p, \lambda) b^\dagger(p, \lambda) e^{-ip_\mu x^\mu}. \quad (3.1.16)$$

We use a multi-index notation where $\alpha = \mu_1 \dots \mu_n$ are spacetime indices of the corresponding representation under $SO(3, 1)$.

Here, the operators $a(p, \lambda)$ and $b^\dagger(p, \lambda)$ are raising and lowering operators as defined in the previous section with the commutation relations (3.1.4).

The object A is the field annihilating the corresponding particle, whereas B is the field creating the anti-particle.

Stated otherwise, fields are always of the form

Quantum Fields

$$\Phi_\alpha(x^\mu) = \sum_\lambda \int dp \left(u_\alpha(p, \lambda) a(p, \lambda) e^{ip_\mu x^\mu} + v_\alpha(p, \lambda) b^\dagger(p, \lambda) e^{-ip_\mu x^\mu} \right). \quad (3.1.17)$$

For integer spin or helicity n , the object on the left transforms as some rank n tensor under $SO(1, 3)_\uparrow$, that is, it transforms under the $(n/2, n/2)$ representation of the Lorentz algebra.

The integral and sum on the right is a superposition of all unitary irreducible representations of one-particle states as classified above via the little groups.

The parameter λ labels spins for massive particles and helicity for massless particles, while the integration over momenta is performed using the invariant measure

$$\int dp = \int d^4p \delta(p^2 - m^2) = \int d^3p \frac{1}{2E_p (2\pi)^3}, \quad E_p = p_0, \quad E_p^2 = \mathbf{p}^2 + m^2. \quad (3.1.18)$$

The wave functions $u_\alpha(p, \lambda)$ and $v_\alpha(p, \lambda)$ in momentum space describe the dynamics of fields in spacetime given that they carry Lorentz indices α .

As we will in the next part, the functions $u_\alpha(p, \lambda)$ and $v_\alpha(p, \lambda)$ need to satisfy certain constraints in order to write down (off-shell) actions in terms of the fields.

The relation between the two sides in (3.1.17) is determined by the *coefficient functions* $e^{ip_\mu x^\mu} u_\alpha(p, \lambda)$ and $e^{-ip_\mu x^\mu} v_\alpha(p, \lambda)$ which carry both Lorentz indices α and x as well as Poincaré representation labels p and λ .

One important remark concerns the form (3.1.14) that we started with which ensures the existence of anti-particles.

Let us explain this in more detail.

First, we note that writing the field $\Phi_\alpha(x^\mu)$ in terms of $A_\alpha(x^\mu)$ and $B_\alpha(x^\mu)$ as in (3.1.14) is essentially required by causality.

Above, we stated that all operators should commute at spacelike separations, cf. Eq. (3.1.10).

That is, for a fixed time and two different locations, we demand that

$$\left[\Phi_{\alpha}(x^i, t), \Phi_{\alpha}^{*}(y^i, t) \right] = 0 . \quad (3.1.19)$$

However, it is impossible for both $A_{\alpha}(x^{\mu})$ and $B_{\alpha}(x^{\mu})$ to satisfy this condition by themselves because

$$\left[A_{\alpha}(x^i, t), A_{\alpha}^{*}(y^i, t) \right] \neq 0 . \quad (3.1.20)$$

These commutators can be explicitly derived from those for the creation and annihilation operators $a(p, \lambda)$ and $a^{\dagger}(p, \lambda)$ as defined in (3.1.4).

Thus, both objects are needed to build fields $\Phi_{\alpha}(x^{\mu})$ satisfying (3.1.19).

This is the requirement for the existence of anti-particles.

If $a(p, \lambda) = b(p, \lambda)$, the particle is simply its own anti-particle.

3.1.3 Field theories and their actions

So far, we have seen that putting special relativity and quantum mechanics together lead us to classifying one-particle states in terms of their masses and spins.

Interactions lead us to the concept of locality and to use fields rather than particles to describe our theory.

In finding all unitary irreducible representations of the Poincaré group, we defined states for fixed $\mathbf{p}^\mu$ over which we have to integrate to get a suitable superposition of eigenstates, i.e., we found fields of the form

$$\Phi_\alpha(x^\mu) = \sum_\lambda \int d^4p \left(u_\alpha(p, \lambda) a(p, \lambda) e^{ip_\mu x^\mu} + v_\alpha(p, \lambda) b^\dagger(p, \lambda) e^{-ip_\mu x^\mu} \right) . \quad (3.1.21)$$

The action S becomes a function of these fields Φ_α and their derivatives, i.e.,

$$S[\Phi_\alpha, \partial\Phi_\alpha] = \int d^4x \mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha] \quad (3.1.22)$$

where $\mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha]$ is the Lagrange density or simply Lagrangian of the theory.

Here, translation invariance forbids the explicit dependence of $\mathcal{L}$ on the coordinates x^μ , i.e., $\mathcal{L} \neq \mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha, x^\mu]$.

The Lagrangian is typically written as a sum of individual terms of the form

$$\mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha] = \sum_i c_i \mathcal{O}_i(\Phi_\alpha, \partial\Phi_\alpha) . \quad (3.1.23)$$

Here, c_i are some “constant” coefficients and $\mathcal{O}_i$ are referred to as operators since they are functions of the fields Φ_α which are themselves operators (as it can be seen from their dependence on the creation and annihilation operators).

The equations of motion for Φ_α are obtained as usual from the Euler-Lagrange equations, i.e.,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_\alpha} - \frac{\partial \mathcal{L}}{\partial \Phi_\alpha} = 0 . \quad (3.1.24)$$

When the field configuration satisfies these classical equations, we say that the field is on its mass shell or on-shell.

Otherwise we say it is off-shell.

Quantization of field theories proceeds most easily through the path integral approach where e.g. the partition function can be written as

$$\mathcal{Z} = \int \mathcal{D}\Phi_\alpha e^{-S[\Phi_\alpha, \partial\Phi_\alpha]} . \quad (3.1.25)$$

Only in the classical limit the on-shell condition is satisfied. Similarly, correlations functions and amplitudes can be straightforwardly computed within this formalism through perturbation theory.

Note that there is an infinite-to-one mapping from actions to on-shell scattering amplitudes: infinitely many actions can give rise to the same on-shell amplitude due to field redefinitions $\Phi \rightarrow \Phi'(\Phi)$.

This begs the question: what is the point of introducing fields in the first place?

First and foremost, they provide us with an organizing principle for interactions among particles governed by symmetries.

What is more, non-perturbative effects, running couplings as well as off-shell correlation functions can be systematically studied.

As Weinberg himself always stresses, quantum fields are “the only way of satisfying the principles of Lorentz invariance plus quantum mechanics plus cluster decomposition”.

Recently, though, there has been much effort towards computing amplitudes directly without the use of Lagrangians.

No details here.

Let us now provide examples of free field theories focussing on spin/helicity states less than one for which the massless and massive states have the same number of degrees of freedom:

- *Free scalar field (spin/helicity 0)*. In order to create and annihilate spinless particles either massive $|m, j=0; p^\mu, j_3=0\rangle$ or massless $|p^\mu, \lambda=0\rangle$ we introduce a Lorentz scalar field $\phi(x^\mu)$ satisfying classically the *Klein-Gordon equation*

$$(\partial^\mu \partial_\mu + m^2)\phi = 0 \tag{3.1.26}$$

which is nothing but the Fourier transformation of the on-shell condition $p^\mu p_\mu = m^2$ in momentum space.

The expansion of ϕ in creation and annihilation operators obtained from (3.1.17) reads

$$\phi(x) = \int dp \left(a(p) e^{ip_\mu x^\mu} + a^\dagger(p) e^{-ip_\mu x^\mu} \right) . \tag{3.1.27}$$

The Lagrangian density that reproduces the Klein-Gordon equation by plugging it into the Euler-Lagrange equations (3.1.24) is

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 . \quad (3.1.28)$$

Notice that the single degree of freedom of a free scalar matches the single one-particle state in both cases, massive and massless.

Since the field is real, the corresponding particle will be its own anti-particle.

Extending to a complex scalar $\Phi = \phi_1 + i \phi_2$ with $\phi_{1,2}$ real scalar fields, is straightforward and they will correspond to two one-particle states: the particle and its anti-particle.

- *Free spin (helicity) 1/2 fermion.* The one-particle states are now for the massive case $|m, j=1/2; p^\mu, j_3 = \pm 1/2\rangle$ and $|p^\mu, \lambda = \pm 1/2\rangle$ for the massless case.

The corresponding field could either be a left-handed ψ_L or right-handed ψ_R spinor.

The free Dirac Lagrangian containing both fields to include a natural mass term is

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - m \bar{\psi} \psi = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R - m (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) . \quad (3.1.29)$$

Here, for completeness, we also wrote the corresponding Dirac spinor ψ satisfying the standard *Dirac equation*

$$(i \not{\partial} - m) \psi = 0 . \quad (3.1.30)$$

Again, this can be obtained from the on-shell mass relation $p^\mu p_\mu = m^2$ after using some algebra for the γ -matrices.

Following the general expression (3.1.17) the Dirac field can be written as

$$\psi(x^\mu) = \sum_{\lambda=\pm\frac{1}{2}} \int dp \left(u(p, \lambda) a(p, \lambda) e^{ip_\mu x^\mu} + v(p, \lambda) b^\dagger(p, \lambda) e^{-ip_\mu x^\mu} \right). \quad (3.1.310)$$

We have omitted spinorial indices in $\psi(x^\mu)$ and the momentum-space wave functions $u(p^\mu)$, $v(p^\mu)$.

Here, on-shell u , v satisfy the Dirac equation in momentum space $(\not{p} - m)u = (\not{p} + m)v = 0$.

Note that for both massless and massive one-particle states, the corresponding multiplet has two spin states.

Adding the same for the corresponding anti-particle, we are left with four independent degrees of freedom (two spin states for each particle and anti-particle) matching the four independent degrees of freedom for a Weyl spinor (two complex-component spinor).

Higher spin/helicity states will be discussed below.

More work is required to write down their actions because we need to be careful about additional constraints that have to be imposed to account for the correct number of physical degrees of freedom.

It turns out that the notion of *symmetries* will be crucial for this process which we introduce in the next section.

3.2 Symmetries in QFT

A guiding principle to understand the structure of quantum field theories are symmetries, that is, transformations of the fields and spacetime coordinates that leave physics invariant.

Up to this point, we have talked extensively about the Poincaré group acting on spacetime coordinates.

On the level of the action, we talk about Poincaré invariance as the statement that Poincaré transformations should leave the action invariant (possibly up to total derivatives).

But there are various other notions of symmetries that play a pivotal role in constructing general field theories.

Here, we give a brief summary of the most relevant types for understanding Standard Model physics.

3.2.1 Coleman-Mandula theorem

The celebrated Coleman-Mandula theorem states that the most general symmetries of the S-matrix are of the form

$$\text{Spacetime} \otimes \text{Internal}. \quad (3.2.1)$$

The left hand side is given by the Poincaré group with generators P^μ , $M^{\mu\nu}$ and has been studied in detail in the previous part.

We recall that under a general Poincaré transformation $\{\Lambda|a\} \in \mathcal{P}(3, 1)$ the states $|\psi\rangle$ in our Hilbert space $\mathcal{H}$ transform with respect to some operator $U(\Lambda, a)$, $\Lambda \in \text{SO}(3, 1)$, $a \in \mathbb{R}^{3,1}$, such that

$$|\psi\rangle \rightarrow U(\Lambda, a) |\psi\rangle . \quad (3.2.2)$$

Operators $\mathcal{O}_i$ in our theory transform according to

$$\mathcal{O}_i \rightarrow U^\dagger(\Lambda, a) \mathcal{O}_i U(\Lambda, a) . \quad (3.2.3)$$

We require that these representations are unitary, i.e.,

$$\langle \psi_1 | \psi_2 \rangle \rightarrow \langle \psi_1 | U^\dagger U | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle , \quad |\psi_1\rangle , |\psi_2\rangle \in \mathcal{H} . \quad (3.2.4)$$

This can also be realized for the parity operator, while the time reversal operator needs to be anti-unitary, $\mathcal{T}^\dagger \mathcal{T} = -1$.

For a field $\Phi_\alpha(x)$, with the index $\mathfrak{v}$ specifying the corresponding representation of the Lorentz group, this implies that

$$\Phi_\alpha(x) \rightarrow U^\dagger(\Lambda, a) \Phi_\alpha(x') U(\Lambda, a) = D_\alpha{}^\beta \Phi_\beta(x') \quad (3.2.5)$$

where $D_\alpha{}^\beta$ are representation matrices for the Lorentz group.

Note the dependence on $x' = \Lambda x + a$ rather than x on the right hand side.

Next, we consider the *internal symmetries* for which local operators transform according to

$$\mathcal{O}_i(x) \rightarrow \mathcal{O}'_i(x) . \quad (3.2.6)$$

We stress that, contrary to the case of spacetime symmetries, the operators are evaluated at the same spacetime point x^μ on both the left and right hand side.

If our theory is invariant under such a transformation, then it is called *internal* and the corresponding transformation U commutes with the Hamiltonian

$$[H, U] = 0. \quad (3.2.7)$$

Under an internal transformation $g(\alpha_a) \in G$ with parameters α_a , $a = 1, \dots, \dim(G)$, a field $\Phi_\alpha^i(x)$ transforms as

$$\Phi_\alpha^i(x) \rightarrow U^\dagger(\alpha_a) \Phi_\alpha^i(x) U(\alpha_a) = g^i{}_j \Phi_\alpha^j(x) \quad (3.2.8)$$

with $g^i{}_j$ matrices representing the internal symmetry group G .

Above, the indices $i, j, \dots$ are associated with the representations under G which have nothing to do with the indices $\alpha, \beta, \dots$ coming from the representation of Lorentz group.

Hence, the left and right hand side of (3.2.8) have both the same index α because the internal transformation does not act on the spacetime components of the field.

If the action S is invariant under those transformations, we speak of symmetries of $\mathcal{L}$ (up to a total derivative).

The Coleman-Mandula theorem is very important in the sense that the fact that the most general symmetry is a direct product (and not a semi-direct or other combination) it forbids non-trivial combinations of internal and spacetime symmetries.

In particular it guarantees that particle states are still labeled by mass and spin/helicity and the internal symmetries could only add extra labels such as electric charge.

3.2.2 Examples of Internal Symmetries

In order to be more explicit regarding internal symmetries, let us consider the Lagrangian for

- A massive Dirac spinor with

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - m \bar{\psi} \psi \quad (3.2.9)$$

which can be written in terms of the left- and right-handed Weyl spinors as

$$\mathcal{L} = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R - m (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) . \quad (3.2.10)$$

In the limit $m \rightarrow 0$, $\mathcal{L}$ is invariant under a symmetry group $U(1)_L \times U(1)_R$ since both fields can be independently altered by a phase

$$\psi_L \rightarrow e^{i\alpha_L} \psi_L , \quad \psi_R \rightarrow e^{i\alpha_R} \psi_R , \quad (3.2.11)$$

which is referred to as *chiral symmetry*.

In the massive case, the Lagrangian is only invariant under a single $U(1)_V$ with $\alpha_V = \alpha_L = \alpha_R$, i.e.,

$$\psi_L \rightarrow e^{i\alpha_V} \psi_L , \quad \psi_R \rightarrow e^{i\alpha_V} \psi_R . \quad (3.2.12)$$

- A massive complex scalar field with quartic interactions

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi^* - m^2 |\phi|^2 - \lambda |\phi|^4. \quad (3.2.13)$$

This Lagrangian is invariant under U(1)-transformations of the form

$$\phi \rightarrow e^{ie\alpha} \phi. \quad (3.2.14)$$

- A massive scalar in the fundamental representation of SU(2)

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi^\dagger - m^2 |\phi|^2, \quad \phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}. \quad (3.2.15)$$

Here, ϕ_1 and ϕ_2 are two complex-valued scalar fields.

In those components, the action reads

$$\mathcal{L} = \sum_{i=1}^2 \left(\partial^\mu \phi^i \partial_\mu (\phi^i)^* - m^2 |\phi^i|^2 \right). \quad (3.2.16)$$

For an SU(2) transformation

$$U(\alpha_a) = \exp \left(-\frac{i}{2} \alpha_a \sigma^a \right) \in \text{SU}(2), \quad \alpha_a \in \mathbb{R}, \quad a = 1, 2, 3 \quad (3.2.17)$$

in terms of the Pauli matrices σ^a , recall (2.1.39) (without the identity), we clearly see that (3.2.15) is invariant since $UU^\dagger = \mathbb{I}_2$.

3.2.3 Types of symmetries

There are many types of internal symmetries that we will consider next.

In general we distinguish the following types of symmetries:

a) *Spacetime or internal*. As we have already mentioned above, a field $\Phi_\alpha(x)$ transforms under a spacetime transformation $x' = \Lambda x + a$ according to

$$\Phi_\alpha^i(x) \rightarrow U^\dagger(\Lambda, a) \Phi_\alpha^i(x') U(\Lambda, a) = D_\alpha{}^\beta \Phi_\beta^i(x') \quad (3.2.18)$$

where $D_\alpha{}^\beta$ are representation matrices for the Lorentz group.

Under an internal transformation $g(\alpha_a) \in G$, $a = 1, \dots, \dim(G)$, a field $\Phi_\alpha^i(x)$ transforms instead through

$$\Phi_\alpha^i(x) \rightarrow U^\dagger(\alpha_a) \Phi_\alpha^i(x) U(\alpha_a) = g^i{}_j \Phi_\alpha^j(x) \quad (3.2.19)$$

where $g^i{}_j$ are representation matrices of the internal symmetry group G .

b) *Continuous or discrete*. E.g., $\phi \rightarrow e^{i\alpha}\phi$ for a $U(1)$ -symmetry or $\phi \rightarrow -\phi$ corresponding to a Z_2 - symmetry.

For instance, consider a real scalar field with

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 + g\phi^3 - \lambda\phi^4. \quad (3.2.20)$$

Imposing the Z_2 -symmetry $\phi \rightarrow -\phi$ forbids the presence of the cubic term $\sim g\phi^3$.

On the other hand, for a $U(1)$ -symmetry $\phi \rightarrow e^{ie\alpha}\phi$ of a complex scalar ϕ , we have a Lagrangian of the form

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2 - \lambda |\phi|^4. \quad (3.2.21)$$

The potential for $m^2 > 0$ is depicted on the left and for the case $m^2 < 0$ on the right in Fig. 3.3.

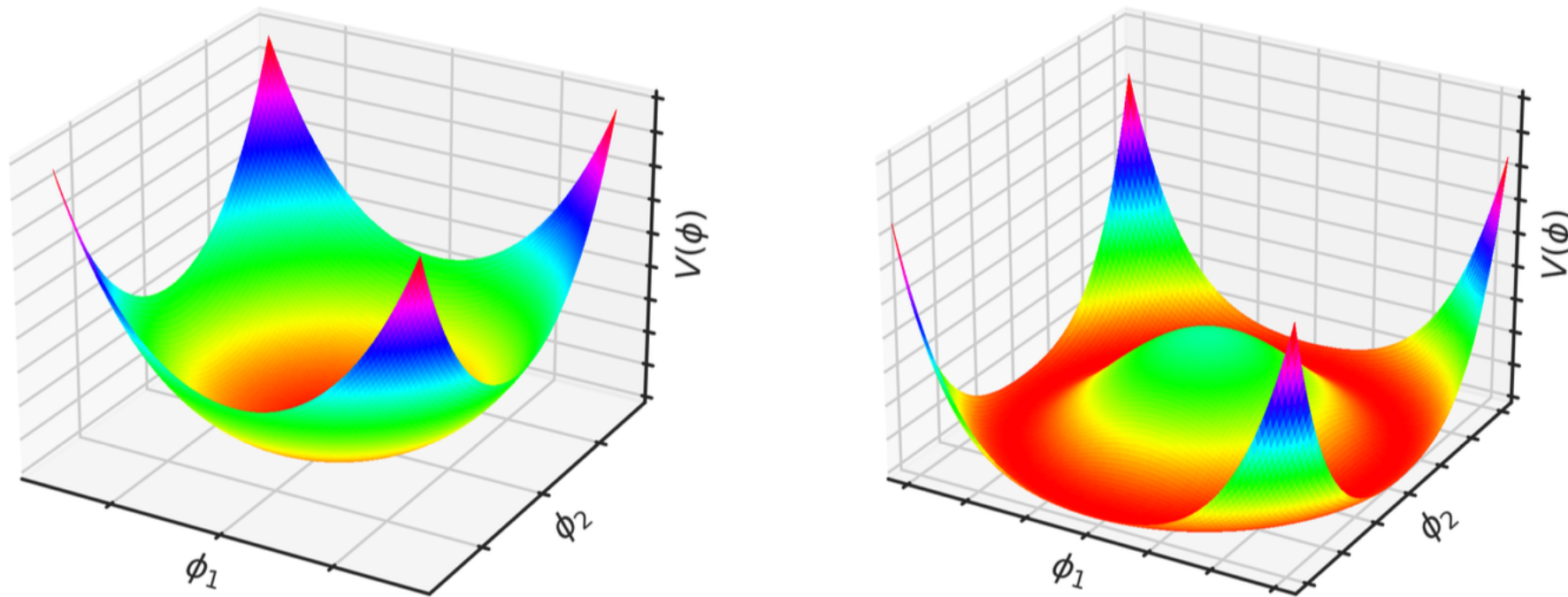


Figure 3.3: Left: Plot of the scalar potential (3.2.21) in the unbroken phase for $m^2 > 0$ with a minimum at the origin. The potential clearly respects the proposed $U(1)$ -symmetry. Right: Spontaneously broken phase for $m^2 < 0$ in (3.2.21) with a maximum at the origin. Projection onto the ϕ -plane shows that the $U(1)$ -symmetry is intact even in the “broken” phase

c) *Global or local*. In the case of a U(1)-symmetry $\phi \rightarrow e^{ie\alpha}\phi$, we distinguish

$$\begin{cases} \alpha = \text{const.} & \text{global symmetry,} \\ \alpha = \alpha(x) & \text{local symmetry.} \end{cases}$$

The latter case of a local symmetry leads to a modification of (3.2.21), since the original kinetic term is not invariant $\partial_\mu \phi \rightarrow e^{ie\alpha} \partial_\mu \phi + ie \partial_\mu \alpha \phi$.

A possible modification of the Lagrangian is to introduce a new field A_μ such that its transformation compensates for the lack of invariance of the kinetic term.

That is, we write instead

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2 |\phi|^2 - \lambda |\phi|^4. \quad (3.2.22)$$

where we define the covariant derivative with an additional term $\sim A_\mu \phi$ as

$$D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi. \quad (3.2.23)$$

Under the U(1)-action, A_μ transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \quad (3.2.24)$$

and $\mathcal{L}$ is modified in order to add kinetic terms for the A_μ fields according to

$$\mathcal{L}' = \mathcal{L} + F^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.2.25)$$

The interactions between A_μ and ϕ are hidden in the kinetic term $D_\mu\phi D^\mu\phi^*$ for ϕ .

Also for Dirac fields, the Lagrangian (3.2.9) gets modified for a local U(1) such that

$$\mathcal{L} = \bar{\psi} \not{D} \psi - m \bar{\psi} \psi \quad (3.2.26)$$

where the kinetic term is again modified to

$$\bar{\psi} \not{D} \psi = \bar{\psi} \not{\gamma}^\mu D_\mu \psi, \quad D_\mu = \partial_\mu - ie A_\mu. \quad (3.2.27)$$

This can be seen by considering the transformation property of $\bar{\psi} \not{D} \psi$, that is

$$\psi(x) \rightarrow e^{ie\alpha(x)} \psi(x) \quad \Rightarrow \quad \bar{\psi} \not{D} \psi \rightarrow \bar{\psi} \not{D} \psi - e(\bar{\psi} \gamma^\mu \psi) \partial_\mu \alpha \quad (3.2.28)$$

which is clearly not invariant.

On the other hand, the modified kinetic term is indeed gauge invariant,

$$\bar{\psi} \not{D} \psi \rightarrow \bar{\psi} \not{D} \psi - e(\bar{\psi} \gamma^\mu \psi) \partial_\mu \alpha + e \bar{\psi} \not{A} \psi + e(\bar{\psi} \gamma^\mu \psi) \partial_\mu \alpha = \bar{\psi} \not{D} \psi. \quad (3.2.29)$$

We will see later that these Lagrangians describe interactions of matter fields ϕ, ψ to spin 1 fields A_μ like the photon.

d) *Manifest or hidden*. A symmetry is manifest if the vacuum state, or state of minimum energy, shares the same symmetries of the theory. Fig. 3.3 shows on the left the scalar potential with a $U(1)$ symmetry where the vacuum state is the origin which respects the symmetry.

On the other hand, a symmetry is hidden if the vacuum state does not share the symmetry of the Lagrangian.

Hidden symmetries are usually called spontaneously broken which may be a misleading term.

The symmetry is not actually broken, but just not respected by the vacuum state.

For an observer living in such a vacuum, the symmetry is in that sense hidden.

However, even though the observer can hardly detect the symmetry, it remains intact from a global point of view, see the plot on the right of Fig. 3.3.

e) *Anomalous or non-anomalous (exact)*. A symmetry is referred to as being anomalous whenever it is realized in the classical theory, but gets broken by quantum corrections.

In that sense, the notion of anomalies is crucial in understanding the quantum theory.

Not surprisingly, the Standard Model itself is free of (gauge) anomalies, which is a very strong consistency test, see section 5.6.

In general, the concept of anomalies is a well established guiding principle in building new theoretical models of particle physics and quantum gravity.

f) *Real or accidental*. It may happen that the Lagrangian consistent with a set of symmetries may accidentally have extra symmetries that were not imposed on it.

This will happen in the Standard Model with symmetries such as the conservation of lepton or baryon number.

g) *Compact or non-compact*. The Poincaré group is non-compact which is forced upon us by special relativity.

Since quantum mechanics tells us to work with unitary representations, we are naturally lead to work with infinite dimensional representations of non-compact groups.

On the other hand, we typically restrict to compact internal symmetry groups which allows us to restrict to finite dimensional representations.

h) *Abelian or non-Abelian*. E.g., $U(1)$ or $SU(N)$, $SO(N)$, $Sp(2N)$, G_2 , F_4 , E_6 , E_8 .

All these groups are characterized by Dynkin diagrams in the Cartan classification of simple Lie groups.

3.2.4 Noether's theorem

An important result related to the presence of symmetries in a theory is *Noether's theorem*.

Noether's theorem

If the action $S = \int d^4x \mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha]$ has a continuous symmetry for $\Phi_\alpha \simeq \Phi_\alpha \rightarrow \Phi'_\alpha$, then there exists a current J^μ that is conserved when the field equations are satisfied (sometimes referred to as the current is conserved on-shell), that is, $\partial_\mu J^\mu = 0$, and the corresponding charge $Q = \int d^3x J^0$, is a constant of motion ($dQ/dt = 0$).

Let us review the arguments entering the proof.

We consider a general field transformation

$$\Phi_\alpha \rightarrow \Phi_\alpha + \Delta_\alpha \quad (3.2.30)$$

that leaves the action invariant, i.e.,

$$\delta S = 0 \quad \Rightarrow \quad \delta \mathcal{L} = \partial_\mu F^\mu(\Phi_\alpha) \quad (3.2.31)$$

where F^μ are some arbitrary functions so that the Lagrangian density transforms as a total derivative.

The variation of the Lagrangian is then given by

$$\delta \mathcal{L} = \left[\frac{\partial \mathcal{L}}{\partial \Phi_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \right] \Delta_\alpha + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \Delta_\alpha \right) \quad (3.2.32)$$

Then, as long as the equations of motion are satisfied, the first term cancels so that

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \Delta_\alpha - F^\mu, \quad \partial_\mu J^\mu = 0. \quad (3.2.33)$$

If the Lagrangian density (and not only the action) is invariant under the symmetry transformations $\delta L = 0$, then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} \Delta_\alpha \quad (3.2.34)$$

and

$$Q = \int d^3x J^0 = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_\alpha} \Delta_\alpha . \quad (3.2.35)$$

This charge Q is conserved since it satisfies

$$\frac{dQ}{dt} = \int d^3x \partial_t J^0 = - \int d^3x \nabla \cdot \mathbf{J} = 0 \quad (3.2.36)$$

where in the last step we have assumed that the current $\mathbf{J}$ vanishes at spatial infinity and falls off fast enough.

Noether's theorem expresses the importance of symmetries for physical observables.

It essentially states that the existence of a continuous symmetry implies a conservation law.

We may identify the most important physical quantities such as energy, momenta, electric charge, etc., as those that are conserved due to the existence of symmetries: time translations for energy, space translations for momenta, rotations for angular momenta.

For electric charge the corresponding symmetry is an internal $U(1)$ symmetry.

Similar observations hold e.g. for baryon and lepton number, although their associated $U(1)$ symmetry is global, whereas for electric charge it is local.

We will discuss these symmetries in future chapters.

3.2.5 Charges as generators

The statement of Noether's theorem above is valid in classical physics.

We will see now that QFT adds another layer of importance to Noether's theorem by interpreting the conserved charges as operators.

It is a general fact that Noether charges themselves generate the symmetry underlying their conservation in a quantum theory which is why Noether's theorem remains indispensable even in QFT.

The proof relies on the fact that in QFT the fields Φ_α are operators determined by the creation and annihilation operators.

It further uses the canonically conjugate fields

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_\alpha} . \quad (3.2.37)$$

The canonical commutation relations among conjugate variables at equal time are

$$[\Phi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} , \quad (3.2.38)$$

$$[\Phi_\alpha(\mathbf{x}, t), \Phi_\beta(\mathbf{y}, t)] = [\Pi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)] = 0 . \quad (3.2.39)$$

They determine Q above as an operator since (3.2.35) implies that the canonical momenta Π_α are essentially the Noether charges.

We can then extract from this the commutation relations between the conserved charges and the fields Φ_α through

$$Q = \int d^3x \Pi_\alpha \Delta_\alpha \quad \Rightarrow \quad [\Phi_\alpha(\mathbf{x}, t), Q] = i\Delta_\alpha(\mathbf{x}, t) . \quad (3.2.40)$$

This shows that the conserved charges Q of Noether's theorem act as generators of the *corresponding symmetry* $\Phi_\alpha \simeq \Phi_\alpha + \Delta_\alpha$.

For a general internal symmetry with Lie group G , we consider the infinitesimal transformation

$$\Phi_\alpha^i \rightarrow \Phi_\alpha^i + i\alpha^a (T_a)^i{}_j \Phi_\alpha^j , \quad (3.2.41)$$

with parameters α_a and generators T_a , $a = 1, \dots, \dim(G)$.

The conserved charges are the operators that act on the fields Φ_α^i according to the transformation generated by the generator T_a .

It is easily verified that the charge associated to the transformation by T_a satisfies

$$[\Phi_\alpha^i, Q_a] = (T_a)^i{}_j \Phi_\alpha^j \quad (3.2.42)$$

with the generators of the symmetry group and act as generators of the corresponding symmetry.

To see it more explicitly, we may exponentiate the above expression to find

$$U^\dagger \Phi_\alpha^i U = \left(e^{i\alpha^a T_a} \Phi_\alpha \right)^i, \quad U = e^{i\alpha^a Q_a}. \quad (3.2.43)$$

For an internal U(1)-symmetry with $\psi \simeq e^{ie\alpha}\psi$, the conserved current is $j_\mu = e\bar{\psi}\gamma_\mu\psi$ and the conserved charge is the electric charge e that can be seen as the generator of the U(1) symmetry.

Consider now spacetime translations $x^\mu \simeq x^\mu + a^\mu$ with current $T^\mu{}_\nu$ being the stress-energy tensor.

The corresponding charges are given by

$$P^0 = E = \int d^3x T^{00}, \quad P^i = \int d^3x T^{0i}. \quad (3.2.44)$$

For rotations, one can compute the charges

$$M^{ij} = \int d^3x \left(x^i T^{0j} - x^j T^{0i} \right). \quad (3.2.45)$$

Thus, the conserved charges themselves play the role of generators of the Poincaré group as we have seen previously in section 2.1.2.

3.3 Effective Field Theories in a nutshell

Up to this point, we collected all the necessary tools to build arbitrary QFTs in 4 dimensions and compute S-matrix elements that lead to observable quantities such as cross sections and decay rates.

Before we come to building up the Standard Model, let us make general remarks about how QFTs can be used to understand physics at different energy scales by introducing *Effective Field Theories* (EFTs).

The introduction of EFTs has been one of the most important theoretical developments in the past 50 years.

It is only relatively recently that their full power has been fully appreciated.

They provide a systematic way to organize our understanding of nature at different energy scales, from low to high energies.

Indeed, we experience the world always through an effective low-energy description.

Even if we know a more fundamental description, we only need to take into account the behavior of the degrees of freedom that are accessible at the energy scale of our experiments when describing low-energy phenomena.

To study the properties of water in everyday life, for instance, we do not need to start from the Lagrangian of QCD or understand quantum gravity.

A more relevant example for the purposes of this lecture is QED where we focus only on photons and electrons, while neglecting all other heavier particles.

This is a valid approximation at energies $E < 2m_e$ with $m_e = 511 \text{ keV}$.

In this case we know the theory at higher energies, at least in a path integral prescription.

If we are interested only in the low-energy physics, we integrate out the heavy states to obtain a theory only for the low-energy states (electrons and photons for instance) defining the corresponding EFT.

More generally, even if we do not know the theory at high energies, we can identify the relevant low-energy states and write down an EFT.

This effective description is then capable of accurately describing physical phenomena at low energies.

While these types of calculations can be explicitly performed in e.g. perturbation theory, in the case of the Standard Model, we do not even know precisely which new degrees of freedom have to be added at higher energies.

Said differently, it is almost impossible to backtrack the process of integrating out modes without knowing the full spectrum in the UV.

Luckily, it is irrelevant for many purposes: at low enough energies, theories can be immensely predictive even if we do not have the full information about a given theory at hand.

In other words, we do not need quantum gravity to describe a cup of tea.

It is this basic principle that makes *effective field theories so powerful*.

3.3.1 Interactions: organizing physics by energy scales

Let us now describe how physics can be organized by energy scales.

Starting from some theory at high energies described by a path integral of the form (3.1.25), we can integrate out all the heavy degrees of freedom above a given energy scale Λ and treat the remaining fields in an effective description.

This typically implies that this description involves less degrees of freedom than the theory from which we originally started in the UV.

In practice, when we write down a quantum field theory that should be predictive at energies $E \leq \Lambda$ below some cutoff Λ , the most basic question we might ask is which operators $\mathcal{O}_i$ to include in the expansion (3.1.23) for the Lagrangian of a QFT.

That is, at low energies, our theory should be represented by an *effective Lagrangian*

$$\mathcal{L}_{\text{eff}}[\Phi_\alpha, \partial\Phi_\alpha] = \sum_i c_i \mathcal{O}_i(\Phi_\alpha, \partial\Phi_\alpha). \quad (3.3.1)$$

Here, c_i are some “constant” coefficients, $\mathcal{O}_i$ operators and $\forall$ all fields in our theory.

The question about which operators $\mathcal{O}_i$ to include in (3.3.1) can be addressed by noticing that there exists an ordering principle for the operators $\mathcal{O}_i$.

Indeed, it turns out that the level of importance of the operators $\mathcal{O}_i$ depends on their dimensionality and the energies which we are interested in exploring.

Since the action S is dimensionless, we can determine the mass dimension of $\mathcal{L}_{\text{eff}}$ as

$$S_{\text{eff}}[\Phi_\alpha, \partial\Phi_\alpha] = \int \mathcal{L}_{\text{eff}}[\Phi_\alpha, \partial\Phi_\alpha] d^4x \quad \Longrightarrow \quad [\mathcal{L}_{\text{eff}}] = 4. \quad (3.3.2)$$

Operators $\mathcal{O}_i$ of dimensions $d_i = [\mathcal{O}_i]$ fall into three categories:

- a) *Irrelevant*: c_i becomes smaller at lower energies: $d_i > 4$, $[c_i] < 0$;
- b) *Relevant*: c_i increases at lower energies: $d_i < 4$, $[c_i] > 0$;
- c) *Marginal*: c_i does not change with the energy scale: $d_i = 4$, $[c_i] = 0$.

The coefficients with negative dimensionality would naturally be suppressed by powers of a UV scale Λ and would then be less relevant if we are interested in the physics at scales $E \ll \Lambda$.

Thus, we call a theory

- *Renormalizable* if

$$[c_i] \geq 0 \quad \forall i. \quad (3.3.3)$$

This is quite restrictive for the simple reason that

$$d_i = [\mathcal{O}_i] = 4 - [c_i] \geq 0. \quad (3.3.4)$$

This implies that in a renormalizable theory only a few c_i are non-zero and, hence, the theory is immensely predictive: only those few coefficients c_i have to be matched with experiments at energies $E \ll \Lambda$.

- *Non-renormalizable* if

$$[c_i] \leq 0 \quad \text{for some } i. \quad (3.3.5)$$

Then, the coefficients c_i scale with the characteristic energy scale Λ of our theory as

$$c_i \sim \Lambda^{4-d_i}. \quad (3.3.6)$$

We distinguish the following scenarios where E is a typical energy of the theory being studied:

- if $E \ll \Lambda$, it is generically sufficient to keep only a few operators $\mathcal{O}_i$ and the EFT becomes predictive.
- if $E \sim \Lambda$, we have to include infinitely many operators and we lose predictive power. Thus, we need a UV completion of our theory.

Clearly, there can be, in principle, infinitely many such coefficients because $d_i = 4 - [c_i] \geq 0$ is always satisfied.

Notice that for a non-renormalizable theory, the scale of new physics Λ may be very large and therefore the theory may be predictive for a large range of energies E as long as $E \ll \Lambda$.

A typical example that illustrates renormalizable and non-renormalizable theories is to consider the following Lagrangian for a real scalar field

$$\mathcal{L}_{\text{eff}} = \underbrace{\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 - g \phi^3 - \lambda \phi^4}_{\text{Renormalisable}} + \underbrace{\frac{\alpha}{\Lambda} \phi^5 + \frac{\beta}{\Lambda^2} \phi^6 + \dots}_{\text{Non-Renormalisable}} \quad (3.3.7)$$

The first four terms define a renormalizable theory which has predictive power due to having only 3 arbitrary couplings m, g, λ .

Beyond that, adding operators of higher dimensionality would make the theory non-renormalizable, while keeping a few terms would define an EFT valid for energies $E \ll \Lambda$.

Otherwise the theory breaks down at energies close to Λ and would need to be substituted by either a new EFT valid at higher energies or, ultimately, by an ultra-violet (UV) complete theory.

Let us mention the 4-Fermi theory as an effective description for the weak interactions at energies $E \ll 80 \text{ GeV}$ as one prominent example that we will discuss in detail in part 6.

3.3.2 General relativity as an EFT*

Einstein's gravity is an example of such theory that needs to be UV completed.

Einstein's gravity can be treated quantum mechanically as long as it is an EFT addressing questions at low energies (meaning $E \ll M_{\text{P}} \simeq 10^{18} \text{ GeV}$).

Einstein's theory of *General Relativity* (GR) is described by the *Einstein-Hilbert (EH)* term

$$\mathcal{L}_{EH} = M_P^2 R^{(4)} \sqrt{-g}, \quad M_P^2 = \frac{\hbar c}{G_N} \quad (3.3.8)$$

in terms of the 4-dimensional Ricci scalar $R^{(4)}$.

The coupling constant G_N has negative mass dimension and is therefore non-renormalizable.

Alternatively, we may expand the Ricci scalar $R^{(4)}$ in terms of fluctuations of the metric around a constant Minkowski background, that is,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu} \Rightarrow M_P^2 R^{(4)} = (\partial h)^2 + \frac{h}{M_P} (\partial h)^2 + \frac{h^2}{M_P^2} (\partial h)^2 + \dots \quad (3.3.9)$$

Thus, an infinite number of counterterms would be necessary in perturbation theory.

Hence, the theory does not admit a continuum limit, but has an intrinsic cutoff set by the Planck scale M_P .

Having said that, non-renormalizability does not constitute an obstruction to making reliable perturbative quantum calculations in gravity as long as we limit our considerations to energies μ well below M_P ,

$$\mu \ll M_P = \sqrt{\frac{\hbar c}{G_N}} \sim 10^{19} \text{GeV}. \quad (3.3.10)$$

APPENDIX A - CROSS SECTIONS AND DECAY RATES

In this appendix we summarize basic formulas to compute cross sections and decay rates which are the final quantities that can be explicitly computed from the full theoretical formalism and at the end are compared with experiments.

This is standard QFT material that is otherwise only partially covered in this class because we did it all in prior classes..

Particle physics experiments are some of the most technically complex machines we ever built.

Yet we ask them very basic questions like

“How frequently does X decay to products $\omega + \varepsilon + \vartheta + \dots$?”

or

“Given N collisions between beams of A and B particles,
how many times do we produce particle X ?”

From these measurements, we determine the free parameters of the Standard Model or even quantify deviations from it.

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A.1 From Correlation Functions to Scattering Amplitudes

As often mention in these lectures, the relevant quantity is the S-matrix for scattering between an initial state α and final state β .

In the case of decays just described, $\alpha = X$ and we are interested in inelastic scattering, where β has different particle content than α .

In general, the S-matrix elements are given by Dyson's formula

$$\langle \beta | S | \alpha \rangle = \lim_{t_{\pm} \rightarrow \pm\infty} \langle \beta | U(t_+, t_-) | \alpha \rangle \quad (\text{A.1.1})$$

with

$$U(t_+, t_-) = \mathcal{T} \exp \left(-i \int_{t_-}^{t_+} dt' H_I(t') \right) . \quad (\text{A.1.2})$$

Here, the S-matrix itself is defined as

$$S = \mathcal{T} \exp \left(i \int d^4x \mathcal{L}_I \right) . \quad (\text{A.1.3})$$

The S matrix can be separated into a boring part (where nothing happens) and an interesting part (the T-matrix)

$$S = 1 + i T . \quad (\text{A.1.4})$$

Due to momentum conservation, we can define the invariant amplitude $\mathcal{M}$ as

$$\langle \beta | S - 1 | \alpha \rangle = (2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) i \mathcal{M}_{\beta\alpha} . \quad (\text{A.1.5})$$

If we expand the S-matrix perturbatively, and perform the spacetime integrals, we are able to pull out the factor of $(2\pi)^4 \delta(p_\beta - p_\alpha) i$ on the right hand side ensuring momentum conservation.

The quantity $\mathcal{M}_{\beta\alpha}$ is defined in momentum space and can be computed to every order in perturbation theory by using the momentum space Feynman rules.

To see this, recall that the LSZ formula certifies that the relevant information in scattering amplitudes is encoded in fully connected correlation functions $\langle . . . \rangle^{\text{conn}}$.

As an example, let us write the LSZ formula for scattering of m scalar particles in $|\alpha\rangle$ into n new particles in $|\beta\rangle$ as

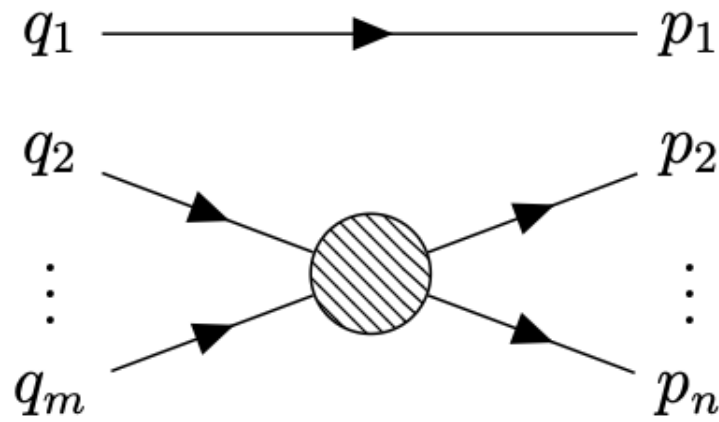
$$\langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_m, \text{in} \rangle = \langle p_1, \dots, p_n, \text{in} | S | q_1, \dots, q_m, \text{in} \rangle = \langle f | S | i \rangle \quad (\text{A.1.6})$$

$$= \text{disconnected terms} + (i)^{n+m} \int d^d y_1 \dots \int d^d y_n \int d^d x_1 \dots \int d^d x_m e^{i \sum_{k=1}^n p_k y_k - i \sum_{l=1}^m q_l x_l}$$

$$(\square_{y_1} + m^2) \dots (\square_{y_n} + m^2) (\square_{x_1} + m^2) \dots (\square_{x_m} + m^2) \langle \Omega | \mathcal{T} \phi(y_1) \dots \phi(y_n) \phi(x_1) \dots \phi(x_m) | \Omega \rangle$$

assuming that all particles have the same mass m .

The first term is associated with disconnected scattering processes where a subset of particles does not participate in the actual scattering such as



The true information about scattering events is encrypted in the fully connected second term.

After plugging in the Fourier transform for the fields, it can be written as

$$\begin{aligned} & \langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_m, \text{in} \rangle^{\text{conn}} \\ &= (i)^{n+m} \prod_{k=1}^n (-p_k^2 + m^2) \prod_{l=1}^m (-q_l^2 + m^2) \langle \Omega | \mathcal{T} \hat{\phi}(p_1) \dots \hat{\phi}(p_n) \hat{\phi}(q_1) \dots \hat{\phi}(q_m) | \Omega \rangle \end{aligned} \quad (\text{A.1.7})$$

Crucially, the momenta appearing on both sides are **on-shell** which means that

$$p_k^2 - m^2 = 0 \quad , \quad q_l^2 - m^2 = 0 . \quad (\text{A.1.8})$$

For the left hand side to be non-zero, the contributing correlation function on the right has to have a very specific pole structure.

Indeed, these are precisely the connected correlation functions which have $m+n$ poles at $+m^2$ because each external line contributes a propagator.

Due to the simple fact that the S-matrix measures quantum mechanical probabilities, the right hand side cannot have more poles than that.

Otherwise, it would be singular.

Thus, our final expression for the LSZ formula is

$$\langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_m, \text{in} \rangle^{\text{conn}} = \langle \Omega | \mathcal{T} \hat{\phi}(p_1) \dots \hat{\phi}(p_n) \hat{\phi}(q_1) \dots \hat{\phi}(q_m) | \Omega \rangle^{\text{conn}} \Big|_{\text{amputated}} \quad (\text{A.1.9})$$

where $\Big|_{\text{amputated}}$ gets rid of all external propagators by cutting off the corresponding lines.

To conclude, we reduced the problem of computing scattering amplitudes to computing amputated, connected correlation functions.

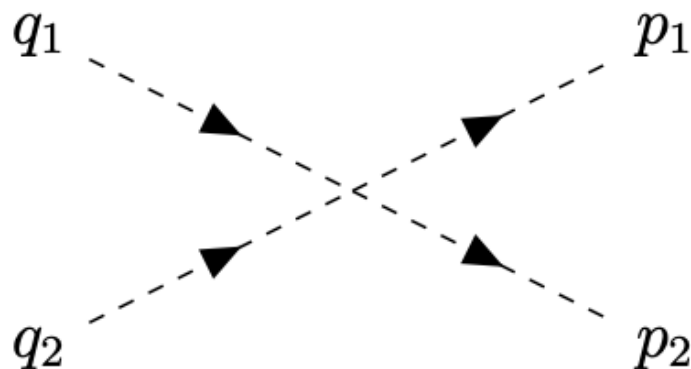
The above results can be shown to hold for more general theories involving different species of particles.

Superficially, this just amounts to introducing new labels for spins, polarizations etc.

As a simple example, we consider 2 - 2 scattering of particles in a ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \quad (\text{A.1.10})$$

At tree level to order $O(\lambda)$, there is a single connected diagram



First, we compute the 4-point correlation function in position space

$$\begin{aligned} \langle \Omega | \mathcal{T} \phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) | \Omega \rangle \Big|_{\lambda} &= \prod_{j=1}^4 \int \frac{d^4 k_j}{(2\pi)^4} e^{-ik_1 x_1} e^{-ik_2 x_2} e^{ik_3 x_3} e^{-ik_4 y_2} \\ &\quad (-i\lambda)(2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_3 - k_4) \prod_{l=1}^4 \frac{i}{k_l^2 - m^2} . \end{aligned} \quad (\text{A.1.11})$$

To find the S-matrix element, we take the Fourier transform

$$\begin{aligned} &\int dy_1 e^{-iy_1 q_1} \int dy_2 e^{-iy_2 q_2} \int dx_1 e^{ix_1 p_1} \int dx_2 e^{ix_2 p_2} \langle \Omega | \mathcal{T} \phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) | \Omega \rangle \Big|_{\lambda} \\ &= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \frac{i}{q_1^2 - m^2} \frac{i}{q_2^2 - m^2} . \end{aligned} \quad (\text{A.1.12})$$

Discarding all the propagators from external lines, we recover the connected component of the scattering amplitude as given by the right hand side of (A.1.9)

$$\langle p_1, p_2, \text{out} | q_1, q_2, \text{in} \rangle^{\text{conn}} \Big|_{\lambda} = (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) . \quad (\text{A.1.13})$$

From (A.1.5), we see that

$$\mathcal{M}_{q_1, q_2 \rightarrow p_1, p_2}^{\text{tree}} = -\lambda. \quad (\text{A.1.14})$$

Recalling (A.1.3) and (A.1.5), we could have simply used that at leading order in the perturbative expansion

$$\mathcal{M}_{\beta\alpha}^{\text{tree}} = \langle f | \mathcal{L}_I | i \rangle \quad (\text{A.1.15})$$

where $\mathcal{L}_I = \lambda\phi^4/4!$, but more generally includes all interaction vertices present in the theory.

It is then only a matter of applying suitable Feynman rules for in and out states which can be summarized as:

- for scalars:

$$\overline{\phi} | \phi(k) \rangle \rightarrow 1, \quad \langle \phi(k) | \phi \rightarrow 1. \quad (\text{A.1.16})$$

- for fermions:

$$\overline{f} | f(k) \rangle \rightarrow u_s(k), \quad \overline{\bar{f}} | \bar{f}(q) \rangle \rightarrow \bar{v}_s(q), \quad \langle \overline{f(k)} | \bar{f} \rightarrow \bar{u}_s(k), \quad \langle \bar{\bar{f}(q)} | \bar{f} \rightarrow v_s(q). \quad (\text{A.1.17})$$

- for vectors:

$$\overline{A_\mu} | A_\mu(k) \rangle \rightarrow \epsilon_\mu(\lambda, k), \quad \langle \overline{A_\mu(q)} | A_\mu \rightarrow \epsilon_\mu^*(\lambda, q). \quad (\text{A.1.18})$$

A.2 Decay Rates

Next, we may ask what is the probability that a state α decays into β .

The probability that we measure $\alpha \rightarrow \beta$ is given by the relevant S matrix element squared over the norm-squared for the initial and final states

$$\mathcal{P}(\alpha \rightarrow \beta) = \frac{|\langle \beta | S - 1 | \alpha \rangle|^2}{\langle \beta | \beta \rangle \langle \alpha | \alpha \rangle} \quad (\text{A.2.1})$$

where

$$\langle \alpha | \alpha \rangle = (2\pi)^3 2p_\alpha^0 \delta^{(3)}(0) = 2p_\alpha^0 V, \quad \langle \beta | \beta \rangle = \prod_{r \in \beta} (2p_r^0 V) \quad (\text{A.2.2})$$

Here, we work in finite spatial volume V to avoid dealing with subtleties regarding with non-normalizable states.

The probability the decay will occur is

$$\mathcal{P}(\alpha \rightarrow \beta) = \frac{|\mathcal{M}_{\beta\alpha}|^2}{2m_i V} (2\pi)^4 \delta^{(4)}\left(p_i - \sum_{r \in f} p_r\right) VT \prod_{r \in f} \frac{1}{2p_r^0 V}. \quad (\text{A.2.3})$$

In experiments, the momentum of the final state is never measured with infinite precision.

This means it needs to be integrated over the region in space corresponding to the precision of the detector or, alternatively, over all possible values for the momenta.

The partial decay rate for the process $\alpha \rightarrow \beta$ is then obtained by dividing the probability (A.2.3) by T and integrating over momenta.

Since the number of 1-particle states in the box with momentum in a momentum-space volume d^3p is $Vd^3p/(2\pi)^3$, the partial decay rate is

$$\Gamma(\alpha \rightarrow \beta) = \frac{1}{T} \int \sum_{\text{spins, pol., ...}} \mathcal{P}(\alpha \rightarrow \beta) \prod_{r \in \beta} \frac{V d^3 p_r}{(2\pi)^3} . \quad (\text{A.2.4})$$

Here, we sum over spin and polarization states which can typically not be measured by detectors directly.

The Lorentz-invariant the integral measure on the phase space for the final state β is given by

$$d\rho_\beta = (2\pi)^4 \delta^{(4)} \left(p_\alpha - \sum_{r \in \beta} p_r \right) \prod_{r \in \beta} \frac{d^3 p_r}{(2\pi)^3} \frac{1}{2p_r^0} . \quad (\text{A.2.5})$$

The partial decay rate then becomes

$$\Gamma(\alpha \rightarrow \beta) = \frac{1}{2m_\alpha} \int \sum_{\text{spins, ...}} |\mathcal{M}_{\alpha\beta}|^2 d\rho_\beta . \quad (\text{A.2.6})$$

The total decay rate is then simply

$$\Gamma_\alpha = \sum_\beta \Gamma(\alpha \rightarrow \beta) = \frac{1}{2m_\alpha} \sum_\beta \int \sum_{\text{spins, ...}} |\mathcal{M}_{\alpha\beta}|^2 d\rho_\beta . \quad (\text{A.2.7})$$

The following identities might come in handy when computing partial decay rates using (A.2.6):

- spin sum rules (with $m^2 = k^2$ and $m^2 = q^2$)

$$\sum_s u_s(k) \bar{u}_s(k) = \not{k} + m, \quad \sum_s v_s(q) \bar{v}_s(q) = \not{q} - m. \quad (\text{A.2.8})$$

- Trace identities

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_n}) = 0 \quad \text{for } n \text{ odd}, \quad (\text{A.2.9})$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}), \quad (\text{A.2.10})$$

$$\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = -4i \epsilon^{\mu\nu\rho\sigma}, \quad (\text{A.2.11})$$

$$\text{Tr} \not{k} \not{q} = 4 k \cdot q, \quad (\text{A.2.12})$$

$$\text{Tr} \gamma^5 \not{k} \not{q} = 0, \quad (\text{A.2.13})$$

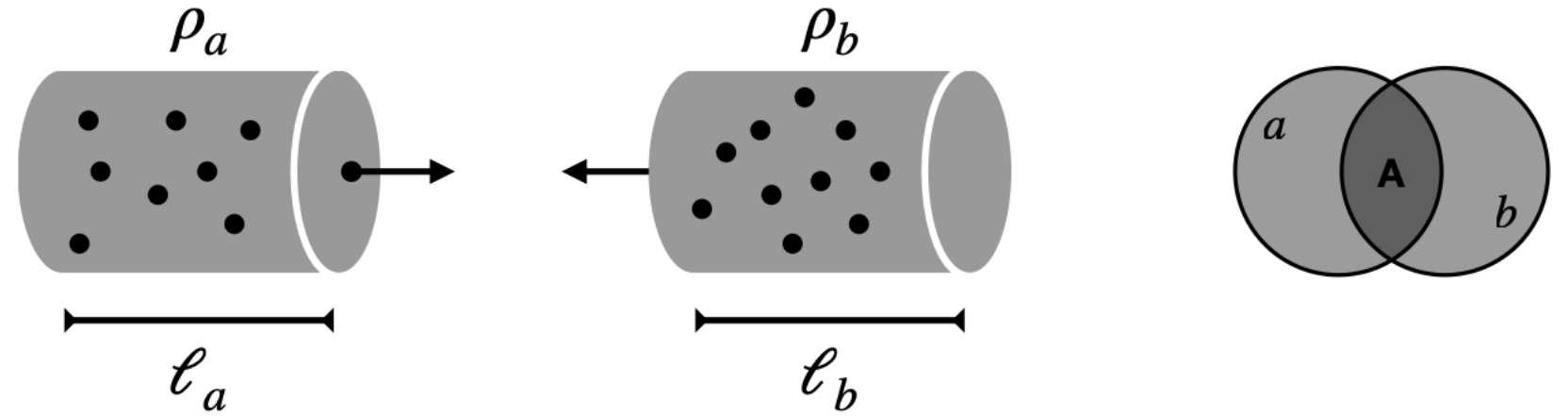
$$\text{Tr} \gamma^\mu = \text{Tr} \gamma^\mu \gamma^5 = 0. \quad (\text{A.2.14})$$

A.3 Cross Sections

In addition to knowing how often particles decay, many experiments in particle physics need to quantify how often particle collides.

Suppose we have two beams of particles and collide them as depicted in Fig. A.1.

Figure A.1: Two bunches of particles, with densities ρ_a, ρ_b and lengths ℓ_a, ℓ_b collide with cross-sectional area A .



The number of collision events can then be estimated through cross sections: they are obtained from the total number of scattering events divided by cross-sectional area of the collision region as well as the densities and size of the beams.

That is, we compute the quantity

$$\sigma = \frac{\# \text{ scattering events}}{\rho_a \ell_a \rho_b \ell_b A} = \frac{N}{F \rho_b V} \quad (\text{A.3.1})$$

where we introduced the number of

- target particles $\rho_b V$ in volume $V = \ell_b A$, and
- incoming particles per unit area per unit time $F = |\vec{v}_a - \vec{v}_b| \rho_a$ (also referred to as *incident flux*) in terms of the relative velocity of the particles in the two beams $|\vec{v}_a - \vec{v}_b|$.⁶⁹

The dimension of the cross-section σ in (A.3.1) is that of an area.

It is traditionally measured in the unit barn with $1 \text{ barn} = 10^{-28} \text{m}^2$

We are mostly interested in the differential cross sections: the beam of outgoing particles will be measured at different angles and we would like to understand the probability distribution of measuring scattered particles e.g. for different solid angle elements.

Specifically, we want to compute the differential probability per unit time of an event $\alpha \rightarrow \beta$.

We divide by the flux of particles through the interaction region.

For the particles moving in the lab frame, the prefactor $1/2m_\alpha$ in Γ in (A.2.6) becomes $1/2E$ for each beam of particles.

Thus, we find

$$\begin{aligned} d\sigma &= \frac{1}{F} \frac{1}{4E_a E_b V} |\mathcal{M}_{\beta\alpha}|^2 d\rho_\beta \\ &= \frac{1}{|\vec{v}_a - \vec{v}_b|} \frac{1}{4E_a E_b} |\mathcal{M}_{\beta\alpha}|^2 d\rho_f \end{aligned} \tag{A.3.2}$$

in terms of

$$dN = \frac{1}{4E_a E_b V} |\mathcal{M}_{\beta\alpha}|^2 d\rho_\beta . \tag{A.3.3}$$

The notion of differential cross sections is useful as it can reveal the substructures of particles just the atomic nucleus in Rutherford scattering.

In the previous case of 2 - 2 scattering in ϕ^4 , the leading order expression for the cross section obtained from (A.1.14) reads

$$\frac{d\sigma}{d\Omega_3} = \frac{1}{2!} \frac{1}{64\pi^2} \frac{1}{s} \lambda^2 \sim \frac{1}{s} \quad (\text{A.3.4})$$

in terms of the Mandelstam variable $s = (p_1 + p_2)^2$.

This result is very characteristic for scattering point-like objects and, in fact, holds more generally:

For a target with no substructure of length $l \geq 1/\sqrt{s}$, the differential cross-section for hard scattering falls off as $1/s$.

This behavior was critical to discover that hadrons have a parton structure in deep inelastic scattering experiments with hadrons.

A.4 π decay*

To conclude this appendix, we discuss the π^- meson decay to electron and anti-neutrino.

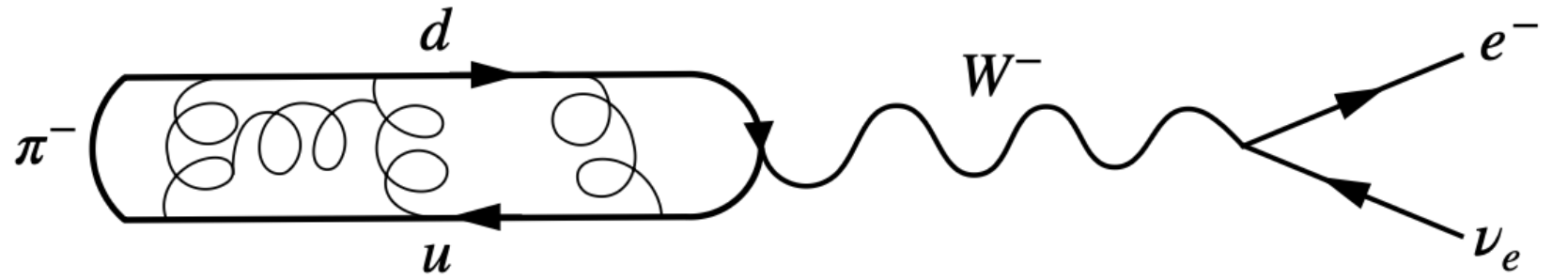
The process is similar to the muon's decay discussed in Sect. 6.4.8 since it is mediated through the charged weak currents in Eq. (6.4.65).

Specifically, a down quark d and an up anti-quark $\bar{u}$ annihilate to a W^- boson, which then promptly decays into electron and anti-neutrino.

Crucially, however, the initial state of the d-type and $\bar{u}$ -type quarks *does not correspond to freely propagating particles*; as we learned in Sect. 7.4, the quark and anti-quark are strongly bound together into a hadronic state which in the above case corresponds to a π^- -meson, or pion.

The decay is visualized in Fig. A.2.

Figure A.2: Weak decay of a pion to electron and anti-neutrino.



Let us start by collecting some important definitions.

Recall the definition (6.4.65) for the charges currents, namely

$$J_\mu^+ = \bar{\nu}_L^i \gamma_\mu e_L^i + \bar{u}_L^i \gamma_\mu d_L^i, \quad J_\mu^- = \bar{e}_L^i \gamma_\mu \nu_L^i + \bar{d}_L^i \gamma_\mu u_L^i. \quad (\text{A.4.1})$$

To isolate the relevant contributions for the above decay, we first define the *leptonic weak current*

$$J_{\text{had}}^\mu = V_{\text{had}}^\mu - A_{\text{had}}^\mu. \quad (\text{A.4.2})$$

This can be used to describe the decay $W^- \leftrightarrow e^- + \bar{\nu}_e$.

In addition, we need the hadronic weak current where it is convenient to separate the current into terms of definite parity, namely

$$J_{\text{had}}^\mu = V_{\text{had}}^\mu - A_{\text{had}}^\mu . \quad (\text{A.4.3})$$

Here, the vector and axial-vector currents in the mass eigenbasis for the quarks (recall (6.4.80)) are

$$\begin{aligned} V_{\text{had}}^\mu &= \bar{u} \gamma^\mu (V_{ud} d + V_{us} s + V_{ub} b) \\ A_{\text{had}}^\mu &= \bar{u} \gamma^\mu \gamma^5 (V_{ud} d + V_{us} s + V_{ub} b) . \end{aligned}$$

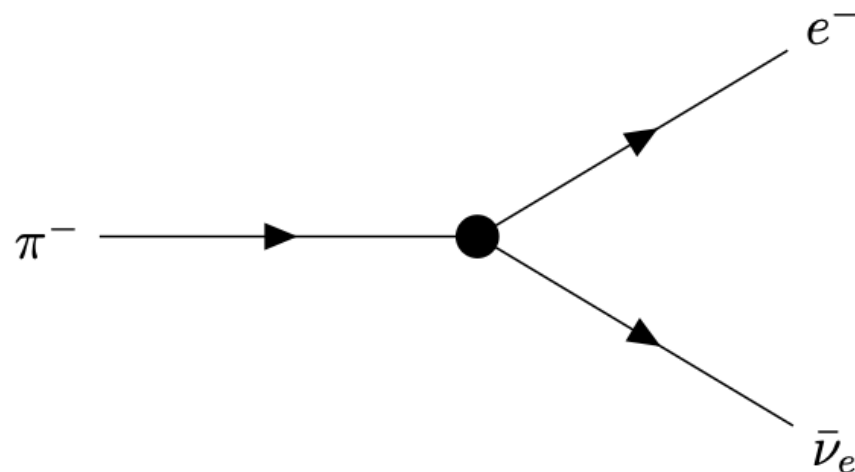
Here, we keep only the terms relevant for the π^- - decay.

For energies $E \ll m_W$, the effective Lagrangian from integrating out the $W^\pm$ - boson becomes

$$\mathcal{L}_W^{\text{eff}} = -\frac{4G_F}{\sqrt{2}} J_{\mu, \text{lept}} J_{\text{had}}^\mu . \quad (\text{A.4.4})$$

This is again of the form as in Eq. (6.2.5) as proposed by Marshak and Sudarshan.

At tree level, the relevant amplitude in the above approximation is simply depicted as



The vertex corresponds to the effective interaction in (A.4.4).

It is computed as follow

$$\begin{aligned}\mathcal{M} &= \langle e^-(k) \bar{\nu}_e(q) | \mathcal{L}_W^{\text{eff}} | \pi^-(p) \rangle \\ &= -\frac{G_F}{\sqrt{2}} \langle e^-(k) \bar{\nu}_e(q) | \bar{e} \gamma_\mu (1 - \gamma^5) \nu_e | 0 \rangle \langle 0 | J_{\text{had}}^\mu | \pi^-(p) \rangle .\end{aligned}\tag{A.4.5}$$

Since the pseudo-scalar meson π^- is by definition parity-odd, while the QCD vacuum is parity-even, the matrix element of J_{had}^μ only picks up the piece A_{had}^μ .

Hence, we obtain

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \bar{u}_e(k) \gamma_\alpha (1 - \gamma^5) v_{\nu_e}(q) \langle 0 | A_{\text{had}}^\alpha | \pi^-(p) \rangle .\tag{A.4.6}$$

QCD is strongly interacting and, in particular, free quarks are forbidden due to confinement .

Hence, approximating the hadronic matrix element perturbatively is a hopeless task.

Instead, we parametrize our ignorance in a single dimensionful parameter, the so-called pion decay constant F_π , such that

$$\langle 0 | V_{ud} \bar{u} \gamma^\alpha \gamma^5 d | \pi^-(p) \rangle = i V_{ud} \sqrt{2} F_\pi p^\alpha .\tag{A.4.7}$$

Due to momentum conservation, we write $p = k + q$ to arrive at (the neutrino is massless)

$$\bar{u}_e(k) \not{k} = \bar{u}_e(k) m_e, \quad \not{q} v_{\nu_e}(q) = 0. \quad (\text{A.4.8})$$

Then we find

$$\mathcal{M} = iG_F F_\pi m_e V_{ud} \bar{u}_e(k) (1 - \gamma^5) v_{\nu_e}(q). \quad (\text{A.4.9})$$

Next, we want to derive the decay rate (A.2.6) for which we need the squared amplitude $|\mathcal{M}|^2$ and sum over all spins.

We therefore compute

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 2|G_F F_\pi m_e V_{ud}|^2 \text{Tr} \left[(\not{k} + m_e) (1 - \gamma^5) \not{q} \right] \quad (\text{A.4.10})$$

having used $(1 - \gamma^5) \gamma^\mu (1 + \gamma^5) = 2(1 - \gamma^5) \gamma^\mu$.

Utilizing the trace identities (A.2.12), (A.2.13) and (A.2.14), we obtain

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 8|G_F F_\pi m_e V_{ud}|^2 k \cdot q. \quad (\text{A.4.11})$$

Therefore the decay rate in the π rest frame is

$$\begin{aligned}
\Gamma_{\pi \rightarrow e \bar{\nu}} &= \frac{1}{2m_\pi} \int \frac{d^3 k}{(2\pi)^3 2k^0} \frac{d^3 q}{(2\pi)^3 2q^0} (2\pi)^4 \delta^{(4)}(p - k - q) \sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{|G_F F_\pi m_e V_{ud}|^2}{m_\pi (2\pi)^2} \int \frac{d^3 k}{k^0} \frac{d^3 q}{q^0} \delta^{(4)}(p - k - q) (k \cdot q) \\
&= \frac{|G_F F_\pi m_e V_{ud}|^2}{4\pi^2 m_\pi} \int \frac{d^3 k}{E |\vec{k}|} \delta(m_\pi - E - |\vec{k}|) (E + |\vec{k}|) |\vec{k}|.
\end{aligned} \tag{A.4.12}$$

We now use the composition rule for the δ -distribution

$$\delta(f(k)) = \sum_i \frac{\delta(k - k_0^i)}{|f'(k_0^i)|}, \tag{A.4.13}$$

where k_0^i are the roots of $f(k) = 0$.

In our case, we have the roots

$$k_0 = \frac{m_\pi^2 - m_e^2}{2m_\pi}, \quad |f'(k_0)| = 1 + \frac{k_0}{E}. \tag{A.4.14}$$

Plugging this into the decay rate, we find

$$\begin{aligned}
\Gamma_{\pi \rightarrow e \bar{\nu}} &= |G_F F_\pi m_e V_{ud}|^2 \frac{1}{4\pi^2 m_\pi} \int_0^\infty \frac{4\pi |\vec{k}|^2 d|\vec{k}|}{E} (E + |\vec{k}|) \frac{\delta(|\vec{k}| - k_0)}{1 + \frac{k_0}{E}} \\
&= \frac{G_F^2 |F_\pi V_{ud}|^2}{4\pi} m_e^2 m_\pi \left(1 - \frac{m_e^2}{m_\pi^2}\right)^2.
\end{aligned} \tag{A.4.15}$$

A similar calculation for $\pi \rightarrow \mu \bar{\nu}_\mu$: yields

$$\Gamma_{\pi \rightarrow \mu \bar{\nu}} = \frac{G_F^2 |F_\pi V_{ud}|^2}{4\pi} m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \quad (\text{A.4.1.16})$$

One can take the ratio where the least well-known quantities cancel

$$\frac{\Gamma(\pi \rightarrow e \bar{\nu}_e)}{\Gamma(\pi \rightarrow \mu \bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2 = 1.28 \times 10^{-4}. \quad (\text{A.4.1.17})$$

Experimentally, the ratio is measured to be $1.230(4) \times 10^{-4}$.

The small deviations must come from *quantum effects*, i.e., loop diagrams.

PART 4 GAUGE THEORIES

Symmetries and Inevitability

There is one common feature that gives both general relativity and the standard model most of their sense of inevitability and simplicity: they obey principles of symmetry.

Steven Weinberg

This chapter presents an overview of local symmetries.

The main message is that the celebrated gauge symmetries are nothing but redundancies of a theory in order to describe massless particles of helicity greater or equal than 1.

We emphasize that, even though there is some freedom in describing interacting theories for spin/helicity 0, 1/2, there are strong constraints for higher helicities (essentially because the dimension of little group representations is always 2), recall Sect. 2.3.

This is important towards a proper description of the Standard Model: we will see in this part that only massless particles of helicities 0, 1/2, 1, 3/2, 2 can exist as interacting theories.

Further, we argue that helicity 1 only allows QED or Yang-Mills theories, whereas helicities 2 only gravity.

This limits substantially the options to build the Standard Model.

Therefore, once we formulate the Standard Model in terms of these theories for massless fields, it is not because we will make a particular choice of theory, but it is the only option we have.

There are no alternatives as long as quantum mechanics and the symmetries of special relativity are valid.

In other words, the beauty of the basic principles behind the Standard Model is not because symmetries are beautiful, it is because there is a sense of inevitability.

Things cannot be otherwise.

In this part we also present a few concepts which may have been introduced in other courses earlier, but which are crucial to construct the Standard Model.

This also allows us to set our notation and conventions.

Throughout, we use natural units, $\hbar = c = 1$.

4.1 The Origin of Gauge (Local) Symmetries

Initially, let us introduce gauge redundancies.

They are essentially the price that we have to pay when trying to describe massless helicity-1 fields in an off-shell formalism.

4.1.1 Gauge symmetries from Lorentz invariance

To begin with, we consider a real spin/helicity-1 field described by

$$A_\mu(x) = \sum_\lambda \int dp \left[\epsilon_\mu a(p^\nu, \lambda) e^{ipx} + \epsilon_\mu^* a^\dagger(p^\nu, \lambda) e^{-ipx} \right]. \quad (4.1.1)$$

Here, ϵ_μ , ϵ_μ^* are the polarization vectors as the objects carrying the Lorentz index and describing the propagation of the fields in spacetime, while λ represents helicity for massless particles or spin for massive ones.

Moreover, recall that we use the notation

$$\int dp \equiv \int \frac{d^4 p}{(2\pi)^4} \delta(p^2 - m^2) \Theta(p^0) = \int \frac{d^3 p}{E_p (2\pi)^3}, \quad E_p^2 = \mathbf{p}^2 + m^2. \quad (4.1.2)$$

As it stands, A_μ has in total 4 degrees of freedom given by $\mu = 0, 1, 2, 3$.

However, we learnt in section 2.3 that the corresponding 1-particle states have either 3 degrees of freedom for massive particles or 2 for massless.

Hence, we need extra constraints:

- Massive case: we impose the Lorentz invariant condition

$$p^\mu \epsilon_\mu = 0 \quad (4.1.3)$$

which reduces the number of degrees of freedom to 3.

This is the only Lorentz invariant quantity that can be written to constrain the polarization vector ϵ_μ and the other ingredient at hand, namely the momenta p_μ .

It successfully reduces the number of degrees of freedom from 4 to 3 agreeing with the degrees of freedom of a massive particle.

- *Massless case*: apart from (4.1.3), there are no more Lorentz invariant constraints.

However, since in the massless case $p^\mu p_\mu = 0$, there is the following ambiguity

$$\epsilon_\mu \underbrace{\equiv}_{\text{same state}} \epsilon_\mu + \hat{\alpha}(p)p_\mu, \quad (4.1.4)$$

where $\hat{\alpha}(p)$ is an arbitrary function of the momenta.

Both sides clearly satisfy (4.1.3).

This implies that actually ϵ_μ is not a Lorentz vector.

The arbitrariness in the parameter $\hat{\alpha}$ reduces the number of degrees of freedom by 1, leading to the 2 degrees of freedom needed to describe massless particles.

This implies a similar equivalence relation for the field in position space (4.1.1) of the form

$$A_\mu(x) \equiv A_\mu(x) + \partial_\mu \alpha(x) \quad (4.1.5)$$

where $\alpha(x)$ is the Fourier transform of $\hat{\alpha}(p)$.

This is referred to as gauge invariance which simply corresponds to a mathematical redundancy in our description of physics. Generally, one can state that

A proper Lorentz invariant description of physical amplitudes
for massless helicity-1 fields *implies* gauge invariance.

Notice that we have already seen gauge transformations $A_\mu(x) \equiv A_\mu(x) + \partial_\mu \alpha(x)$ in Sect. 3.2 in the context of local symmetries.

For matter fields $\psi(x)$ transforming as $\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$, this defines a U(1) transformation.

Their gauge invariant kinetic energy is $\bar{\psi} i \gamma^\mu D_\mu \psi$ involves the covariant derivative $D_\mu = \partial_\mu - ie A_\mu$ provided that A_μ transforms through gauge transformations as above.

This determines the coupling of the matter field ψ to the gauge field A_μ , namely $A_\mu J^\mu = \bar{\psi} \gamma^\mu A_\mu \psi$ with J^μ a conserved current.

The physical quantity to look at is in fact the field strength tensor (just as in electrodynamics) given by

$$F_{\mu\nu}(x) = i \sum_{\lambda=\pm 1} \int dp [\epsilon_\mu p_\nu - \epsilon_\nu p_\mu] a(p^\mu, \lambda) e^{ipx} + \text{h.c.} \quad (4.1.6)$$

which is invariant under $\epsilon_\mu \rightarrow \epsilon_\mu + \hat{\alpha}(p)p_\mu$ and, after using (4.1.1), amounts to

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) . \quad (4.1.7)$$

In electromagnetism $F_{0i} = E_i$ and $\epsilon_{ijk}F_{jk} = B_i$ with E_i, B_i the components of the electric and magnetic fields, respectively. $F_{\mu\nu}$ carries the degrees of freedom of a helicity 1 field.

This is already apparent from the decomposition $(1/2, 1/2) \otimes (1/2, 1/2) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1)$ where the latter has 2 indices and $(1, 0) \oplus (0, 1)$ is the anti-symmetric component corresponding to $F_{\mu\nu}$ that carries helicity ± 1 (the $(0, 0)$ state is a scalar of helicity 0 and the $(1, 1)$ state is a helicity 2 state).

Notice that neither ϵ_μ nor A_μ are Lorentz vectors since

$$A_\mu \rightarrow \Lambda_\mu{}^\nu A_\nu + \partial_\mu \alpha \quad (4.1.8)$$

is only a vector up to a gauge transformation.

In contrast, $F_{\mu\nu}$ is a proper Lorentz tensor since it transforms as

$$F_{\mu\nu} \rightarrow \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta} . \quad (4.1.9)$$

and as we saw above, it is invariant under gauge transformations.

However, in order to consider interactions of gauge fields with matter fields, we cannot just concentrate on $F_{\mu\nu}$ but need to include also A_μ .

For amplitudes, interactions involving helicity-1 fields that do not vanish at 0-momentum need to be described with A_μ rather than just $F_{\mu\nu}$.

This allows us to introduce the Lagrangian

$$\mathcal{L} = \int \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu + \dots \right) \quad (4.1.10)$$

for some current J^μ , e.g., $J^\mu = \bar{\psi} \gamma^\mu \psi$.

The equivalence $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$ implies that the current must be conserved

$$\partial_\mu J^\mu = 0. \quad (4.1.11)$$

In general, the matrix elements involved in the amplitude written in terms of the fields A_μ and their corresponding polarizations ϵ_μ are of the form

$$\mathcal{M}(p_i^\mu, \lambda_i) = \epsilon^\mu \mathcal{M}_\mu \quad (4.1.12)$$

In order for the amplitude to be Lorentz invariant, it has to be invariant under the shift of polarizations $\epsilon_\mu \sim \epsilon_\mu + \hat{\alpha}(p) p_\mu$ (gauge redundancy).

Hence, it has to satisfy the following constraint:

Ward identity

$$p^\mu \mathcal{M}_\mu = 0. \quad (4.1.13)$$

This so-called *Ward identity* is an important condition that is solely determined by the requirement of Lorentz invariance and the fact that ϵ_μ is not a Lorentz vector but enjoys the equivalence relation

$$\epsilon_\mu \sim \epsilon_\mu + \hat{\alpha}(p)p_\mu \quad .$$

We continue our discussion about gauge theories in Sect. 4.4 after a brief detour into soft theorems.

4.1.2 Gravity from helicity-2 states

We consider massless particles of helicity $\lambda = \pm 2$ described by a two-index symmetric field $h_{\mu\nu}$ written in terms of polarization vectors $\epsilon_{\mu\nu}$ that also have only two degrees of freedom or polarization states:

$$h_{\mu\nu}(x) = \sum_{\lambda} \int dp \left[\epsilon_{\mu\nu} a(p^\nu, \lambda) e^{ipx} + \epsilon_{\mu\nu}^* a^\dagger(p^\nu, \lambda) e^{-ipx} \right] . \quad (4.1.14)$$

As for the case of helicity 1, the Lorentz invariant constraint $q^\mu \epsilon_{\mu\nu} = 0$ leaves a gauge redundancy

$$\epsilon_{\mu\nu} \rightarrow \epsilon_{\mu\nu} + \hat{\alpha}_\mu(q) q_\nu + \hat{\alpha}_\nu(q) q_\mu \quad (4.1.15)$$

or equivalently

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \alpha_\nu(x) + \partial_\nu \alpha_\mu(x) . \quad (4.1.16)$$

which can be identified with the local version of a general coordinate transformation.

This is the underlying symmetry that defines General Relativity.

For completeness, we also need to impose $\epsilon^\mu{}_\nu = 0$ and $q^\mu \epsilon_{\mu\nu} = 0$.

In total, a symmetric tensor has 10 degrees of freedom.

The constraints reduce those by $5 = 1 + 4$ in the massive and $8 = 1 + 4 + 3$ in the massless case.

Notice that $\varepsilon^\mu{}_\nu = 0$ requires $p^\mu \delta_\mu = 0$ resulting in an additional constraint on the components $\hat{\alpha}_\mu$ which is why the number of degrees of freedom is only reduced by 3 for massless spin 2 particles.

Therefore we can confirm that a massless helicity-2 particle has only $10 - 8 = 2$ independent degrees of freedom.

As discussed in the part, an invariant action (the Einstein-Hilbert action (3.3.8)) can be written in terms of the field $h_{\mu\nu}(x)$ taken as the perturbation of the metric $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$.

4.2 Soft Theorems

In this section, we demonstrate the true power of Lorentz invariance in terms of the constraints it imposes on the S-matrix of four dimensional theories.

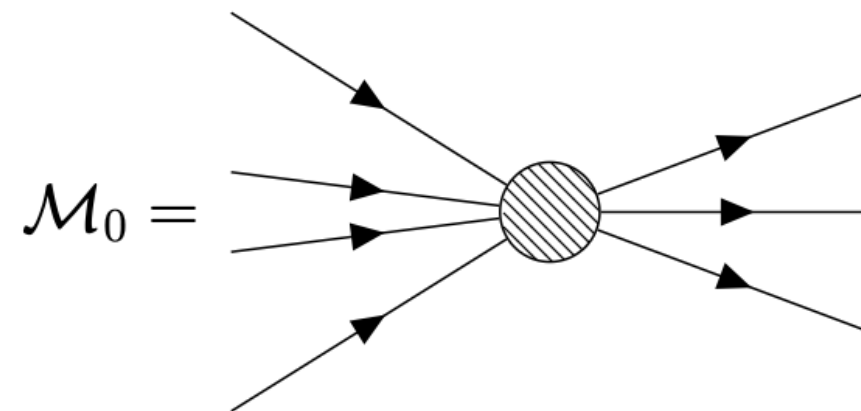
The idea is to use so-called *soft particles* which simply means considering particles whose momenta are taken to zero.

As we will see, this approach directly leads to powerful statements like conservation of charge or the absence of long-range interactions involving massless particles with helicities > 2 .

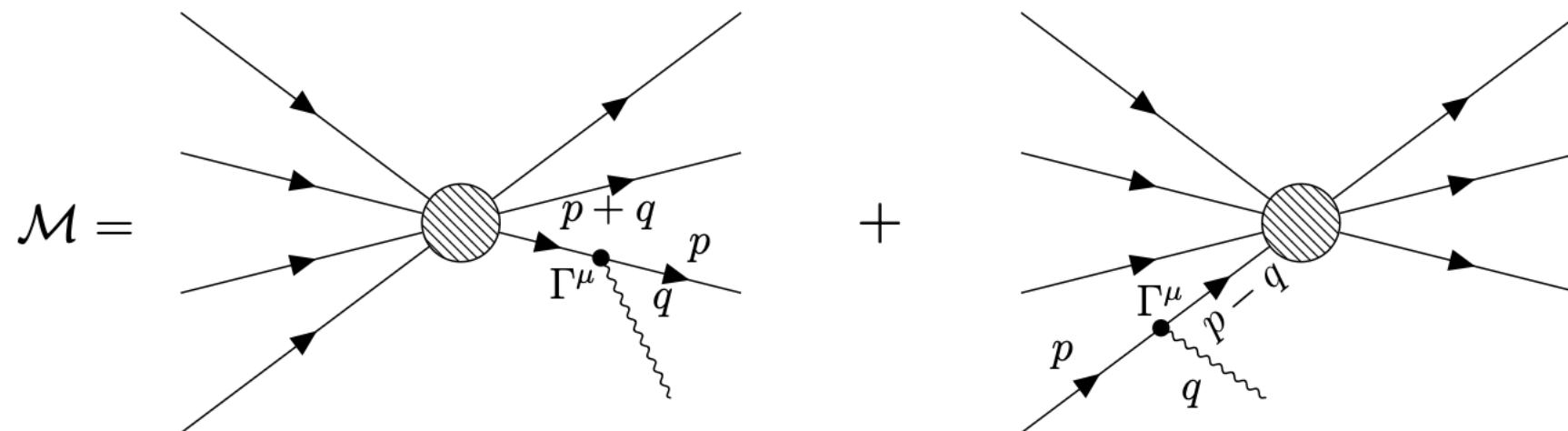
4.2.1 Charge conservation

In this section, we show that charge conservation already follows from Lorentz invariance.

Thus, we consider the following scattering diagram



Let us add a soft photon with momentum q^μ for which we want to take the soft limit $q^\mu \rightarrow 0$ before as well as after the interactions encoded below



The interaction vertex $\Gamma^\mu = \Gamma^\mu(p, q)$ can be written in general as

$$\Gamma^\mu(p, q) = p^\mu F(p^2, q^2, p \cdot q) + q^\mu G(p^2, q^2, p \cdot q) \quad (4.2.1)$$

Since the final amplitude is of the form $\epsilon^\mu \Gamma_\mu$ and $\epsilon^\mu q_\mu = 0$, we can forget G .

Since p^2 and $q^2 = 0$, we find

$$\Gamma^\mu = p^\mu F \left(\frac{p \cdot q}{m^2} \right) \quad (4.2.2)$$

by dimensionality.

If we have one soft photon per external line, one finds

$$\mathcal{M} = \mathcal{M}_0 \left(\sum_{\text{incoming}} \frac{p_i^\mu \epsilon_\mu}{2p_i^\mu q_\mu} F_i(0) - \sum_{\text{outgoing}} \frac{p_i^\mu \epsilon_\mu}{2p_i^\mu q_\mu} F_i(0) \right) \quad (4.2.3)$$

where we used that for $q^\mu \rightarrow 0$ the propagators for incoming and outgoing particles can be written as

$$\frac{1}{(p_i + q)^2 - m^2} \sim \frac{1}{2p_i^\mu q_\mu}, \quad \frac{1}{(p_i - q)^2 - m^2} \sim -\frac{1}{2p_i^\mu q_\mu}. \quad (4.2.4)$$

Invariance under $\epsilon_\mu \downarrow \epsilon_\mu + \alpha q_\mu$ then implies

$$\sum_{\text{incoming}} F_i(0) - \sum_{\text{outgoing}} F_i(0) = 0 \quad (4.2.5)$$

where $F_i(0)$ is nothing but the charges of the particles involved.

Hence, one finds that charge is conserved:

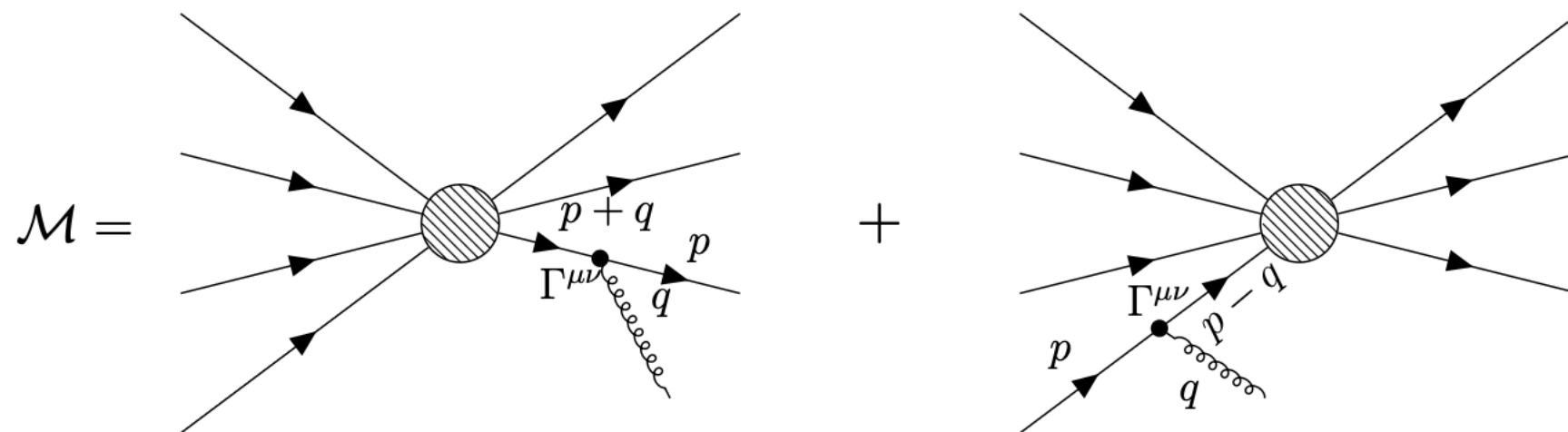
$$\sum_{\text{incoming}} Q_i = \sum_{\text{outgoing}} Q_i. \quad (4.2.6)$$

This is already a remarkable observation as we can see that the well known result that electric charge is a conserved quantity can be derived directly by just arguments of Lorentz invariance.

4.2.2 The equivalence principle and (no) helicities > 2

One can play the same game for gravity.

That is, consider scattering processes with soft gravitons of the form



This results in an amplitude

$$\mathcal{M} = \mathcal{M}_0 \left(\sum_{\text{incoming}} \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{2p_i^\mu q_\mu} F_i(0) - \sum_{\text{outgoing}} \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{2p_i^\mu q_\mu} F_i(0) \right). \quad (4.2.7)$$

We then find that

$$\sum_{\text{incoming}} F_i(0)p_i^\nu - \sum_{\text{outgoing}} F_i(0)p_i^\nu = 0 \quad (4.2.8)$$

which can only be satisfied for arbitrary p_i^μ if all $F_i(0)$ are the same, i.e.,

$$F_i(0) = \kappa, \quad \forall i. \quad (4.2.9)$$

Thus, the coupling to gravity must be universal implying the principle of equivalence.

It is hard to overemphasize the importance of this result.

The principle of equivalence is the basic premise behind Einstein's General theory of relativity.

Here it is not assumed but derived from basic principles of quantum mechanics and special relativity.

The whole concept of gravity reduces to be the unique theory that describes the interaction of massless particles of helicity ± 2 .

This somehow enhances the beauty of the theory not because of the symmetries behind (which are only redundancies) but because of its inevitability.

It also provides a different perspective on what gravity is.

In the search of a more fundamental theory describing gravity at the quantum level, the concrete requirement is to describe interactions of particles of helicity ± 2 .

We note however that, as we mentioned above, interactions can be described in terms of an EFT for the corresponding helicity ± 2 field $h_{\mu\nu}(x)$.

Considering helicity $\lambda = \pm 3$ particles results in a constraint

$$\sum_{\text{incoming}} F_i(0) p_i^\mu p_i^\nu = \sum_{\text{outgoing}} F_i(0) p_i^\mu p_i^\nu \quad (4.2.10)$$

which is only satisfied by $F_i(0) \equiv 0 \ \forall \ i$.

Thus, there are no interacting massless particles of helicity greater than 2.

This is again a crucial result eliminating an infinite number of possibilities for interacting elementary particles.

Therefore we conclude, by only using arguments of Lorentz invariance that any theory that can describe interacting massless particles can only include a handful of particles:

Massless particle content of any interacting QFT

$\lambda = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2$ are all possible
massless particle states in an interacting theory.

In nature we have examples of particles of precisely these helicities, the Higgs particle is the example for $\lambda = 0$, quarks and leptons for $\lambda = \pm 1/2$, photons, gluons, W- and Z-particles for $\lambda = \pm 1$ and the graviton for $\lambda = \pm 2$ with the (so far) only exception of $\lambda = \pm 3/2$.

A proper interactive theory of these particles (known individually as the gravitino) is only consistent in supersymmetric theories where they have to couple to gravity.

Their study and potential impact in nature is beyond the scope of these lectures.

4.3 Non-Abelian gauge theories from scattering amplitudes

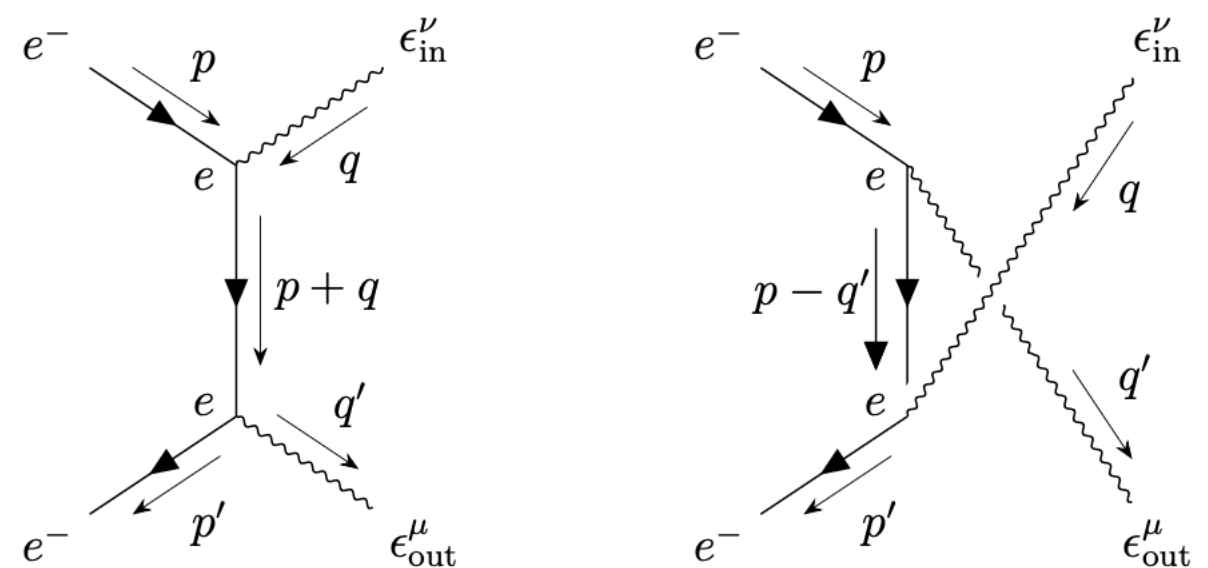
Above, we focussed on theories with a single type of field A_μ .

Here, we would like to understand how gauge symmetries are modified in the presence of several species of massless helicity-1 particles.

Instead of stating the answer right away (Yang-Mills theory is the proper theoretical description of helicity-1 particles), we provide a brief derivation of the structure underlying non-Abelian gauge theories.

First, we consider *Compton scattering* in QED ($e^- \gamma \rightarrow e^- \gamma$) as shown in Fig. 4.1.

Figure 4.1: The two Feynman diagrams contributing to Compton scattering $e^- \gamma \rightarrow e^- \gamma$ in QED.



Using the standard Feynman rules, the amplitude can be written as

$$\begin{aligned}\mathcal{M} &= -ie^2 \bar{u}(p', \sigma') \left(\frac{\gamma_\mu (\not{p} + \not{q} + m) \gamma_\nu}{(p+q)^2 - m^2} + \frac{\gamma_\nu (\not{p} - \not{q}' + m) \gamma_\mu}{(p-q')^2 - m^2} \right) u(p, \sigma) \epsilon_{\text{in}}^\nu \epsilon_{\text{out}}^\mu \\ &= \mathcal{M}_{\mu\nu} \epsilon_{\text{in}}^\nu \epsilon_{\text{out}}^\mu\end{aligned}\tag{4.3.1}$$

where, as usual $\not{p} = \gamma^\mu p_\mu$, etc.

Using basic identities like

$$p + q = p' + q', \quad (\not{p} - m)u = 0, \quad \bar{u}(\not{p}' - m) = 0,\tag{4.3.2}$$

we can check the Ward identity (4.1.13) by considering how the amplitude changes under $\epsilon^{\nu}_{\text{in}} \rightarrow \epsilon^{\nu}_{\text{in}} + q^\nu$

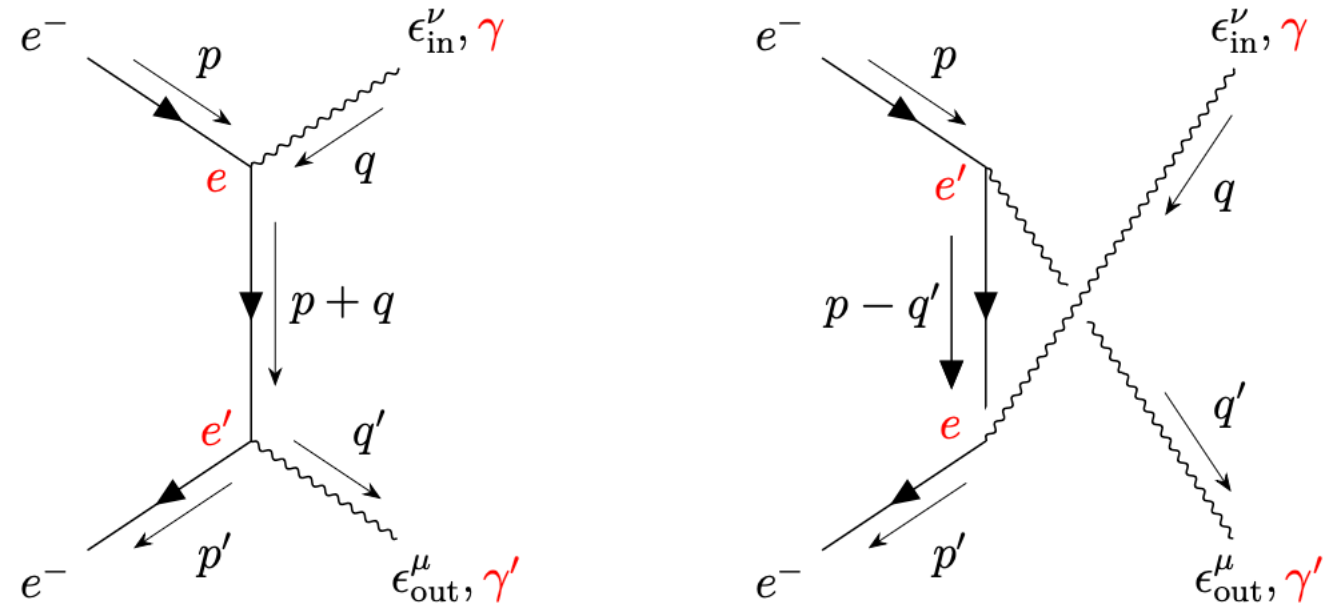
$$\begin{aligned}\mathcal{M}_{\mu\nu} q^\nu \epsilon_{\text{out}}^\mu &= -ie^2 \bar{u}(p', \sigma') \left(\frac{\not{\epsilon}_{\text{out}} (\not{p} + \not{q} + m) \not{q}}{(p+q)^2 - m^2} + \frac{\not{q} (\not{p}' - \not{q} + m) \not{\epsilon}_{\text{out}}}{(p'-q)^2 - m^2} \right) u(p, \sigma) \\ &= -ie^2 \bar{u}(p', \sigma') \not{\epsilon}_{\text{out}} u(p, \sigma) \left(\frac{2p_\mu q^\mu}{(p+q)^2 - m^2} + \frac{2p'_\mu q^\mu}{(p'-q)^2 - m^2} \right) \\ &= 0\end{aligned}\tag{4.3.3}$$

using (4.2.4) in the last step.

Similarly we may consider having two different types of photons with couplings e and e' .

We consider the process $e^- \gamma \rightarrow e^- \gamma'$ shown diagrammatically in Fig. 4.2 generalizing the Compton scattering in Fig. 4.1.

Figure 4.2: Generalized Compton scattering $e^- \gamma \rightarrow e^- \gamma'$ for two types of photons.



The only difference to the previous amplitude are the two different couplings in each of the vertices.

Going through the algebra from above, we find that (4.3.3) becomes

$$ee' - ee' = 0. \quad (4.3.4)$$

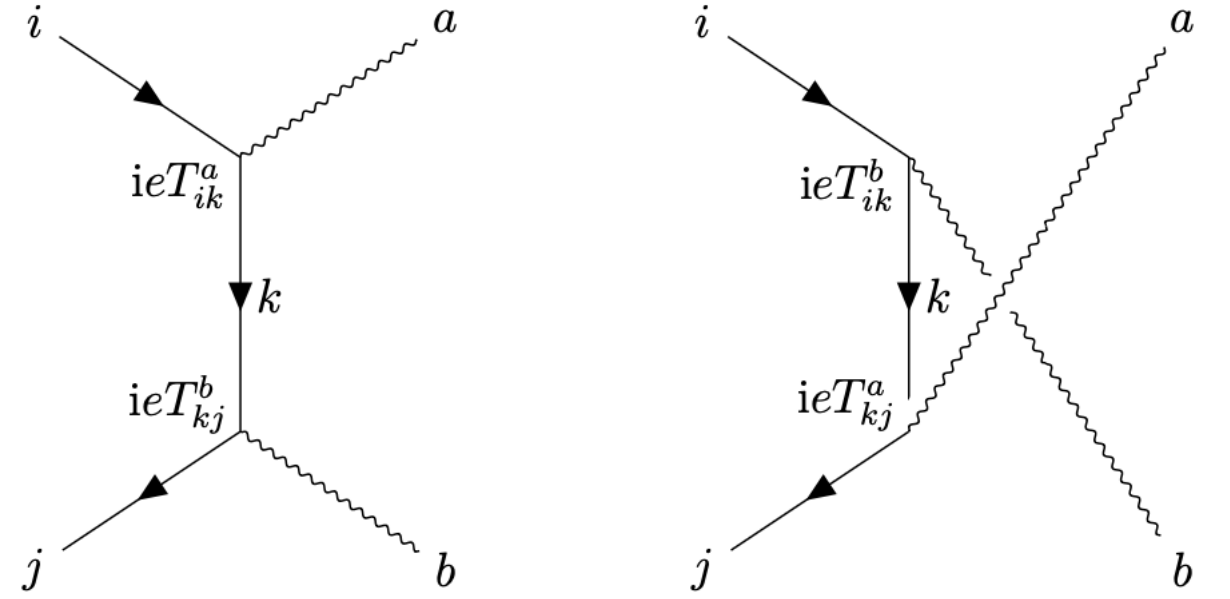
This just means that any value of the two couplings are allowed and hence that the matter fields can be charged under two independent $U(1)$'s.

Now, for an arbitrary number of matter particles e_i and gauge particles γ^a , we look at the process

$$e_i \gamma^a \rightarrow e_j \gamma^b, \quad i, j = 1, \dots, N \quad a, b = 1, \dots, D. \quad (4.3.5)$$

The Feynman diagrams are shown in Fig. 4.3

Figure 4.3: Generalized Compton scattering $e_i \gamma^a \rightarrow e_j \gamma^b$ for many species of matter and gauge fields with couplings T^a_{ij} between two matter and one gauge particle.



where we introduced coupling constants T^c_{lk} between two matter particles e^l, e^k and one gauge particle γ^c .

We compute the diagrams in Fig. 4.3

$$\begin{aligned} \mathcal{M}^{ab}_{ij} &= -ie^2 \bar{u}(p', \sigma') \left(T^a_{ik} \frac{\gamma_\mu (\not{p} + \not{q} + m) \gamma_\nu}{(p+q)^2 - m^2} T^b_{kj} + T^b_{ik} \frac{\gamma_\nu (\not{p} - \not{q}' + m) \gamma_\mu}{(p-q')^2 - m^2} T^a_{kj} \right) u(p, \sigma) \epsilon_{\text{in}}^\nu \epsilon_{\text{out}}^\mu \\ &= (\mathcal{M}^{\mu\nu})^{ab}_{ij} \epsilon_{\text{in}}^\nu \epsilon_{\text{out}}^\mu \end{aligned} \tag{4.3.6}$$

which gives rise to

$$q^\nu (\epsilon_{\text{out}}^\mu) (\mathcal{M}^{\mu\nu})^{ab}_{ij} \propto T^a_{ik} T^b_{kj} - T^b_{ik} T^a_{kj} = 0. \tag{4.3.7}$$

Said differently, the Ward identity implies the vanishing commutator

$$[T^a, T^b] = 0. \quad (4.3.8)$$

This means we have simply D copies of QED or, more precisely, a $U(1)^D$ gauge theory coupled to N charged scalar fields.

This is true unless there exist self-interactions among the gauge bosons.

That is, if there exists an interaction vertex $\sim f^{abc}$ coupling three gauge particles to each other, then there is a third diagram shown in Fig. 4.4 that contributes to the process $e_i \gamma^a \rightarrow e_j \gamma^b$.

Contrary to the diagrams in Fig. 4.3, there is now a gauge particle exchanged between the two vertices.

It turns out that the structure of such a three point vertex is highly constrained by e.g. permutation symmetries of external particles.

For the sake of brevity, we leave a more detailed discussion to App. B where we present the full argument that non-abelian Yang-Mills theory is the unique description theories with many helicity-1 fields.

Crucially, this argument works *without having to impose any gauge symmetry to begin with: the underlying Lie-algebra structure arises as a consistency condition of the Ward identity (or Lorentz invariance)*.

To make a long story short, combining the contribution from the diagram in Fig. 4.4 with (4.3.6) and testing the Ward identity analogously to (4.3.7) amounts to requiring

$$T_{ik}^a T_{kj}^b - T_{ik}^b T_{kj}^a = if^{abc} T_{ij}^c \quad (4.3.9)$$

or equivalently

$$[T^a, T^b] = if^{abc} T^c. \quad (4.3.10)$$

This is nothing but a *non-Abelian algebra* which in turn gives rise to the notion of *non-Abelian gauge symmetries* and the associated *Yang-Mills theories*.

It is important to appreciate the significance of this statement which is sometimes taken for granted when starting from a given gauge group with underlying Lie algebra: the couplings of helicity-1 fields to matter $\sim T_{ij}^a$ and among themselves $\sim f^{abc}$ satisfy the non-linear relationship (4.3.10) (an algebra) *because* the Ward identity needs to be imposed.

Said differently, the Lie algebra structure of the underlying theory *emerges from Lorentz invariance and unitarity*.

It is in fact inevitable!

To summarize, we conclude that a system with many gauge fields is either

- a) a theory with many photon-like gauge bosons, that is, $G = U(1)^n$
- b) or a non-Abelian Yang-Mills system with G being some non-Abelian group $SU(N)$, $SO(N)$ etc.

The structure constants f^{abc} appearing in (4.3.10) satisfy the *Jacobi identity*

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \quad (4.3.11)$$

due to

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \quad (4.3.12)$$

In general this describes the algebra of a Lie group.

The group elements are obtained from exponentiating

$$U = e^{i\theta^a T^a} = \mathbb{1} + i\theta^a T^a + \dots \quad (4.3.13)$$

with T^a the generators and θ^a some parameter.

The Lie group G itself corresponds to a smooth manifold with coordinates θ^a , $a = 1, \dots, D$.

We call D the dimension of G , while the rank r corresponds to the number of generators that commute.

This is the general structure of Yang-Mills theories that we describe next.

4.4 Yang-Mills theory

Previously, we derived the structure of scattering amplitudes or rather the properties of couplings between different species of helicity-1 particles from first principles.

Now, let us see how this translates into theories of fields.

We briefly review the Abelian case complementing the treatment of section 4.1.1 before we put the lessons learned in the preceding section about non-Abelian gauge theories to good use.

4.4.1 The Abelian case

Recall that for Abelian gauge fields A_μ transforming as $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$ we have a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \quad (4.4.1)$$

with J^μ a conserved current $\partial_\mu J^\mu = 0$.

Coupling this field to a matter spin 1/2 field, the current is $J^\mu = e\bar{\psi}\gamma^\mu\psi$ associated to the symmetry $\psi \rightarrow e^{ie\alpha'}\psi$ with $\alpha' \in \mathbb{R}$ some constant.

Plugging this current into the Lagrangian together with the kinetic term $\bar{\psi}i\not{D}\psi$ for ψ we can see that the Lagrangian can be written as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu + \bar{\psi}i\not{D}\psi \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}i\not{D}\psi \end{aligned} \quad (4.4.2)$$

with $D_\mu \equiv \partial_\mu - ieA_\mu$ the covariant derivative.

Note that the conserved current J^μ and the kinetic term for the fermions together give rise to D_μ .

In this form, the Lagrangian is invariant under the local ($\alpha' = \alpha(x)$) gauge transformation

$$\psi \rightarrow e^{ie\alpha}\psi, \quad D_\mu\psi = (\partial_\mu - ieA_\mu)\psi \rightarrow e^{ie\alpha}D_\mu\psi, \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha. \quad (4.4.3)$$

The field strength can be written as

$$F_{\mu\nu} = \frac{i}{e} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (4.4.4)$$

and it is invariant under the gauge transformation.

Note that

$$\partial_\mu\psi \rightarrow e^{ie\alpha}(\partial_\mu\psi + ie(\partial_\mu\alpha)\psi) \quad (4.4.5)$$

is not covariant, but D_μ transforms covariantly in the sense that it transforms with a phase like ψ does, i.e., $D_\mu\psi \rightarrow e^{ie\alpha}D_\mu\psi$.

This brings us to the standard argument (that we did not follow here) that promoting a global symmetry ($\alpha' = \text{constant}$) to a local symmetry $\alpha' = \alpha(x)$ motivates introducing a gauge field A_μ to turn the standard derivative into a covariant derivative containing A_μ .

The latter transforms appropriately under gauge transformations such that the Lagrangian is gauge invariant for the matter fields ψ and gauge fields A_μ .

This is a simple prescription to follow.

However, since this is an arbitrary logic (why to impose that the symmetry is local? Is the introduction of A_μ unique? etc.), we have preferred to actually **derive** the existence of the gauge symmetry and the covariant derivative rather than imposing it, in the sense that we asked the question of how to consistently construct a theory for helicity $|\lambda| = 1$ massless vector fields coupled to matter fields of spin/helicity 0, 1/2.

The coupling of matter to the gauge field is determined by the coupling of the conserved current to the gauge field $A^\mu J_\mu$ which gives precisely the contribution that turns normal derivatives into covariant derivatives.

4.4.2 The general non-Abelian case

Let us now generalize this to the non-Abelian case.

For a general Lie group with generators T^a_R in some representation R that we keep implicit here, a field ψ transforms as

$$\psi \rightarrow U\psi, \quad U = e^{i\theta^a T^a_R}. \quad (4.4.6)$$

The covariant derivative transforms also covariantly in the sense that

$$D_\mu \psi \rightarrow U D_\mu \psi \quad (4.4.7)$$

where now in terms of the coupling constant g

$$D_\mu = \partial_\mu - igA_\mu \quad (4.4.8)$$

or, more explicitly, in components

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig A_\mu^a (T_R^a)_{ij} \quad (4.4.9)$$

with the generators being in the same representation R of ψ .

Let us now impose that (4.4.7) is true.

Then we want to know how A_μ transforms which is why we compute

$$\begin{aligned} D_\mu \psi &\rightarrow \partial_\mu (U \psi) - ig A'_\mu U \psi \\ &= U \partial_\mu \psi + (\partial_\mu U) \psi - ig A'_\mu U \psi \\ &= U D_\mu \psi + \left(ig U A_\mu + (\partial_\mu U) - ig A'_\mu U \right) \psi. \end{aligned} \quad (4.4.10)$$

In order to ensure (4.4.7), we need to impose

$$ig U A_\mu + (\partial_\mu U) - ig A'_\mu U = 0 \quad (4.4.11)$$

which amounts to the following general and infinitesimal transformations:

$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}, \quad A_\mu^a \rightarrow A_\mu^a + \frac{1}{g} \partial_\mu \theta^a - f^{abc} \theta^b A_\mu^c. \quad (4.4.12)$$

Here we used (4.3.10) for the commutator for T_R^a .

The field strength for non-Abelian groups is most easily found by considering

$$[D_\mu, D_\nu]\psi(x) = \left(-ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]\right)\psi(x) \quad (4.4.13)$$

so that

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \quad (4.4.14)$$

We call A_μ^a the gauge connection and $F_{\mu\nu}^a$ the curvature.

The latter transforms under gauge transformations as

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}, \quad F_{\mu\nu}^a \rightarrow F_{\mu\nu}^a - f^{abc}\theta^b F_{\mu\nu}^c. \quad (4.4.15)$$

Notice, as the index structure indicates, gauge fields always transform in the adjoint representation which is the one for which the generators are the structure constants themselves $T_{bc}^a = f_{bc}^a$.

The most general gauge invariant, renormalizable Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4}g_{ab}F_{\mu\nu}^a F^{b,\mu\nu} + \mathcal{L}_M(\psi, D_\mu\psi) + \Theta F_{\mu\nu}^a \tilde{F}^{a,\mu\nu} \quad (4.4.16)$$

where g_{ab} , $a, b = 1, \dots, \dim(G)$, is a metric on the group manifold.

In components, we write

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \quad \tilde{F}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a. \quad (4.4.17)$$

In order to have only physically propagating particles (positive kinetic energy), we restrict to groups for which g_{ab} is positive definite.

This implies that the group is compact, simple or semi-simple and eliminates all non-compact groups.

This again is a strong argument by which we can eliminate an infinite number of potential symmetry groups.

We can thus safely restrict our attention to the compact groups classified by Cartan.

These groups (unlike the non-compact ones) allow for finite dimensional unitary representations which makes them suitable to describe physical interactions.

We typically normalize the metric as $g_{ab} = \kappa \delta_{ab}$ with $\kappa = 1$ since κ may be absorbed in a rescaling of $F_{\mu\nu}$ and A_μ .

This rescaling allows us to move the coupling constant from the kinetic term for gauge fields to its appearance in the definition of the covariant derivative and the $F_{\mu\nu}$ fields.

For instance, rescaling $A_\mu \rightarrow A_\mu/g$ amounts to rescale $F_{\mu\nu} \rightarrow F_{\mu\nu}/g$ and have the covariant derivative independent of g .

But then g appears in the kinetic term for the gauge fields as $F^{\mu\nu} F_{\mu\nu} /g^2$.

Therefore the arbitrariness in rescaling the metric g_{ab} amounts to the freedom in where to include the coupling g in the Lagrangian.

As long as we are consistent, the physical results are unaffected by this rescaling, but it illustrates the need to have the free parameter g .

Notice that the last term in (4.4.16) can be written as

$$\Theta F_{\mu\nu}^a \tilde{F}^{a,\mu\nu} = 2\Theta \partial_\mu \left(\epsilon^{\mu\nu\rho\sigma} A_\nu^a F_{\rho\sigma}^a \right) . \quad (4.4.18)$$

Hence, being a total derivative, this term has no immediate effect on the classical equations of motion, but is indispensable in a full quantum theory.

In fact, one can show using canonical quantization that the term associated to Θ needs to be taken into account when working in a basis of gauge invariant physical states.

Θ is arbitrary in the sense that no physical principle determines the value of Θ .

However, Θ does not change under local gauge-invariant perturbations nor under time evolution.

Hence, Θ labels different sectors of the theory and it corresponds to a different choice of vacuum.

In fact, this can be formulated in terms of a superselection rule: *quantizing non-Abelian gauge theories requires a definite choice of Θ restricting the Hilbert space of states in a specific way.*

Once Θ has been fixed, one cannot reach states of the full Hilbert space with another value of Θ .

It therefore is a new fundamental constant which is required to specify the dynamics of quantum Yang-Mills theory.

We assume here implicitly that Θ is constant.

For fun, we will briefly introduce *axions* in section 9.2.1 in which case Θ becomes a dynamical field itself.

But even for constant Θ , it can have non-trivial effects so that research has shown that magnetic monopoles have non-integer valued electric charge.

This effect was called Witten effect.

Beyond that, for the specific value of $\Theta = \pi$, the resulting theory describes a new state of matter corresponding to topological insulators.

Such topological states of matters have been constructed in experiments and have surprisingly interesting properties.

For instance, if we place an electric charge in a vacuum with $\Theta = 0$ opposite of a material with $\Theta = \pi$, the resulting induced electromagnetic field mirrors that of a magnetic monopole placed at the same distance to the boundary (rather than that of a mirror electric charge as for a conventional metal).

Field equations and Bianchi identities

Ignoring Θ for the moment, we mostly work with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \left(F_{\mu\nu}^a \right)^2 + \mathcal{L}_M(\psi, D_\mu \psi) \tag{4.4.19}$$

where $\mathcal{L}_M(\psi, D_\mu\psi)$ stands for a general Lagrangian density describing interactions between the gauge fields and some matter fields ψ .

The field equations are determined as usual from the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^a)} = \frac{\partial \mathcal{L}}{\partial A_\nu^a} . \quad (4.4.20)$$

One finds that

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^a)} = -\partial_\mu F^{a,\mu\nu} , \quad \frac{\partial \mathcal{L}}{\partial A_\nu^a} = -gf^{abc} A_\mu^b F^{c,\nu\mu} - i \frac{\partial \mathcal{L}_M}{\partial(D_\nu\psi)} T^a \psi \quad (4.4.21)$$

and thus

$$\partial_\mu F^{a,\mu\nu} = -J^{a,\nu} \quad (4.4.22)$$

in terms of the current

$$J^{a,\nu} = -gf^{abc} A_\mu^b F^{c,\nu\mu} - i \frac{\partial \mathcal{L}_M}{\partial(D_\nu\psi)} T^a \psi . \quad (4.4.23)$$

The current is conserved

$$\partial_\nu J_a^\nu = 0 , \quad (4.4.24)$$

as expected from Noether's theorem.

But this equation is not gauge covariant.

In terms of covariant derivatives, one can write this equation as

Yang-Mills equations

$$D_\mu F^{a,\mu\nu} = -j^{a,\nu} . \quad (4.4.25)$$

The associated current

$$j^{a,\nu} = -i \frac{\partial \mathcal{L}_M}{\partial (D_\nu \psi)} T^a \psi \quad (4.4.26)$$

corresponding to the contribution to the total current $J^{a,\nu}$ coming from the matter fields.

In terms of $j^{a,\nu}$ we can write a gauge covariant equation

$$D_\nu j_a^\nu = 0 \quad (4.4.27)$$

meaning that the matter current $j^{a,\nu}$ is not conserved but only “covariantly” conserved.

This is a manifestation of the fact that the total current $J^{a,\nu}$ includes the matter part $j^{a,\nu}$ but also a gauge field dependent part, since for non-Abelian gauge theories the gauge fields transform non-trivially under the gauge transformations (and therefore are self-interacting).

This is similar to the situation in General Relativity in which the total energy momentum tensor $T_{\mu\nu}$ is conserved but the matter part (the right hand side of Einstein's equations) is only covariantly conserved due to the fact that gravity is also self-interactive and contributes a term to the total energy momentum tensor.

In addition, the curvature $F^a_{\mu\nu}$ also satisfies the *Bianchi identity*

Bianchi identity

$$D_\mu F^a_{\nu\lambda} + D_\nu F^a_{\lambda\mu} + D_\lambda F^a_{\mu\nu} = 0 . \quad (4.4.28)$$

This can also be written in terms of the dual field strength $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ as

$$D^\mu \tilde{F}_{\mu\nu} = 0 . \quad (4.4.29)$$

Notice that Bianchi identities and field equations are interchanged when we inter-change $\tilde{F}_{\mu\nu} \leftrightarrow F_{\mu\nu}$.

This has interesting consequences because in the language of differential forms we can construct solutions $F \pm \star F$ which are called (anti-)self-dual instanton solutions.

For them, the Bianchi identity automatically implies Yang-Mills equations.

In fact, $F \pm \star F$ is an even simpler system of equations which is only of first order, whereas Yang-Mills equation is of order 2.

These field configurations are topologically non-trivial (because $\int_{\mathbb{R}^4} \text{Tr}(F \wedge F)$ is a topological invariant for 4-dimensional manifolds) and localized both in space and time – hence the name *instanton*.

Such states cannot be described in the usual perturbative manner through Feynman diagrams, but they are inherently non-perturbative.

We will not have time to discuss instantons in detail in these lectures.

General Comments

Let us make some general technical comments about the previous ideas:

- a) In comparison with QED, the kinetic term $\sim (F^a_{\mu\nu})^2$ includes a cubic coupling $\sim g(\partial A)AA$ as well as a quartic coupling g^2AAAA .

Hence, the gauge fields have self-interactions unlike photons.

This is as predicted in the previous section when the need for Yang-Mills was demonstrated.

In particular, this non-linearity implies that, unlike for light beams as in QED, propagating waves of non-Abelian fields interact among each other.

- b) For QED $F_{\mu\nu}$ is invariant, but for Yang-Mills $F^a_{\mu\nu}$ is only covariant.

The field equations and Bianchi identities are very similar in both cases and the Yang-Mills case reduces to the Maxwell case when the group is Abelian, i.e., a $U(1)$.

c) For QED, there exists a conserved, gauge invariant current.

For Yang-Mills, of both currents J^a_μ and j^a_μ one is conserved, but not gauge invariant and the other is gauge invariant but not conserved.

d) Analogy with gravity. There is an interesting similarity between the structure of Yang-Mills theories and gravity as summarized in table 4.1.

	YM	gravity
Connections	A^a_μ	$\Gamma^\rho_{\mu\nu}$
Covariant derivative	$D_\mu\psi = (\partial_\mu - igA_\mu)\psi$	$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$
Curvature	$\frac{i}{g}[D_\mu, D_\nu]\psi = F_{\mu\nu}\psi$	$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\alpha\mu\nu} V^\alpha$

Table 4.1: Comparison between Yang-Mills theory and General Relativity.

Notice that we can define $F_{ab\mu}\epsilon = F_{c\mu}\epsilon f_{cab}$ which describes the change parallel transported around infinitesimal loops in internal space $V= V^aT_a$ for the generators T_a .

Also recall that in gravity the stress energy tensors are only covariantly conserved and the Bianchi identity holds.

Yang-Mills theories may be considered as symmetries in an extended spacetime, adding compact dimensions to our 4-dimensional spacetime.

The simplest realization is a fifth dimension corresponding to a circle which would naturally have the U(1) symmetry.

Symmetries of higher dimensional compact manifolds would correspond to Yang-Mills groups.

This is the idea behind Kaluza-Klein theories of extra dimensions.

- e) Weinberg-Witten theorem (stated without proof): There can never be a conserved Lorentz covariant current in a theory with massless $\lambda = 1$ particles with non-vanishing charges associated to that current.

The version of this theorem for gravity states: A theory with a conserved Lorentz covariant energy momentum tensor cannot have a massless particle of helicity 2.

This illustrates the uniqueness of both Yang-Mills and gravity theories.

Properties of compact Lie groups

Recall the following about compact Lie groups:

- a) Compact Lie groups are classified as summarized in table 4.2.

Table 4.2: Compact simple Lie groups

Group	Rank	Dimension
$SU(N)$	$N - 1$	$N^2 - 1$
$SO(N)$	$\frac{N}{2}, \frac{N-1}{2}$	$\frac{N(N-1)}{2}$
$SP(N)$	N	$N(2N + 1)$
Exceptional G_2, F_4, E_6, E_7, E_8	2, 4, 6, 7, 8	14, 52, 78, 133, 248

- b) A group is called
- *Simple* if there exists no non-trivial ideal (invariant sub-algebra),
 - *Semi-simple* if it can be written as a product of simple groups and $U(1)$'s.

For these cases there is always finite-dimensional irreducible Hermitian representations of the algebra lifting to unitary representations of the group.

c) Standard representations:

- *Fundamental*: this is the smallest non-trivial representation and denoted as

$$\phi_i \rightarrow \phi_i + i\alpha^a (T_F^a)_{ij} \phi_j \quad (4.4.30)$$

where T_F^a are the generators T^a in the corresponding representation. For $SU(N)$, this representation is N -dimensional.

- *Anti-fundamental*: The anti-fundamental representation is related to the fundamental via

$$T_{AF}^a = -(T_F^a)^* . \quad (4.4.31)$$

Similarly, one then finds (using the hermiticity of T^a)

$$\phi_i^* \rightarrow \phi_i^* + i\alpha^a (T_{AF}^a)_{ij} \phi_j^* = \phi_i^* - i\alpha^a (T_F^a)_{ji} \phi_j^* . \quad (4.4.32)$$

- *Adjoint*: For the adjoint representation for $SU(N)$, one can choose the generators

$$(T_A^a)^{bc} = -if^{abc} \quad (4.4.33)$$

corresponding to a $(N^2 - 1)$ -dimensional representation.

d) Normalization: for $SU(N)$, we normalize the generators T^a such that

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$$

where

$$T^a T^b = \frac{1}{2N} \delta^{ab} + \frac{1}{2} d^{abc} T^c + \frac{1}{2} i f^{abc} T^c$$

in terms of the symmetric coefficients

$$d^{abc} = 2 \text{Tr} \left(T^a \{T^b, T^c\} \right) .$$

For a specified representation R , we then have

$$\text{Tr}(T_R^a T_R^b) = T(R) \delta^{ab}$$

for some index $T(R)$ - $T(R)$ is the *Dynkin index* of the representation, i.e., $T(R)$ is equal to $1/2$ for generators in the fundamental representation. The quadratic Casimir is

$$C(R) = T_R^a T_R^a$$

or the identity operator by Schur's lemma. For the fundamental representation, one finds

$$C_F = \frac{N^2 - 1}{2N}$$

and for the adjoint

$$C_A = N .$$

Part 5 BROKEN SYMMETRIES

Symmetries we may not see

It suddenly came home to us that there is much more symmetry in the laws of nature than one would guess merely by looking at the properties of elementary particles. The reality we observe in our laboratories is only an imperfect reflection of a deeper and more beautiful reality, the reality of the equations that display all the symmetries of the theory.

Steven Weinberg

In this part, we introduce the fundamental concepts of symmetries that may be hidden from an observer – symmetries upheld by the Lagrangian but not apparent in physical observables.

This phenomenon is commonly known as *Spontaneous Symmetry Breaking* (SSB).

As will be explored in later chapters, SSB is a critical component for understanding the weak interactions through the Higgs mechanism.

It also plays a significant role in theories with global symmetries, superconducting materials, and the broader comprehension of quantum Yang-Mills theories.

Additionally, we examine anomalies, a distinct way symmetries can be broken.

Anomalies arise when quantum effects disrupt a symmetry present in classical field theory, and in some cases – such as gauge symmetries – they can make the theory inconsistent.

This is another crucial factor in ensuring the consistency of the Standard Model, with notable consequences such as charge quantization.

5.1 Unitarity problems with massive spin-1 fields

Let us see where we stand and provide a brief motivation to study SSB to begin with.

So far, we have been able to identify the following states labelled by spin or helicity:

- a) Helicity or spin $0, \pm 1/2$ particles which can be massless or massive.
- b) Helicity 1 massless particles implying gauge invariance with Abelian (QED) or Non-Abelian (Yang-Mills) gauge symmetries.
- c) Helicity 2 gravity.

Historically, Pauli criticized the idea of Yang and Mills: Massless non-Abelian gauge fields should have been seen, like the photon, but had not at the time.

Hence, he suggested to forget about Yang-Mills theory right from the beginning since he had found the same result but discarded it as unphysical and did not publish it.

What about massive particles?

Massive scalar fields (spin 0) and fermion fields (spin 1/2) can be consistently described in QFT, but what about spin-1 fields?

Recall that these fields possess 3 polarization states constrained by $p^\mu \epsilon_\mu = 0$. Furthermore, they satisfy $p^\mu p_\mu = m^2$ and the polarizations can be normalized such that

$$\epsilon_\mu^* \epsilon^\mu = -1 . \tag{5.1.1}$$

The field and the equation of motion read

$$(\square + m^2)A_\mu = 0, \quad A_\mu = \int \frac{d^3p}{(2\pi)^3} a_i(p) \epsilon_\mu^i e^{ipx} + \text{h.c.} \quad (5.1.2)$$

But what could possibly be wrong with such fields?

The answer is simple and really just depends on what we mean with consistent.

As an EFT below some cutoff scale Λ , the theory is totally valid and predictive.

However, there is a subtle issue when taking the energy to too large values as we now explain.

Let us consider momenta in the z-direction

$$p^\mu = (E, 0, 0, p_z), \quad E^2 - p_z^2 = m^2. \quad (5.1.3)$$

The transverse polarizations can be defined as

$$\epsilon_1^\mu = (0, 1, 0, 0), \quad \epsilon_2^\mu = (0, 0, 1, 0) \quad (5.1.4)$$

and the longitudinal as

$$\epsilon_L^\mu = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m} \right). \quad (5.1.5)$$

At high energies $E \gg m$, we may write the latter as

$$\epsilon_L^\mu \sim \frac{E}{m} (1, 0, 0, 1). \quad (5.1.6)$$

As we have seen before, scattering amplitudes $\mathcal{M}$ are proportional to the polarization vectors and hence this implies for an amplitude with two external massive particles of spin 1

$$g^2 \epsilon_L^0 \epsilon_L^3 \sim g^2 \frac{E^2}{m^2}.$$

What this actually implies is that probabilities (obtained from cross-sections) blow-up at $E \gg m$.

Perturbative unitarity is broken and therefore the theory of massive spin-1 fails at $E \gg m$.

For instance, having $m \sim 100\text{GeV}$ with $g \sim 0.1$ results in a scale $E \sim 1\text{TeV}$ at which the theory fails to make sense.

This is a signal that the theory needs to be replaced by a different theory which often involves new degrees of freedom.

We then typically speak of a UV completion.

To summarize, there are two problems associated with the above:

- a) For massless particles: No massless Yang-Mills field observed.
- b) For massive particles: Theories of massive spin-1 *fail perturbative unitarity*.

We will see that the the solution to both problems is to consider *spontaneously broken symmetries*.

We will next describe this phenomenon in detail.

5.2 Spontaneous breaking of a discrete symmetry

Let us consider the simplest system of a real scalar field $\phi(x)$ with a discrete Z_2 symmetry $\phi \rightarrow -\phi$.

The most general renormalizable Lagrangian is of the form

$$\mathcal{L}[\phi] = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V_\pm(\phi) \quad (5.2.1)$$

where the scalar potential can take one of two forms

$$V_\pm(\phi) = \pm \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 + \kappa_\pm. \quad (5.2.2)$$

We need $\lambda > 0$ for stability, i.e., the scalar potential should be bounded from below.

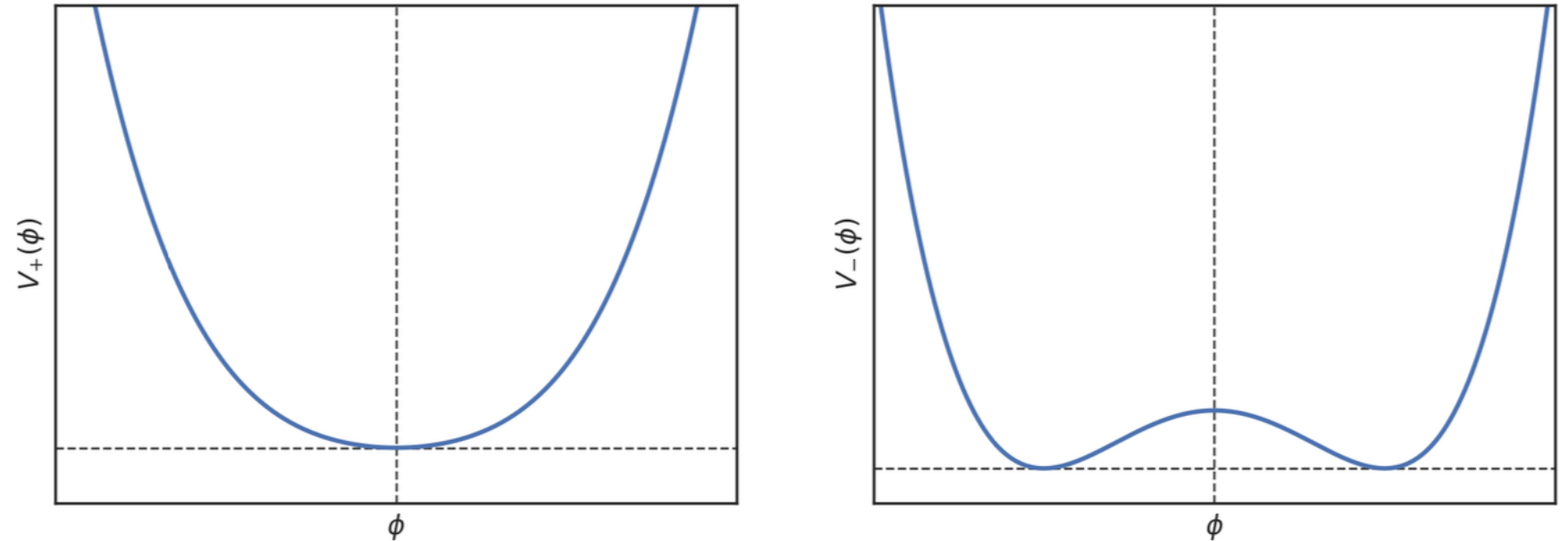
The constant $\kappa_\pm$ is chosen such that the potential vanishes at the minimum which in the absence of gravity is not physically important because the relevant quantity is the difference of energies.

The main feature of these two potentials is the quadratic piece or mass term for which the sign is not determined.

We consider now each case separately since they have different physical implications.

- a) V_+ (with $m^2 > 0$): The scalar potential $V_+(\phi)$ has a classical minimum at $\phi_0 = 0$, cf. the left panel of Fig. 5.1.

Figure 5.1: Left: Scalar potential V_+ with symmetric minimum. Right: Double well potential V_- with two degenerate minima exhibiting spontaneous symmetry breaking.



Clearly, the minimum is invariant under $\phi \rightarrow -\phi$ so that the symmetry is manifest.

In a quantum theory, we consider the vacuum expectation value (VEV)

$$\langle \phi \rangle = \langle 0 | \phi | 0 \rangle = \int \mathcal{D}\phi \, \phi \, \exp \left(\frac{i}{\hbar} \int \mathcal{L} d^4x \right) \quad (5.2.3)$$

normalized with respect to $\langle 0 | 0 \rangle = \int \mathcal{D}\phi e^{\frac{i}{\hbar} \int \mathcal{L} d^4x}$.

In the limit $\hbar \rightarrow 0$, the path integral is dominated by the stationary value of the action and hence

$$\langle \phi \rangle = \phi_0 = 0. \quad (5.2.4)$$

Perturbations around the minimum can be written as $\phi = \phi_0 + \sigma(x)$, $\sigma \simeq 1$, so that

$$\mathcal{L}[\phi] \rightarrow \mathcal{L}_{\text{pert}}[\sigma] = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m^2 \sigma^2 + \frac{\lambda}{4} \sigma^4. \quad (5.2.5)$$

In the following, we denote the Lagrangian for the perturbations around the true vacuum as $\mathcal{L}_{\text{pert}}$.

This Lagrangian describes nothing but a heavy particle of mass $m^2 > 0$ with $p_\mu p^\mu = m^2$ and

$$m^2 = \left. \frac{\partial^2 V}{\partial \phi^2} \right|_{\phi=\phi_0}. \quad (5.2.6)$$

So the parameter m^2 in the potential is actually the physical mass-squared of the *corresponding particle*.

b) V+ (with $m^2 > 0$): in this case the potential can be written as:

$$V_-(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 \quad (5.2.7)$$

in terms of

$$v = \sqrt{\frac{m^2}{\lambda}}. \quad (5.2.8)$$

There are two degenerate vacua $|0_\pm\rangle$ corresponding to $\phi_0 = \pm v$ as can be seen from the right panel of Fig. 5.1.

Let us consider perturbations around any one of these minima by setting

$$\phi = \pm v + h(x) \quad (5.2.9)$$

to write the Lagrangian for the fluctuation $h(x)$

$$\mathcal{L}[\phi] \rightarrow \mathcal{L}_{\text{pert}}[h] = \frac{1}{2} \partial_\mu h \partial^\mu h - V_{\text{pert}}(h) \quad (5.2.10)$$

where the scalar potential reads

$$V_-(\phi) \rightarrow V_{\text{pert}}(h) = \lambda v^2 h^2 \pm \lambda v h^3 + \frac{\lambda}{4} h^4. \quad (5.2.11)$$

The first term implies that h describes a massive particle of mass

$$m_h^2 = 2\lambda v^2 = 2m^2 > 0. \quad (5.2.12)$$

Equivalently, this can be obtained from the original potential V_+ by taking the second derivative and evaluating at the correct minimum, that is,

$$m_h^2 = \left. \frac{\partial^2 V_-(\phi)}{\partial \phi^2} \right|_{\phi^2=v^2} = 2\lambda v^2. \quad (5.2.13)$$

Note that the physical mass-squared m_h^2 differs from the original m^2 parameter in the potential.

This illustrates the fact that the mass can be read directly from the potential only if the vacuum state corresponds to $\phi = 0$ as in the case for V_+ above.

Note also that if we had expanded around the symmetric point $\phi = 0$, the particle spectrum would consist of a particle of negative mass squared, $-m^2$, which is called *tachyon* signalling the wrong expansion.

That is, expanding around a maximum instead of a minimum as becomes evident from the right panel in Fig. 5.1.

As we had mentioned in part 2 in the context of representations of the Poincaré group, tachyon states ($p^\mu p_\mu < 0$) are allowed by special relativity, but the apparent presence of a tachyon only indicates an instability of the corresponding expansion point (the maximum) towards the true vacuum of the theory which might be degenerate as in the case above.

Let us now comment on the apparent “breaking” of the symmetry.

The cubic term in (5.2.11) seems to break the original Z_2 symmetry which is why it is hidden.

Importantly, however, the symmetry is not actually broken, but is still present due to the symmetry under

$$h \pm v \rightarrow -(h \pm v) \quad \Rightarrow \quad h \rightarrow -h \mp 2v . \quad (5.2.14)$$

However once we expand around one of the two vacua, the symmetry is not manifest in the expansion.

In this sense the word hidden is more appropriate.

Quantum mechanically we could have considered the vacuum state to be a superposition of the two vacua $|0_+> \pm |0_->$ but for large systems (where the infinite volume limit can be considered) locality or cluster decomposition implies that any two Hermitian operators $\mathcal{O}_1(x, t)$, $\mathcal{O}_2(x, t)$ representing physical observables commute at fixed time and large separations and matrix elements factorize $\langle 0 | \mathcal{O}_1(x, t) \mathcal{O}_2(0, t) | 0 \rangle = \langle 0 | \mathcal{O}_1(x, t) | 0 \rangle \langle 0 | \mathcal{O}_2(0, t) | 0 \rangle$.

This condition is satisfied if the system is at one of the two vacuum states, but not at a superposition.

5.2.1 Topological defects

In the context of spontaneously broken discrete symmetries, an interesting phenomenon takes place.

In physical 3-dimensional space we will find regions in which the system is in vacuum $|0_+>$ and other regions in which the system is in the second vacuum $|0_\uparrow>$.

Going from a region for which $\langle \phi \rangle = +v$ to the region for which $\langle \phi \rangle = -v$ we need to pass through $\langle \phi \rangle = 0$ where the potential is actually non-vanishing.

The boundary that separates the two regions is a 2-dimensional wall called a *domain wall*.

This is a topological defect of the system reflecting the fact that the space of vacua is disconnected, consisting of two points.

Static domain wall solutions can be easily found for the system by solving the field equations

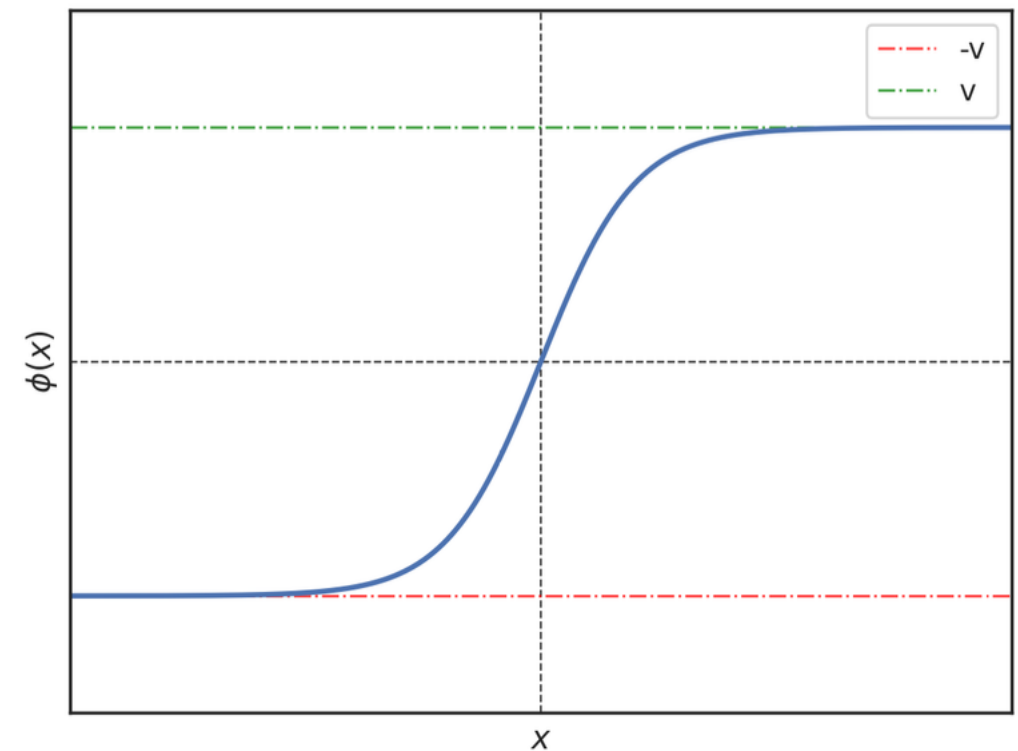
$$\square\phi + V'(\phi) = 0 \quad (5.2.15)$$

for which $\phi(x)$ (independent of y, z) satisfies $\phi'' = \lambda\phi(\phi^2 - v^2)$ that, even though it is a non-linear equation, it has a closed solution with a profile of the form

$$\phi(x) = v \tanh \rho x, \quad \rho^2 \equiv \frac{\lambda v^2}{2} = \frac{m^2}{2} \quad (5.2.16)$$

which interpolates between the two vacua in the limits $x \rightarrow \pm \infty$ (see Fig. 5.2).

Figure 5.2: Domain wall solution interpolating between the two minima $\phi(x) = \pm v$ of the scalar potential.



The domain wall would extend through the y, z directions.

Domain walls are physical entities that carry energy and could play an important role in the dynamics of the system.

We can explicitly compute the energy density in this case from the $\phi(x)$ profile above.

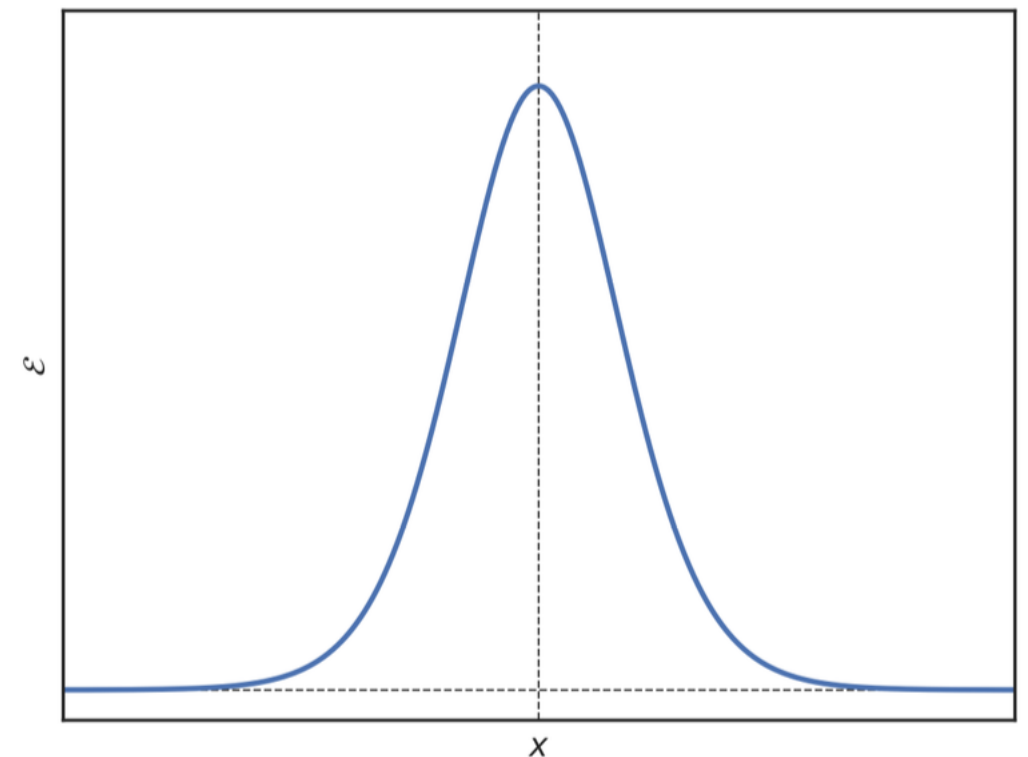
The energy density $\mathcal{E}$ of the wall can be computed by evaluating it at $\phi(x) = v \tanh \rho x$ for which we get

$$\mathcal{E} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V_\pm(\phi) = \frac{\lambda v^4}{2 \cosh^4 \rho x} . \quad (5.2.17)$$

It can be seen in Fig. 5.3 that this profile is highly localized at $x \sim 0$ and y, z arbitrary.

The value of ρ determines the thickness of the wall (or membrane or 2-brane using current terminology)

Figure 5.3: Energy density profile for the domain wall solution illustrating that the domain wall solution actually corresponds to a physical object carrying energy (and tension) which is localized in the x direction and arbitrary in the yz directions which is the standard picture of a wall (or 2-brane).



5.2.2 Symmetry restoration at high temperature

Symmetry breaking can be understood as a phase transition.

This model captures the physics of several systems such as ferromagnetic materials.

In this case the parameter m^2 of the scalar potential corresponds to a temperature difference

$$m^2 \propto (T - T_c) . \quad (5.2.18)$$

That is, for high temperatures $T > T_c$ the potential is V_+ with one single minimum and unbroken symmetry.

In this case the expectation value of the ferromagnetic material vanishes meaning that all different directions of magnetization are realized and the average is zero.

While the system cools down, we enter the region for which $T < T_c$ where a phase transition occurs and the potential is now of the form V_- exhibiting SSB.

The two possible expectation values indicate two opposite directions of the magnets which are polarized pointing at only one direction in one vacuum and the opposite direction in the other vacuum (the average value or expectation value of the field is non-zero now).

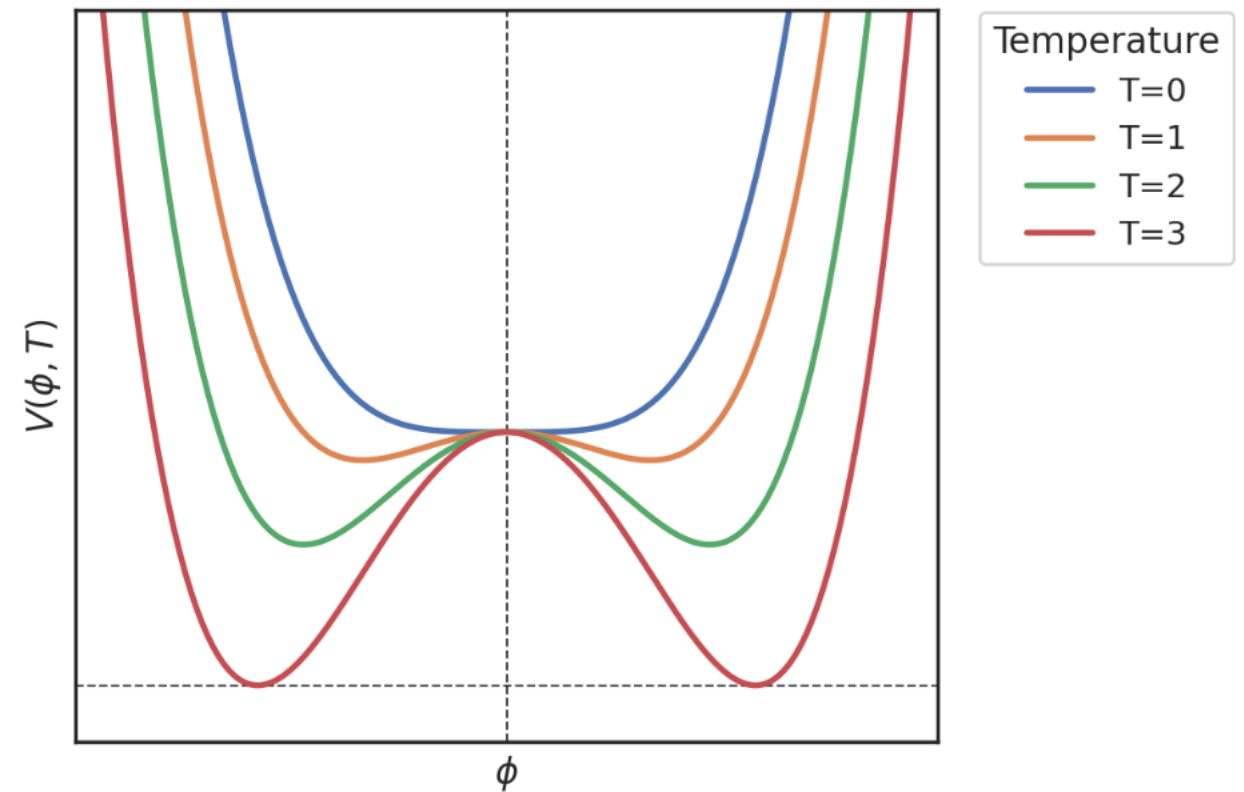
In cosmology, the early universe has a high temperature.

If there was a scalar field with a potential of the above type, the quadratic term in the scalar potential would read $-m^2\phi^2 + T^2\phi^2$.

At high temperatures ($T \gg m$) the effective coefficient of ϕ^2 is positive and the system is in the unbroken phase with a minimum at $\phi = 0$.

As the universe cools down while expanding, it reaches a critical temperature $T_c = m$ and, for temperatures smaller than T_c , the coefficient of ϕ^2 becomes negative such that the universe enters the symmetry breaking phase, see Fig. 5.4.

Figure 5.4: Temperature dependent potential. At high temperature the unbroken phase. When temperature gets reduced (universe expanding gets colder) the broken symmetry phase is realized.



In this regime, the phase transition may occur towards a broken phase.

This is usually referred to as *symmetry restoration* in the early universe.

In this case, domain walls, if present, may have a significant impact on the evolution of the universe by contributing a large amount to the energy density of the universe which may over close it.

5.3 Spontaneous symmetry breaking (SSB) of continuous global symmetries

Let us begin our generalization to the case of spontaneous breaking of continuous global symmetries with a simple example, namely that of an N -component real scalar field $\phi = (\phi_1, \dots, \phi_N)^T$.

The Lagrangian is given by

$$\mathcal{L}[\phi] = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - V_\pm(\phi) \quad (5.3.1)$$

with

$$V_\pm(\phi) = \pm \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4, \quad \lambda > 0, \quad \phi^2 = \phi \cdot \phi, \quad \phi^4 = (\phi^2)^2. \quad (5.3.2)$$

The Lagrangian is invariant under global $O(N)$ transformations of the field which naturally leave the inner product on $\mathbb{R}^N$ invariant.

We are primarily interested in the case $V_-(\phi)$ because it leads to SSB.

We can replace the potential (up to an irrelevant constant term) by

$$V_-(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2, \quad v^2 = \frac{m^2}{\lambda} > 0. \quad (5.3.3)$$

This potential is often called the Mexican hat potential and has the shape of a wine bottle as shown on the right of Fig. 5.5.

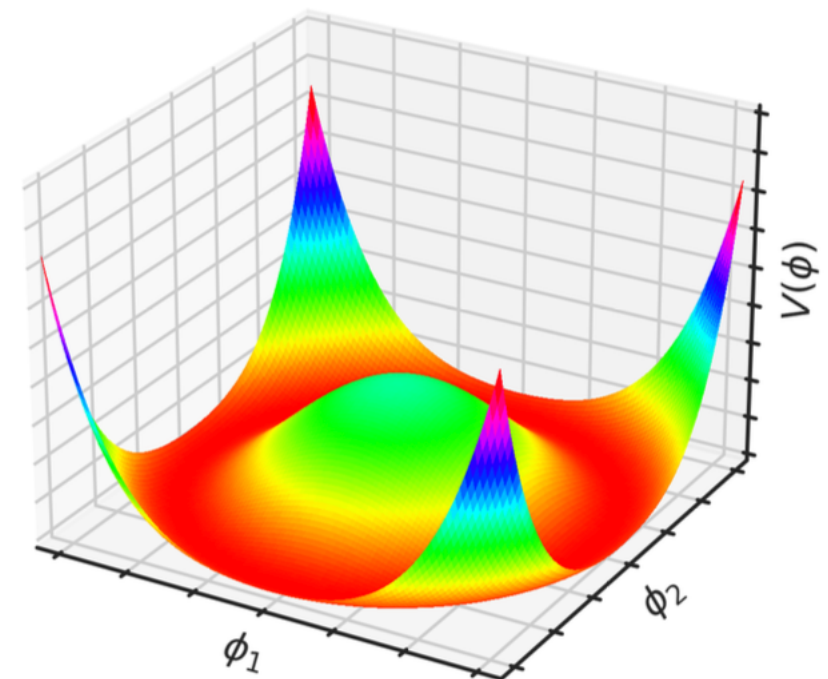
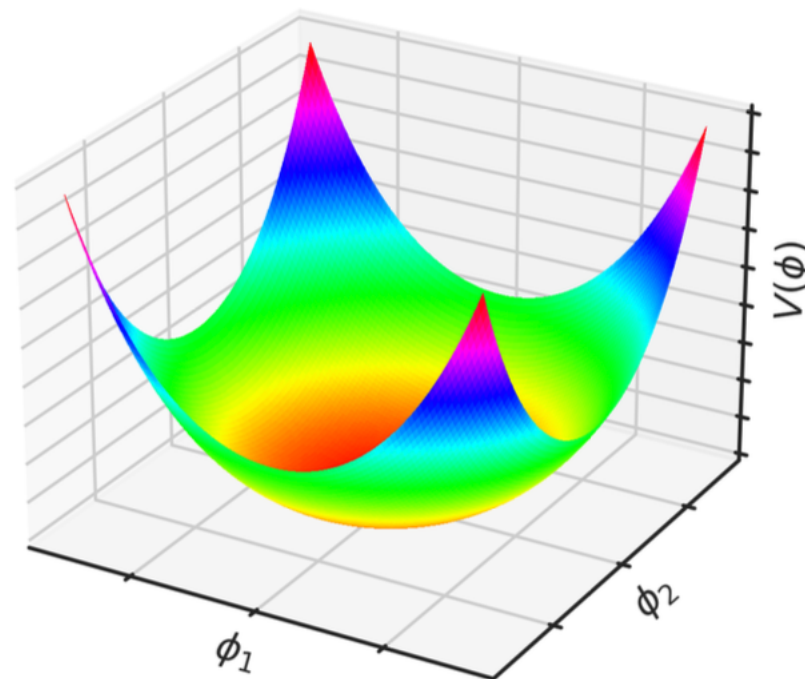


Figure 5.5: Potential in the symmetric or unbroken phase with $m^2 > 0$ (left) and in the spontaneously broken phase with $m^2 < 0$ (right).

The vacua are defined by the equation

$$\langle \phi \rangle^2 = \phi_0^2 = v^2 \quad (5.3.4)$$

resembling the defining equation of an (N - 1)-sphere.

Without loss of generality let us pick one possible solution to (5.3.4)

$$\langle \phi \rangle = \phi_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix}. \quad (5.3.5)$$

This VEV breaks the original symmetry O(N) to the subgroup O(N - 1) because ϕ_0 is invariant only under rotations in N - 1 dimensions.

When performing fluctuations around the minimum with fluctuating fields $\pi_i(x)$ and $\sigma(x)$, we write

$$\phi(x) = \begin{pmatrix} \pi_1(x) \\ \pi_2(x) \\ \vdots \\ \pi_{N-1}(x) \\ v + \sigma(x) \end{pmatrix}. \quad (5.3.6)$$

The Lagrangian for those fluctuations becomes

$$\mathcal{L}_{\text{pert}}[\pi_i, \sigma] = \frac{1}{2} \partial_\mu \pi_i \partial^\mu \pi_i + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V_{\text{pert}}(\pi_i, \sigma) \quad (5.3.7)$$

with scalar potential

$$V_{\text{pert}}(\pi_i, \sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \lambda v (\sigma^2 + \pi_i^2) \sigma + \frac{\lambda}{4} (\sigma^2 + \pi_i^2)^2 . \quad (5.3.8)$$

The (diagonalized) mass matrix schematically looks like

$$\left. \frac{\partial^2 V_-(\phi)}{\partial \phi_i \partial \phi_j} \right|_{\phi=\langle \phi \rangle} = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 2\lambda v^2 \end{pmatrix} \quad (5.3.9)$$

with the eigenvalues

$$m_\sigma = \sqrt{2\lambda v^2}, \quad m_{\pi_i} = 0 . \quad (5.3.10)$$

The only non-zero eigenvalue corresponds to the mass of σ .

We obtain $N - 1$ massless fields $\pi_i, i = 1, \dots, N \rightarrow 1$, which are called **Goldstone bosons** for reasons to become clear below.

It turns out that the theory after SSB possesses a manifest $O(N - 1)$ symmetry.

This makes sense intuitively from the wine-bottle shape of the potential (Fig. 5.5): radial excitations come with a large energy penalty, whereas excitations in the field which locally seek to transform the field to another of the equivalent vacua can be made to have arbitrarily small energy difference from the vacuum.

5.4 Goldstone's theorem

Next, we introduce Goldstone's theorem which makes general statements about the number of massive and massless fields (or occasionally also synonymously referred to as modes/excitations/fluctuations) after SSB for general continuous groups G .

We distinguish the classical version formulated simply in terms of a scalar potential and the quantum version phrased in terms of quantum mechanical Hilbert spaces and path integrals.

5.4.1 The classical version

In general, if a Lagrangian is invariant under G , then a *non-zero vacuum expectation value* (VEV) $\phi_0 = \langle \phi \rangle$ for a field ϕ breaks $G \rightarrow H \subset G$.

The *associated vacuum manifold* $\mathcal{M}_0$ to ϕ_0 is defined as

$$\mathcal{M}_0 = \{ \phi_0 : V(\phi_0) = V_{\min} \} . \quad (5.4.1)$$

In the former case of $O(N)$, $\mathcal{M}_0$ is determined by condition (5.3.4) implying that $\mathcal{M}_0 \cong S^{N-1}$

The invariant group H

The invariant or *stability group* H_{ϕ_0} is the subgroup of G that leaves the vacuum ϕ_0 invariant, that is,

$$H_{\phi_0} = \{h \in G : h\phi_0 = \phi_0\} . \quad (5.4.2)$$

The different vacua themselves are linked by transformations in G , i.e., for $\phi_0, \phi_0' \in \mathcal{M}_0$ there exists $g \in G$ such that

$$\phi_0' = g\phi_0 . \quad (5.4.3)$$

Let us prove that the stability groups H_{ϕ_0} for different $\phi_0 \in \mathcal{M}_0$ are isomorphic.

As before, let $\phi_0' \in \mathcal{M}_0$ and $g \in G$ be such that $\phi_0' = g\phi_0$ and define

$$H_{\phi_0'} = \{h \in G : h\phi_0' = \phi_0'\} . \quad (5.4.4)$$

Then we can write for $h \in H_{\phi_0'}$

$$h\phi_0' = \phi_0' \quad \Rightarrow \quad hg\phi_0 = g\phi_0 \quad \Rightarrow \quad g^{-1}hg\phi_0 = \phi_0 \quad (5.4.5)$$

implying that

$$g^{-1}hg \in H_{\phi_0} \quad \Rightarrow \quad H_{\phi_0} = g^{-1}H_{\phi_0'}g . \quad (5.4.6)$$

Hence, the stability groups H_{ϕ_0} are indeed isomorphic.

Therefore, in the following, we will denote the stability group of the vacua collectively as H .

The coset G/H and Goldstone modes

The elements $g \in G$ mapping one vacuum to another belong to the coset G/H and fall into equivalence classes $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2 h$.

That is, if there exists two elements $g_1, g_2 \in G$ such that

$$\phi'_0 = g_1 \phi_0 = g_2 \phi_0, \quad (5.4.7)$$

then

$$g_2^{-1} g_1 \in H. \quad (5.4.8)$$

We can associate to $\phi'_0 \in \mathcal{M}_0$ an equivalence class

$$\mathcal{M}_0 \cong G/H. \quad (5.4.9)$$

To derive Goldstone's theorem, let us consider infinitesimal transformations around a field ϕ in a representation R of a group G of dimension $\dim(R) = N$ so that

$$g\phi = \phi + \delta\phi, \quad (g\phi)_r = \phi_r + \delta\phi_r \quad (5.4.10)$$

with

$$\delta\phi_s = i\alpha^a T_{sr}^a \phi_r, \quad a = 1, \dots, \dim(G), \quad r, s = 1, \dots, N. \quad (5.4.11)$$

For a scalar potential invariant under G , we have

$$V(g\phi) = V(\phi + \delta\phi) = V(\phi) . \quad (5.4.12)$$

Expanding around ϕ amounts to

$$V(\phi + \delta\phi) - V(\phi) = \mathrm{i}\alpha^a (T^a \phi)_r \frac{\partial V}{\partial \phi_r} = 0 \quad (5.4.13)$$

Differentiating (5.4.13) once again leads to

$$\mathrm{i}\alpha^a \left[(T^a)_{rs} \frac{\partial V}{\partial \phi_r} + (T^a \phi)_r \frac{\partial^2 V}{\partial \phi_r \partial \phi_s} \right] = 0 \quad (5.4.14)$$

and evaluating at $\phi = \phi_0$ gives rise to

$$(T^a \phi_0)_r M_{rs}^2 = 0 \quad (5.4.15)$$

where we defined the mass matrix M_{rs}^2 as

$$M_{rs}^2 = \left. \frac{\partial^2 V}{\partial \phi_r \partial \phi_s} \right|_{\phi=\phi_0} . \quad (5.4.16)$$

We then distinguish the following cases:

- if the symmetry is unbroken and the vacuum unique in the sense that $g\phi_0 = \phi_0$ for all $g \in G$, then $\delta\phi = 0$ and thus

$$(T^a\phi_0)_r = 0 \quad (5.4.17)$$

for all $a = 1, \dots, \dim(G)$ and $r = 1, \dots, N$.

- if there exists $g \in G$ and $a \in \{1, \dots, \dim(G)\}$ with $T^a\phi_0 \neq 0$, then by (5.4.15) $T^a\phi_0$ is an eigenvector of the mass matrix M_{rs}^2 with zero eigenvalue.

The question is how many of such massless states exist?

Let us assume that G is compact and semi-simple and let us split the generators T^a as

$$T^a = (\tilde{T}^i, R^\alpha) \quad (5.4.18)$$

with $\tilde{T}^i \in \mathfrak{h}$ so $i = 1, \dots, \dim \mathfrak{h}$ with

$$\tilde{T}^i\phi_0 = 0 \quad (5.4.19)$$

and the orthogonality condition

$$\text{Tr}(\tilde{T}^i R^\alpha) = 0 \quad (5.4.20)$$

with $\alpha = 1, \dots, \dim(G/\mathfrak{h})$.

Each vector $R^\alpha\phi_0$ is a unique eigenvector of eigenvalue zero to M_{rs}^2 and therefor are

$$\dim(G/H) = \dim(G) - \dim(H) = \dim(\mathcal{M}_0) \quad (5.4.21)$$

massless modes called *Goldstone bosons*.

Since M^2_{rs} is a $N \times N$ -matrix, then there are at most $N - \dim(G/H)$ massive modes.

This is Goldstone's theorem:

Goldstone's theorem

Every spontaneously broken continuous global symmetry of a QFT gives rise to massless states.

For an internal symmetry with breaking pattern $G \rightarrow H$ it leads to

$$\dim(G/H) = \dim(G) - \dim(H) = \dim(\mathcal{M}_0)$$

massless particles: the *Goldstone bosons*

Coming back to the example of the $O(N)$ model, we found $N - 1$ massless fields and that the unbroken symmetry group is $O(N - 1)$.

We now understand that

$$\dim(O(N)/O(N - 1)) = \frac{N(N - 1)}{2} - \frac{(N - 1)(N - 2)}{2} = N - 1 \quad (5.4.22)$$

is the number of Goldstone bosons and that there is exactly one massive field σ .

For spacetime symmetries, their breaking also leads to massless states but the counting is not given by $\dim(G) - \dim(H)$.

Examples include the phonons in condensed matter physics and the domain walls we have just discussed.

In this case translation invariance is clearly broken and the free motion of the wall corresponds to a Goldstone mode, but the number of these modes does not match the number of broken symmetries (that include boosts and translations).

5.4.2 Quantum aspects of SSB

Since this is such an important theorem with far reaching implications, we will now provide a different quantum perspective of Goldstone's theorem and SSB in general.

Order parameter for SSB

The Noether charge associated to some symmetry is defined as

$$Q^a = \int d^3x J_0^a . \quad (5.4.23)$$

As discussed in the previous section, the charges themselves act as quantum operators such that

$$[\phi_i, Q^a] = iT_{ij}^a \phi_j . \quad (5.4.24)$$

The order parameter of SSB is given by the VEV of the field operator ϕ

$$\langle 0|\phi|0\rangle = \langle \phi \rangle \begin{cases} = 0 & \text{unbroken,} \\ \neq 0 & \text{broken.} \end{cases} \quad (5.4.25)$$

If the symmetry is spontaneously broken, i.e., if $\langle \phi \rangle \neq 0$, then it follows that

$$\langle 0|[\phi, Q^a]|0\rangle = \langle 0|\phi Q^a - Q^a \phi|0\rangle \neq 0. \quad (5.4.26)$$

Hence, we deduce that

$$Q^a |0\rangle \begin{cases} = 0 & \text{unbroken,} \\ \neq 0 & \text{broken.} \end{cases} \quad (5.4.27)$$

It is then usually stated that the condition for an unbroken symmetry is that the corresponding generator annihilates the vacuum.

Equivalently, this can be formulated in terms of generators of the underlying symmetry group.

That is, let

$$U = e^{i\alpha_a T_a} \quad (5.4.28)$$

such that for the vacuum state ϕ_0

$$U_{ij}(\phi_0)_j = \left(\delta_{ij} + i\alpha^a T_{ij}^a \right) (\phi_0)_j \equiv (\phi_0)_i. \quad (5.4.29)$$

This implies that

$$T_{ij}^a(\phi_0)_j = 0 . \quad (5.4.30)$$

Degenerate energies and Goldstone modes

In quantum mechanics, if

$$|\psi\rangle = Q |\chi\rangle \quad (5.4.31)$$

and Q is conserved,

$$[Q, H] = 0 , \quad (5.4.32)$$

then one can show that

$$E_\psi |\psi\rangle = H |\psi\rangle = HQ |\chi\rangle = QH |\chi\rangle = E_\chi Q |\chi\rangle = E_\chi |\psi\rangle . \quad (5.4.33)$$

Thus, the energy levels E_ψ and E_χ are the same,

$$E_\psi = E_\chi , \quad (5.4.34)$$

and are degenerate energies.

But in field theories where SSB occurs, the particles need not be degenerate necessarily.

Consider for example two fields related by a symmetry transformation, that is,

$$i\phi_1 = [\phi_2, Q] \quad (5.4.35)$$

for some fields ϕ_1, ϕ_2 .

Then, for the corresponding particle states, we obtain

$$|1\rangle = a_1^\dagger |0\rangle = i[a_2^\dagger, Q] |0\rangle = ia_2^\dagger Q |0\rangle - iQa_2^\dagger |0\rangle = -iQ |2\rangle + ia_2^\dagger Q |0\rangle . \quad (5.4.36)$$

So the two particle states corresponding to the two fields are directly related $|1\rangle \leftrightarrow Q |2\rangle$ only if $Q |0\rangle = 0$.

This is not true for SSB.

In this case energy degeneracy of physical states will not hold.

However, for SSB, if $Q^a |0\rangle \neq 0$, we find a new result

$$H(Q^a |0\rangle) = Q^a H |0\rangle = E_0(Q^a |0\rangle) . \quad (5.4.37)$$

Hence, $Q^a |0\rangle$ is degenerate with the vacuum $|0\rangle$ and both have energy E_0 .

Let us now define the momentum states

$$|\pi^a(p)\rangle = K \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} J_0^a(x) |0\rangle \quad (5.4.38)$$

of energy $E(\mathbf{p}) + E_0$ with $E^2(\mathbf{p}) = \mathbf{p}^2 + m^2$.

Since

$$|\pi^a(0)\rangle = K Q^a |0\rangle \quad (5.4.39)$$

has energy E_0 , then $E(\mathbf{p}) \rightarrow 0$ for $|\mathbf{p}| \rightarrow 0$.

This implies that the states $|\pi^a\rangle$ are massless and correspond to the Goldstone modes (one per broken symmetry) - this can be seen as a **quantum version of the Goldstone's theorem**.

Quantum effective action

Let us now consider the description in terms of path integrals and quantum effective actions.

Recall from the path integral formulations of QFT that we can define the functional $W[J]$, usually called *effective action*, as

$$e^{iW[J]} = \int \mathcal{D}\phi e^{i\int(\mathcal{L}+J\phi)} \quad (5.4.40)$$

which is the generating functional for all fully connected Green's functions.

We can define

$$\frac{\delta W}{\delta J} = \frac{\int \mathcal{D}\phi \phi e^{i\int(\mathcal{L}+J\phi)}}{\int \mathcal{D}\phi e^{i\int(\mathcal{L}+J\phi)}} = \frac{\langle 0|\phi|0\rangle}{\langle 0|0\rangle} = \phi_c(x). \quad (5.4.41)$$

which is nothing but the 1-point function of a field $\phi(x)$ in the presence of sources.

Like in statistical field theory, we define the Legendre transformed functional

$$\Gamma[\phi_c] = W[J] - \int d^4x J(x)\phi_c(x), \quad \frac{\delta\Gamma}{\delta\phi_c(x)} = -J(x) \quad (5.4.42)$$

which is the *1PI effective action* generating 1PI (one-particle irreducible) connected amputated Green's functions $\Gamma(n)(x_1, \dots, x_n)$.

Expanding in momenta, we can write:

$$\Gamma[\phi_c] = \int d^4x \left[-V_{\text{eff}}(\phi_c) + \frac{1}{2}(\partial_\mu \phi_c)^2 Z(\phi_c) + \dots \right] \quad (5.4.43)$$

with the effective potential $V_{\text{eff}}(\phi_c)$.

This potential is different from the (classical) potential $V(\phi)$: it knows about the full quantum effects in the theory.

The inverse propagator is

$$\Delta^{-1} = \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \quad (5.4.44)$$

which, at vanishing momentum, is the mass matrix

$$\Delta^{-1} \Big|_{p=0} = \frac{d^2 V_{\text{eff}}}{d\phi_c^2} \Big|_0. \quad (5.4.45)$$

This provides the standard interpretation as mass of a particle in terms of the location of the poles of the propagator.

SSB occurs when the classical field $\phi_c \neq 0$ in the absence of a current $J = 0$, i.e.,

$$\frac{\delta \Gamma}{\delta \phi_c} = 0, \quad \phi_c \neq 0 \quad (5.4.46)$$

so that at zero momentum

$$\frac{dV_{\text{eff}}}{d\phi_c} = 0 \quad (5.4.47)$$

for $\phi_c \neq 0$.

Goldstone modes correspond to

$$\frac{\delta^2 V_{\text{eff}}}{\delta\phi_r \delta\phi_s} (T^a \phi)_s = 0.$$

This clearly promotes the original claim to the full quantum domain since the effective potential reduces to the classical potential at leading order.

5.5 Spontaneous breaking of gauge symmetries

We started this part pointing out two problems: massless Yang-Mills fields have not been observed and massive spin-1 fields fail perturbative unitarity.

But rather than making progress in addressing these problems, what we achieved so far is only adding yet another apparent problem: the non-observation of Goldstone modes.

In summary, we now have to deal with three problems:

- a) Yang-Mills fields are massless and have not been observed.
- b) Goldstone bosons are massless and have not been observed.
- c) Theories with massive spin 1 fields are not valid at high energies.

In this section, we will understand how all of these problems can be cured at once.

The key point is the Higgs mechanism which is based on the simple idea of SSB in the presence of gauge theories

5.5.1 The Abelian Higgs model

The first model that we like to consider is the *Abelian Higgs model* for a complex scalar field ϕ coupled to a U(1) gauge field A_μ with Lagrangian

$$\mathcal{L}[A_\mu, \phi] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\phi^\dagger - V(\phi^*\phi) \quad (5.5.1)$$

where

$$D_\mu = \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5.5.2)$$

The Lagrangian is invariant under gauge transformations

$$\phi \rightarrow e^{i\alpha(x)}\phi, \quad A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha. \quad (5.5.3)$$

The scalar potential is assumed to be of the form

$$V(\phi^*\phi) = \frac{\lambda}{4} \left(|\phi|^2 - v^2 \right)^2. \quad (5.5.4)$$

The minimum is clearly given by

$$|\phi_0|^2 = v^2, \quad (5.5.5)$$

so we can choose ϕ_0 to be real by setting

$$\langle\phi\rangle = \phi_0 = v. \quad (5.5.6)$$

Fluctuations around the minimum can be included by considering

$$\phi(x) = e^{i\xi(x)} (\eta(x) + v) \quad (5.5.7)$$

so that the kinetic term for ϕ becomes

$$D^\mu \phi D_\mu \phi^* = \partial^\mu \eta \partial_\mu \eta + (\eta + v)^2 (\partial^\mu \xi + e A^\mu)^2 \quad (5.5.8)$$

and similarly for the potential

$$V(\eta) = \frac{\lambda}{4} \left((\eta + v)^2 - v^2 \right)^2 = \lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4. \quad (5.5.9)$$

At this point, we count a single massive field η of mass

$$m_\eta = \sqrt{2\lambda v^2} \quad (5.5.10)$$

as well as a massless field ξ since there is no quadratic term in ξ in the scalar potential.

Actually the only way that ξ appears in the Lagrangian is in the combination $\partial_\mu \xi + e A_\mu$ as can be seen from (5.5.8).

Since we are working with a U(1) gauge theory with unfixed gauge, we can use this freedom to modify the above result accordingly.

That is, we fix the gauge (known as *unitary gauge*) by redefining

$$A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \xi \quad (5.5.11)$$

which gives rise to a Lagrangian

$$\mathcal{L}[A_\mu, \phi] \rightarrow \mathcal{L}_{\text{pert}}[A_\mu, \eta] = \mathcal{L}^{\text{quadratic}} + \mathcal{L}^{\text{interaction}} \quad (5.5.12)$$

with

$$\mathcal{L}^{\text{quadratic}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial^\mu\eta\partial_\mu\eta - \lambda v^2\eta^2 + \frac{1}{2}e^2v^2A^\mu A_\mu, \quad (5.5.13)$$

$$\mathcal{L}^{\text{interaction}} = -\lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{2}(\eta^2 + 2v\eta)A^\mu A_\mu. \quad (5.5.14)$$

The first line encodes a massive vector field, but no massless Goldstone mode ξ !

The spectrum is therefore given by

- a massive gauge field with three degrees of freedom with two from the transverse polarisation from the original massless spin 1 vector boson and an additional from the “Goldstone” mode acting now as longitudinal polarisation
- and a massive scalar η which we call the *Higgs boson*.

This is the Higgs mechanism.

Higgs Mechanism

For a spontaneously broken gauge symmetry the transverse degrees of freedom of the broken symmetry gauge fields $A_\mu(x)$ combine with the Goldstone bosons $\xi(x)$ to complete the degrees of freedom of a massive vector field with the Goldstone modes becoming the longitudinal components.

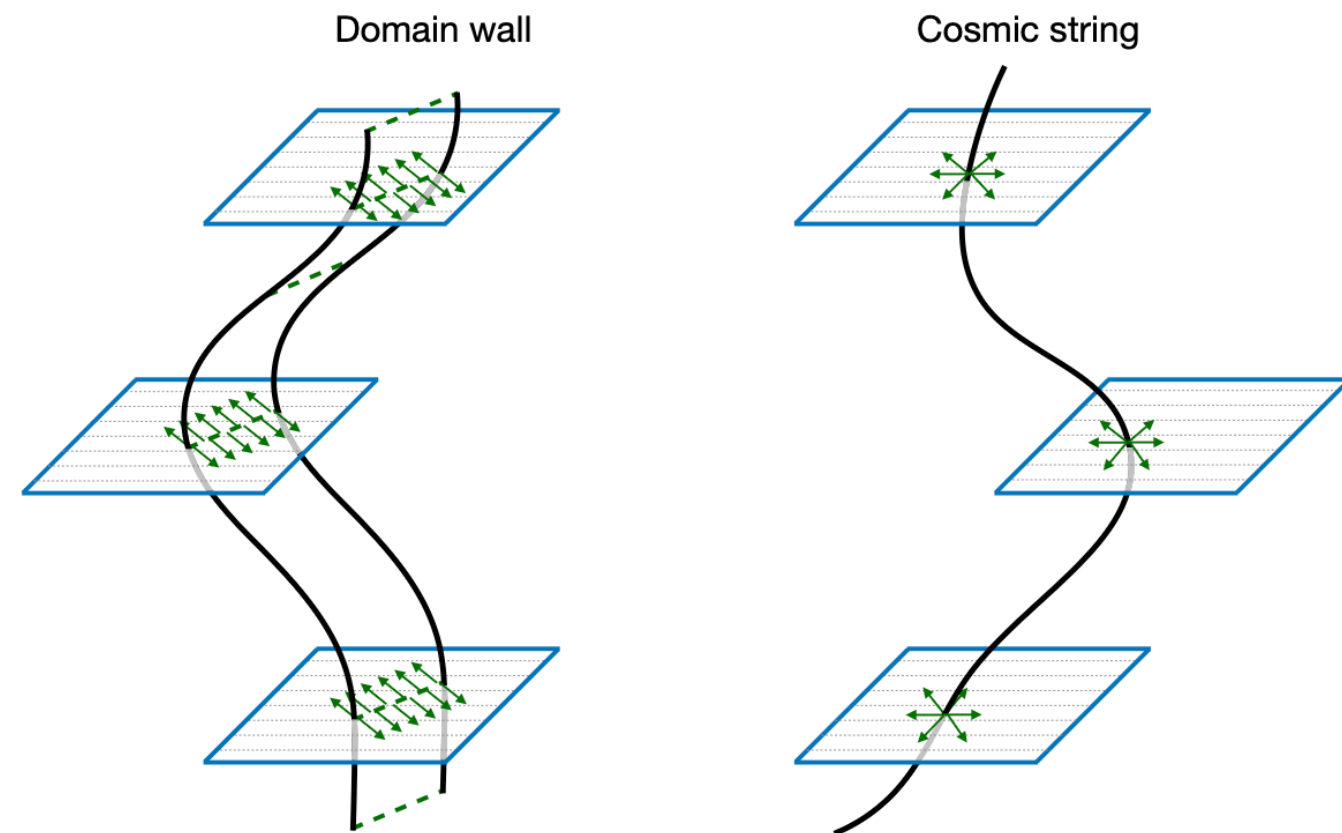
The gauge fields of the unbroken symmetries remain massless.

The remaining scalar degrees of freedom are massive corresponding to the Higgs bosons.

Aside: Cosmic strings

In general, similar to the case of discrete symmetry breaking, there may be topological defects corresponding not to domain walls, but to what is called as cosmic strings, see Fig. 5.6.

Figure 5.6: In our 3D physical space there appear topological defects separating regions of different vacua (represented here by the arrows as in ferromagnetism). For an SSB potential with two vacua the domain walls separate the different domains. These are topological defects interpolating between different vacua for a discrete set of vacua. In the figure on the right, we illustrate the case for the continuous symmetry breaking with continuous vacuum degeneracy. The corresponding topological defects are one-dimensional and are known as cosmic strings or vortices.



As discussed earlier, domain walls arise from the presence of two different vacua where some regions in physical 3-dimensional space could be either in one vacuum or the other.

The domain wall then corresponds to the separation between the two phases.

In the current example of a continuous group, the vacuum manifold is not only two points, but a whole circle containing infinitely many vacua.

Then in a given plane we may have vacua in all directions, giving rise to a string since the topology of the vacuum manifold (a circle) is non-trivial.

Cosmic strings may be relevant in early universe cosmology.

Aside: Superconductivity

Even though the Abelian Higgs model provides the simplest concrete example of the Higgs mechanism at work, just by itself it is not yet particularly interesting for particle physics, but it turns out to be relevant for superconductivity.

In this case the Abelian Higgs model has been shown to be an effective description of superconductivity.

Superconductor

Definition: A superconductor is a material for which the U(1) of electromagnetism is spontaneously broken.

In terms of an effective field theory, known as Landau-Ginzburg, a scalar field ϕ with $\langle \phi \rangle \neq 0$ is identified with the Cooper pair of two electrons $\psi_e \psi_e$ moving in the material.

Even though electrons are fermions, a scalar can be obtained from a pair of electrons.

At low temperatures this composite field may condense $\langle \phi \rangle = \langle \psi_e \psi_e \rangle \neq 0$, thereby breaking the electromagnetic U(1) to a discrete Z_2 symmetry (since the electric charge of the condensing field is $2e$).

It then gives a mass to the photon and induces a new phase for the corresponding material in which there is a current with essentially no resistance.

In fact, a magnetic field inside the material is energetically unfavorable (since the $A_\mu A_\mu$ term adds a substantial component to the energy so it minimizes at $A_\mu = 0$ which in turn implies zero magnetic field $B = 0$).

This is known as the *Meissner effect*.

Also, the relevant scales in the superconductor can be interpreted in terms of this model: the physical penetration depth of the magnetic field in the material can be estimated to be proportional to the mass of the photon and the correlation length corresponds to the mass of the Higgs field η .

Cosmic string defects correspond to vortices in the material, etc

Let us briefly try to see why there is a superconducting behavior.

The Lagrangian can be written as a function of $A_\mu + \partial_\mu \xi/e$.

The electric current and charge density are:

$$J^i = \frac{\partial \mathcal{L}}{\partial A^i}, \quad J^0 = \frac{\partial \mathcal{L}}{\partial A^0} = e \frac{\partial \mathcal{L}}{\partial \dot{\xi}}. \quad (5.5.15)$$

We can see that $\dot{\xi}$ acts as a conjugate variable to J^0 .

Therefore, we find

$$\dot{\xi} = e \frac{\partial \mathcal{H}}{\partial J^0} \propto \text{voltage} \quad (5.5.16)$$

in terms of the Hamiltonian $\mathcal{H}$ since voltage is the variation of energy with respect to the charge density at a given point.

From here we can conclude that we can have a time-independent configuration with stationary current with $\dot{\xi} = 0$ implying zero voltage which is essentially the definition of superconductivity.

5.5.2 A Non-Abelian Example

Let us briefly see how the Higgs mechanism can be straightforwardly extended to the non-Abelian case.

Let us consider the $N = 3$ of the $O(N)$ model discussed above.

We promote the model to a gauge theory with generators $T_{jk}^i = -i\epsilon_{ijk}$

$$\mathcal{L}[A_\mu^i, \phi] = -\frac{1}{4} \left(F_{\mu\nu}^i \right)^2 - \frac{1}{2} D_\mu \phi \cdot D^\mu \phi - V(\phi) \quad (5.5.17)$$

with $\phi^2 = \phi \cdot \phi$; $\phi^4 = (\phi^2)^2$ and $(D_\mu)_{ij} = \delta_{ij} \partial_\mu + g\epsilon_{ijk} A_\mu^k$ with $i, j, k = 1, 2, 3$.

We consider the potential

$$V_\pm(\phi) = \pm \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4, \quad \lambda > 0. \quad (5.5.18)$$

For V_+ the minimum sits at $\phi = (0, 0, 0)^T$ describing a standard $O(3)$ Yang-Mills theory with three massless gauge fields A_μ^i with $i, j = 1, 2, 3$ and three real massive scalars ϕ_i with a total of $2 \times 3 + 3 = 9$ degrees of freedom.

For V_- , the minimum sits at a non-zero VEV which we can choose as

$$\langle \phi \rangle_0 = \phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}. \quad (5.5.19)$$

This breaks the symmetry to O(2) or U(1).

Again, we can consider fluctuations around the minimum

$$\phi = \exp \left[\frac{i}{v} (\xi_1(x)T_1 + \xi_2(x)T_2) \right] \begin{pmatrix} 0 \\ 0 \\ v + \eta(x) \end{pmatrix}. \quad (5.5.20)$$

We can go to unitary gauge by setting

$$\phi \rightarrow \phi' = \exp \left[-\frac{i}{v} (\xi_1 T_1 + \xi_2 T_2) \right] \phi = \begin{pmatrix} 0 \\ 0 \\ v + \eta(x) \end{pmatrix}. \quad (5.5.21)$$

The Lagrangian then becomes

$$\mathcal{L}_{\text{pert}}[A_\mu^i, \eta] = -\frac{1}{4} \left(F_{\mu\nu}^i \right)^2 - \frac{1}{2} \partial^\mu \eta \partial_\mu \eta - m^2 \eta^2 - \frac{g^2 v^2}{2} \left(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu} \right) + \mathcal{L}^{int.} \quad (5.5.22)$$

with a massive Higgs field η , massive vector fields A_μ^1, A_μ^2 and a massless gauge field A_μ^3 corresponding to the unbroken symmetry.

In summary we have seen that the problems of massless gauge bosons and massless Goldstone bosons solve each other by the Higgs mechanism.

Furthermore the presence of the physical, massive, Higgs field $\eta(x)$ takes care of the perturbative unitarity problem mentioned before in the sense that given that gauge theories are renormalizable even after symmetry breaking is implemented, the corresponding left-over theory is UV complete.

This means that a low-energy theory of massive vector fields obtained after SSB is different from a low-energy theory of massive vector fields not based on gauge invariance.

For an explicit calculation to illustrate how perturbative unitarity is recovered by an spontaneously broken gauge theory we have to wait for the next part.

Before that, we will consider another SSB example and then the crucial aspect of chiral gauge theories known as anomalies.

5.5.3 SSB in an SU(2) gauge theory*

For illustrative purposes, let us consider another example, namely SU(2) gauge theory coupled to a two component complex scalar field

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{5.5.23}$$

The generators of SU(2) are given by the Pauli matrices

$$\tau^a = \frac{1}{2} \sigma^a . \tag{5.5.24}$$

The associated Lagrangian is defined as

$$\mathcal{L} = -\frac{1}{4}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{Tr}((D^\mu\phi)^\dagger(D_\mu\phi)) - \frac{1}{2}\lambda\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2. \quad (5.5.25)$$

The bold face notation on the sheet simply corresponds to putting the components F^a_μ and A^a_μ into a vector with the conventional definition of scalar and cross product.

We work in conventions where

$$D_\mu = \partial_\mu \mathbb{1}_2 + igA^a_\mu \tau^a \quad (5.5.26)$$

$$F_{\mu\nu} = F^a_{\mu\nu} \tau^a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \right) \tau^a. \quad (5.5.27)$$

The fact that the indices run over $a = 1, 2, 3$ and that ϵ^{abc} appears in the last term as the structure constants of SU(2) explains the appearance of the cross product on the sheet.

Spontaneous symmetry breaking

Regardless of the notation, we are interested in understanding the breaking pattern of the theory at minima of the scalar potential $V(\phi)$.

Hence, we need to have a closer look at

$$V(\phi) = \frac{\lambda}{2} \left(\phi^\dagger\phi - \frac{1}{2}v^2 \right)^2. \quad (5.5.28)$$

The first derivative is given by

$$\frac{dV}{d\phi_i} = \lambda \phi_i^* \left(|\phi_1|^2 + |\phi_2|^2 - \frac{1}{2}v^2 \right) . \quad (5.5.29)$$

The two stationary points are given by

$$\tilde{\phi}_0 = 0 , \quad \phi_0^\dagger \phi_0 = \frac{v^2}{2} . \quad (5.5.30)$$

One easily verifies that the Hessian matrix at $\tilde{\phi}_0$ has only negative eigenvalues for $\lambda, v > 0$ corresponding to a maximum.

In contrast, at ϕ_0 the Hessian has eigenvalues

$$\nu_1 = \nu_2 = \nu_3 = 0 , \quad \nu_4 = \lambda |\phi_0|^2 = \frac{\lambda v^2}{2} . \quad (5.5.31)$$

This implies that the Hessian is positive semi-definite and thus ϕ_0 describes a local minimum.

Now that we have established that the minimum is given by

$$\phi_0^\dagger \phi_0 = \frac{v^2}{2} \quad (5.5.32)$$

we can discuss the breaking pattern in more detail.

First, keep in mind that the potential in (5.5.28) is non-negative for $\lambda > 0$ and also

$$V(\phi)\Big|_{\phi=\phi_0} = 0 \quad (5.5.33)$$

is the minimal energy.

The vacuum manifold is given by

$$\mathcal{M}_0 = \left\{ \phi_0 \text{ such that } \phi_0^\dagger \phi_0 = \frac{v^2}{2} \right\}. \quad (5.5.34)$$

Since $\phi^\dagger \phi \geq 0$, we must have $v^2 \geq 0$ for any symmetry breaking to occur.

At the minimum (5.5.32), we e.g. set

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (5.5.35)$$

Before we start any actual calculation, we should first look at what Goldstone's theorem tells us.

We can understand the constraint for the vacuum manifold

$$\frac{v^2}{2} = \phi_0^\dagger \phi_0 = a^2 + b^2 + c^2 + d^2 \quad (5.5.36)$$

as the defining equation for a 3-sphere S^3 .

Hence, we deduce

$$\mathcal{M}_0 = S^3 \quad \Rightarrow \quad \dim(\mathcal{M}_0) = 3 = \dim(G) - \dim(H) \quad (5.5.37)$$

and we expect 3 massless Goldstone bosons after spontaneous symmetry breaking.

We started with $G = \text{SU}(2)$ with $\dim(G) = 3$ which is why

$$\dim(H) = 0 \quad (5.5.38)$$

and the symmetry is completely broken.

The three massless Goldstone modes become the longitudinal degree of freedom of the new massive gauge bosons.

This means that, whenever SSB occurs in a gauge theory, $\dim(\mathcal{M}_0)$ counts the number of massive gauge bosons after SSB in unitary gauge.

As it stands, our theory lives in a state expanded around $\tilde{\phi}_0 = 0$ corresponding to a saddle point as shown above.

We would like to understand the theory from the perspective of an observer at the minimum (5.5.35).

Hence, we consider fluctuations around (5.5.35) by defining

$$\phi = \begin{pmatrix} a + ib \\ c + id \end{pmatrix} \quad (5.5.39)$$

as well as some element $U \in \text{SU}(2)$ as

$$U = \exp \left(i\alpha_i \tau^i \right) . \quad (5.5.40)$$

By definition, U has 3 degrees of freedom associated with the individual generators.

These can be used to remove 3 degrees of freedom in ϕ which then fixes a gauge.

This allows us to write

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + f \end{pmatrix} . \quad (5.5.41)$$

for some *real* scalar field f .

It is crucial to keep in mind that *only because we fix a particular gauge, we write ϕ in the form (5.5.41)*.

This gauge is referred to as unitary gauge, and is the gauge in which a particular subset of gauge fields absorb the massless Goldstone bosons to become massive.

It is also important to remember that the the gauge is completely fixed and we cannot perform any more gauge transformations.

This is as it should be: gauge invariance is only the sign for a mathematical redundancy, but does not have any physical significance.

That is, to discuss physical phenomena, one first needs to specify a certain gauge.

Masses and interactions after SSB

To obtain the masses explicitly, we examine the quadratic piece of the Lagrangian.

First, we compute for (5.5.41) the terms $\sim f^2$

$$\phi^\dagger \phi = \frac{1}{2} (v^2 + 2vf + f^2) \quad (5.5.42)$$

and therefore

$$V(\phi) = \frac{\lambda}{2} \left(vf + \frac{f^2}{2} \right)^2 = \frac{\lambda}{2} v^2 f^2 + \frac{\lambda}{2} v f^3 + \frac{\lambda}{8} f^4. \quad (5.5.43)$$

We find that f has mass

$$m_f^2 = \lambda v^2 \quad (5.5.44)$$

as expected from the eigenvalues of the Hessian (5.5.31) (up to factors of 2).

Next, we look at the covariant derivative term for the complex doublet ϕ which reads

$$D_\mu \phi = \partial_\mu \phi + ig A_\mu^a \tau^a \phi \frac{1}{2\sqrt{2}} \begin{pmatrix} ig(A_\mu^1 - iA_\mu^2)(v + f) \\ 2\partial_\mu f - igA_\mu^3(v + f) \end{pmatrix}. \quad (5.5.45)$$

We are then able to compute (keeping in mind that v, f and A_μ^a are real-valued)

$$(D^\mu \phi)^\dagger D_\mu \phi = \frac{1}{8} \left[g^2 ((A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^3)^2) (v^2 + 2vf + f^2) + 4(\partial_\mu f)^2 \right]. \quad (5.5.46)$$

We then read off

- the kinetic term $\sim (\partial_\mu f)^2$ for the massive real scalar f ,
- the mass terms for **all** the gauge fields A_μ^a ,
- a cubic interaction vertex $\sim f(A_\mu^a)^2$ of coupling strength

$$\lambda_3 = \frac{g^2 v}{2}, \quad (5.5.47)$$

- and a quartic interaction vertex $\sim f^2(A_\mu^a)^2$ of coupling strength

$$\lambda_4 = \frac{g^2}{2}. \quad (5.5.48)$$

The masses for the gauge particles are

$$m_A^2 = \frac{g^2 v^2}{4}. \quad (5.5.49)$$

Each gauge boson becomes massive and the symmetry is completely broken.

Even though the symmetry is completely broken in the vacuum, there remains a trace of its original presence.

If we look closely at the cubic (5.5.47) and quartic coupling (5.5.48), we obtain

$$\frac{\lambda_3}{\lambda_4} = v = \frac{m_f}{\sqrt{\lambda}}. \quad (5.5.50)$$

Measuring m_f and λ , i.e., the mass of f and its quartic self interaction determines the ratio of both couplings which would not be true for generic cubic and quartic interactions!

Observations of this kind can therefore be seen as a hint for spontaneous symmetry breaking.

5.6 Anomalies

Finally, let us discuss aspects of symmetries in quantum theories which are different from classical theories.

It was a profound observation in the 1960's and 1970's that not all classical symmetries survive in a quantum theory.

This leads us to the notion of anomalies:

Anomalies

Classical symmetries are called anomalous if they are broken in the quantum theory.

To understand how anomalies arise, recall that in the quantum theory we work with path integrals of the form

$$\int \mathcal{D}\Phi_\alpha e^{iS[\Phi_\alpha, \partial\Phi_\alpha]} . \quad (5.6.1)$$

Let us assume that the classical action $S[\phi_\alpha, \partial\phi_\alpha]$ is invariant under $\phi_\alpha \rightarrow \phi'_\alpha$.

For this symmetry to survive at the quantum level, the path integral above also needs to be invariant.

This necessitates in particular that the measure $\mathcal{D}\phi_\alpha$ itself is invariant.

Otherwise, we speak of the presence of an anomaly.

In general, we distinguish two types of anomalies:

- *Anomalies of a global symmetry* which lead to a breaking of the symmetry in the quantum theory.

This implies that the corresponding classical conservation law does not hold in the quantum theory.

A typical example in the Standard Model is baryon number which is not conserved in nature, i.e., it is anomalous, while the difference of baryon and (total) lepton number is non-anomalous.

We will discuss these anomalies in more detail later in Sect. 8.3.2.

- *Anomalies for local symmetries*: the associated current is not conserved and the Ward identity correspondingly violated (recall our discussion in Sect. 4.1.1).

This leads to unphysical polarizations of the gauge field and the loss of Lorentz invariance.

In this sense, the theory becomes inconsistent.

As we will see, the notion of anomalies is most relevant for chiral theories where so-called “chiral anomalies” appear.

In the subsequent section, we explain in detail how anomalies arise in the presence of gauge fields by studying QED as an example.

We explicitly derive the anomaly there from first principles.

We then generalize anomalies to non-Abelian gauge theories.

We apply our insights to the Standard Model later in Sect. 8.3 proving that, despite being a chiral theory, the Standard Model is gauge anomaly free.

As another proof of how powerful the machinery of anomalies actually is, we also show in Sect. 8.3 that the charges of electron and proton are exactly equal implying charge quantization in nature which also guarantees the existence of electrically neutral atoms – everything that we are made of.

5.6.1 Abelian gauge theories – derivation of the anomaly

Let us consider QED as an illustrative example

$$\mathcal{L}_{QED}[A, \psi] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} (\mathrm{i}\not{\partial} - e\not{A} - m) \psi . \quad (5.6.2)$$

The Dirac spinor ψ can be separated into left- and right-handed components such that

$$\bar{\psi} (\mathrm{i}\not{\partial} - e\not{A} - m) \psi = \bar{\psi}_L (\mathrm{i}\not{\partial} - e\not{A}) \psi_L + \bar{\psi}_R (\mathrm{i}\not{\partial} - e\not{A}) \psi_R - m\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L . \quad (5.6.3)$$

In the limit $m \rightarrow 0$, the theory is invariant under the two transformations

$$\psi \rightarrow \mathrm{e}^{\mathrm{i}\alpha} \psi , \quad \psi \rightarrow \mathrm{e}^{\mathrm{i}\beta\gamma_5} \psi \quad (5.6.4)$$

or equivalently for ψ_L, ψ_R

$$\psi_L \rightarrow \mathrm{e}^{\mathrm{i}(\alpha-\beta)} \psi_L , \quad \psi_R \rightarrow \mathrm{e}^{\mathrm{i}(\alpha+\beta)} \psi_R . \quad (5.6.5)$$

The conserved currents associated with these two symmetries are the *vector current*

$$J_v^\mu = \bar{\psi} \gamma^\mu \psi \quad (5.6.6)$$

and the *axial current*

$$J_{ax}^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi . \quad (5.6.7)$$

In the limit $m = 0$, both currents are conserved

$$\partial_\mu J_\nu^\mu = \partial_\mu J_{ax}^\mu = 0, \quad (5.6.8)$$

while for $m \neq 0$

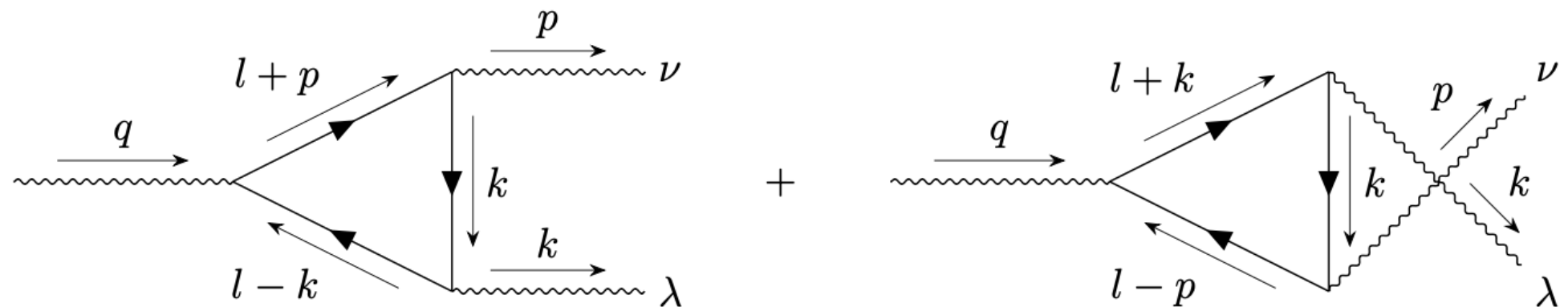
$$\partial_\mu J_\nu^\mu = 0, \quad \partial_\mu J_{ax}^\mu = 2im\bar{\psi}\gamma^5\psi. \quad (5.6.9)$$

So, only the vector current is actually conserved in both limits.

We will see now that even in the massless limit the axial current conservation will be broken in the quantum theory.

This was originally seen by considering the triangle diagrams shown in Fig. 5.7 which, when computed, lead to $\partial_\mu J_{ax}^\mu \neq 0$.

Figure 5.7: Triangle diagrams of a three photon process containing a single loop of fermions.



For gauge currents the corresponding Ward identities fail to cancel in a way proportional to the divergence of the axial current, signaling the breaking of gauge symmetry and inconsistency of the theory.

However there is a very general way to understand the anomaly directly from the path integral.

I choose to present this proof which illustrates the generality of the problem and the power of path integral methods.

This was done by Fujikawa in 1979.

To derive the anomaly, let us consider the path integral for massless QED

$$\mathcal{Z}_{QED} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{iS_{QED}[A,\psi]}, \quad S_{QED}[A,\psi] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi} \not{D} \psi \right) \quad (5.6.10)$$

with the classical symmetry

$$\psi \rightarrow e^{i\alpha} \psi, \quad \psi \rightarrow e^{i\beta \gamma_5} \psi, \quad S_{QED} \rightarrow S_{QED}. \quad (5.6.11)$$

To understand the transformation properties of the measure, we consider the general transformations

$$\psi \rightarrow \Delta \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \Delta_c \quad (5.6.12)$$

such that

$$\mathcal{D}\bar{\psi} \mathcal{D}\psi \rightarrow (\mathcal{J}_c \mathcal{J})^{-1} \mathcal{D}\bar{\psi} \mathcal{D}\psi \quad (5.6.13)$$

with Jacobians

$$\mathcal{J} = \det(\Delta), \quad \mathcal{J}_c = \det(\Delta_c). \quad (5.6.14)$$

More explicitly, the Jacobian for Δ can be written as

$$\mathcal{J} = \det(\Delta) = e^{\text{Tr}(\ln(\Delta))} = e^{\int d^4x \langle x | \text{Tr}(\ln(\Delta(x))) | x \rangle} . \quad (5.6.15)$$

For $\Delta = e^{i\alpha}$, (5.6.13) remains invariant since the phases from $\mathcal{J}$ and $\mathcal{J}_c$ cancel, while for the axial transformations $\Delta = e^{i\beta\gamma^5}$ the phases add up.

Therefore, under the axial transformation the measure transforms as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \rightarrow \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i \int d^4x \beta(x) \mathcal{A}(x)} \quad (5.6.16)$$

where the anomaly coming from the Jacobian is:

$$\mathcal{A} = -2\text{Tr} \left[\gamma_5 \delta^4(x - x) \right] . \quad (5.6.17)$$

Note that even though $\text{Tr}(\gamma_5) = 0$, the $\delta^4(x \rightarrow x)$ factor is not well defined and we will need to regularize this term.

But before we get to this, let us first understand what this implies.

The path integral transforms as

$$\mathcal{Z}_{QED} \rightarrow \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp \left(i \int d^4x [\mathcal{L}_{QED} - J_{ax}^\mu \partial_\mu \beta + \beta(x) \mathcal{A}(x)] \right) \quad (5.6.18)$$

This means that infinitesimally the integral only over the fermions transforms as

$$\delta \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_{\text{QED}}} \rightarrow i \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \beta(x) (\partial_\mu J_{ax}^\mu + \mathcal{A}(x)) \exp \left(i \int d^4x [\mathcal{L}_{\text{QED}}] \right) \quad (5.6.19)$$

Therefore, instead of having the classical Noether current conservation law for the axial current $\partial_\mu J_{ax}^\mu = 0$, we have

$$\langle \partial_\mu J_{ax}^\mu \rangle = -\mathcal{A}(x) \quad (5.6.20)$$

In order to compute the anomaly let us introduce a regulator

$$\mathcal{A} = -2\text{Tr} \left[\gamma_5 \delta^4(x - x) \right] = -2 \lim_{y \rightarrow x} \text{Tr} \left\{ \gamma_5 F \left(-\frac{\not{D}^2}{\Lambda^2} \right) \right\} \delta^4(x - y) \quad (5.6.21)$$

where F is a well-behaved function such as a Gaussian.

Concretely we normalize it to $F(0) = 1$ and impose that F and its derivatives vanish at infinity.

D is the covariant derivative $D_\mu \equiv \partial_\mu - ieA_\mu(x)$ and Λ a cut-off scale.

Now, we apply a Fourier transformation

$$\begin{aligned}
\mathcal{A} &= -2 \int \frac{d^4 k}{(2\pi)^4} \lim_{y \rightarrow x} \text{Tr} \left\{ \gamma_5 F \left(-\frac{\not{D}^2}{\Lambda^2} \right) \right\} e^{ik \cdot (x-y)} \\
&= -2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left\{ \gamma_5 F \left(-\frac{(i\not{k} + \not{D})^2}{\Lambda^2} \right) \right\} \\
&= -2\Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left\{ \gamma_5 F \left(-\left(i\not{k} + \frac{\not{D}}{\Lambda} \right)^2 \right) \right\} .
\end{aligned} \tag{5.6.22}$$

We used the fact that $\mathcal{D}$ is a differential operator and rescaled $k \rightarrow \Lambda k$.

The argument of the function F is

$$-\left(i\not{k} + \frac{\not{D}}{\Lambda} \right)^2 = k^2 - \frac{ik \cdot D}{\Lambda} - \left(\frac{\not{D}}{\Lambda} \right)^2 . \tag{5.6.23}$$

In the Taylor expansion we can see that terms with less than four powers of γ matrices vanish because their trace vanishes.

This takes care of the positive powers of Λ in the limit $\Lambda \leftrightarrow \infty$.

Also higher factors than 4 powers of $1/\Lambda$ will vanish in this limit.

We are then left with

$$\mathcal{A} = - \int \frac{d^4 k}{(2\pi)^4} F''(k^2) \text{Tr} (\gamma_5 \not{D}^4)$$

We can now evaluate the ordinary integral

$$\int d^4 k F''(k^2) = i \int_0^\infty 2\pi^2 k^3 F''(k^2) dk = i\pi^2 \int_0^\infty ds s F''(s) = -i\pi^2 \int_0^\infty ds F'(s) = i\pi^2$$

where we have used the assumption that the function F is such that $sF''(s)$ and $F'(s)$ vanish at $s = \infty$.

Now, before calculating the trace we will need the following result

$$\not{D}^2 = \frac{1}{4} \{D^\mu, D^\nu\} \{\gamma_\mu, \gamma_\nu\} + \frac{1}{4} [D^\mu, D^\nu] [\gamma_\mu, \gamma_\nu] = D^2 - \frac{ie}{4} F^{\mu\nu} [\gamma_\mu, \gamma_\nu] . \quad (5.6.24)$$

Using the identity

$$\text{Tr} \{ \gamma_5 [\gamma_\mu, \gamma_\nu] [\gamma_\rho, \gamma_\sigma] \} = 16 i \epsilon_{\mu\nu\rho\sigma} , \quad (5.6.25)$$

we finally arrive at

$$\mathcal{A}(x) = -\frac{e^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(x) F^{\rho\sigma}(x) \quad (5.6.26)$$

Therefore, from the path integral above, we find that within the QED background the axial anomaly is given by

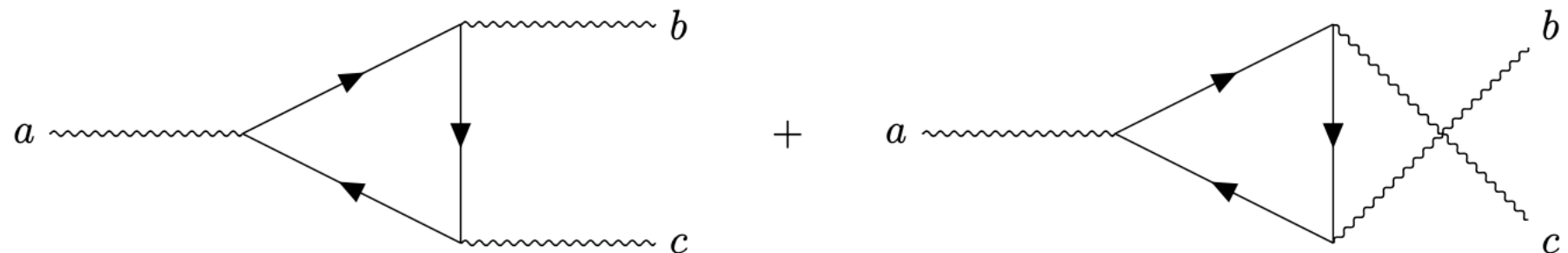
Axial Anomaly

$$\langle \partial_\mu J_{ax}^\mu \rangle = -\frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} . \quad (5.6.27)$$

More precisely, this is called the *Adler-Bell-Jackiw anomaly*.

It can be shown that this result is valid to all loop orders so the one-loop triangle diagrams below happen to capture the whole structure of the anomaly.

Figure 5.8: Triangle diagrams with loops



In general we can write the anomaly as

$$\partial_\mu \langle J_{ax}^\mu \mathcal{O} \rangle = -\frac{e^2}{16\pi^2} \langle \epsilon^{\alpha\lambda\beta\nu} F_{\alpha\beta} F_{\lambda\nu} \mathcal{O} \rangle . \quad (5.6.28)$$

What does this result imply?

Essentially the anomaly tells us that the axial current is not conserved.

The anomaly happens to be a total derivative itself.

That is, defining the quantity $G^\mu = \epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}$, known as the **Chern-Simons term**, we can see that $\partial_\mu G^\mu = 1/2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ and so we may construct a conserved quantity

$$\partial^\mu K_\mu = 0, \quad K^\mu = \langle J_{ax}^\mu \rangle + \frac{e^2}{8\pi^2} G^\mu. \quad (5.6.29)$$

However, notice that since G^μ depends explicitly on A_ν it is not gauge invariant.

Also if we had used $F(-\not{\partial}^2)$ instead of $F(-\not{D}^2)$ as a regulator, we would have obtained a vanishing anomaly term.

But again, this would not have been gauge invariant.

We may then say that either the anomaly breaks the chiral symmetry or the gauge symmetry, but we cannot find a way to preserve both.

The study of anomalies provides useful techniques to eliminate inconsistent and identify consistent theories.

Anomalies are relevant particularly for chiral theories such as the Standard Model.

This will be explained momentarily in more detail below.

5.6.2 Anomalies in non-Abelian gauge theories

Let us next look at the generalization of the above results to non-Abelian gauge theories.

Now, we consider the diagrams in Fig. 5.8 with different gauge fields as external legs.

The resulting amplitude $\mathcal{A}(a, b, c)$ is proportional to the divergence of the current as mentioned above, but with constant of proportionality depending on the couplings.

In gauge theories, the latter are proportional to the charges for Abelian theories and the generators of the corresponding gauge group for the non-Abelian case, that is

$$\mathcal{A}(a, b, c) \sim A^{abc} \dots \quad (5.6.30)$$

where we defined the proportionality constants

$$A^{abc} = \text{Tr}(T^a T^b T^c) + \text{Tr}(T^a T^c T^b) = \text{Tr}(T^a \{T^b, T^c\}) = A(R) d^{abc}. \quad (5.6.31)$$

Here, d^{abc} is the group invariant defined in the previous part (cf. (4.4.36)), but normalized for the fundamental representation, i.e., $A(R) = 1$ for the fundamental and $A(R) = -1$ for the anti-fundamental representation.

Altogether, the non-Abelian generalization of Eq. (5.6.27) reads

Anomaly in non-Abelian gauge theories

$$\partial_\mu J_{ax}^{a,\mu} = \left(\sum_{\text{left}} A(R_l) - \sum_{\text{right}} A(R_r) \right) \frac{g^2}{128\pi^2} d^{abc} \epsilon^{\alpha\lambda\beta\nu} F_{\alpha\beta}^b F_{\lambda\nu}^c. \quad (5.6.32)$$

The structure of this result reveals an important fact mentioned several times before: if left- and right-handed representations within a given theory are the same, i.e., the theory is non-chiral, the anomaly cancels automatically.

Vice versa, this means that cancellation of anomalies puts non-trivial constraints on chiral theories like the Standard Model or its possible extensions.

But there is more: if R is (pseudo) real $(T^a)^* = -S T^a S^{-1}$, we find that the right hand side of (5.6.32) vanishes because

$$A^{abc} = \text{Tr}(\{(T^c)^*, (T^b)^*\}(T^a)^*) = -A^{abc} = 0. \quad (5.6.33)$$

It so happens that most groups arising in physics exhibit pseudo-real representations and therefore the associated gauge theories have no anomalies.

This applies to the following compact groups: $SO(2n + 1)$, $SO(4n)$ for $n \geq 2$, $Usp(2n)$ for $n \geq 3$, G_2 , F_4 , E_7 , E_8 .

Other groups, such as E_6 also have $A_{abc} = 0$.

This leaves only $SU(n)$ and products of $U(1)$'s as potentially dangerous.

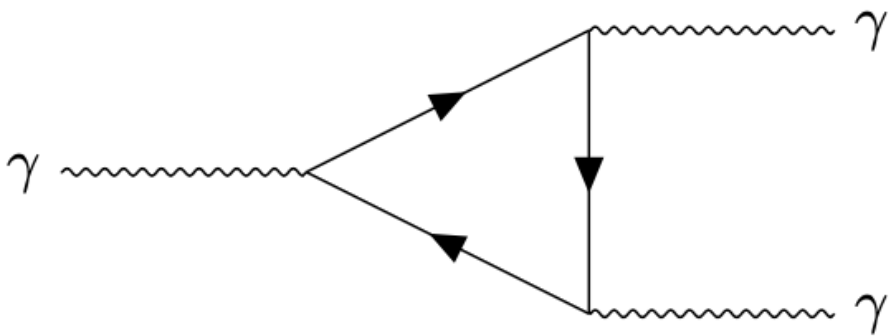
These are precisely the groups relevant for the Standard Model.

In some way, nature likes to live dangerously!

But for physicists this is actually great news: *the spectrum of the Standard Model is highly constrained and far from arbitrary.*

Let us mention two examples before arguing that the Standard Model is free of (gauge) anomalies later in Sect. 8.3.

First, for $U(1)^3$ the resulting constraint reads:

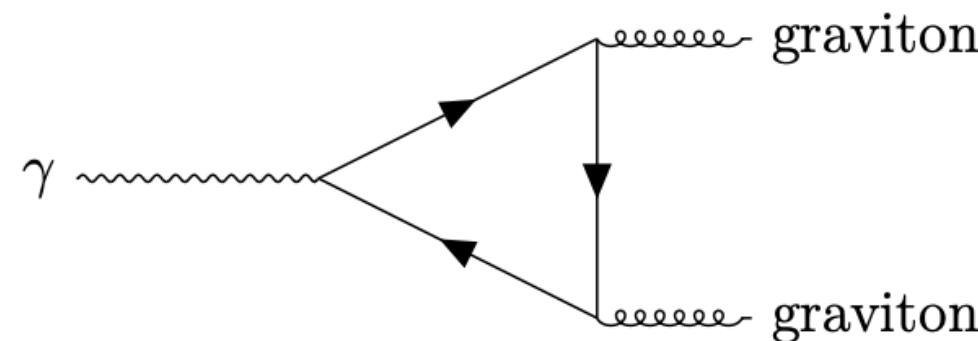


A Feynman diagram showing a triangle loop of fermions. An incoming photon line (wavy) on the left splits into two fermion lines (solid with arrows) that form a triangle. The top and bottom fermion lines meet at a vertex from which two outgoing photon lines (wavy) emerge, labeled γ .

$$\sim A \sim \text{Tr}(Q^3) \begin{cases} = 0 & \text{consistent} \\ \neq 0 & \text{inconsistent} \end{cases}$$

Hence, the sum over the cubed charges has to vanish.

For gravity, we get graviton



A Feynman diagram showing a triangle loop of fermions. An incoming photon line (wavy) on the left splits into two fermion lines (solid with arrows) that form a triangle. The top and bottom fermion lines meet at vertices from which two outgoing graviton lines (coiled) emerge, labeled "graviton".

$$\sim A \sim \text{Tr}(Q) \begin{cases} = 0 & \text{consistent} \\ \neq 0 & \text{inconsistent} \end{cases}$$

This means that the sum over charges needs to vanish, i.e., it implies charge conservation.

PART 6 ELECTROWEAK INTERACTIONS

Electroweak interactions

The search for a renormalizable theory of weak interactions was the right strategy but, as it turned out, not for the reasons we originally thought.

Steven Weinberg

We now apply the theoretical framework developed in previous parts to describe weak interactions.

Up to this point, we have mainly relied on the fact that the underlying theories – quantum mechanics and special relativity – define the fundamental constituents of matter and their interactions through irreducible representations of the Poincaré group.

To describe the interactions among these elementary particles, the principles of locality and unitarity guided us to quantum field theory as the fundamental framework.

We have argued that, to ensure interactions consistent with Lorentz invariance, local gauge symmetries are necessary: abelian symmetries lead to the well-known case of QED, while non-abelian or Yang-Mills theories have a more complex structure.

When coupled to scalar fields, renormalizable Lagrangians allow for at least two distinct phases of the corresponding gauge theory: unbroken gauge theories or those with spontaneous symmetry breaking (SSB).

In the SSB phase, Yang-Mills gauge fields can acquire mass through the Higgs mechanism and mediate short-range interactions.

The goal here is to build a gauge theory for the weak interactions of nature, using the Higgs mechanism to give mass to the weak gauge bosons.

Specifically, we will emphasize some of the key decisions that were essential in shaping the Standard Model as we understand it today.

To make tangible progress, we must now rely not only on theoretical consistency but also on crucial experimental input — after all, physics is fundamentally an experimental science.

6.1 Weak, but powerful

Weak interactions were discovered essentially once radioactivity was found.

The prime example of a process involving the weak interactions is β -decay ($n \rightarrow p + e^- + \bar{\nu}_e$).

The name weak was given to differentiate them from the strong interaction.

We know that the electromagnetic interactions are responsible to keep electrons bound to nuclei to make atoms and the strong interactions are responsible to keep the quarks inside hadrons together and indirectly the protons and neutrons bound in the nuclei of all the elements.

Weak interactions do not lead to bound states.

However they are crucial for our existence since they are responsible for the leading interactions that give rise to the thermonuclear fusion in stars, including the Sun, from which we receive the energy to live.

As a side note, this process is a beautiful illustration of the 4 interactions at play: gravity dominating at large distances and being attractive induces the formation of stars and galaxies, the electromagnetic and strong interactions compete in the process of having two protons close enough to then allow the weak interaction to start the fusion chain reaction.

Starting with the fusion of two protons (nuclei of Hydrogen) $p + p \rightarrow d + e^- + \nu_e$ with $d = pn$ the deuteron or heavy Hydrogen nucleus.

Two of these processes produce an α particle (2 protons and 2 neutrons) or nucleus of Helium, releasing energy in the form of neutrinos and photons.

This starts the proton chain reaction of fusion interactions that make the stars burn and also create the heavy elements like Oxygen, Carbon, Nitrogen, etc. from which we are all made.

Weak interactions are the source of any interaction involving neutrinos and also play an important role in the early history of the Universe.

Describing weak interactions within a consistent QFT was very challenging and the success in achieving it will remain as one of the greatest scientific achievements of all time.

The electroweak theory is attributed primarily to Steven Weinberg and Abdus Salam with important work earlier by Sheldon Glashow and John Ward.

They all built from the original work of Enrico Fermi, George Sudarshan, Robert Marshak, Murray Gell-Mann, Richard Feynman and others.

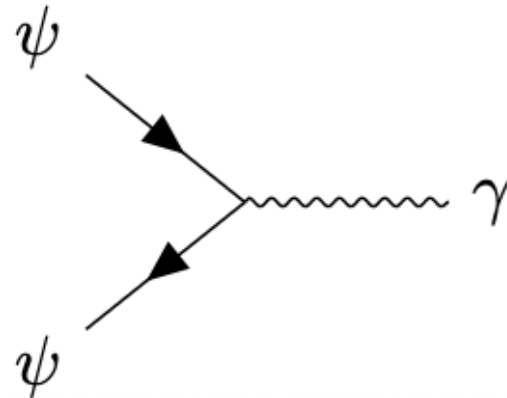
6.2 Electromagnetic and weak processes

Before we study the weak interactions, let us briefly summarize some basic facts about QED and compare with the weak interactions that were already established at the time of developing the electroweak theory.

6.2.1 Electromagnetic interactions

As we said several times before, QED is based on an unbroken U(1) gauge theory, leading to long-range interactions mediated by the corresponding gauge field: the *photon* γ .

The basic interaction vertex between photons and matter fields of spin 1/2 like electrons

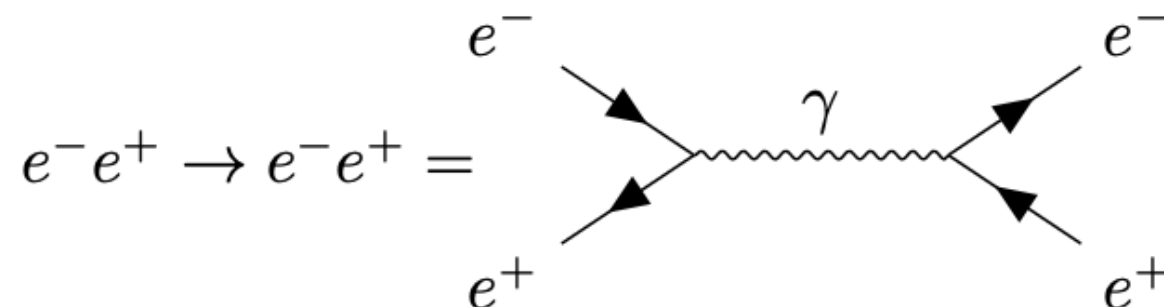


involves two fermions and one photon associated with the conserved current

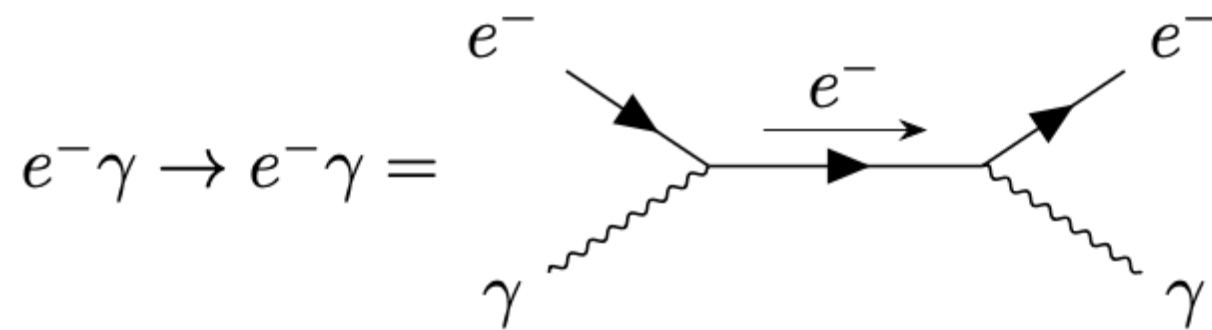
$$J^\mu = \bar{\psi} \gamma^\mu \psi . \quad (6.2.1)$$

In the action (5.6.10), the interaction term is as usual represented by $A_\mu J^\mu$.

Concrete physical processes include electron-positron annihilation



or Compton scattering



The power of QED is simply that it successfully describes all these different interactions observed in nature with calculations that can be contrasted with experiments.

6.2.2 Weak interactions

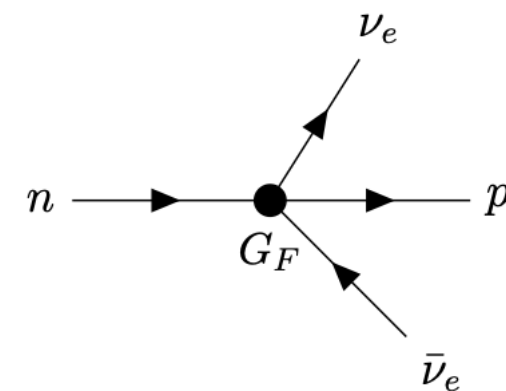
Next, let us try to answer the following question: *can the weak interactions be described in a way similar to QED with one or more mediating fields playing the role of the photons?*

Fermi, V-A theory and chirality

For the Weak Interactions, let us first take on a historical perspective.

In the early 1900's, many physical processes had been observed that guided physicists towards a proper description of a consistent theory of the weak interactions.

One typical interaction was the β - or n-decay $n \rightarrow p + \bar{\nu}_e +$



(6.2.2)

Fermi proposed a concrete formalism to describe these interactions.

This is the famous *Fermi-interaction* with effective Lagrangian description

$$\mathcal{L}_{\text{Fermi}} = G_F \psi_1 \psi_2 \psi_3 \psi_4 . \quad (6.2.3)$$

Here, the ψ_i represent each of the particles in the interaction and G_F determines the coupling.

The fermion fields ψ_i have mass dimension $[\psi] = 3/2$ which is why the mass dimension of G_F is

$$[G_F] = -2 . \quad (6.2.4)$$

Experimentally the coupling had been found to be $G_F = 1.164 \times 10^{-5} \text{ GeV}^{-2}$.

Clearly, this interaction is non-renormalizable which is why this is only a good description at energies $E \ll G_F^{-1/2}$.

Given the absence of an underlying theory at the time, people considered the most general Lorentz invariant 4-fermion interactions which take the form

$$\mathcal{L} = \sum_i g_i J^i J_i , \quad J^i = \bar{\psi} \mathcal{O}^i \psi , \quad g_i \sim G_F \quad (6.2.5)$$

with the currents J^i written in terms of operators $\mathcal{O}^i$.

Here i is a generic index that labels the operators listed below.

These operators are classified depending on how they transform under Lorentz transformations as follows:

- $\mathcal{O} = 1$ scalar
- $\mathcal{O} = \gamma^5$ pseudo-scalar
- $\mathcal{O} = \gamma^{\mu\nu}$ tensor
- $\mathcal{O} = \gamma^\mu$ vector (V)
- $\mathcal{O} = \gamma^\mu \gamma^5$ axial vector (A)

A detailed analysis of several experiments in the 1950's led Marshak and Sudarshan to identify the correct combination that describes all the weak interaction processes as V - A.

This included interactions, e.g., for the β or n—decay

$$g\bar{\psi}_p\gamma^\mu(1-\gamma^5)\psi_n\bar{\psi}_e\gamma^\mu(1-\gamma^5)\psi_\nu + h.c.. \quad (6.2.6)$$

The V - A theory describes an important concept for the weak interactions, namely chirality (parity violation).

This difference between left- and right-handed fermions, as can be seen from the presence of only $1-\gamma^5$ and not $1+\gamma^5$ in the operators, is a very important property of the weak interactions that comes from observations.

Any theory describing weak interactions has to have this property, as Lee and Yang had observed earlier.

Therefore, from the dependence on $1 - \gamma^5$, it is usually said that weak interactions are left-handed and so chiral.

This was the first successful description of weak interactions at low energies.

But it became apparent pretty quickly that this cannot be the right theory to describe weak interactions at all energies, because the theory is non-renormalizable.

Moreover, even though the calculation of physical quantities, like cross sections and decay rates fit well with experiments at energies $E \ll G_F^{-1/2}$, for higher energies the theory gives diverging results which are clearly against experiments.

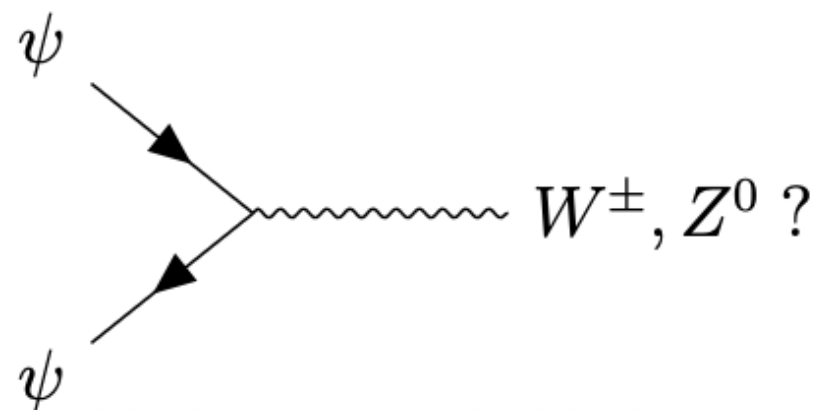
This suggested that the four-fermion vertex with dimensionful coupling G_F vertex should be replaced by a three-point interaction and propagator for mediator particles as in QED.

We need a consistent description that unlike the Fermi theory, is valid at all energies.

This will turn out to be unique: *a spontaneously broken gauge theory*.

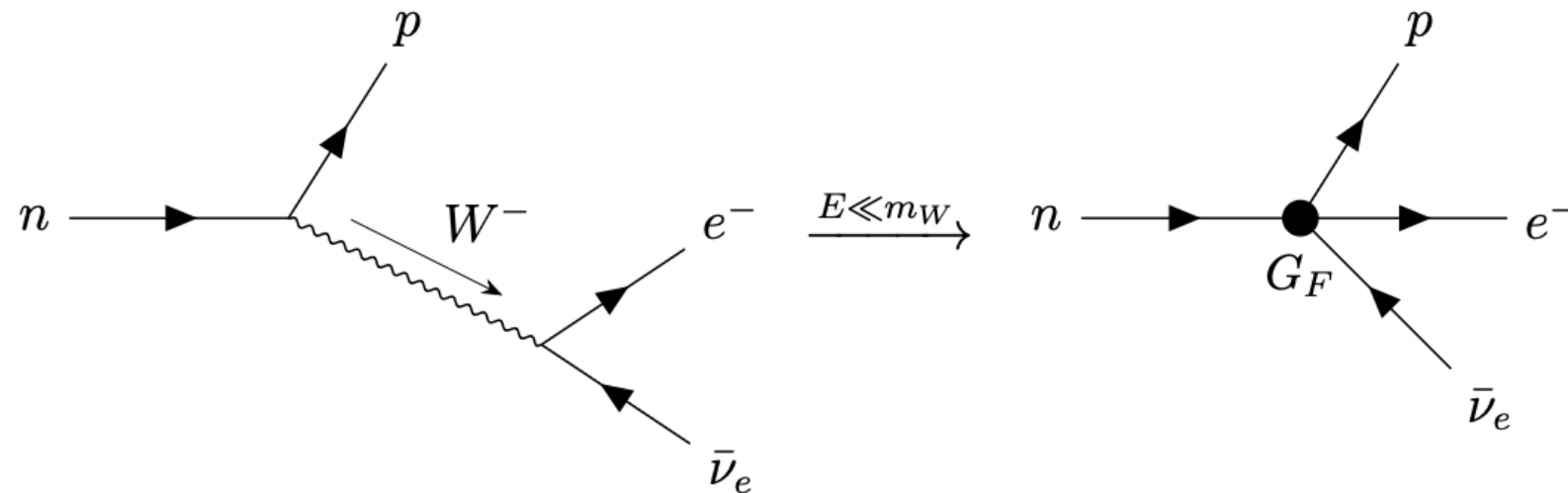
Mediators for weak interactions

As in QED, we expect a mediator and a basic interaction vertex of the form



Contrary to QED the mediating fields may have an electric charge in order to have charge conservation at each vertex and we may name them W^+ , W^- and Z^0 with the superscript representing the electric charge.

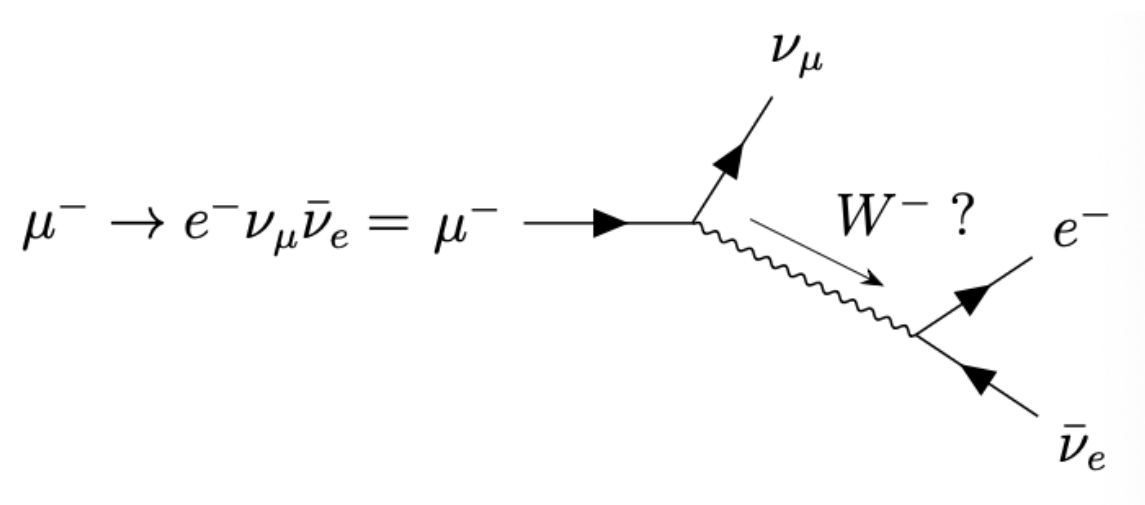
The β^- - or n-decay process may then be seen as:



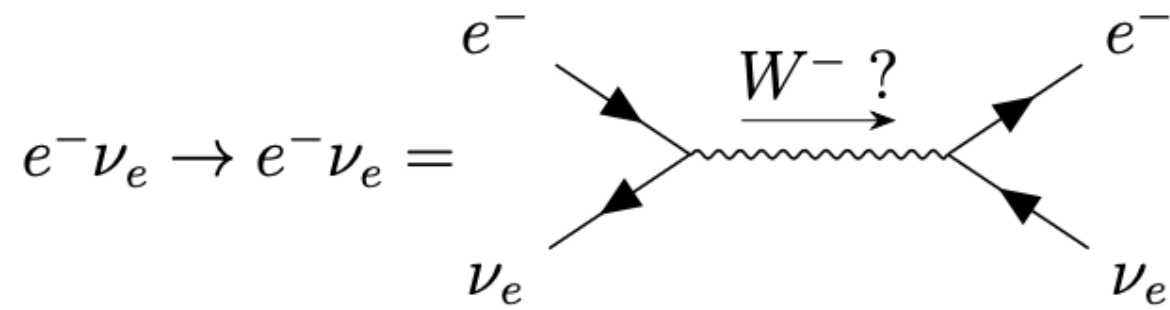
Similarly, other weakly interacting processes may be considered.

We distinguish processes according to the particles involved:

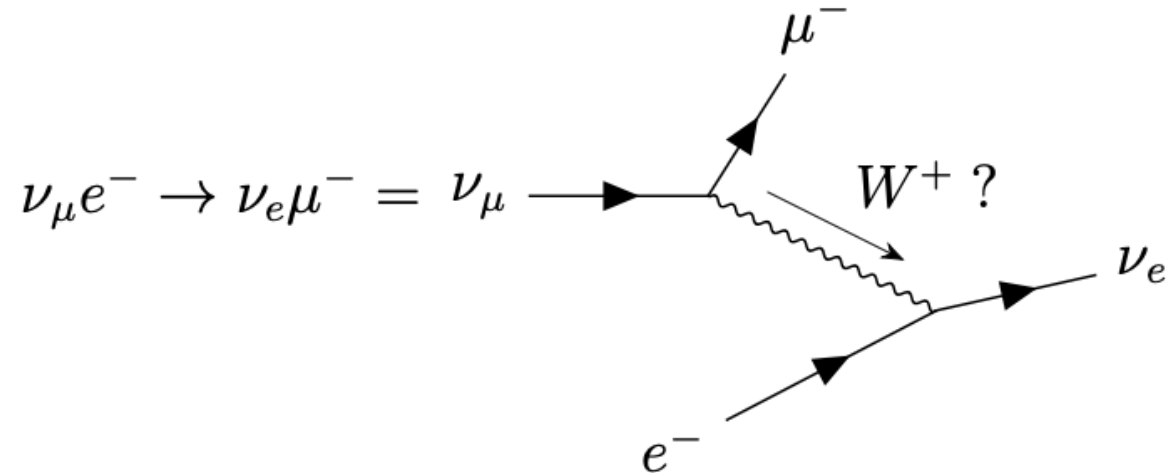
- a) *Leptonic*: processes that include only leptons in the initial and final states that include *charged current* processes such as



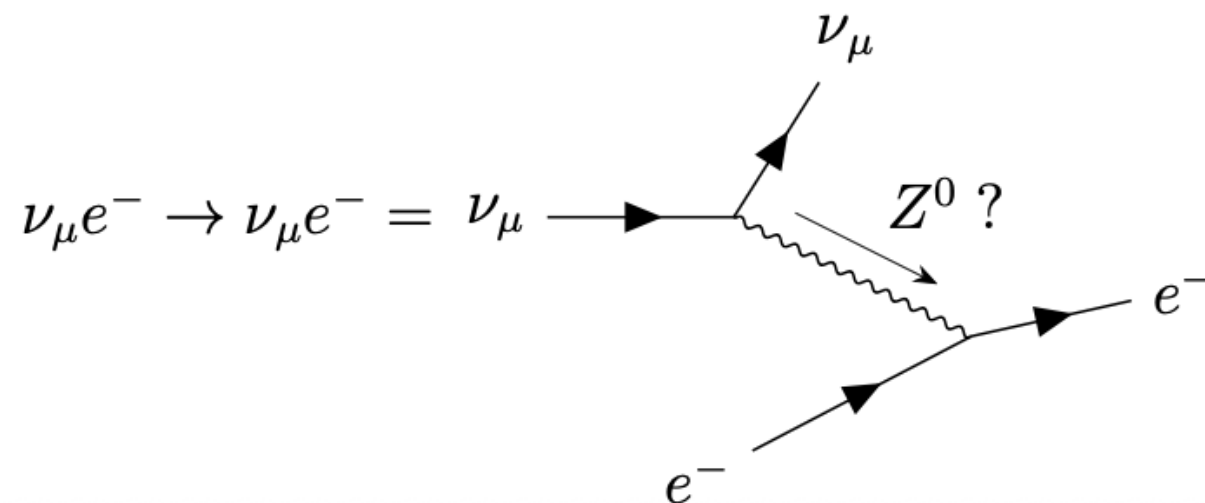
and



and

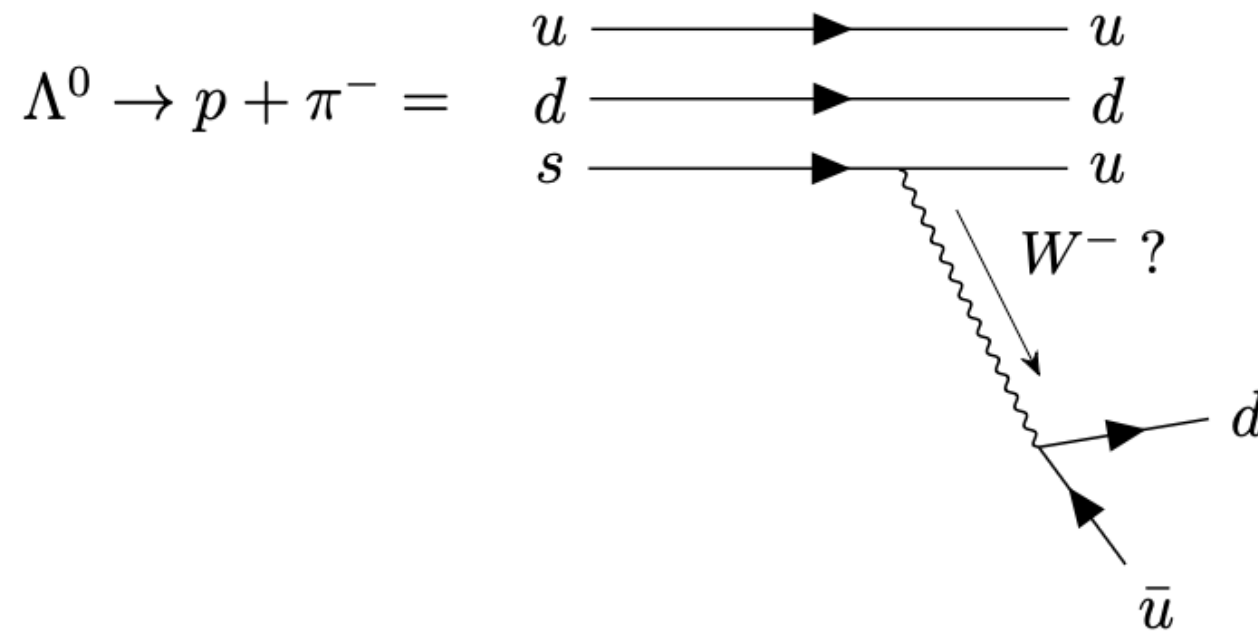


and non-charged or *neutral* current processes as



- b) *Semi-leptonic*, that is weak interactions that involve leptons and hadrons in initial or final states, e.g., β^- or n-decay.

- c) *Non-leptonic*, that is weak interactions that do not involve leptons in the initial or final state, e.g.,



Here we write the hadrons (Λ^0 and proton) in terms of their component quarks.

Again, neutral and charged mediators are needed.

Some comments are in order.

At the time when these processes were first hypothesized or even measured, it remained unclear what the mediators of the weak force would be.

Charge conservation suggests that there should be three types of particles with electric charge 0 or ± 1 .

But then, as we already discussed in Sect. 5.1, we face problems with the loss of unitarity due to the presence of massive spin-1 fields.

As we will see in this part of the lectures, this is where spontaneous symmetry breaking enters the stage.

E.g. at the perturbative level, we can show explicitly how the aforementioned unitarity problem is resolved through SSB in Sect. 6.4.2.

In all of these processes, we identify a potential mediating particle that we called $W^\pm, Z^0$, all with question marks since up to this point we cannot specify their nature but their name already indicates the corresponding value of their electric charge, assuming charge is conserved on each interaction.

We immediately observe that, contrary to electromagnetic interactions in which there is only one mediating particle, the photon, weak interactions require at least three particles.

Interactions mediated by $W^\pm$ are called *charged current* interactions and those mediated by Z^0 are called *neutral current* interactions (only charged currents were observed before the Glashow-Weinberg-Salam theory was developed).

Since the weak interactions are short-ranged (and the decay rates are such that the corresponding decaying particles have long lifetimes), the mediating particles $W^\pm, Z^0$ are expected to be very massive (recall Yukawa theory in which a mediating particle of mass m would give rise to a force of the type $\sim e^{-mr}/r^2$ which reduces to the standard $1/r^2$ when $m = 0$ as in electromagnetism).

For $m \neq 0$ the interaction decays exponentially fast with distance and would therefore be short-range as observed for the weak interactions).

At small energies, that is energies much smaller than the mass of the corresponding mediating particle, propagators involving heavy gauge bosons can be replaced by an effective 4-fermion interaction as shown above.

6.3 Identifying the model for SSB

In this section, we turn to the question of what the structure of the gauge theory needs to be.

Rather than simply writing down the answer given by the *Electroweak Theory*, we try to discuss and justify the proper framework for the gauge fields and fermions in that sector.

For simplicity, we consider only the neutrino-electron system with 4-component Weyl spinors

- electron e_L, e_R
- neutrino ν_L .

Here, the right- and left-handed electron fields are defined as

$$e_{L,R} = \frac{1}{2} \left(1 \mp \gamma^5 \right) e \quad (6.3.1)$$

in terms of the Dirac spinor e .

To identify the gauge group, Lorentz invariance requires to put those fields with the same Lorentz transformation properties into a single representation.

Hence, we split the fields into left- and right-handed content

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad e_R. \quad (6.3.2)$$

The largest possible group allowing for such representations is

$$G = \text{SU}(2)_L \times \text{U}(1)_L \times \text{U}(1)_R \quad (6.3.3)$$

with associated generators T^a , $a = 1, 2, 3$, Q_L and Q_R .

The actions of the individual generators on the fields can be described as follows:

- $T^a = \sigma^a/2$ acts on the doublet field $(\nu_L, e_L)^T$ in the fundamental of $\text{SU}(2)$, but *not* on e_R which is only a singlet of $\text{SU}(2)$, i.e., in the trivial representation.
- The individual $\text{U}(1)$ -generators act on the fields in the following way

$$Q_L \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad Q_L e_R = 0$$

$$Q_R \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = 0, \quad Q_R e_R = e_R.$$

We define *hypercharge* Y as the combination

$$Y = -Q_R - Q_L \quad (6.3.4)$$

and the *electron lepton number* L_e as

$$L_e = 2Q_L + Q_R. \quad (6.3.5)$$

These two operators act on the fields in the following way

$$\begin{aligned} Y \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, & Y e_R &= -e_R \\ L_e \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} &= \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, & L_e e_R &= e_R. \end{aligned} \tag{6.3.6}$$

Since the eigenvalues of L_e are all equal to 1, it means that L_e counts the number of leptons.

Finally, we observe that the combination

Electric charge

$$Q = T^3 + Y \tag{6.3.7}$$

gives

$$Q \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} 0 \\ -e_L \end{pmatrix}, \quad Q e_R = -e_R \tag{6.3.8}$$

where the action of Q on the fields is understood in the associated representation of the fields.

Since both left- and right-handed electrons have eigenvalue -1 and the neutrino zero eigenvalue under Q , it is identified with the electric charge.

Altogether, we found the group

$$G = \text{SU}(2)_L \times \text{U}(1)_Y \times \text{U}(1)_{L_e} . \quad (6.3.9)$$

Since there is no evidence for the existence of a gauge field associated with $\text{U}(1)_{L_e}$, we will forget about it for the moment, but come back to it later, cf. section 9.2.1.

6.4 Glashow-Weinberg-Salam Model

Let us now start with the core part of this part of the lectures which is the Weinberg-Salam model with useful earlier work of Glashow, who shared the Nobel prize with them and hence the name *GSW model*.

This is the description of the weak interactions by means of a gauge symmetry with SSB.

This is probably the most complex component of the Standard Model.

We will construct it by following several steps to make it more comprehensible.

We promise that the effort to follow all the details pays off by the impressive success of this model which is theoretically sound and experimentally tested with great precision.

Following the previous section, we concentrate on building-up a gauge theory based on the group

$$G_{EW} = \text{SU}(2)_L \times \text{U}(1)_Y \quad (6.4.1)$$

with general group elements $U \in G_{EW}$ defined as

$$U = e^{i\alpha^a T^a} e^{i\beta Y} \quad a = 1, 2, 3 . \quad (6.4.2)$$

The gauge fields are defined as

- $W_\mu = W_\mu^a T^a$ for $SU(2)_L$ with field strength

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu] . \quad (6.4.3)$$

Under infinitesimal gauge transformations, they behave as

$$\delta W_\mu^a = \frac{1}{g} \partial_\mu \alpha^a - \epsilon^{abc} \alpha^b W_\mu^c . \quad (6.4.4)$$

- B_μ for $U(1)_Y$ with field strength

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (6.4.5)$$

transforming as

$$\delta B_\mu = \frac{1}{g'} \partial_\mu \beta . \quad (6.4.6)$$

Keep in mind that g and g' are two independent gauge couplings associated with either $SU(2)_L$ or $U(1)_Y$.

These are free parameters of the theory that eventually have to be determined experimentally.

Now, let us remind ourselves what requirements we want the above gauge theory to satisfy:

- SSB phase with 3 massive gauge bosons.
- Chirality: only left-handed fields feel the weak interactions.
- Massless gauge field corresponding to the photon.
- ...

6.4.1 Bosonic Lagrangian and SSB

In order to consider the possibility of spontaneous symmetry breaking, we add to the theory a scalar field transforming non-trivially under the action of G_{EW} .

We then introduce a complex scalar H as a doublet under $SU(2)_L$ and hypercharge conventionally chosen as $Y_H = 1/2$ so that

$$H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}_{Y_H=1/2} \quad (6.4.7)$$

NOTE: The choice of $Y_H = 1/2$ is at the moment arbitrary but we will see how well it fits when we discuss the couplings of H to matter fields.

with H_+ and H_0 complex components of the scalar field H .

We then start with six degrees of freedom coming from W^a_μ (the two polarization degrees of freedom for each value of a) and four degrees of freedom from H (a doublet with complex entries).

The purely bosonic part of the renormalizable Lagrangian is then

$$\mathcal{L}_B[W_\mu^a, B_\mu, H] = -\frac{1}{4} (W_{\mu\nu}^a)^2 - \frac{1}{4} B_{\mu\nu}^2 + D_\mu H (D^\mu H)^\dagger - V(H) \quad (6.4.8)$$

with scalar potential

$$V(H) = \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2, \quad v^2 = \frac{m^2}{\lambda}. \quad (6.4.9)$$

The gauge covariant derivative acting on H is here given by

$$D_\mu H = \partial_\mu H - ig W_\mu^a T^a H - \frac{i}{2} g' B_\mu H \quad (6.4.10)$$

where now the two last terms appear because of the product structure of G and the 1/2 in the last term corresponds to the hypercharge of H.

The gauge symmetry is broken for a non-trivial $\text{VEV}(H) \neq 0$.

For the scalar potential in (6.4.9), we have extrema at

$$\partial_H V = 2\lambda H^\dagger \left(H H^\dagger - \frac{v^2}{2} \right) = 0. \quad (6.4.11)$$

For SSB, the terms in bracket have to cancel.

As usual, let us pick one direction

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (6.4.12)$$

with v positive and expand around the vacuum as

$$H = \frac{1}{\sqrt{2}} e^{-i\xi^a(x)T^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \quad (6.4.13)$$

with $\xi^a(x)$ the 3 fields that will correspond to the Goldstone modes and one Higgs boson $h(x)$.

Plugging this back into the Lagrangian (do it!), we find, as in the previous chapter, that the $\xi^a(x)$ only appear in the combination

$$\partial_\mu \xi^a T^a + g W_\mu^a T^a + \frac{1}{2} g' B_\mu \quad (6.4.14)$$

which is why we can redefine the gauge fields using gauge transformations such that the Goldstone modes ξ^a are being absorbed.

This is nothing but fixing the gauge to be the unitary gauge where the massless fields ξ^a give rise to the longitudinal polarization of the massive spin-1 particles.

After SSB, the bosonic Lagrangian can be separated into two pieces

$$\mathcal{L}_B[W_\mu^a, B_\mu, H] \xrightarrow{\text{SSB}} \mathcal{L}_B^{\text{quadratic}}[W_\mu^a, B_\mu, h] + \mathcal{L}_B^{\text{interaction}}[W_\mu^a, B_\mu, h]. \quad (6.4.15)$$

Below, we carefully analyze the different contributions to this Lagrangian.

Bosonic Lagrangian – Quadratic Pieces

Let us begin with the discussion of the quadratic terms, namely

$$\begin{aligned}\mathcal{L}_B^{\text{quadratic}}[W_\mu^a, B_\mu, h] = & -\frac{1}{4} \left(W_{\mu\nu}^a\right)^2 - \frac{1}{4} B_{\mu\nu}^2 + \frac{1}{2} \partial^\mu h \partial_\mu h - m^2 h^2 \\ & + \frac{g^2 v^2}{8} \left[(W_\mu^1)^2 + (W_\mu^2)^2 + \left(\frac{g'}{g} B_\mu - W_\mu^3 \right)^2 \right].\end{aligned}\tag{6.4.16}$$

We diagonalize the mass matrix for the gauge fields by defining

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right)\tag{6.4.17}$$

$$Z_\mu = W_\mu^3 \cos(\theta_W) - B_\mu \sin(\theta_W)\tag{6.4.18}$$

$$A_\mu = W_\mu^3 \sin(\theta_W) + B_\mu \cos(\theta_W)\tag{6.4.19}$$

in terms of the Weinberg angle or weak mixing angle θ_W defined as

$$\cos(\theta_W) = \frac{g}{\sqrt{g^2 + (g')^2}}, \quad \sin(\theta_W) = \frac{g'}{\sqrt{g^2 + (g')^2}}.\tag{6.4.20}$$

NOTE: Z_μ is usually referred to as Z_μ^0 to specify that it has 0 electric charge. We will not write explicitly the superscript 0 for ease of notation.

The mass spectrum can be identified from (6.4.16) to be

$$\begin{array}{ll}
 m_h = \sqrt{2\lambda}v & \text{Higgs boson mass, experimentally } m_h \approx 125.2\text{GeV} \\
 m_{W_\mu^\pm} = \frac{vg}{2} & \text{W-mass, experimentally } m_W \approx 80.38\text{GeV} \\
 m_{Z_\mu} = \frac{v}{2}\sqrt{g^2 + (g')^2} & \text{Z-mass, experimentally } m_Z \approx 91.19\text{GeV} \\
 m_{A_\mu} = 0 & \text{Photon, experimentally } m_\gamma < 10^{-18}\text{eV} .
 \end{array} \tag{6.4.21}$$

Note that

$$m_W = m_Z \cos(\theta_W) < m_Z \tag{6.4.22}$$

is a prediction of the theory which is indeed confirmed by experiment.

The first question that comes to mind is why is A_μ massless?

And why does it correspond to the photon?

We know the answer to the first question: the symmetry group is not completely by $\langle H \rangle$.

To see this, we write $U = \mathbb{1} + i\alpha^a T^a + i\beta Y + \dots$ for $U \in \text{GEW}$ and consider

$$\delta\langle H\rangle = U\langle H\rangle - \langle H\rangle = \frac{iv}{2\sqrt{2}} \begin{pmatrix} \alpha^1 - i\alpha^2 \\ \beta - \alpha^3 \end{pmatrix}. \quad (6.4.23)$$

The unbroken symmetry group is then defined as $\delta(H) = 0$ and so $\alpha^1 = \alpha^2 = 0, \alpha^3 = \beta$.

That is, $U \in H \subset G_{EW}$ can be written as $U = e^{i\alpha^a T^a} e^{i\beta Y} = e^{i\beta(T^3+Y)} = e^{i\beta Q} \in U(1)$ in terms of the electric charge (6.3.7).

This then allows us to identify the unbroken gauge group as the electromagnetic $U(1)_{EM}$.

The breaking pattern is given by

SSB of the Electroweak theory

$$SU(2)_L \times U(1)_Y \xrightarrow{\langle H \rangle \neq 0} U(1)_{EM}. \quad (6.4.24)$$

This also fits the predictions of Goldstone's theorem since

$$\# \text{ Goldstone modes } \xi^a = \dim(SU(2)_L \times U(1)_Y) - \dim(U(1)_{EM}) = 3. \quad (6.4.25)$$

Next, we need to work out the charges of the physical fields.

Thereto, we consider global rotations $U \in U(1)_{\text{EM}}$ with $U = e^{i\beta Q} \sim 1 + i\beta Q$.

For the Higgs, we have then $\delta H = i\beta QH$ implying

$$\delta H = \left(\frac{i\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} H_+ \\ H_0 \end{pmatrix} = i\beta \begin{pmatrix} H_+ \\ 0 \end{pmatrix}. \quad (6.4.26)$$

We deduce that H_+ has electric charge +1 and H_0 charge 0 justifying their names.

For the gauge fields, we obtain

$$\delta W_\mu^1 = \frac{\partial_\mu \alpha^1}{g} - \epsilon^{132} \alpha^3 W_\mu^2 = \beta W_\mu^2, \quad \delta W_\mu^2 = \frac{\partial_\mu \alpha^2}{g} - \epsilon^{231} \alpha^3 W_\mu^1 = -\beta W_\mu^1 \quad (6.4.27)$$

where we used $\alpha^1 = \alpha^2 = 0, \alpha^3 = \beta$.

This implies

$$\delta W_\mu^\pm = \pm i\beta W_\mu^\pm \quad (6.4.28)$$

and therefore the charges of $W_\mu^\pm$ are $QW^\pm = \pm W_\mu^\pm$ justifying their definition.

Also, since $QW_\mu^3 = QB_\mu = 0$, the charge of Z_μ and A_μ is zero.

In order to define experimentally meaningful parameters, we observe that in $D_\mu H$ the terms gW_μ^3 and $g'B_\mu$ lead to terms of the form $g \sin(\theta_W)A_\mu + \dots$ and $g'\cos(\theta_W)A_\mu + \dots$

This motivates the definition of the *electromagnetic coupling*

$$e = g \sin(\theta_W) = g' \cos(\theta_W) . \quad (6.4.29)$$

We now treat the original parameters m, λ, g, g' in the original Lagrangian (6.4.16) for e, θ_W, m_h, m_W which need to be measured experimentally and with the remaining observables being predictions of the theory.

The 4 free parameters are determined experimentally to have the values

$$e = 0.303 , \quad \sin^2(\theta_W) = 0.223 , \quad g = \frac{e}{\sin(\theta_W)} = 0.64 , \quad g' = \frac{e}{\cos(\theta_W)} = 0.34 . \quad (6.4.30)$$

In terms of the fields A_μ, Z_μ^0 and $W_\mu^\pm$, the quadratic Lagrangian reads

$$\begin{aligned} \mathcal{L}_B^{\text{quadratic}}[W_\mu^a, B_\mu, h] = & -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}Z_{\mu\nu}^2 - \frac{1}{2} \left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \right) \left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- \right) \\ & + \frac{1}{2}m_Z^2 Z^\mu Z_\mu + m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} \partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 . \end{aligned} \quad (6.4.31)$$

This is a Lagrangian for one massless A_μ and three massive spin 1 fields $W_\mu^\pm, Z_\mu^0$ and one massive scalar h as a function of the 4 arbitrary parameters.

Note that here m_Z is not a free parameter, but determined by m_W and θ_W (recall that $m_W = m_Z \cos \theta_W$).

The total number of degrees of freedom now is three for each of the massive vector fields $W^\pm_\mu$ and Z_μ , two for the photon A_μ and one for the Higgs h adding up to the total of twelve degrees of freedom which matches our counting for the original fields W^a_μ , B_μ and H .

It is actually remarkable that this attempt to describe the physics of weak interactions leads not only to a consistent theory for the weak interactions, but also, as a bonus, the theory includes the electromagnetic interactions in a unified way.

Both interactions, mediated either by A_μ giving rise to QED or by $W^\pm_\mu$, Z^0_μ giving rise to the weak interactions, come from one and the same underlying theory, a spontaneously broken $SU(2)_L \times U(1)_Y$ gauge theory.

This unification of two interactions in one single theory is an achievement that may be comparable with Newton's unification of terrestrial and celestial gravitational interactions and the unification of electric and magnetic interactions within electromagnetism by Maxwell and Faraday.

For this reason this theory is often referred to as the electroweak theory.

Propagator for a massive vector field

Since we have found that both the Z_μ and $W^\pm_\mu$ bosons are massive, before we consider the interactions coming from the cubic and quartic terms, let us compute explicitly the propagator of a massive vector field that is constructed from the quadratic piece of the Lagrangian.

In order to be as general as possible we will compute the propagator for any massive vector field.

It would correspond in particular to the propagators for both Z_μ and $W^\pm_\mu$ in **unitary gauge**.

Let us start with the Lagrangian density for the massive vector field X_μ

$$\mathcal{L} = -\frac{1}{4}X^{\mu\nu}X_{\mu\nu} + \frac{1}{2}m^2X^\mu X_\mu. \quad (6.4.32)$$

In order to extract the propagator let us manipulate this expression as follows (in which we use integration by parts)

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial^\mu X^\nu - \partial^\nu X^\mu)(\partial_\mu X_\nu - \partial_\nu X_\mu) + \frac{1}{2}m^2X^\mu X_\mu \\ &= \frac{1}{2}X^\mu \left[(\partial^\alpha \partial_\alpha + m^2) \eta_{\mu\nu} - \partial_\mu \partial_\nu \right] X^\nu \\ &\equiv \frac{1}{2}X^\mu \mathcal{D}_{\mu\nu} X^\nu. \end{aligned} \quad (6.4.33)$$

Therefore, starting from the matrix $\mathcal{D}_{\mu\nu}$ and going to momentum space, we can read off the corresponding propagator as

$$\Delta_{\mu\nu} = D_{\mu\nu}^{-1}(p) = -\frac{1}{p^2 - m^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right). \quad (6.4.34)$$

Note that the massive case is in some sense simpler than the massless case since in the massless case the matrix $\mathcal{D}_{\mu\nu}$ has one zero eigenvalue and extracting the propagator is more difficult as we know from QED.

Also remember that for an arbitrary massive vector this is the propagator but if the massive vector comes from a broken gauge symmetry this propagator is only valid in unitary gauge.

This will play an important role later on.

Bosonic Lagrangian – Cubic and Quartic Interactions

Now that we have full control of the quadratic part of the bosonic Lagrangian, we can consider the interactions which can also be separated into two contributions depending on the number of interacting fields

$$\mathcal{L}_B^{\text{interaction}}[W_\mu^a, B_\mu, h] = \mathcal{L}_B^{\text{cubic}}[W_\mu^a, B_\mu, h] + \mathcal{L}_B^{\text{quartic}}[W_\mu^a, B_\mu, h] \quad (6.4.35)$$

where the cubic interactions are (written in terms of the physical fields $W_\mu^\pm, Z_\mu, A_\mu, h$):

$$\begin{aligned} \mathcal{L}_B^{\text{cubic}}[W_\mu^\pm, Z_\mu, A_\mu, h] &= ie \cot(\theta_W) \left[Z^{\mu\nu} W_\mu^+ W_\nu^- - \left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \right) Z^\mu W^{-\nu} + \left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- \right) Z^\mu W^{+\nu} \right] \\ &+ ie \left[F^{\mu\nu} W_\mu^+ W_\nu^- - \left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \right) A^\mu W^{-\nu} + \left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- \right) A^\mu W^{+\nu} \right] \\ &- g \frac{m_h^2}{4m_W} h^3 + \frac{2h}{v} \left[m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} m_Z^2 Z_\mu^2 \right]. \end{aligned} \quad (6.4.36)$$

The first line encodes the interactions between the massive gauge bosons, the second line the interactions between the massless photon with the charged, massive bosons $W^\pm_\mu$ and the third row the interactions involving the Higgs.

The quartic interactions are given by

$$\begin{aligned}
\mathcal{L}_B^{\text{quartic}}[W_\mu^a, B_\mu, h] = & \frac{1}{2} \frac{e^2}{\sin^2(\theta_W)} \left[W^{+\mu} W_\mu^+ W^{-\nu} W_\nu^- - W^{+\mu} W_\mu^- W^{+\nu} W_\nu^- \right] \\
& + e^2 \left[A^\mu W_\mu^+ A^\nu W_\nu^- - A_\mu^2 W^{+\nu} W_\nu^- \right] + e^2 \cot^2(\theta_W) \left[Z^\mu W_\mu^+ Z^\nu W_\nu^- - Z_\mu^2 W^{+\nu} W_\nu^- \right] \\
& + e^2 \cot(\theta_W) \left[A^\mu Z^\nu W_\mu^+ W_\nu^- + A^\mu Z^\nu W_\mu^- W_\nu^+ - 2W^{+\mu} W_\mu^- A^\nu Z_\nu \right] \\
& - \frac{g^2}{32} \frac{m_h^2}{m_W^2} h^4 + \left(\frac{h}{v} \right)^2 \left[m_W^2 W^{+\mu} W_\mu^- + \frac{1}{2} m_Z^2 Z_\mu^2 \right] .
\end{aligned} \tag{6.4.37}$$

As before, the first row encodes quartic interactions among the $W^\pm_\mu$, the second interactions of the $W^\pm_\mu$ with one of the gauge fields A_μ, Z_μ , the third interactions involving all gauge fields and the last row interactions between the Higgs itself as well as $W^\pm_\mu, Z_\mu$.

Notice that despite the length of the Lagrangians and the presence of many couplings between the individual fields, there are only four arbitrary parameters, i.e., e, θ_W, m_h, m_W which is why many of the predictions of this theory can be tested.

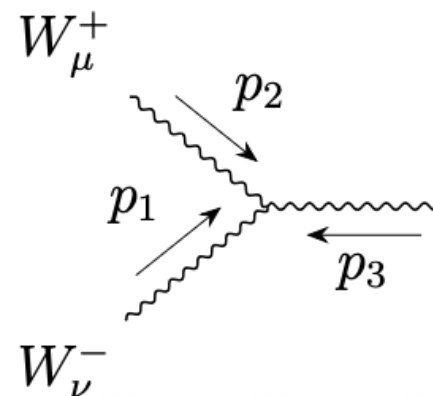
This is completely general observation: in an arbitrary theory involving e.g. a scalar field h , its cubic and quartic interactions need not be related to each other, while in the case of SSB (as in (6.4.36) and (6.4.37)) one observes that their ratio

$$\frac{\lambda_3^2}{\lambda_4} = 2m_h^2 \quad (6.4.38)$$

is proportional to the mass of h .

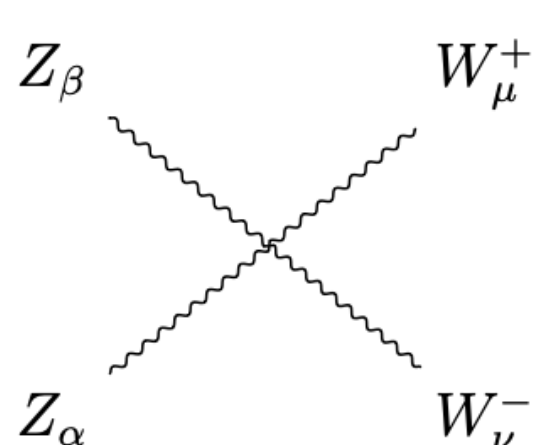
Let us work out *some* of the Feynman rules

- Cubic interactions $\sim W_\mu^+ W_\nu^- Z_\lambda$ originating from the first row in (6.4.36)



$$Z_\lambda = -ie \cot(\theta_W) \left[\eta^{\mu\nu} (p_1 - p_2)^\lambda + \eta^{\nu\lambda} (p_2 - p_3)^\mu + \eta^{\lambda\mu} (p_3 - p_1)^\nu \right] .$$

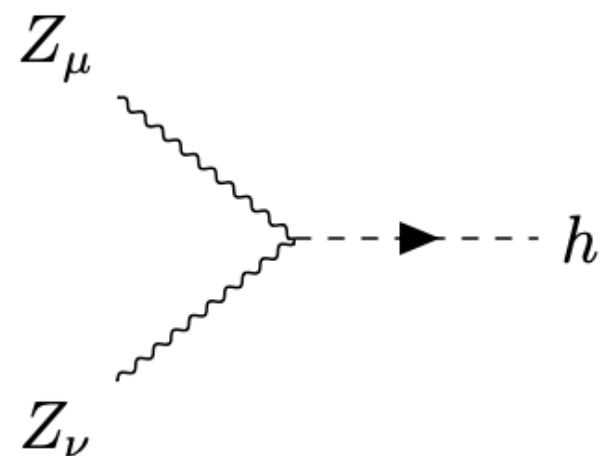
- Quartic interactions among the heavy gauge fields $\sim W_\mu^+ W_\nu^- Z_\alpha Z_\beta$ as obtained from the second term in the second row of (6.4.37) . We have to distinguish between the



interaction $\sim h W_\mu^+ W_\nu^-$

$$= ie^2 \cot^2(\theta_W) \left[\eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu} - 2\eta^{\alpha\beta} \eta^{\mu\nu} \right] .$$

and $\sim h Z_\mu Z_\nu$



$$= i \frac{e}{\sin(\theta_W)} \frac{m_Z^2}{m_W} \eta^{\mu\nu} = i \frac{e}{\sin(\theta_W) \cos^2(\theta_W)} m_W \eta^{\mu\nu}.$$

6.4.2 $W Z \rightarrow W Z$ Scattering and Perturbative Unitarity

Before we get to fermions, let us consider the amplitude for scattering longitudinally polarized gauge bosons $W^\pm$ and Z .

First, we introduce Mandelstam variables.

Recall that for any 2-2 scattering with external momenta p_1, p_2, p_3, p_4 it is convenient to work with the Mandelstam variables s, t, u

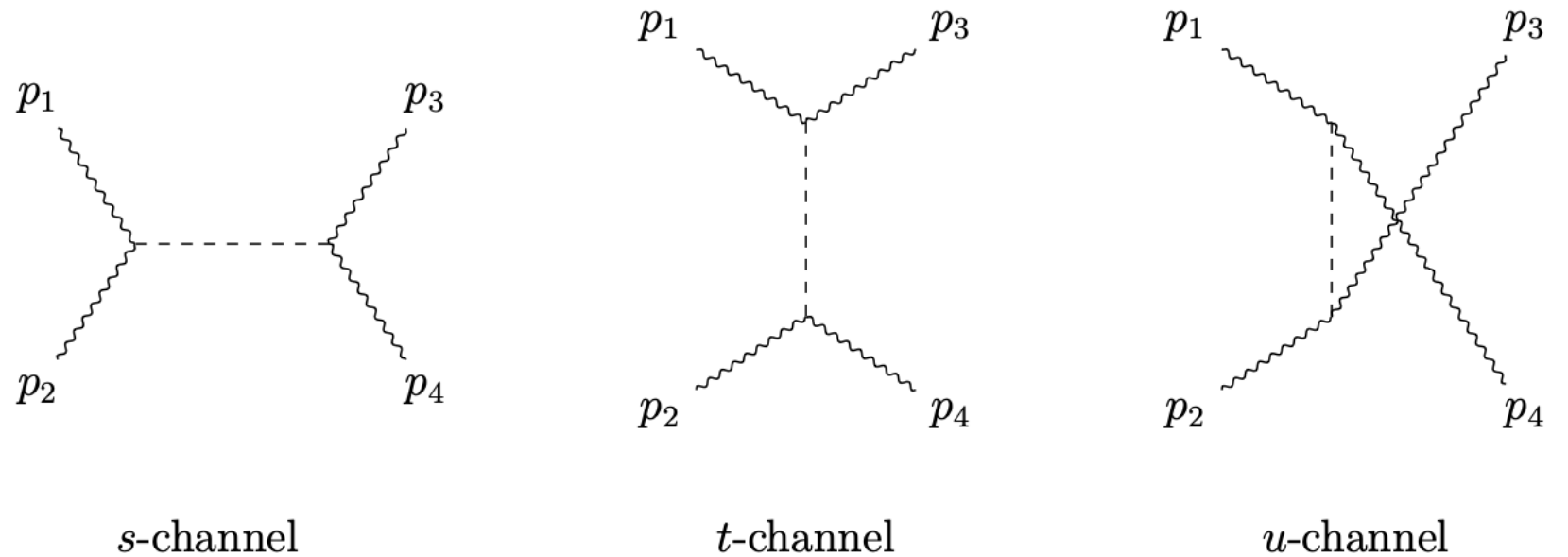
$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2, \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned} \tag{6.4.39}$$

satisfying the useful identity

$$s + t + u = \sum_{k=1}^4 m_k^2. \tag{6.4.40}$$

The corresponding Feynman diagrams, as depicted in Fig. 6.1, follow the s, t, u channels respectively.

Figure 6.1: Mandelstam variables and the corresponding s, t, u channels for any 2-2 scattering amplitude.



Let us now consider all the channels that contribute to the $W_L Z_L \rightarrow W_L Z_L$ scattering where here the sub-index L stands for longitudinal.

More explicitly, we compute the amplitude

$$\mathcal{M}(W_L Z_L \rightarrow W_L Z_L) = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \dots \quad (6.4.41)$$

channel by channel.

We will narrow down the missing pieces contributing to... in (6.4.41) below.

Since the amplitudes depend explicitly on the polarization vectors, let us try to choose a suitable basis that captures the fact that we are interested only on the longitudinal modes.

Recall that for a massive particle we can pick a frame such that $p^\mu = (E, 0, 0, p_z)$ ($E^2 - p_z^2 = m^2$) and so the polarization vectors are

$$\epsilon_{T1}^\mu = (0, 1, 0, 0), \quad \epsilon_{T2}^\mu = (0, 0, 1, 0), \quad \epsilon_L^\mu = \left(\frac{E}{m}, 0, 0, \frac{p_z}{m} \right). \quad (6.4.42)$$

where the subscripts T and L stand for transverse and longitudinal respectively.

In the limit $E \gg m$ the longitudinal vector is approximately $\epsilon_L^\mu \sim E/m(1, 0, 0, 1)$.

But since this gives $\epsilon_L \cdot p \neq 0$ we have to have an expression that is valid beyond leading order in a m/E expansion.

A convenient set of approximate (unnormalized) longitudinal polarization vectors for each particle satisfying $\epsilon_k \cdot p_k = 0$ can be constructed as

$$\begin{aligned} \epsilon_1^\mu &= \frac{1}{m_W} \left(p_1^\mu + \frac{2m_W^2}{t - 2m_W^2} p_3^\mu \right), & \epsilon_2^\mu &= \frac{1}{m_Z} \left(p_2^\mu + \frac{2m_Z^2}{t - 2m_Z^2} p_4^\mu \right) \\ \epsilon_3^\mu &= \frac{1}{m_W} \left(p_3^\mu + \frac{2m_W^2}{t - 2m_W^2} p_1^\mu \right), & \epsilon_4^\mu &= \frac{1}{m_Z} \left(p_4^\mu + \frac{2m_Z^2}{t - 2m_Z^2} p_2^\mu \right) \end{aligned} \quad (6.4.43)$$

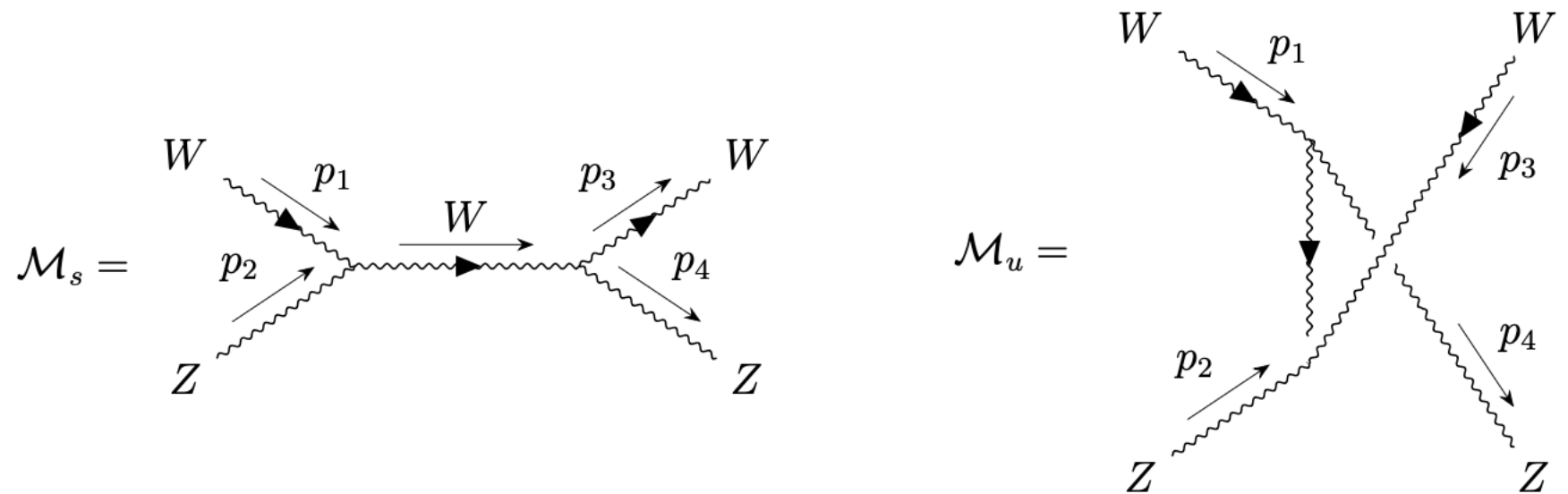
where t is the Mandelstam variable defined above.

Now we can compute the contribution of each channel to the amplitude (6.4.41):

- *s-channel*: Looking at the diagram on the left in Fig. 6.2, we can use the Feynman rules derived above as well as the expressions for the polarisation tensors in (6.4.43) to write down the expression for the *s*-channel contribution, namely

$$\begin{aligned}
\mathcal{M}_s &= (ie \cot \theta_W)^2 \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\alpha \epsilon_4^\beta \frac{1}{s - m_W^2} \left(-\eta_{\lambda\kappa} + \frac{(p_1 + p_2)_\lambda (p_1 + p_2)_\kappa}{m_W^2} \right) \\
&\quad \times [\eta_{\mu\nu} (p_1 - p_2)_\lambda + \eta_{\nu\lambda} (p_1 + 2p_2)_\mu - \eta_{\lambda\mu} (2p_1 + p_2)_\nu] \\
&\quad \times [\eta_{\alpha\beta} (p_3 - p_4)_\kappa + \eta_{\beta\kappa} (p_3 + 2p_4)_\alpha - \eta_{\kappa\alpha} (2p_3 + p_4)_\beta] \\
&= \frac{e^2 \cot^2 \theta_W}{4m_W^2 m_Z^2} \left[2su + s^2 - 2m_W^2 \frac{3su + u^2}{s + u} + 2m_Z^2 \frac{s^2 - 3su - 2u^2}{s + u} \right. \\
&\quad \left. - \frac{m_Z^4}{m_W^2} s + \mathcal{O}(1) \right]. \tag{6.4.44}
\end{aligned}$$

Figure 6.2: Left: *s*-channel contribution $\mathcal{M}_s$ to $W Z \rightarrow W Z$. Right: *u*-channel contribution $\mathcal{M}_u$ to $W Z \rightarrow W Z$.



Specifically, we used the concrete expression for the $W + W \rightarrow Z$ vertex and also the propagator of the massive vector as computed above, cf. (6.4.34).

Note that we kept only the leading order terms in an m/E expansion.

Since both $s, u \sim E^2$ we can see that the first two terms diverge like $O(E^4/m^4)$ where m stands for either m_W, m_Z , the next three terms are $O(E^2/m^2)$ and we have not computed the finite terms which are $O(1)$ and below.

This is because we are interested in the potentially divergent contributions to the amplitude that would imply the break down of perturbative unitarity.

- *t-channel*: This contribution vanishes since it requires a cubic vertex for the Z_μ boson that does not exist,

$$\mathcal{M}_t = 0. \quad (6.4.45)$$

- *u-channel*: The u -channel contribution is the same as the s -channel by just changing $s \leftrightarrow u$ in the expression for the s -channel,

$$\mathcal{M}_u(s, u) = \mathcal{M}_s(u, s). \quad (6.4.46)$$

It will then have also quartic and quadratic divergences in E/m .

Notice that the combination of both processes above give

$$\mathcal{M}_s + \mathcal{M}_u = \frac{e^2 \cot^2 \theta_W}{4m_W^2 m_Z^2} \left[4su + s^2 + u^2 - 2(m_W^2 + m_Z^2) \frac{6su + s^2 + u^2}{s + u} - \frac{m_Z^4}{m_W^2} (s + u) + \mathcal{O}(1) \right]. \quad (6.4.47)$$

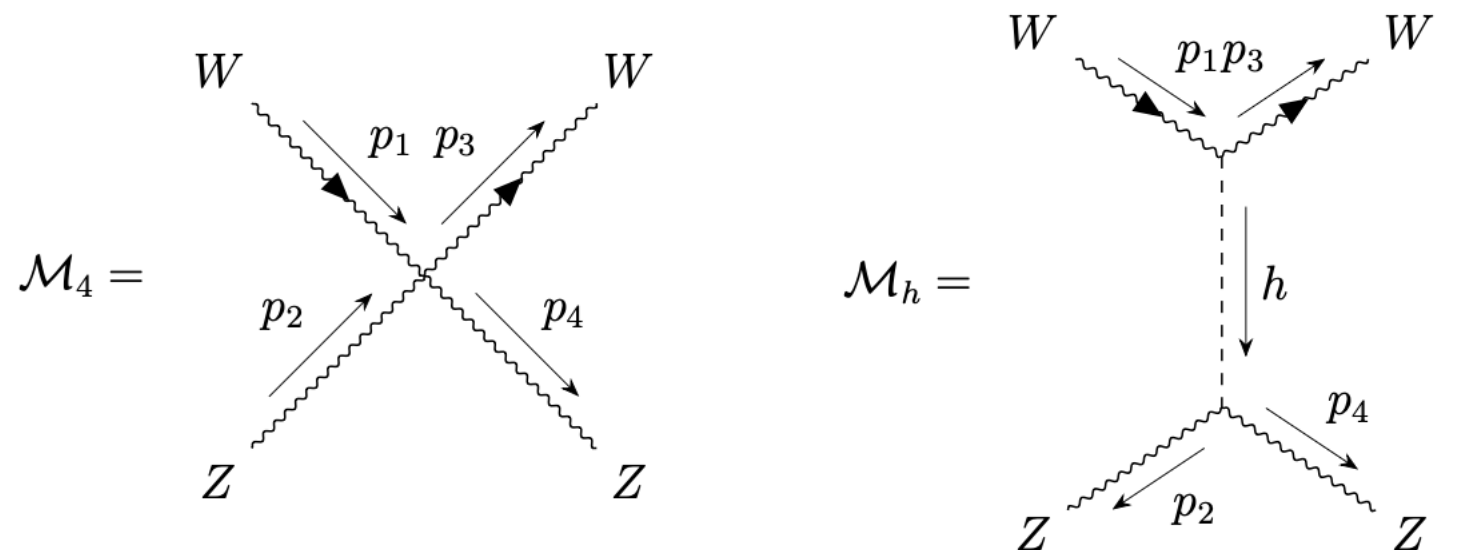
If these were the only contributions, the amplitude (6.4.41) would diverge in the limit $s, u \rightarrow \infty$

$$\mathcal{M}(W_L Z_L \rightarrow W_L Z_L) \xrightarrow{s, u \rightarrow \infty} \infty. \quad (6.4.48)$$

However, we are clearly not done yet.

Remembering (6.4.37), there are two additional diagrams that contribute to $W Z \rightarrow W Z$, see Fig. 6.3.

Figure 6.3: Left: 4-point vertex contribution $\mathcal{M}_4$ to $W Z \rightarrow W Z$. Right: Higgs contribution $\mathcal{M}_h$ to $W Z \rightarrow W Z$.



The contribution on the left stems from the 4-point vertex $\sim W^2 Z^2$ in (6.4.37) which can be computed as

$$\begin{aligned} \mathcal{M}_4 &= e^2 \cot^2 \theta_W \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\alpha \epsilon_4^\beta (\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - 2\eta_{\mu\alpha} \eta_{\nu\beta}) \\ &= \frac{e^2 \cot^2 \theta_W}{4m_W^2 m_Z^2} \left[-s^2 - 4su - u^2 + 2(m_W^2 + m_Z^2) \frac{s^2 + 6su + u^2}{s + u} + \mathcal{O}(1) \right] \end{aligned} \quad (6.4.49)$$

which has quartic and quadratic divergences (quartic from the first three terms (recall, s, t, u scale like $\mathcal{O}(E^2)$) and quadratic from the next term).

However, we notice something remarkable: if we compute the combination $\mathcal{M}_s + \mathcal{M}_u + \mathcal{M}_4$, the most dangerous quartic divergences in (6.4.49) precisely cancel those of the s- and u-channels.

While non-trivial at first, it is an important consequence of SSB in the presence of gauge fields.

As we mentioned several times before, there are remain traces of the original gauge invariance even after SSB which manifest themselves through relations among the quartic and cubic vertices of the theory.

This is an important test for our theory since amplitudes for generic massive vectors would have quartic divergences because there are no such relationships between the various quartic and cubic vertices.

Notice that, even though the quartic divergences cancel, the quadratic divergences are only partially cancelled and remain of order

$$\mathcal{M}_s + \mathcal{M}_u + \mathcal{M}_4 = \frac{t}{v^2} = \mathcal{O}(E^2) \quad (6.4.50)$$

where we have used the relation of the Higgs VEV v in terms of m_W , m_Z , θ_W .

It might not really come as a surprise that this last dangerous piece is cancelled by an additional contribution coming from interactions involving the Higgs.

Specifically, we have a Higgs channel diagram shown on the right in Fig. 6.3 which amounts to

$$\begin{aligned}
\mathcal{M}_h &= -\frac{e^2}{\sin^2 \theta_w \cos^2 \theta_W} \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\alpha \epsilon_4^\beta (\eta_{\alpha\mu} \eta_{\beta\nu}) \frac{m_W^2}{t - m_h^2} \\
&= -\frac{e^2}{4m_Z^2 \sin^2 \theta_w \cos^2 \theta_W} \frac{t^2(t - 4m_W^2)(t - 4m_Z^2)}{(t - m_h^2)(t - 2m_W^2)(t - 2m_Z^2)} \\
&= -\frac{t}{v^2}
\end{aligned} \tag{6.4.51}$$

This contribution clearly diverges as $\mathcal{O}(E^2/m^2)$, but it exactly cancels the quadratic divergence of the sum of the other amplitudes!

To summarize, putting all the pieces together, we find that

$$\mathcal{M}(WZ \rightarrow WZ) = \mathcal{M}_s + \mathcal{M}_u + \mathcal{M}_4 + \mathcal{M}_h = \mathcal{O}(1). \tag{6.4.52}$$

Therefore, the total amplitude is finite and perturbative unitarity is recovered from the Higgs.

Let us pause here for a moment to appreciate how important this result is.

Recall that when we discussed massive spin-1 fields we pointed out the perturbative unitarity problem of increasing probability amplitudes with energy, that at high energies would give rise to probabilities bigger than one breaking unitarity, cf. section 5.1.

Here we see explicitly how SSB theories solve this problem by having the Higgs field recovering unitarity.

Before the Higgs discovery, this was the main argument to justify LHC and its energy range since, in order to recover unitarity the Higgs mass should be smaller than 1 TeV.

This was referred to as the *no-loose argument* when arguments in favor of the construction of LHC were given.

The discovery of the Higgs at the 125 GeV spectacularly confirmed theoretical expectations.

Higgs particle and perturbative unitarity

The dangerous potential loss of unitarity in the scattering of massive vector fields is precisely cancelled by their interaction with the Higgs field.

This illustrates that the Higgs mechanism provides a UV completion for an effective theory of massive vectors.

This perturbative unitarity problem is present for a generic massive vector field, despite the naive appearance that a theory for a massive vector seems renormalizable by simple power counting argument (canonical kinetic term plus a mass term and a quartic term).

In fact, the theory is non-renormalizable in the following sense: for high energies or momenta p , as we have seen, the propagator of a generic massive vector field goes like $\Delta \sim O(1/m^2)$ for $p \gg m$, which is very different from the massless case in which $\Delta \sim O(1/p^2)$.

Even for massive fermions $\Delta \sim O(1/p)$ and massive scalars $\Delta \sim O(1/p^2)$, the propagator decreases with increasing p .

Given a particular non-divergent loop diagram and adding an internal line of the vector field to add an extra loop, the level of the divergence may increase because of the extra integral corresponding to the new loop without any falloff at high energies.

In contrast, if the propagator vanishes for large momenta p such as for massive fermions or scalars, it compensates the level of the divergence and keeps the theory

But if the propagator does not vanish with increasing p , as for the case of the massive vectors, then the divergence is not cancelled.

This will generically make the theory behave as a non-renormalizable theory in which an infinite number of terms would be required to renormalize the theory.

A typical example is that of a box diagram with four external legs.

If the four internal lines correspond to fermions, the loop will have an integral over momenta with the four propagators contributing an $O(p^{-4})$ in total that combine with the line element $p^3 dp$ to contribute a harmless logarithmic behavior.

This situation remains the same if we add a new internal line corresponding to a massless vector, like the photon, since the two-loop diagram now will have two integrals over momenta with six internal fermion lines $O(p^{-6})$ that combine with the $1/p^2$ behavior of the photon propagator to keep the integral logarithmic.

But if instead of a photon we add a massive vector internal line, the contribution of the propagator would be $1/m^2$ instead of $1/p^2$ and the integral would become quadratically divergent.

Massive vectors coming from spontaneously broken gauge theories avoid this problem since in that case the $\Delta \sim O(1/m^2)$ behavior is just an artifact of the unitary gauge.

Going to other gauges the behavior $\Delta \sim O(1/p)$ is recovered and the divergences disappear.

This explains why in the unitary gauge that we used above we needed several cancellations to obtain a finite result but these are not actually miraculous cancellations, but a reflection of the underlying gauge symmetry.

In summary, the only renormalizable theories of massive vector fields are those derived from gauge theories with spontaneous symmetry breaking.

They include not only the massive vectors, but also a massive scalar, the Higgs, that can be seen as providing the UV completion of theories with only massive vectors.

A final comment: as stressed many times before, the Goldstone modes provide the longitudinal degrees of freedom of the massive vector field.

Therefore, computing scattering of longitudinal gauge bosons is therefore the same as computing the scattering of Goldstone bosons (using a gauge different from the unitary gauge).

It is often useful to make directly the calculation in terms of the Goldstone bosons since calculations with scalars are usually easier than calculations with vectors.

This is known as the *Goldstone equivalence theorem*.

6.4.3 Lagrangian for boson and fermion couplings

So far, we have focussed exclusively on the bosonic fields in the GSW model.

But we know that matter is formed by fermions like the electron and up-/down-type quarks.

In this section, we collect all of the couplings involving fermions and their corresponding interactions with the Higgs before discussing their properties after SSB.

We denote the fermionic fields as follows:

- Left-handed leptons transforming as doublets under $SU(2)_L$:

$$L_L^i = \left\{ \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix} \right\}_{Y_L=-1/2} \quad (6.4.53)$$

with $\nu_{e,L}, \nu_{\mu,L}, \nu_{\tau,L}$ and e_L, μ_L, τ_L left-handed **Weyl spinors** written as 4-component Dirac spinors with only the top two components non-vanishing as we saw in part 2.

The index i labels the *families* or *generations*.

- Right-handed leptons transforming as singlets under $SU(2)_L$

$$e_R^i = \{e_R, \mu_R, \tau_R\}_{Y_e=1}, \quad \left(\nu_R^i = \{\nu_{e,R}, \nu_{\mu,R}, \nu_{\tau,R}\}_{Y_\nu=0} \right). \quad (6.4.54)$$

- Left-handed quarks transforming as doublets under $SU(2)_L$

$$Q_L^i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}_{Y_Q=1/6}. \quad (6.4.55)$$

The index i labels again the three families or generations. There are 6 quark flavours u, d, c, s, t, b (two flavours per family).

- Right-handed quarks transforming as singlets under $SU(2)_L$

$$u_R^i = \{u_R, c_R, t_R\}_{Y_u=2/3}, \quad d_R^i = \{d_R, s_R, b_R\}_{Y_d=-1/3}. \quad (6.4.56)$$

The assignments for the hypercharges are not arbitrary, in the absence of right-handed neutrinos, they are essentially uniquely specified by anomaly cancellation.

We will show this in great detail in section 8.3.

The weak interactions for the fermions can be split into two terms

$$\mathcal{L}_F = \mathcal{L}_F^{\text{kin}} + \mathcal{L}_F^{\text{Yukawa}} \quad (6.4.57)$$

where the kinetic terms for the fermions are

$$\mathcal{L}_F^{\text{kin}}[D] = i\bar{L}_L^i \not{D} L_L^i + i\bar{Q}_L^i \not{D} Q_L^i + i\bar{e}_R^i \not{D} e_R^i + i\bar{\nu}_R^i \not{D} \nu_R^i + i\bar{u}_R^i \not{D} u_R^i + i\bar{d}_R^i \not{D} d_R^i. \quad (6.4.58)$$

The covariant derivative for our gauge group $G_{EW} = SU(2)_L \times U(1)_Y$ can be written as (recall (6.4.29))

$$D_\mu = \partial_\mu - igW_\mu^a T^a - ig'B_\mu Y \mathbb{1} \quad (6.4.59)$$

$$= \partial_\mu - ig \left(W_\mu^+ T^+ + W_\mu^- T^- \right) - iZ_\mu \left(g \cos(\theta_W) T^3 - g' \sin(\theta_W) Y \mathbb{1} \right) - ieA_\mu Q \mathbb{1}$$

where

$$T^\pm = \frac{T^1 \pm iT^2}{\sqrt{2}}. \quad (6.4.60)$$

As always, the action of D_μ on a field is understood to be in the corresponding representation.

Hence, D_μ acts differently for each field, e.g., for d_R^i we have

$$D_\mu d_R^i = \partial_\mu d_R^i - ig'B_\mu Y d_R^i = \partial_\mu d_R^i - ie(A_\mu - \tan(\theta_W)Z_\mu) Q d_R^i. \quad (6.4.61)$$

It is occasionally useful to separate the interactions terms between fermions and gauge fields from the kinetic terms by writing

$$\mathcal{L}_F^{\text{kin}}[D] = \mathcal{L}_F^{\text{kin}}[\partial] + eA_\mu J_\mu^{EM} + \frac{e}{\sin(\theta_W)} Z_\mu J_\mu^Z + \frac{e}{\sqrt{2}\sin(\theta_W)} \left(W_\mu^+ J_\mu^+ + W_\mu^- J_\mu^- \right) \quad (6.4.62)$$

where $\mathcal{L}_F^{\text{kin}}[D]$ is given by the kinetic terms in (6.4.58) with the covariant derivatives D_μ replaced by ordinary derivatives ∂_μ .

In (6.4.62), we extract easily the expressions for the various current interactions involving combinations of the fermion fields.

The *electromagnetic current* from couplings to the photon A_μ are given by

$$J_\mu^{EM} = \sum_i Q_i \left(\bar{\psi}_i^L \gamma_\mu \psi_i^L + \bar{\psi}_i^R \gamma_\mu \psi_i^R \right) . \quad (6.4.63)$$

Here, Q_i denotes the electric charge under (6.3.7) and $\psi^{L,R}_i$ denotes all left- and right-handed fermions from above.

The *neutral current* from couplings to Z_μ reads

$$J_\mu^Z = \frac{1}{\cos(\theta_W)} \left(\sum_i \bar{\psi}_i^L \gamma_\mu T^3 \psi_i^L - \sin^2(\theta_W) J_\mu^{EM} \right) . \quad (6.4.64)$$

The *charged currents* $J_\mu^\pm$ from the couplings to $W_\mu^\pm$ in the Lagrangian above are

$$J_\mu^+ = \bar{\nu}_L^i \gamma_\mu e_L^i + \bar{u}_L^i \gamma_\mu d_L^i , \quad J_\mu^- = \bar{e}_L^i \gamma_\mu \nu_L^i + \bar{d}_L^i \gamma_\mu u_L^i . \quad (6.4.65)$$

Lastly, we consider the interactions involving fermions which can be written a

$$\mathcal{L}_F^{\text{Yukawa}} = \mathcal{L}_F^{\text{Higgs-leptons}}[L_L^i, e_R^i, H] + \mathcal{L}_F^{\text{Higgs-quarks}}[Q_L^i, u_R^i, d_R^i, H] . \quad (6.4.66)$$

The quark contribution is

$$\mathcal{L}_F^{\text{Higgs-quarks}}[Q_L^i, u_R^i, d_R^i, H] = -y_{ij}^d \bar{Q}_L^i H d_R^j - y_{ij}^u \bar{Q}_L^i \tilde{H} u_R^j + \text{h.c.} \quad (6.4.67)$$

where y^d, y^u are free parameters called Yukawa couplings and

$$\tilde{H} = i\sigma^2 H^* . \quad (6.4.68)$$

Notice that these combinations are all gauge invariant, since they are uncharged under $U(1)_Y$ where

$$\bar{Q}_L^i H d_R^j \rightarrow -\frac{1}{6} + \frac{1}{2} - \frac{1}{3} = 0, \quad \bar{Q}_L^i \tilde{H} u_R^j \rightarrow -\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0. \quad (6.4.69)$$

They are also $SU(2)_L$ singlets (recall the product of two $SU(2)$ doublets gives a triplet plus a singlet).

Further, knowing that left-handed quarks transform as $\mathbf{3}$ of $SU(3)_c$ (as mentioned in part 1 and to be properly introduced in the next part of the lectures) and right-handed quarks as $\bar{\mathbf{3}}$, these couplings are also $SU(3)_c$ invariant.

It is remarkable that the hypercharge assignments imposed by anomaly cancellations are precisely what are needed in order to have non-zero Yukawa couplings.

This is important because these couplings are the ones responsible to give the quarks a mass.

Note that the masses for the fermions arise only after the Higgs gets a $\text{VEV}(H) \neq 0$.

There are **no direct mass terms for the quarks which are forbidden by gauge (chiral) symmetry**.

Therefore, it is the Yukawa couplings and the Higgs VEV that give mass to fermions.

Without right-handed neutrinos, the Lagrangian for the Yukawa interactions between the leptons and the Higgs includes only the following couplings

$$\mathcal{L}_F^{\text{Higgs-leptons}}[L_L^i, e_R^i, H] = -y_{ij}^e \bar{L}_L^i H e_R^j + \text{h.c.} . \quad (6.4.70)$$

Contrary to (6.4.67) for the quarks involving two different types of Yukawa couplings for Q_L^i to u_R^i and d_R^i respectively, here there is only a single Yukawa term.

As we will see below, this implies that there is only a mass term for the electrons, muons and tauons, but not to their neutrinos.

Since there are only left-handed neutrinos inside L^i , but no right-handed neutrinos, the neutrinos remain massless in this case after SSB.

6.4.4 Quarks: mass matrix and weak couplings

Let us come back to the Yukawa couplings between the Higgs and the quarks, cf. (6.4.67).

As mentioned before, these couplings give rise to the masses for the quarks after SSB.

Indeed, we can see this explicitly by replacing the Higgs by (6.4.13) in the Yukawa couplings above which leads to

$$\mathcal{L}_F^{\text{Higgs-quarks}}[Q_L^i, u_R^i, d_R^i, H] = \mathcal{L}_F^{\text{Higgs-boson-quarks}}[Q_L^i, u_R^i, d_R^i, h] + \mathcal{L}_F^{\text{mass}}[Q_L^i, u_R^i, d_R^i] \quad (6.4.71)$$

in terms of the Higgs-boson-quark interactions

$$\mathcal{L}_F^{\text{Higgs-boson-quarks}}[Q_L^i, u_R^i, d_R^i, h] = -\frac{h}{\sqrt{2}} \left[\bar{d}_L^i y_{ij}^d d_R^j + \bar{u}_L^i y_{ij}^u u_R^j \right] + \text{h.c.} \quad (6.4.72)$$

and the mass term for the quarks

$$\mathcal{L}_F^{\text{mass}}[Q_L^i, u_R^i, d_R^i] = -\frac{v}{\sqrt{2}} \left[\bar{d}_L^i y_{ij}^d d_R^j + \bar{u}_L^i y_{ij}^u u_R^j \right] + \text{h.c.} \quad (6.4.73)$$

with the mass matrices encoded by y_{ij}^d and y_{ij}^u .

This is the source of the standard claim that the Higgs field is responsible to give a mass to all the other particles since the masses are proportional to the Higgs VEV v .

This claim will be reconsidered in the next part of the lectures.

Note that the mass matrices given by $v y_{ij}^d$ and $v y_{ij}^u$ are in general non-diagonal.

In order to read the masses of the physical particles, we need to diagonalize these matrices.

In general, any matrix can be diagonalized using two unitary matrices U, K

$$y^d = U_d M^d K_d^\dagger, \quad y^u = U_u M^u K_u^\dagger \quad (6.4.74)$$

with M^d, M^u real and diagonal leads to a new basis of fields

$$d'_L = U_d^\dagger d_L, \quad d'_R = K_d^\dagger d_R, \quad u'_L = U_u^\dagger u_L, \quad u'_R = K_u^\dagger u_R. \quad (6.4.75)$$

The Lagrangian for the mass terms in this new basis reads

$$\mathcal{L}_F^{\text{mass}} = -\frac{v}{\sqrt{2}} \left[\bar{d}'_L{}^i M_{ii}^d d'_R{}^i + \bar{u}'_L{}^i M_{ii}^u u'_R{}^i \right] + \text{h.c.} \quad (6.4.76)$$

implying that the quark masses are given by

$$m_{d'_i} = \frac{v}{\sqrt{2}} M_{ii}^d, \quad m_{u'_i} = \frac{v}{\sqrt{2}} M_{ii}^u. \quad (6.4.77)$$

Therefore, we managed to define a basis for the up and down quarks for which the mass matrix is diagonal leading to six free parameters $m_{d'_i}$ and $m_{u'_i}$.

In this basis, the interactions (6.4.72) of this Higgs bosons to the quarks reads

$$\mathcal{L}_F^{\text{Higgs-boson-quarks}}[Q_L^i, u_R^i, d_R^i, h] = -\frac{h}{v} \left[m_{d'_i} \bar{d}'_L{}^i d'_R{}^i + m_{u'_i} \bar{u}'_L{}^i u'_R{}^i \right] + \text{h.c.} \quad (6.4.78)$$

Thus, the coupling scales proportional to the masses of the quarks which is why the decay $h \rightarrow b\bar{b}$ has the highest probability.

From now-on, we will drop the primes on u_i and d_i and work in this basis unless stated otherwise.

It is, however, important to keep in mind that the basis that diagonalizes the mass matrix is such that in general the couplings to the gauge fields coming from the kinetic terms are not diagonal in this basis.

To see this, let us write the couplings (6.4.62) in the **mass eigenstate** basis

$$\mathcal{L}_F^{\text{kin}+\text{Higgs}} = \mathcal{L}_F^{\text{kin}}[\partial] + eA_\mu J_\mu^{EM} + \frac{e}{\sin(\theta_W)} Z_\mu J_\mu^Z + \frac{e}{\sqrt{2}\sin(\theta_W)} (W_\mu^+ J_\mu^+ + W_\mu^- J_\mu^-) \quad (6.4.79)$$

$$- m_j^d (\bar{d}_L^j d_R^j + \bar{d}_R^j d_L^j) - m_j^u (\bar{u}_L^j u_R^j + \bar{u}_R^j u_L^j)$$

Both J_μ^{EM} and J_μ^Z defined in (6.4.63) and (6.4.64) are manifestly diagonal in this new basis since they come from diagonal generators of the gauge group that do not mix up and down quarks.

But the charged currents $J_\mu^\pm$ in (6.4.65) are not diagonal since they mix up- and down-type quarks in a generation.

This is because the different quark flavors are diagonalized by different matrices U_u and U_d which implies that in the mass eigenbasis the $J_\mu^\pm$ read (for the quarks only)

$$J_\mu^+ = \bar{u}_L^i \gamma_\mu V^{ij} d_L^j, \quad J_\mu^- = \bar{d}_L^i \gamma_\mu (V^\dagger)^{ij} u_L^j. \quad (6.4.80)$$

Here, the mixing between the different quark flavors is encoded in the *Cabibbo-Kobayashi-Maskawa (CKM) matrix*

$$V = V_{CKM} = U_u^\dagger U_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (6.4.81)$$

This matrix being unitary has 9 free parameters which in principle need to be determined experimentally (3 real angles and 6 phases).

However, we can still reduce the number of independent parameters as follows.

Note that there is a remnant $U(1)^6$ global symmetry of the mass terms

$$d_{R,L}^i \rightarrow e^{i\alpha_i} d_{R,L}^i, \quad u_{R,L}^i \rightarrow e^{i\beta_i} u_{R,L}^i. \quad (6.4.82)$$

This symmetry can be used to eliminate 5 phases (only phase difference work, since the overall $U(1)$ for which all the parameters α_i and β_i are equal is a symmetry of the whole Lagrangian and corresponds to baryon number).

There remain $9 - 5 = 4$ free parameters within V_{CKM} : 3 real parameters (the three standard rotation angles in three dimensions $\theta_{12}, \theta_{13}, \theta_{23}$) and 1 phase.

One parametrization of this matrix is as follows (writing $c_{ij} = \cos(\theta_{ij})$, $s_{ij} = \sin(\theta_{ij})$)

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (6.4.83)$$

Another parametrization (known as *Wolfenstein's parametrization*) can be written as

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} \quad (6.4.84)$$

with parameters A, ρ, λ, η .

Approximately, $s_{12} = \lambda$ is determined by the Cabibbo angle θ_{12} , $\sin(\theta_{12}) \approx 0.22$ so for small λ and A, ρ, η of order one we can see that there is a hierarchy in the matrix elements showing that to leading order in λ the matrix is diagonal and the mixing between the first two families is stronger than the mixing of each of them with the third family.

Mass eigenstates $\neq$ “weak or flavor eigenstates”

This statement simply implies that if we work with quarks from different families, and diagonalize the mass matrix in flavor space (a fixed mass for each up and down quark), then the couplings to the $W^\pm_\mu$ fields are not diagonal.

This means that the charged currents $J^\pm_\mu$ connect fermions of different flavors.

This generation mixing is responsible for weak interactions that change flavor.

This suggests in particular that initial states from one family can decay into final states from a different family as in the muon decay, see section 6.4.8 below.

Glashow-Iliopoulos-Maiani (GIM) Mechanism

Unlike the charged currents $J^\pm_\mu$ that mix different generations, the neutral current J^Z_μ is flavor diagonal.

This implies that there are no flavor changing neutral currents (FCNC).

In the 1960's and early 1970's only three quarks were known u, d, s .

If we go back and compute the neutral current with only these three quarks then it would not be diagonal as it can be easily checked.

But interactions that would be mediated by a neutral current such as $s \rightarrow d + e^+ + e^-$ were not observed and there was no explanation.

This was the reason for Glashow, Iliopoulos and Maiani to predict the existence of a fourth quark (charm c) that leads to the absence of FCNC and forbids such decays.

This was spectacularly confirmed with the discovery of the J/ψ resonance in 1974.

From the current perspective there is a compelling need for the existence of c once s was discovered.

Anomaly cancellations require both members of a family for consistency.

CP-violation

The presence of a phase δ in V_{CKM} implies CP-violation.

This can be shown explicitly by analyzing the behavior of the currents under time reversal T (since CPT is an exact symmetry non-invariance under T is equivalent to CP violation).

If there were only two families, the CKM matrix would have only one real parameter (the Cabibbo angle) and no phases; therefore no CP-violation.

So the observation of CP-violation led Kobayashi and Maskawa to predict a third family.

Experimentally, the four parameters in the CKM matrix were measured to be

$$\begin{aligned}\theta_{12} &= 13.02 \pm 0.04, & \theta_{23} &= 2.56 \pm 0.03, \\ \theta_{13} &= 0.20 \pm 0.02, & \delta &= 69 \pm 5.\end{aligned}\tag{6.4.85}$$

Baryon number

The couplings in $\mathcal{L}_F$ defined in (6.4.57) have an accidental $U(1)_B$ symmetry

$$B(Q_L^i) = B(u_R^i) = B(d_R^i) = \frac{1}{3}, \quad B(\bar{Q}_L^i) = B(\bar{u}_R^i) = B(\bar{d}_R^i) = -\frac{1}{3}\tag{6.4.86}$$

and all others $B = 0$.

This $U(1)_B$ symmetry corresponds to baryon number (charge $B = +1$ for baryons and $B = -1$ for anti-baryons whereas leptons have charge $B = 0$).

We will discuss this baryon number symmetry later in section 8.2.

Unitarity Triangle

The fact that the CKM matrix is unitary $VV^\dagger = 1$ can be expressed nicely in terms of its components as the statement

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} + 1 = 0\tag{6.4.87}$$

where the indices refer to the generations and where we divided by the factor $V_{cd}V_{cb}^*$.

This expression is in general the sum of three numbers of which two can be complex.

If so, they would produce a triangle in the complex plane.

If these numbers end up being real, the triangle collapses.

Therefore a measure of the existence of CP violation is to measure the area of this triangle and a test of unitarity of the CKM matrix is to determine experimentally the sides of the triangle and check that it closes.

Using these components of the CKM matrix we can construct the invariant quantity known as the *Jarlskog invariant* which is a measure of the area of the unitarity triangle.

More specifically, it is a convenient way to express the non-vanishing area of the triangle by the existence of a non-vanishing phase in the CKM matrix

$$J \equiv \text{Im} (V_{ud}V_{ub}^*V_{tb}V_{td}^*) = s_{12}s_{23}s_{31}c_{12}c_{23}c_{31}^2 \sin \delta = (2.96 \pm 0.20) \times 10^{-5} . \quad (6.4.88)$$

Measuring $J \neq 0$ guarantees the presence of CP violation in weak interactions.

J is invariant under any of the phase rotations that gave rise to the CKM matrix.

A simple way to see it is that each of the indices u, d, t, b appear in one V_{ij} and in one V_{ij}^* component so the phase rotation of the corresponding quark cancels.

It can be easily shown that J is twice the area of the unitarity triangle (using that the area generated by two vectors can be computed by computing their cross product).

6.4.5 Leptons: mass matrix and weak couplings

We now repeat the same analysis of the weak couplings of leptons to the gauge fields and the Higgs as we just did for the quarks.

Since right-handed neutrinos have not been observed, even though it has been confirmed that neutrinos have a non-zero mass, we will consider first the possibility of no right-handed neutrinos ν_R .

In the subsequent section, we add right-handed neutrinos and discuss the implications.

Without right-handed neutrinos, the Higgs-lepton interactions are given by (6.4.70), namely

$$\mathcal{L}_F^{\text{Higgs-leptons}} = -y_{ij}^L \bar{L}_L^i H e_R^j - (y^\dagger)_{ij}^L \bar{e}_R^i H L_L^j. \quad (6.4.89)$$

After SSB, we set the Higgs field in unitary gauge to

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \quad (6.4.90)$$

in terms of the Higgs boson h .

Plugging this into (6.4.89), we find

$$\mathcal{L}_F^{\text{Higgs-leptons}} = -\frac{y_{ij}^L}{\sqrt{2}} \bar{e}_L^i (v + h) e_R^j - \frac{(y^\dagger)_{ij}^L}{\sqrt{2}} \bar{e}_R^i (v + h) e_L^j. \quad (6.4.91)$$

As before, we can find a field basis in which the Yukawa couplings y_{ij}^L are diagonal.

The eigenvalues are non-negative and will be denoted $m_i\sqrt{2}/v$ in the following.

In this basis, the interaction Lagrangian can be written a

$$\begin{aligned}\mathcal{L}_F^{\text{Higgs-leptons}} &= -\frac{m_i}{v} \bar{e}_L^i (v + h) e_R^i - \frac{m_i}{v} \bar{e}_R^i (v + h) e_L^i \\ &= -m_i \left(\bar{e}_L^i e_R^i + \bar{e}_R^i e_L^i \right) - \frac{m_i}{v} h \left(\bar{e}_L^i e_R^i + \bar{e}_R^i e_L^i \right) .\end{aligned}\tag{6.4.92}$$

The first term is a Dirac mass term for the leptons e, μ, τ .

Let us define the Dirac spinors

$$\ell^i = e_L^i + e_R^i = \begin{pmatrix} \tilde{e}_L^i \\ \tilde{e}_R^i \end{pmatrix}\tag{6.4.93}$$

in terms of 2-component Weyl spinors $\tilde{e}_{L,R}^i$ so that

$$\mathcal{L}_F^{\text{Higgs-leptons}} = -m_i \bar{\ell}^i \ell^i - \lambda_i h \bar{\ell}^i \ell^i\tag{6.4.94}$$

in terms of the couplings

$$\lambda_i^{\text{Higgs-leptons}} = \frac{m_i}{v} .\tag{6.4.95}$$

As in the case of the Yukawa couplings for the quarks, a non-zero VEV for the Higgs field, $\langle H \rangle = v \neq 0$, will provide mass terms for the leptons.

But because there is only a single Yukawa coupling, there are only mass terms for the electrons, muons and taus, but not for their neutrinos.

This is because there are only left-handed neutrinos inside L^i , but no right-handed neutrinos which is why the neutrinos remain massless even after SSB.

Further, this implies that the mass matrix for e, μ, τ can be diagonalized without affecting the mixing of the couplings coming from the kinetic terms.

Therefore, in the absence of right-handed neutrinos, the weak and mass eigenstates of leptons are the same and there is no analogue of the CKM matrix.

This diagonal property of the quadratic Lagrangian for leptons also implies separate conservations of lepton numbers for each family: three independent global $U(1)$ symmetries corresponding to conservation of L_e, L_μ, L_τ , e.g., $L_e(e) = L_e(\nu_e) = 1$, other zero, etc.

Since neutrinos are massive, we know this cannot be the full story.

The natural way to proceed is to introduce the right-handed neutrinos that we will do next.

However, the fact that there is no mass term for neutrinos may be just a limitation of the fact that we are imposing the Lagrangian to be renormalizable.

Once we relax this condition there will be neutrino masses generated from couplings such as $HHLL$ which is of dimension five and therefore non-renormalizable.

Once the Higgs gets a VEV this will induce neutrino masses.

We will discuss this in part 9.

If neutrinos were massless as above, then the fact that the mass matrix can be diagonalized implies that there would be independently conserved family lepton numbers L_e, L_μ, L_τ corresponding to three accidental $U(1)$ global symmetries acting as independent phase rotations $\psi_k \rightarrow e^{i\alpha_k} \psi_k$ where ψ_k $k = 1, 2, 3$ represents each of the three families of leptons.

However, since neutrinos are massive, these symmetries are not actually there, but there remains the overall accidental symmetry $L = L_e + L_\mu + L_\tau$ which counts the overall lepton number which is similar to baryon number for quarks.

Except that, if there is a Majorana mass for right-handed neutrinos, it also breaks the overall lepton number L , see below.

Note that both lepton and baryon number are accidental global symmetries.

6.4.6 Z-boson decay and the number of light neutrinos

One of the big successful results from the LEP experiment in the 1990s was the study of the decay modes of the Z-boson.

The total decay rate (width of the Z-boson resonance) Γ_Z is well determined given all the possible decay channels as

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\text{hadrons}} + N_\nu \Gamma_{\nu\nu} \quad (6.4.96)$$

Since the width was measured very precisely, from this the number of neutrinos lighter than the Z boson can be determined accurately as we now demonstrate.

To this end, let us compute the decay of the Z-boson into leptons.

We initially have to identify the interactions in the Lagrangian relevant for the decay.

These are obtained from the kinetic terms for the leptons after SSB, namely

$$\mathrm{i}\bar{L}_L \gamma^\mu D_\mu L_L + \mathrm{i}\bar{e}_R \gamma^\mu D_\mu e_R = -\frac{g}{2\cos(\theta_W)} J_n^\mu Z_\mu \quad (6.4.97)$$

in terms of the neutral current

$$J_n^\mu = \bar{L}_L \gamma^\mu \left(\cos(\theta_W)^2 \sigma_3 + \sin(\theta_W)^2 \mathbb{1}_2 \right) L_L + 2\sin(\theta_W)^2 \bar{e}_R \gamma^\mu e_R. \quad (6.4.98)$$

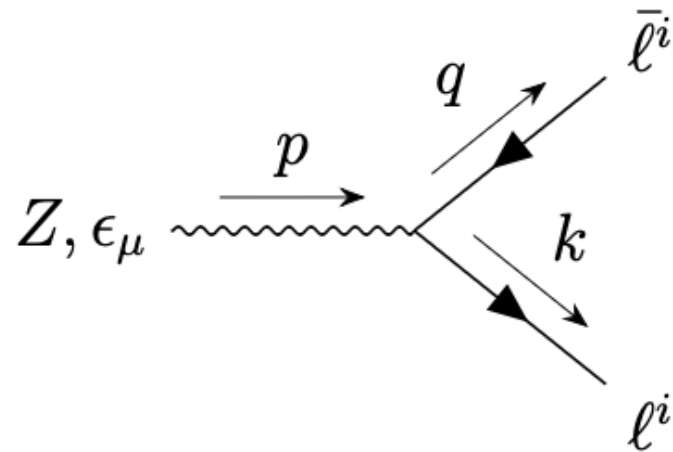
To treat all decays into leptons simultaneously, it is convenient to write

$$J_n^\mu = \sum_{\ell \in \{e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau\}} \bar{\ell} \gamma^\mu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) \ell. \quad (6.4.99)$$

Here, we distinguish between vector couplings v_ℓ and axial couplings a_ℓ where

$$v_i = 2 \sin(\theta_W)^2 - \frac{1}{2}, \quad a_i = -\frac{1}{2}, \quad v_{\nu_i} = a_{\nu_i} = \frac{1}{2}, \quad i \in \{e, \mu, \tau\}. \quad (6.4.100)$$

Diagrammatically, the decay of the Z-bosons into leptons can be visualized as



The associated tree level S-matrix element for this decay is

$$\mathcal{M} = \langle \ell(k) \bar{\ell}(q) | \mathcal{L}_{Z \rightarrow \bar{\ell}^i \ell^i} | Z(p, \epsilon) \rangle \quad (6.4.101)$$

where ϵ_μ is the polarization of the Z-boson.

Momentum conservation implies

$$m_Z^2 = p \cdot k + p \cdot q, \quad p \cdot k = m_{\ell_i}^2 + k \cdot q = p \cdot q, \quad 2k \cdot q = m_Z^2 - m_{\ell_i}^2. \quad (6.4.102)$$

To compute $\mathcal{M}$, we recall the Feynman rules for in and out states listed in Appendix A.

Hence, we have

$$\mathcal{M} = \mathcal{N} \bar{u}_\ell(k) \gamma^\mu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) v_{\bar{\ell}}(q) \epsilon_\mu(p), \quad \mathcal{N} = \frac{g}{2 \cos(\theta_W)}. \quad (6.4.103)$$

As usual, we compute the square of the matrix element and sum over all spins of the leptons and polarizations for the Z-boson,

$$\begin{aligned} \sum_{\text{spins}, \dots} |\mathcal{M}|^2 &= \mathcal{N}^2 \sum_{\text{spins}, \dots} \bar{u}_\ell(k) \gamma^\mu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) v_{\bar{\ell}}(q) \epsilon_\mu(p) \epsilon_\nu^*(p) \bar{v}_{\bar{\ell}}(q) \gamma^\nu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) u_\ell(k) \\ &= \mathcal{N}^2 \text{Tr} \left[(\not{k} + m_\ell) \gamma^\mu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) (\not{q} - m_\ell) \gamma^\nu (v_\ell \mathbb{1}_4 - a_\ell \gamma^5) \right] \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \right) \end{aligned}$$

where we used

$$\sum_{\text{polarisations}} \epsilon_\mu(p) \epsilon_\nu^*(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}. \quad (6.4.104)$$

This identity gives exactly the numerator of the propagator of a massive spin-1 boson, cf. (6.4.34).

In a first approximation, one can neglect the lepton masses which are small compared to the mass of the Z-boson so that

$$\sum_{\text{spins}, \dots} |\mathcal{M}|^2 = \mathcal{N}^2 \text{Tr} \left[(v_\ell^2 + a_\ell^2) \not{k} \gamma^\mu \not{q} \gamma^\nu - 2a_\ell v_\ell \not{k} \gamma^\mu \not{q} \gamma^\nu \gamma^5 \right] \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \right) \quad (6.4.105)$$

where we used various commutation relations for the γ -matrices listed in App. A

Next, we apply (A.2.11) to argue that the term $\sim a_\ell v_\ell$ vanishes due to antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

Then, we utilize (A.2.10) to write

$$\sum_{\text{spins}, \dots} |\mathcal{M}|^2 = \frac{g^2}{\cos(\theta_W)^2} (v_\ell^2 + a_\ell^2) \left[k \cdot q + \frac{2}{m_Z^2} (k \cdot p)(q \cdot p) \right] . \quad (6.4.106)$$

Ignoring the lepton masses, the conditions (6.4.102) yield

$$\sum_{\text{spins}, \dots} |\mathcal{M}|^2 = \frac{g^2}{\cos(\theta_W)^2} (v_\ell^2 + a_\ell^2) m_Z^2 . \quad (6.4.107)$$

Using (A.2.6) for the partial decay rate and including an averaging factor of 1/3 for the initial spins of the Z-bosons, we have to compute

$$\begin{aligned} \Gamma(Z \rightarrow \ell \bar{\ell}) &= \frac{1}{2m_Z} \int \frac{d^3k}{(2\pi)^3 2k^0} \frac{d^3q}{(2\pi)^3 2q^0} (2\pi)^4 \delta^{(4)}(p - k - q) \frac{1}{3} \sum_{\text{spins}, \dots} |\mathcal{M}_{\alpha\beta}|^2 \\ &= \frac{g^2 (v_\ell^2 + a_\ell^2) m_Z}{96\pi^2 \cos(\theta_W)^2} \int \frac{d^3k}{k^0} \frac{d^3q}{q^0} \delta^{(4)}(p - k - q) . \end{aligned} \quad (6.4.108)$$

In the rest frame of the Z-boson and for massless leptons, we find

$$\Gamma(Z \rightarrow \ell \bar{\ell}) = \frac{g^2 (v_\ell^2 + a_\ell^2) m_Z}{96\pi^2 \cos(\theta_W)^2} \int \frac{d^3k}{|\mathbf{k}|^2} \delta(m_Z - 2|\mathbf{k}|) . \quad (6.4.109)$$

After evaluating the final integral, this become

$$\Gamma(Z \rightarrow \ell\bar{\ell}) = \frac{G_F}{\sqrt{2}} \frac{m_Z^3(v_\ell^2 + a_\ell^2)}{16\pi}, \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_Z^2 \cos(\theta_W)^2} \quad (6.4.110)$$

where we introduced *Fermi's constant* G_F .

Looking back at (6.4.96), we find that in this approximation

$$\Gamma_{ee} = \Gamma_{\mu\mu} = \Gamma_{\tau\tau}, \quad \frac{\Gamma_{\nu\nu}}{\Gamma_{ee}} \approx 1.977. \quad (6.4.111)$$

Experimentally, it was found that

$$\frac{\Gamma_Z}{\Gamma_{ee}} \approx 29.7366, \quad \frac{\Gamma_{\text{hadrons}}}{\Gamma_{ee}} \approx 20.80. \quad (6.4.112)$$

Thus, one estimates that the number of light neutrino species is

$$N_\nu \approx \frac{29.74\Gamma_{ee} - 20.80\Gamma_{ee} - 3\Gamma_{ee}}{1.977\Gamma_{ee}} \approx 3.005. \quad (6.4.113)$$

While we made crude approximations here to arrive at this result, this is strikingly close to the experimentally measured value.

The various partial decay rates for the different sectors have been measured very precisely from which the number of neutrinos lighter than the Z boson was determined to be

$$N_\nu = 2.9840 \pm 0.0082. \quad (6.4.114)$$

This is a very strong indication that there are no more than three families of quarks and leptons although it does not rule out the possibility of extra neutrinos heavier than the Z boson.

This result is beautifully complemented with cosmological observations of the cosmic microwave background which limit the number of neutrino-like particles to be no more than 3.

In both measurements, the precision was such that in order to fit the experiments it would not be possible to have more decay channels including more than the three known neutrinos.

6.4.7 Neutrino masses and lepton flavor mixing

Right-handed neutrinos and neutrino masses

Let us now include right handed neutrinos ν_R^i .

But before we start, we stress that these particles *have not been detected* and strictly speaking do not have to be part of the Standard Model.

However, the fact that all other matter particles have a left-handed and a right-handed part and that neutrinos have been found to be massive, hints at their existence.

These fields are sterile in the sense that they are singlets under all non-Abelian gauge groups and also have vanishing hypercharges $Y_{\nu_R} = 0$ making them electrically neutral $Q = 0$.

This means that they only feel gravitational interactions, explaining why they might have escaped detection so far.

However, as we will see momentarily, they can play an important role in generating neutrino masses.

The mass terms for the leptons include the two Yukawa terms similar to the quark case plus an extra term that is possible only for neutrinos corresponding to a direct mass term for right-handed neutrinos

$$\mathcal{L}_F^{\text{leptons}} = \mathcal{L}_F^{\text{kinetic, leptons}} + \mathcal{L}_F^{\text{Higgs-leptons}} + \mathcal{L}_F^{\text{Majorana-mass}} \quad (6.4.115)$$

where $\mathcal{L}_F^{\text{kinetic, leptons}}$ encodes the standard kinetic terms for the leptons and

$$\mathcal{L}_F^{\text{Higgs-leptons}} = -y_{ij}^e \bar{L}_L^i H e_R^j - y_{ij}^\nu \underbrace{\bar{L}_L^i \tilde{H} \nu_R^j}_{\text{Dirac}} + \text{h.c.}, \quad \mathcal{L}_F^{\text{Majorana-mass}} = M_{ij}^\nu \underbrace{\nu_R^i \nu_R^j}_{\text{Majorana}}. \quad (6.4.116)$$

In principle the number of right-handed neutrinos is not limited to coincide with the number of families of the other particles due to the fact that they are not charged under any of the gauge symmetries.

So there could be more or less than three ν_R 's since they are sterile.

The Majorana mass M^ν may be very large and these particles may have only indirect implications at low energies.

Note that this Majorana mass is the only mass term allowed for the fermions of the Standard Model.

So M^ν are the only mass parameters together with the Higgs mass.

Another question that we should address concerns the nature of neutrinos, namely Dirac or Majorana neutrinos.

Recall that, if ν were massless then the state with helicity $\lambda = +1/2$ is called neutrino ν_L and the state with helicity $\lambda = -1/2$ is the anti-neutrino $\bar{\nu}_L$ (or ν_R).

The CPT action is such that $|\nu\rangle \neq \text{CPT} |\nu\rangle$ but gives $\bar{\nu}_L$.

If however ν is massive, then it has spin $j = 1/2$ with the two states $j_3 = \pm 1/2$.

If $|\nu\rangle \neq \text{CPT} |\nu\rangle$ the corresponding particle is a *Dirac neutrino*, while $|\nu\rangle = \text{CPT} |\nu\rangle$ it is called a *Majorana neutrino*, described by a Majorana spinor and the reality condition implies the particle is the same as the anti-particle.

So far there is not enough information to settle if the neutrinos are best represented by Dirac or Majorana spinors.

Dirac and Majorana masses

The first two terms in $\mathcal{L}_F^{\text{leptons}}$ are Yukawa couplings for both electrons and neutrinos coupled to the Higgs.

Again once the Higgs gets a VEV ($\langle H \rangle \neq 0$) they will give rise to mass terms for both electron and neutrinos proportional to the electron and neutrino Yukawa couplings and the VEV of the Higgs v .

This gives rise to standard Dirac mass $m \bar{\psi}_L \psi_R$.

The third term is the only explicit mass term allowed in the Standard Model.

This is due to the fact that the right-handed neutrinos are by themselves invariant under any gauge transformation and therefore a Lorentz invariant quadratic term is allowed by the gauge symmetries and gives an explicit mass to the right-handed neutrinos.

Since ν_R are written as a 2-component Weyl spinor, the Lorentz invariant combination is then $M^{\nu}_{ij} \nu^i_R \nu^j_R$.

Expressed in terms of 4-component spinors this term is of the type $M \bar{\psi}_R \psi_R$ (contrary to the $\bar{\psi}_L \psi_R$ coming from the Yukawa coupling).

A simple way to understand Majorana masses as opposed to Dirac masses is as follows.

Recall our discussion of spinor types in part 2.

Let us consider two 2-component Weyl spinors ψ_α and χ_α and their conjugates.

From them we may construct Lorentz invariant quadratic or mass terms of the form $\psi\chi$, $\psi\psi$, $\chi\chi$ and their complex conjugates.

The first term $\psi\chi + \text{h.c.}$ is the standard Dirac mass $\bar{\Psi}_D \Psi_D$ once the two Weyl spinors are combined in a Dirac fermion $\psi_D = (\psi_\alpha, \bar{\chi}_{\dot{\alpha}})^T$.

The $\psi\psi$ and $\chi\chi$ terms may be directly obtained if we combine ψ and χ in two Majorana spinors.

Namely $\Psi_M^1 = (\psi^\alpha, \bar{\psi}_{\dot{\alpha}})^T$ and $\Psi_M^2 = (\chi^\alpha, \bar{\chi}_{\dot{\alpha}})^T$ and so the Majorana mass terms can be obtained from $\bar{\Psi}_M^1 \Psi_M^1$ and $\bar{\Psi}_M^2 \Psi_M^2$ and the Dirac mass term from $\bar{\Psi}_M^1 \Psi_M^2$.

Notice that if ψ or χ carry some charge (e.g. electric charge), we could not combine them in Majorana spinors.

Right-handed neutrinos are unique in that sense.

From the discussion above we may identify the left-handed Weyl spinor ψ with ψ_L and the right-handed conjugate of χ with χ_R .

See-saw mechanism

In the presence of right-handed neutrinos, the mass matrix for ν 's can be written as

$$m\bar{\psi}_L\psi_R + \frac{1}{2}M\bar{\psi}_R\psi_R \quad \Rightarrow \quad \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \quad (6.4.117)$$

focussing here on a single generation.

The mass matrix has eigenvalues

$$\lambda = \left(1 \pm \sqrt{1 + \frac{m^2}{4M^2}}\right) \frac{M}{2}, \quad |\lambda| \cong M, \frac{m^2}{M} \quad \text{for } M \gg m. \quad (6.4.118)$$

Therefore, if M is large (say close to the Planck scale, $M_{\text{Planck}} \sim 10^{19} \text{ GeV}$), this may be an explanation for the smallness of neutrino masses since m^2/M may be naturally very small.

Note that a negative mass eigenvalue (corresponding to the negative root) for a fermion can always be rotated away to get a positive physical mass.

This mechanism is called see-saw since if we increase M then m decreases and vice versa.

Flavour Mixing

In the presence of right-handed neutrinos, we have a mixing similar to the quark case in the sense that both mass and weak eigenstates do not coincide.

Diagonalizing the mass matrix leads to the equivalent of the CKM matrix for the lepton sector, known as the PMNS matrix.

To understand the flavor mixing in the mass eigenstate basis, we have to study the current interactions:

- The neutral currents are given by

$$\mathcal{L}_{nc} = Z^\mu J_\mu^Z, \quad J_\mu^Z = -\frac{g}{2 \cos(\theta_W)} J_n^\mu \quad (6.4.119)$$

with J_n^μ previously defined in (6.4.98). This shows that they remain flavour diagonal similar to the case of quarks.

- In contrast, the charged currents coupling to $W_\mu^\pm$ are *not* flavour diagonal. Indeed, they can be written as

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} U_{PMNS}^{ij} \left(\bar{e}_L^i \gamma^\mu W_\mu^- \nu_L^j + \text{h.c.} \right) \quad (6.4.120)$$

and are determined in terms of the *PMNS matrix* (named after Pontecorvo-Maki-Nakagawa-Sakata)

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \text{diag} \left(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2} \right) \quad (6.4.121)$$

where $c_{ij} = \cos \beta_{ij}$ and $s_{ij} = \sin \beta_{ij}$, for the 3 angles β_{12} , β_{23} , and β_{13} , and 3 phases δ , α_{21} , α_{31} . Note that α_{21} , α_{31} appear if there are right-handed neutrinos. The reason being that contrary to the quark case, here we can reduce the original 9 parameters of the matrix to 6 and *not* 4 because there are only three U(1)'s that can be used to reduce the number of parameters, corresponding to a phase rotation to the electron fields. There are no U(1) symmetries for neutrinos because the Majorana mass term is not symmetric. Therefore the number of free parameters of the *PMNS* matrix is $9 - 3 = 6$. Measurements show that

$$\sin^2(2\beta_{12}) = 0.857(4), \quad \sin^2(2\beta_{23}) = 0.95, \quad \sin^2(2\beta_{13}) = 0.098(13), \quad (6.4.122)$$

but the phases have not been measured yet.

The value of neutrino masses is still unknown, the only information available experimentally is the difference between their squared masses

$$m_2^2 - m_1^2 = (7.5 \pm 0.2) \cdot 10^{-5} \text{eV}^2, \quad m_3^2 - m_2^2 = 0.00252 \pm 0.00012 \text{eV}^2. \quad (6.4.123)$$

This range of masses hints at extremely small neutrino masses and can be compared with the top quark mass 173GeV, illustrating the huge range of masses within the Standard Model fermions.

Neutrino oscillations

If neutrinos are massive, they can oscillate.

This means that they can be produced in a reaction as “weak” eigenstates $|\nu_i\rangle$ as opposed to mass eigenstates $|\nu_\alpha\rangle$ and, while they travel large enough distances, they may change flavors, oscillating between different flavors.

Both are related by the mixing matrix $U_{\alpha i}$

$$|\nu_\alpha\rangle = \sum_{i=1}^3 U_{\alpha i}^* |\nu_i\rangle, \quad |\nu_i\rangle = \sum_{\alpha=1}^3 U_{\alpha i} |\nu_\alpha\rangle \quad (6.4.124)$$

while they travel for a time t

$$|\nu_\alpha(t)\rangle = \sum U_{\alpha i}^* |\nu_i(t)\rangle. \quad (6.4.125)$$

The mass eigenstates $|\nu_i\rangle$ have plane wave solutions (like standard solutions of Klein-Gordon equations) that determine easily the time dependence

$$|\nu_i(t)\rangle = e^{-i(E_i t - p_i \cdot x)} |\nu_i(0)\rangle . \quad (6.4.126)$$

Since neutrinos are very light they move close to the speed of light and we can take the ultra-relativistic limit

$$p_i \gg m_i, \quad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \simeq E + \frac{m_i^2}{2E} . \quad (6.4.127)$$

Setting the speed of light $c = 1$ we can approximate

$$t \simeq L, \quad p_i \cdot x \simeq EL \quad (6.4.128)$$

with L the distance travelled by the neutrino at time t .

Then we can write

$$|\nu_i(L)\rangle = e^{-im_i^2 L/(2E)} |\nu_i(0)\rangle . \quad (6.4.129)$$

At some point when they are detected they will also be in terms of a weak interaction and therefore the relevant question is: What is the probability of a neutrino being produced at a flavor eigenstates ν_α and detected in ν_β at a distance L ?

Using the equations above we can make an estimate of this probability

$$P_{\alpha\beta} = | \langle \nu_\beta | \nu_\alpha(t) \rangle |^2 = \sum_{ij} e^{-i(m_i^2 - m_j^2)L/2E} U_{\beta i} U_{\beta j}^* U_{\alpha j} U_{\alpha i}^* \quad (6.4.130)$$

where L is the distance travelled and E the energy.

Then studying neutrinos traveling long distances would allow us to determine if they oscillate between different flavors.

Oscillations can be observed if the neutrinos travel distances L large enough so that the argument of the exponential is relevant.

Concretely, if $\Delta m^2 \neq 0$, $L \geq \lambda(E)$ where

$$\lambda(E) = \frac{2E}{\Delta m^2} = 500m \left(\frac{E}{1\text{GeV}} \right) \left(\frac{1\text{eV}^2}{\Delta m^2} \right) . \quad (6.4.131)$$

This means that for energies in the GeV region and mass differences in the eV² region oscillations can be detected after traveling distances of order of kilometers.

This has been observed in many experiments and it is the best evidence we have for neutrino masses *different from zero*.

The evidence comes from three sources:

- *Solar ν -problem.* The thermo-nuclear reactions in the Sun in which Hydrogen nuclei (protons) fuse to produce Helium followed by fusion of Helium to heavier elements produce ν_e . The main source of electron neutrinos is the $p + p \rightarrow d + \bar{\nu} + e^+$ reaction (with $d = p + n$ the deuteron) which account for more than 90% of the neutrinos produced in the Sun. The flux of neutrinos can be computed with much confidence from our understanding of the weak interactions, but the amount of electron neutrinos detected on earth was only 35%. This was known as the solar neutrino problem for many years which was a mystery since only ν_e were possible to detect. Now it is understood in terms of neutrino oscillations in the sense that the other 65% is a combination of ν_μ and ν_τ). It was a big triumph to identify that this problem was a fundamental physics issue rather than an astrophysical issue regarding the physics of the solar model and that the Standard Model with oscillating massive neutrinos actually matches the observations.
- *Atmospheric neutrinos.* Also neutrino oscillations from neutrinos produced in the atmosphere from pions produced by cosmic rays ν : $\pi^- \rightarrow \mu^- \bar{\nu}_\mu \rightarrow (e^- \nu_e \nu_\mu) \bar{\nu}_\mu$ not 2 : 1 ratio. Illustrates the oscillation
- *Neutrino oscillations in the laboratory.* On earth, neutrinos can be produced in nuclear reactors and particle accelerators ν 's can give rise to oscillations as long as they are allowed to travel substantial distances. For this the analysis of neutrino oscillations mentioned above applies and need to have detectors very far away from the source. There are several experimental facilities being developed that can detect neutrino oscillations.

The experimental evidence for neutrino masses comes from neutrino oscillations.

From the equations above, it becomes clear though that the only information we can extract is not about the neutrino masses directly, but only about mass differences, explaining the experimental limits quoted above.

6.4.8 4-Fermi Theory and muon decay*

We recall from the beginning of this part of the lectures that the 4-Fermi theory was the first successful description of the weak interactions at low energies through an interaction vertex involving four fermions $G_F \psi\psi\psi\psi$.

After having written down the full electroweak theory, we are now ready to reconsider how this effective description arises from the GSW model at low energies.

We know that the GSW model contains charge current interactions of the form (recall (6.4.62))

$$\mathcal{L}_W = \frac{e}{\sqrt{2} \sin \theta_W} \left(W^{+\mu} J_\mu^+ + W^{-\mu} J_\mu^- \right) + \frac{e}{\sin \theta_W} Z^\mu J_\mu^Z . \quad (6.4.132)$$

For instance, the lepton sector contributes to $J_\mu^\pm$

$$J_\mu^+ = \bar{\nu}_{eL} \gamma_\mu e_L + \bar{\nu}_{\mu L} \gamma_\mu \mu_L + \bar{\nu}_{\tau L} \gamma_\mu \tau_L , \quad (6.4.133)$$

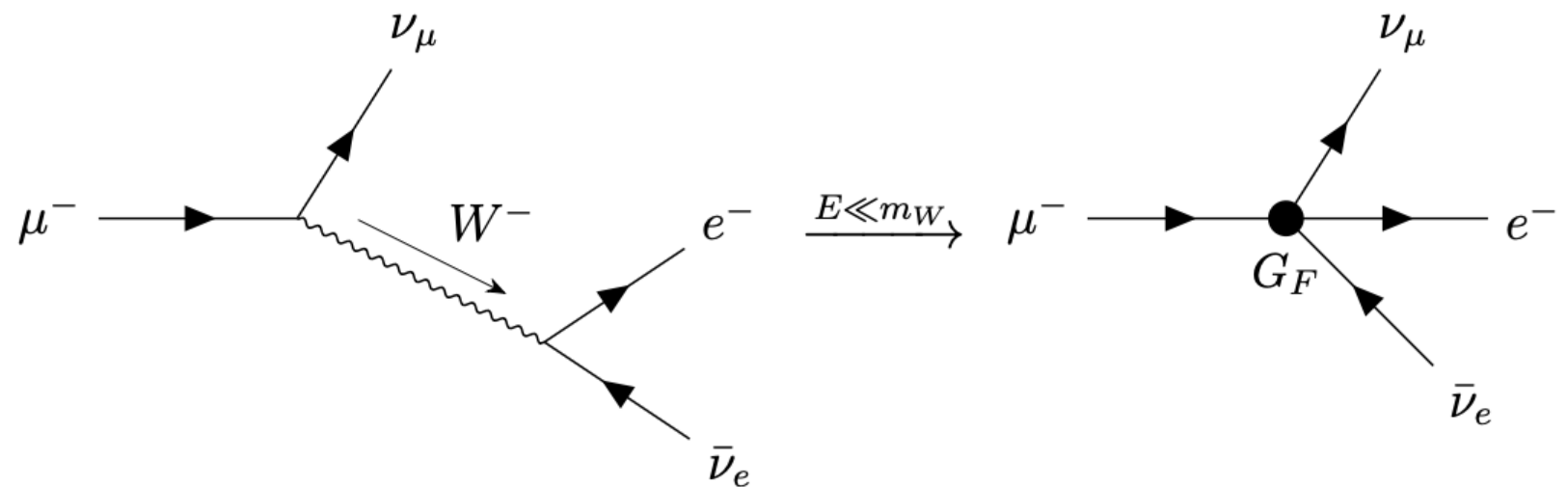
$$J_\mu^- = \bar{e}_L \gamma_\mu \nu_{eL} + \bar{\mu}_L \gamma_\mu \nu_{\mu L} + \bar{\tau}_L \gamma_\mu \nu_{\tau L} , \quad (6.4.134)$$

where for concreteness we are assuming massless neutrinos and no-mixing issues.

From here and knowing the propagator for massive vectors such as $W^\pm$ we can see that for energies $E \ll m_W$ these interactions give rise to amplitudes that can be directly obtained from a 4-Fermi interaction.

Let us describe this in more detail by computing the cross section for muon decay depicted in Fig. 6.4.

Figure 6.4: The muon decay $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ can be approximated at low energies ($E \ll m_W$) through a 4-point fermion vertex in the 4-Fermi effective theory.



In this case, we can focus on interactions for the first two families of leptons (muons, electrons and their associated neutrinos).

From the form of the propagator for massive spin-1 fields in (6.4.34), we can see that for energies $E \ll m_W$ the contribution from the $W^\pm_\mu$ -propagator reduces to

$$\Delta_{\mu\nu}^{W^\pm} = -\frac{1}{p^2 - m_W^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2} \right) \approx \frac{\eta_{\mu\nu}}{m_W^2}. \quad (6.4.135)$$

Thus, the process can be written as a 4-Fermi interaction coming from a Lagrangian

$$\mathcal{L}_{4F} = -\frac{4G_F}{\sqrt{2}} (\bar{e}\gamma_\mu P_L \nu_e + \bar{\mu}\gamma_\mu P_L \nu_\mu) (\bar{\nu}_e \gamma^\mu P_L e + \bar{\nu}_\mu \gamma^\mu P_L \mu) \quad (6.4.136)$$

with

$$\frac{4G_F}{\sqrt{2}} = \frac{e^2}{2m_W^2 \sin^2 \theta_W} = \frac{2}{v^2}, \quad P_L = \frac{1 - \gamma^5}{2}. \quad (6.4.137)$$

Here we have written the left handed fermions in the Dirac notation including the projector P_L .

This allows us to see directly that the coupling we get after setting the limit $E \ll m_W$ is the V - A combination mentioned at the beginning of this part as proposed before the GSW model by Marshak and Sudarshan in 1958, see in particular Eq. (6.2.5).

This can be seen as another success of the GSW model in the sense that it explains why the 4-Fermi model was successful to describe weak interactions at low energies.

It also can relate the Fermi coupling G_F to fundamental parameters of the GSW model.

In this sense the GSW model is a UV completion of the 4-Fermi model.

As an application of the above observation and in order to illustrate how physical amplitudes are computed and to be as explicit as possible, let us now compute in full detail the decay rate for the muon decay shown in Fig. 6.4.

More specifically, we will consider the decay $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ at low energies ($E \ll m_W$) and then start from the 4-Fermi effective theory.

This is justified since $m_W(80.385 \text{ GeV}) \gg m_\mu(105.6583745 \text{ MeV})$.

The decay rate $\Gamma(\alpha \rightarrow \beta)$ from the initial α to final β states depend on the interaction matrix $\mathcal{M}$ as

$$\Gamma(\alpha \rightarrow \beta) = \frac{1}{2m_\alpha} \int \sum_{\text{spins}, \dots} |\mathcal{M}_{\alpha\beta}|^2 d\rho_\beta, \quad (6.4.138)$$

where $d\rho_f$ is the phase space measure

$$d\rho_\beta = (2\pi)^4 \delta^{(4)} \left(p_\alpha - \sum_{r \in \beta} p_r \right) \prod_{r \in \beta} \frac{d^3 p_r}{(2\pi)^3} \frac{1}{2p_r^0}. \quad (6.4.139)$$

where as usual we sum over final moment and the delta function imposes energy-momentum conservation.

For this 4-Fermi case the interaction matrix in terms of the particle (u) and antiparticle (v) wave functions is

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} \bar{u}_e(k) \gamma^\alpha (1 - \gamma^5) v_{\nu_e}(q) \bar{u}_{\nu_\mu}(q') \gamma_\alpha (1 - \gamma^5) u_\mu(p). \quad (6.4.140)$$

Next we need to compute the square of the amplitude summing over spin states of the final particles

$$\begin{aligned}
\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{G_F^2}{4} \sum_{\text{spins}} \left[\bar{u}_e(k) \gamma^\alpha (1 - \gamma^5) v_{\nu_e}(q) \bar{v}_{\nu_e}(q) \gamma^\beta (1 - \gamma^5) u_e(k) \right] \\
&\times \left[\bar{u}_{\nu_\mu}(q') \gamma_\alpha (1 - \gamma^5) u_\mu(p) \bar{u}_\mu(p) \gamma_\beta (1 - \gamma^5) u_{\nu_\mu}(q') \right] \\
&\equiv \frac{G_F^2}{4} S_e^{\alpha\beta} S_{\mu,\alpha\beta} .
\end{aligned} \tag{6.4.141}$$

To compute $S^{\alpha\beta}_e$ depending on the electrons and the muon dependent $S_{\mu,\alpha\beta}$, we need to use the fact that the wave functions u, v solve the Dirac equation in momentum space and use the completeness conditions for the sum over spins

$$\sum u^s(p) \bar{u}^s(p) = \not{p} + m, \quad \sum v^s(p) \bar{v}^s(p) = \not{p} - m . \tag{6.4.142}$$

We will also assume that the neutrino masses are negligibly small and set them to zero.

Then we can write

$$S_e^{\alpha\beta} = \text{Tr} \left[(\not{k} + m_e) \gamma^\alpha (1 - \gamma^5) \not{q} \gamma^\beta (1 - \gamma^5) \right] , \tag{6.4.143}$$

$$S_{\mu,\alpha\beta} = \text{Tr} \left[\not{q}' \gamma_\alpha (1 - \gamma^5) (\not{p} + m_\mu) \gamma_\beta (1 - \gamma^5) \right] . \tag{6.4.144}$$

Now we can use the standard gamma matrix identities (A.2.9), (A.2.10) and (A.2.11) to find

thFinally, from here we can easily compute the contraction $S^{\alpha\beta}_1 S_{2,\alpha\beta}$ to get

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = 64 G_F^2 (p \cdot q) (k \cdot q') . \quad (6.4.145)$$

This finishes the calculation of the integrand in (6.4.138).

The remaining task is to compute the integral.

We start by plugging the expression (6.4.145) into the partial decay rate (6.4.138)

$$\begin{aligned} \Gamma &= \frac{1}{2m_\mu (2\pi)^5} \int \frac{d^3 k}{2k^0} \int \frac{d^3 q}{2q^0} \int \frac{d^3 q'}{2q'^0} \delta^{(4)}(p - k - q - q') \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 \\ &= \frac{G_F^2}{8\pi^5 m_\mu} \int \frac{d^3 k}{k^0} \frac{d^3 q}{q^0} \frac{d^3 q'}{q'^0} \delta^{(4)}(p - k - q - q') (p \cdot q) (k \cdot q') \end{aligned} \quad (6.4.146)$$

where we are using that the electron's momentum is k^μ and the neutrinos have momenta q^μ, q'^μ .

In order to perform the integral, let us consider the following object

$$I_{\mu\nu}(Q) = \int \frac{d^3 q}{|\vec{q}|} \frac{d^3 q'}{|\vec{q}'|} \delta^{(4)}(Q - q - q') q_\mu q'_\nu . \quad (6.4.147)$$

Given that the final result should be a function of Q_μ , the only possible tensorial dependence of $I_{\mu\nu}$ is the following

$$I_{\mu\nu} = aQ_\mu Q_\nu + bQ^2\eta_{\mu\nu} \quad (6.4.148)$$

with unknown coefficients a, b.

In order to determine them, we can contract this equation with $\eta^{\mu\nu}$ and also with $Q^\mu Q^\nu$.

Using the fact that $q^2 = q'^2 = 0$ and that inside the integral we can use the delta function condition $Q = q + q'$, we get

$$a + 4b = \frac{I}{2}, \quad a + b = \frac{I}{4} \quad (6.4.149)$$

where

$$I \equiv \int \frac{d^3q}{|\vec{q}|} \frac{d^3q'}{|\vec{q}'|} \delta^{(4)}(Q - q - q') = \int \frac{d^3q}{|\vec{q}|^2} \delta(Q^0 - 2|\vec{q}|) = 4\pi \int d|\vec{q}| \delta(Q^0 - 2|\vec{q}|) = 2\pi. \quad (6.4.150)$$

Here we used the fact that I is Lorentz invariant and could evaluate it for $Q = (Q^0, 0, 0, 0)$ with no loss of generality.

Therefore, the coefficients a, b lead to

$$I_{\mu\nu} = \frac{\pi}{3}Q_\mu Q_\nu + \frac{\pi}{6}Q^2\eta_{\mu\nu}. \quad (6.4.151)$$

Plugging this into the integral for Γ in (6.4.146), we find

$$\Gamma = \frac{G_F^2}{3m_\mu(2\pi)^4} \int \frac{d^3k}{k^0} \left[2p \cdot (p - k) k \cdot (p - k) + p \cdot k (p - k)^2 \right] . \quad (6.4.152)$$

Now we choose the rest frame of the muon where $p = (m_\mu, 0, 0, 0)$.

Further, we can safely take $k = (E, \mathbf{k})$ with $|\mathbf{k}| \sim E$ since the electron is much lighter than the muon.

Therefore, Γ can be approximated by

$$\Gamma = \frac{2G_F^2 m_\mu}{3m_\mu(2\pi)^3} \int_0^{m_\mu/2} dE E^2 (3m_\mu - 4E) = \frac{G_F^2 m_\mu^5}{192\pi^3} \quad (6.4.153)$$

which is our final result.

The limits of integration correspond to the two extreme cases: the electron at rest and neutrinos in opposite direction $E = 0$ and the neutrinos moving in the same direction and the electron in the opposite for which momentum conservation implies $E = m_\mu/2$.

Experimentally the muon lifetime has been well measured to be

$$\tau_\mu = \frac{1}{\Gamma} \approx 2.1970 \times 10^{-6} \text{ seconds} . \quad (6.4.154)$$

From this and the knowledge of the muon mass we can obtain the value of

$$G_F = 1.164 \times 10^{-5} \text{ GeV}^{-2} . \quad (6.4.155)$$

Of course, the value of G_F can be determined by other transitions.

The fact that they all agree is a further confirmation of the validity of the theory.

Knowing the value of G_F also fixes the VEV of the Higgs

$$v^2 = \frac{1}{\sqrt{2}G_F} \Rightarrow v = 247 \text{ GeV} \quad (6.4.156)$$

which is very close to the experimentally measured value

$$\text{experimentally: } v = \frac{2m_W}{g} \approx 246.22 \text{ GeV} . \quad (6.4.157)$$

Also, even without knowing the value of θ_W , already having this value for v and using its relations to m_W and m_Z would then require

$$m_W = \frac{ev}{2 \sin \theta_W} > 37.4 \text{ GeV} , \quad m_Z = \frac{m_W}{\cos \theta_W} > m_W . \quad (6.4.158)$$

This information was important to know how heavy the W and Z particles could be when experiments were designed to search them.

This concludes this part of the lectures which is at the core of the physics of the Standard Model, in which two of the fundamental interactions known in nature are successfully described in terms of a simple unified gauge theory with many experimental successes: proper explanation of a huge number of accumulated experimental evidence regarding weak interactions, spectacularly confirmed predictions (neutral currents, $W^\pm$, Z , Higgs, etc.), plus precision experimental tests – an outstanding triumph for fundamental physics.

PART 7 STRONG INTERACTIONS

Confinement

Color particles like quarks and gluons can never be isolated. This has never been proved. . . , but since it is true I for one am happy to leave the proof to the mathematicians.

Steven Weinberg

In the previous chapters, we have seen that interacting theories of helicity ± 1 particles give rise to gauge theories.

So far, we investigated the following two phases:

- a) *Coulomb phase*: QED with $G = U(1)_{EM}$ mediated by a massless field A_μ leading to long range interactions.
- b) *Higgs phase*: GSW model (Glashow-Salam-Weinberg) with breaking pattern $G = SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}$ with massive mediators $W^\pm, Z$ and short range interactions at low energies (in addition to the Coulomb phase).

But there still remain several questions unanswered:

- a) Can we describe strong interactions in terms of a gauge theory?
- b) Can a similar theory to the electroweak theory describe strong interactions that are also short range?
- c) Are there other phases of gauge theories that can describe short range interactions (different from the Higgs phase)?

Recall that in the 1930's Yukawa predicted strong interactions to be mediated by the later discovered π -mesons or pions that having a mass (intermediate between protons and electron, hence the name mesons) could explain the fact that the interactions are short ranged.

This is because the scalar potential would decay exponentially with the mass $V \propto e^{-mr}/r$ compared with the long-range interactions in QED that gives $V \propto 1/r$.

However, even though non-relativistic descriptions of the force were partially successful, the efforts to describe it in terms of relativistic QFT failed, mostly due to the need to deal with strong interactions which rendered the standard perturbative expansions used in QED and the electroweak theory useless.

The answer to the second question above is then No.

Actually, when Weinberg discovered his model for the electroweak interactions, he was trying unsuccessfully to describe the strong interactions.

In this part of these lectures, we will see however that the answer to the first and third questions is Yes

7.1 Motivation for SU(3)

Let us list a couple of arguments why an $SU(3)_c$ gauge theory is an appropriate ansatz to model the strong interactions:

- *Color and Eightfold way.* As mentioned in the introduction, cf. Fig. 1.2, the eightfold way model was very successful in classifying the strongly interacting particles: mesons and baryons (hadrons) in terms of representations of an approximate symmetry $SU(3)_f$ (not to be confused with color $SU(3)_c$) and the proposal of Gell-Mann and Zweig to consider the fundamental representations of this group to be the fundamental components of all hadrons hit the spot by the prediction and subsequent detection of the Ω^- - particle.

However this particle also poses a problem in the sense that its composition is $|\Omega^- \rangle = |s \rangle \otimes |s \rangle \otimes |s \rangle$ for the strange quark $|s \rangle$.

The problem is (again!) the Pauli exclusion principle since this would require three identical quarks in the same quantum state.

The solution (Greenberg, Nambu and Han) was to introduce a new quantum number *color* such that

$$|\Omega^- \rangle = \epsilon_{ijk} |s \rangle^i \otimes |s \rangle^j \otimes |s \rangle^k \quad i, j, k = r, g, b$$

with r, g, b three colors (for red, green, blue, but of course this has nothing to do with real colors except for the name).

- *QCD*. Later on it was suggested by Harald Fritzsch, Gell-Mann and, independently, Julius Wess that this new quantum number would be due to an exact gauge symmetry corresponding to $SU(3)_c$ which was coined Quantum Chromodynamics (QCD).

Given the fact there were three and only three colors the options for a gauge group are limited to those that acted on a three-dimensional space.

$SU(3)$ is the only viable choice of gauge group since:

- $SO(3)$ cannot distinguish quarks and anti-quarks so, if $|q\rangle \otimes |\bar{q}\rangle$ exists, then also $|q\rangle \otimes |q\rangle$ which are states of fractional charge, which is not how nature behaves,
- $U(3)$ could not work either since $U(1) \subset U(3)$ would mediate long range interactions just like in QED.

That singled out $SU(3)_c$.

- *Color blindness*. However, the fact that the theory is formulated in terms of fields (gluons and quarks) that are not the observed degrees of freedom in nature (hadrons) made it difficult to see how this gauge theory could make contact with the observed spectrum.
- *Leptons*. Leptons do not carry color which is why it seems unnatural to assign to them a gauge theory with gauge group $SU(3)_c$ to describe strong interactions that are not experienced by leptons - so they should be singlets under this symmetry.

- *Experimental support.* But the proposal gained support experimentally. – Deep Inelastic Scattering (DIS).

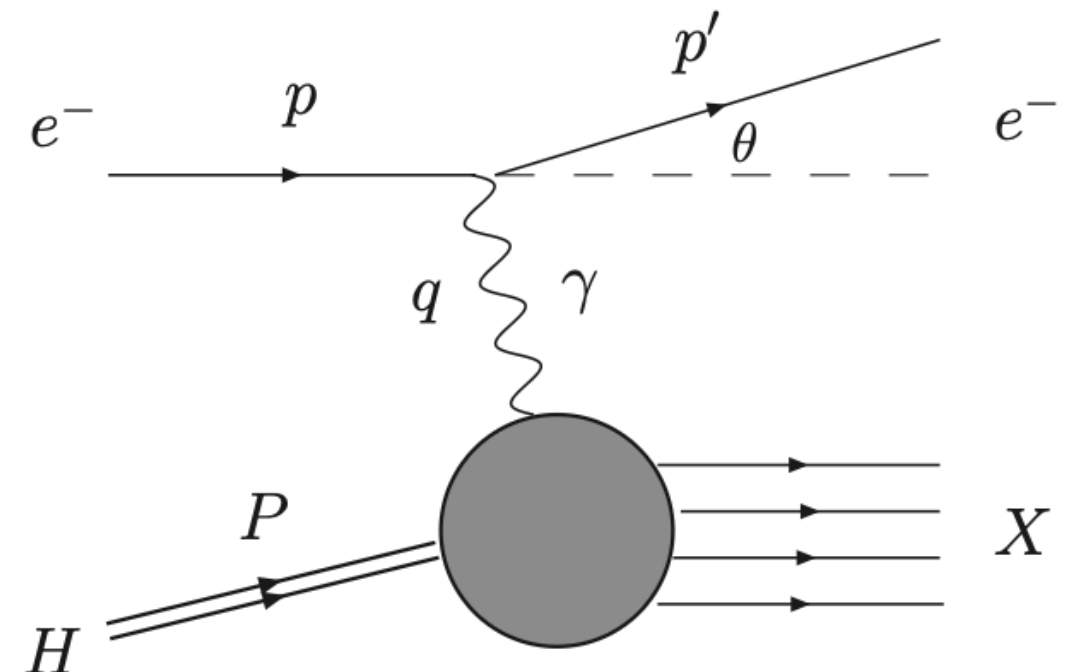
In 1969 experiments at SLAC probed the structure of protons by bombarding them with electrons at high energies.

This was the modern version of the Rutherford experiments that uncovered the structure of the atom.

Instead of α -particles scattering off gold atoms, this time high energy electrons bombarded protons and the result was equally spectacular.

The experiments showed that *hadrons, like protons, behave as composed particles experimentally*, cf. Fig. 7.1.

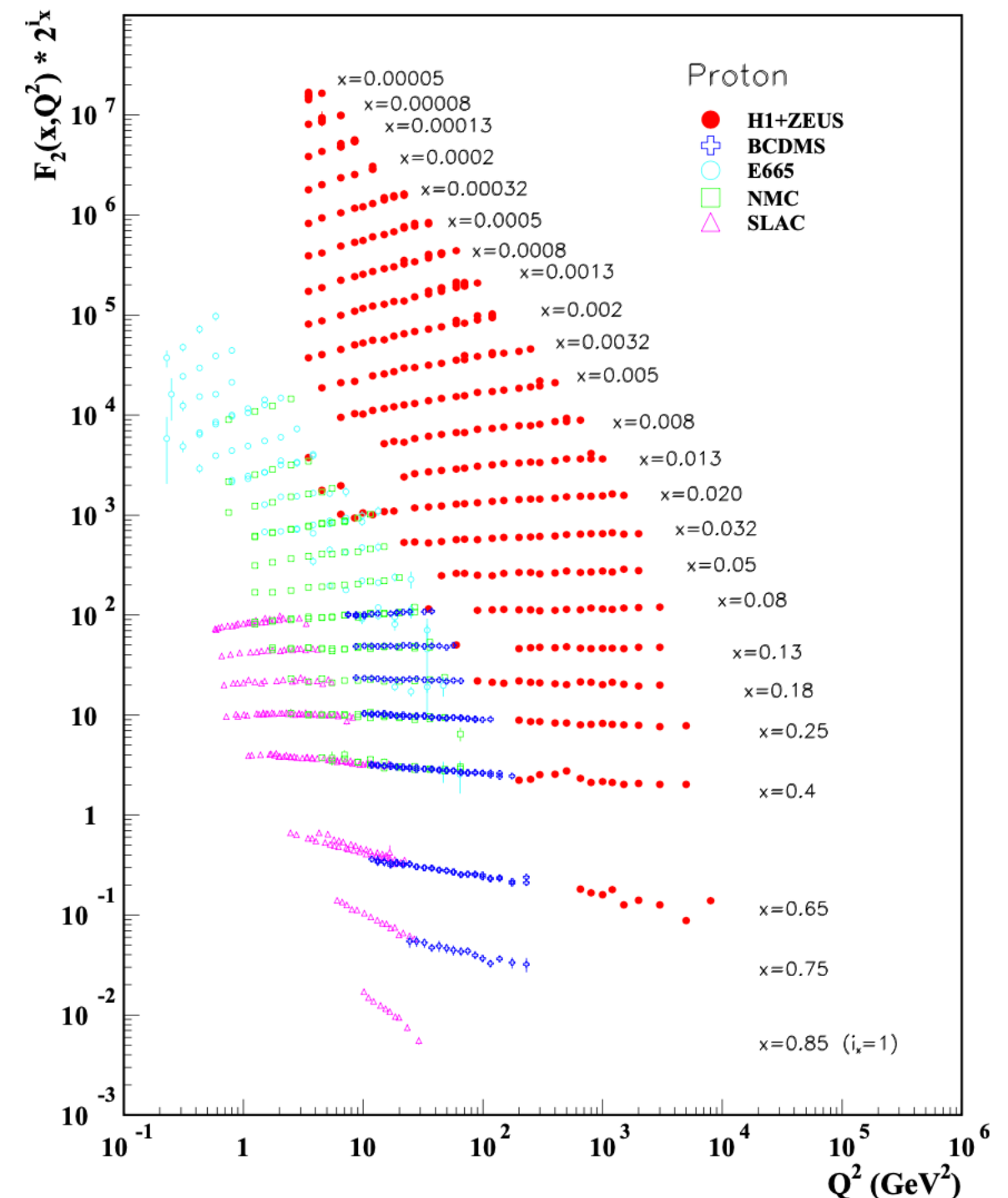
Figure 7.1: Deep inelastic scattering (DES). Scattering of electrons off protons exposed the composite structure of the protons. Analogous to the old Rutherford experiment to uncover the structure of atoms. In this case a high energy electron e^- of momentum p scatters off a hadron H (which in this case is a proton) with center of mass momentum P . The interaction is mediated by a photon γ of momentum q . The electron is scattered at an angle θ . The proton absorbs a good part of the energy and emits further hadrons X (hence the inelastic nature of the event). This event can be explained if both H and X are composed of quarks and gluons.



Not only that, these experiments also showed that the proton's components are weakly interacting once the energy is increased.

This is manifested by the fact that the structure of the proton remained the same independent of how hard it was struck.

In the graph in Fig. 7.2, it can be illustrated by the presence of an approximate scale invariance (flat curves) of the proton structure function.

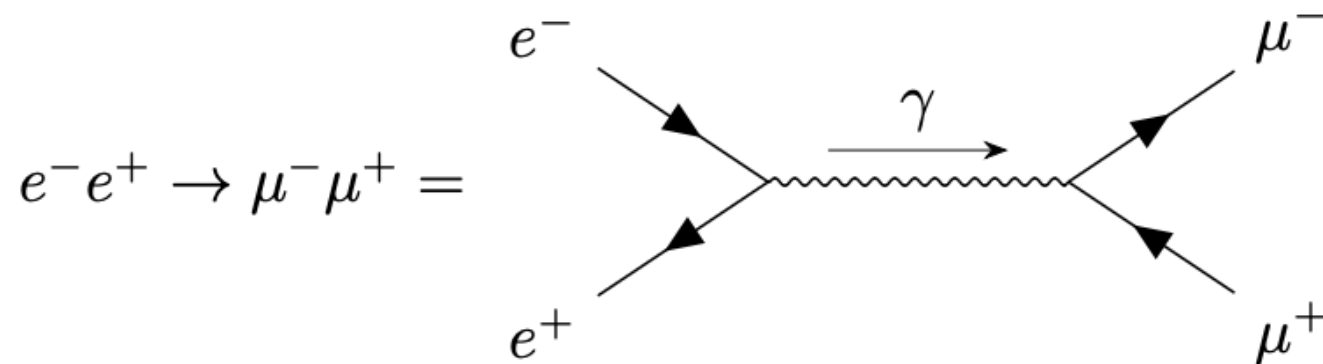


It hinted at the protons being composed of more fundamental particles that interact very weakly (to explain the approximate scaling, known as *Bjorken scaling*).

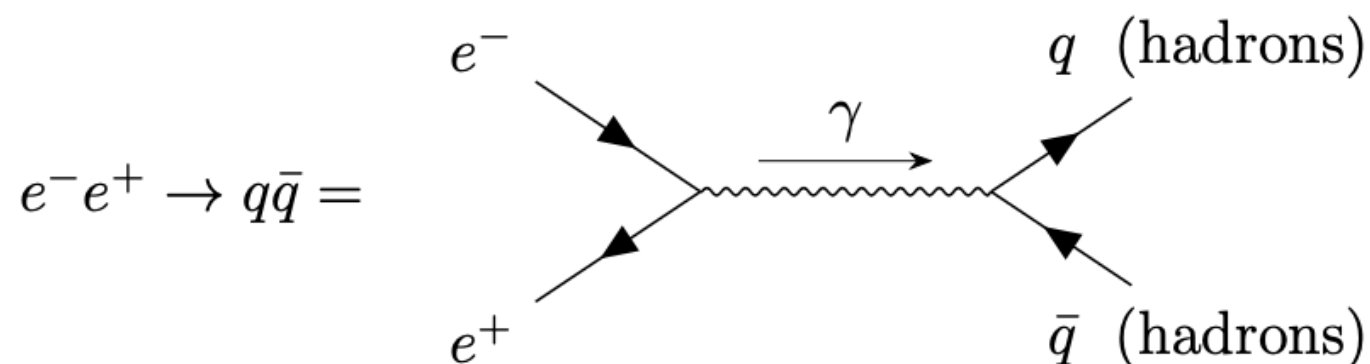
These components were called *Partons* and only several years later were identified with the quarks and gluons of QCD.

- *Three colors*. An impressive source of support for the three color hypothesis can be obtained by considering the following ratio of two processes.

The first is the annihilation of an electron-positron into $\mu^-\mu^+$,



and the second corresponds to the decay into hadrons



These are simple electromagnetic interactions with the vertices just differing by the electric charge of the outgoing particles.

Therefore, when taking the ratio, one can probe the electric charges involved in the two different processes.

That is, we can define the *R-factor*

$$R = \frac{\sigma(e^-e^+ \rightarrow \text{hadrons})}{\sigma(e^-e^+ \rightarrow \mu^-\mu^+)} \sim N_c \sum Q_q^2 \quad (7.1.1)$$

with N_c the number of colors and Q_q^2 the charges of quarks below the top mass which are the energies available for electron-positron colliders so that

$$3 \sum Q_q^2 = 2 \left(\frac{2}{3}\right)^2 + 3 \left(-\frac{1}{3}\right)^2 = \frac{11}{3}. \quad (7.1.2)$$

More generally, we have depending on the probed energies

$$N_c \sum Q_q^2 = \begin{cases} \frac{2}{3} N_c & uds \text{ light}, \\ \frac{10}{9} N_c & udsc \text{ light}, \\ \frac{11}{9} N_c & udscb \text{ light}. \end{cases} \quad (7.1.3)$$

This fits experiment with $N_c = 3$ to high precision.

These numbers are such that it is impossible to have $N_c = 2, 4, \dots$ or any other integer when compared to experiments.

Furthermore, as illustrated in Fig. 7.3, the experiments show a series of plateaus with R increasing once the threshold for the mass of one of the quarks is reached (charm, bottom for instance).

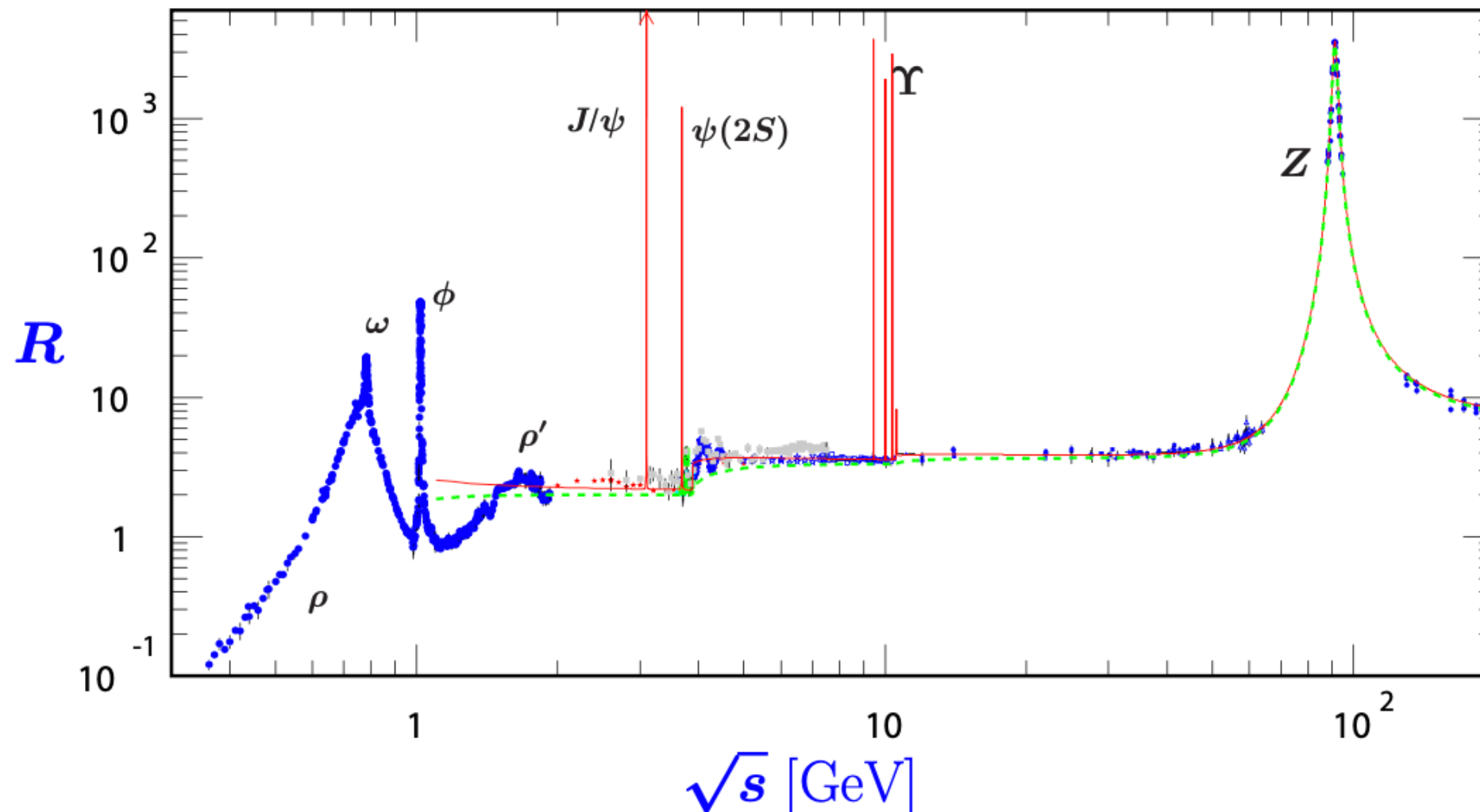


Figure 7.3: Evidence for three colors. Electron-positron scattering to muons and hadrons allows the definition of the R factor as in Eq. (7.1.1). Depending on the masses of quarks there are different plateaus (besides the resonances) at values corresponding to the masses of the quarks that fit below the threshold. Notice that at the energies shown the top quark does not contribute. Furthermore once the S particle threshold is reached there is the new channel corresponding to Z boson exchange together with the original photon-exchange. The value of R fits only if $N_c = 3$. Figure from the particle data group, see this link.

As it can be observed in Fig. 7.3, one finds three regimes:

- a) the horizontal green line is the plateau corresponding to contributions coming only from the u, d, s quarks ($R = 2$),
- b) then at energies of order 4 GeV (after the $\psi(2s)$ resonance), there is a plateau in which the c quark can be produced $R = 10/3$,
- c) and then at energies close to 10 GeV, after the Υ resonance there is the next plateau $R = 11/3$ when the b quarks can also be produced.

This includes 5 quarks, the top quark threshold (top mass ≈ 174 GeV) needs higher energies.

However, at higher energies there is also the Z resonance that can also be seen in Fig. 7.3, that opens a new decay channel with the Z boson as the mediator instead of the photon.

If we compute the above ratio R for γ and Z as mediators, we find theoretically $R = 20.09$ which agrees well with the experimental value $R = 20.79 \pm 0.04$ and the agreement improves when loop corrections are included.

This is compelling evidence of the three color hypothesis and therefore for a $SU(3)_c$ gauge theory of the strong interactions.

Having established the $SU(3)_c$ structure of strong interactions and given our knowledge of gauge theories so far, we will then start studying QCD as a gauge theory of the symmetry group $SU(3)_c$.

The most general Lagrangian for the $SU(3)_c$ gauge theory with fermions in the fundamental representation is

$$\mathcal{L} = -\frac{1}{4} \left(G_{\mu\nu}^A \right)^2 + i\bar{q}_i \not{D}_{ij} q_j - m_i \bar{q}_i q_i + \theta G_{\mu\nu}^A \tilde{G}_{\mu\nu}^A . \quad (7.2.1)$$

The indices i, j are color indices.

The field strength is defined as

$$G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A + g_s f^{ABC} G_\mu^B G_\nu^C \quad (7.2.2)$$

and the covariant derivative reads as usual

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig_s G_\mu^A T_{ij}^A . \quad (7.2.3)$$

The mediators $G^{A\mu}$, $A = 1, \dots, 8$, $\dim(SU(3)) = 8$, are called *gluons*.

A convenient choice of generators for $SU(3)$ is

$$T_{ij}^A = \frac{1}{2} \lambda_{ij}^A \quad (7.2.4)$$

where the matrices λ^A are the Gell-Mann matrices which are the analogues of the Pauli matrices of SU(2) for SU(3):

$$\begin{aligned} \lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{7.2.5}$$

These matrices satisfy

$$\text{Tr}(T^A T^B) = \frac{1}{2} \delta^{AB}. \tag{7.2.6}$$

Notice that their structure is very similar to that of the Pauli matrices, the first three are just the Pauli matrices illustrating an SU(2) embedding within SU(3).

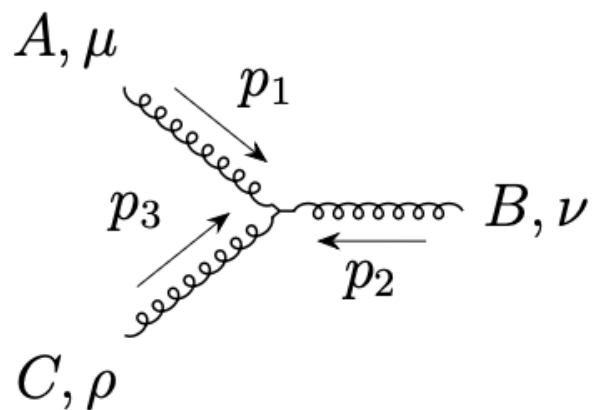
Also, contrary to SU(2) in which only σ_3 is diagonal, here both λ^3 and λ^8 are diagonal, illustrating the fact that SU(3) has rank two.

This implies that two of the generators can be diagonalized simultaneously and the weight and root diagrams are 2-dimensional as we have seen for the eightfold way.

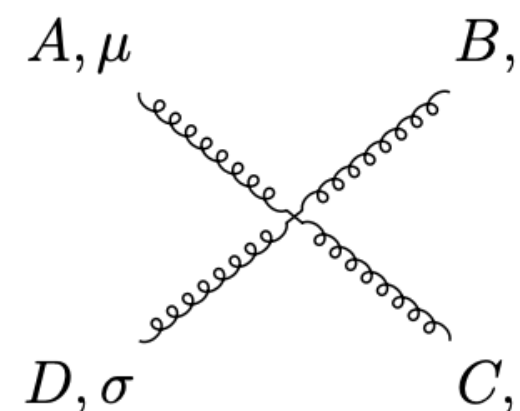
The QCD Lagrangian is a particular case of the general Yang-Mills case.

As we have discussed, the kinetic term for the gluons gives rise to cubic and quartic self-couplings.

The vertices involved are

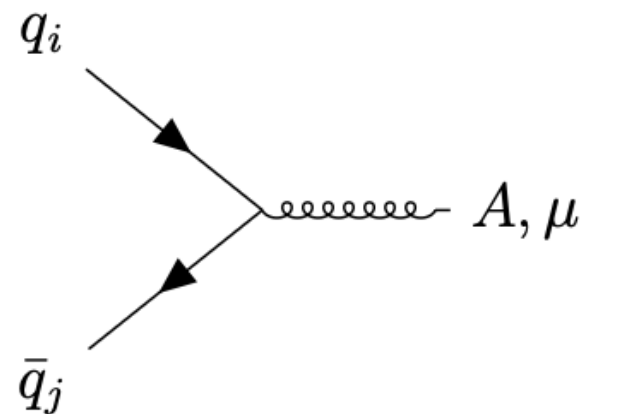


$$= ig_s f^{ABC} [\eta^{\mu\nu}(p_1 - p_2)^\rho + \eta^{\nu\rho}(p_2 - p_3)^\mu + \eta^{\rho\mu}(p_3 - p_1)^\nu]$$



$$= ig_s^2 f^{ABE} f^{CDE} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + ig_s^2 f^{ACE} f^{BDE} (\eta^{\mu\nu} \eta^{\sigma\rho} - \eta^{\mu\sigma} \eta^{\nu\rho}) + ig_s^2 f^{ADE} f^{BCE} (\eta^{\mu\nu} \eta^{\sigma\rho} - \eta^{\mu\rho} \eta^{\nu\sigma})$$

Also the coupling of the gauge field to fermions coming from the covariant derivative takes the form



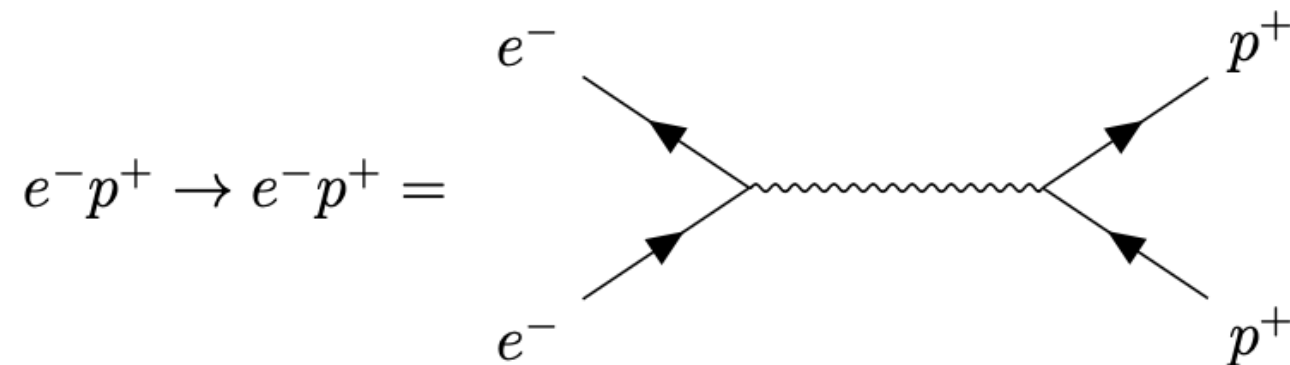
$$= ig_s \gamma^\mu T_{ij}^A$$

7.3 Interaction potentials

Let us next try to get a better grasp of what the strong interactions actually are.

It is always helpful to compare with the well known case of QED.

We know in QED the study of the scattering of $e^- p^+ \rightarrow e^- p^+$ (time running upwards)



can be used to determine the Coulomb interaction potential.

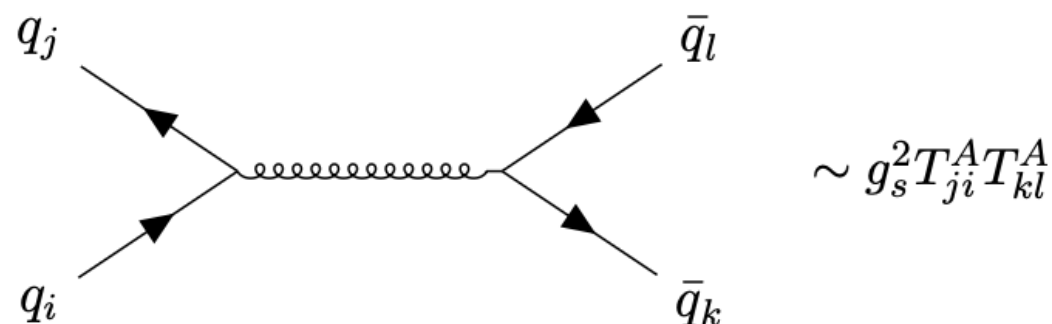
Indeed, one can show that

$$|\mathcal{M}| \propto -\frac{e^2}{|\mathbf{p}|^2} = \tilde{V}(|\mathbf{p}|^2) \quad (7.3.1)$$

that (after a Fourier transform) in position space reads

$$V(r) \propto -\frac{e^2}{4\pi r} \quad (7.3.2)$$

For QCD, a typical process for quark-antiquark scattering looks like



This process is slightly more involved than the analogue for QED above: the incoming quarks and anti-quarks are in $\mathbf{3} \otimes \bar{\mathbf{3}}$ representations of SU(3) that couple naturally to the gluons who are in the adjoint representation (and produce gauge singlets) and the same for the outgoing quark-antiquark pair.

For each case incoming and outgoing quark-antiquark pair, we know from group theory that

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1} \quad (7.3.3)$$

where the octet is just equivalent to the adjoint representation given by the Gell-Mann matrices and the singlet would be the combination proportional to the trace of the matrix $q_i \bar{q}^j$: ($|1\rangle \propto |1\bar{1}\rangle + |2\bar{2}\rangle + |3\bar{3}\rangle$).

Using the Gell-Mann matrices above, it is easy to check that for the octets $T^A T^A < 0$ meaning a repulsive force, whereas for the singlet combination $T^A T^A > 0$ is attractive.

This is similar to the Coulomb interactions.

More explicitly, we can compute the potential as follows:

- **Singlet state.**

Let us consider first the original quark/anti-quark pair to be the singlet

$$|q\bar{q}\rangle_s = \frac{1}{\sqrt{3}} \left(|R\bar{R}\rangle + |G\bar{G}\rangle + |B\bar{B}\rangle \right) \quad R, G, B = 1, 2, 3. \quad (7.3.4)$$

That is, the incoming quarks have the same color $i = k$ (red and anti-red, green and anti-green, blue and anti-blue) and using the explicit representation of the Gell-Mann matrices we have for each

$$T_{j1}^A T_{1l}^A = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad T_{j2}^A T_{2l}^A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad T_{j3}^A T_{3l}^A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \quad (7.3.5)$$

The strength of the interaction can be computed by taking the trace (summing over all possibilities) of each of these matrices as indicated by the expression of $lq\bar{q} \leftrightarrow S$ above, which gives for each color $+(4/3)$ (times 3 for the 3 colors and times $1/(-3)^2$ from the normalization) to give

$$V(r) = -\frac{4}{3} \frac{g_s^2}{4\pi r} = -\frac{4}{3} \frac{\alpha_s}{r}. \quad (7.3.6)$$

The minus sign indicates that, similar to the electromagnetic case, this is an attractive force.

It means in particular that bound states can exist (just like an atom or positronium for electromagnetism).

- **Octet state.**

Now for the octet $j \neq k$, it is easy to show

$$T_{ij}^A T_{kl}^A = -\frac{1}{6} \delta_{ij} \delta_{kl}, \quad j \neq k. \quad (7.3.7)$$

This illustrates that the colours of the outgoing particles are the same as the colors of the incoming particles as it should.

The interaction potential is

$$V(r) = +\frac{1}{6} \frac{g_s^2}{4\pi r} = +\frac{1}{6} \frac{\alpha_s}{r} . \quad (7.3.8)$$

Since the sign is positive, the interaction is repulsive.

This means we cannot create bound states from the octets.

This goes a long way towards explaining why mesons are color singlets.

This result also can be extended for bound states of three quarks.

Since

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \quad (7.3.9)$$

It can be seen again that only the singlet combination gives an attractive interaction.

This is expected since recall that $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$ so the singlet in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ also comes from the singlet in the $\mathbf{3} \otimes \bar{\mathbf{3}}$ product.

In this case it corresponds to the invariant antisymmetric combination

$$|qqq\rangle_s = \frac{1}{\sqrt{6}} (|RGB\rangle + |BRG\rangle + |GBR\rangle - |BGR\rangle - |RBG\rangle - |GRB\rangle) . \quad (7.3.10)$$

This is a first indication that quarks and anti-quarks attract each other in color singlets that can be identified with mesons, whereas the color octets being repulsive do not form bound states.

Similarly out of three quarks we can form another color singlet $\varepsilon_{ijk}\psi_i\psi_j\psi_k$ which are the baryons.

In summary, color singlets are hadrons which can be distinguished in

- $q_i\bar{q}_i$: *mesons*
- $\varepsilon_{ijk}q_iq_jq_k$: *baryons*

Notice that this is consistent with the fact that only color singlets are observed in nature, but it does not explain it.

Contrary to QED, in which both bound states and isolated fundamental states, such as free electrons exist; in QCD quarks on the other hand only live in bound states.

Below, we will present arguments towards understanding why this is the case

7.4 Asymptotic freedom

The most important property of QCD and many Yang-Mills systems is asymptotic freedom that we will discuss now.

It is known from general QFT that proper renormalisation techniques lead to the fact that coupling constants actually change with energy.

For instance, quantum contributions to vacuum polarization for the gauge fields include loops of matter and gauge fields which implies that the gauge coupling g_s becomes energy dependent.

This is well familiar from e.g. QED.

For general Yang-Mills, due to the additional self interactions of the gauge fields, the one-loop diagram involves not only fermions and scalars in the loops but also the gauge fields themselves.

The QCD β -function is given by

$$\beta(\alpha_s) = \mu \frac{d}{d\mu} \alpha_s, \quad \alpha_s = \frac{g_s^2}{4\pi}. \quad (7.4.1)$$

To leading order, one finds

$$\beta(\alpha_s) = -\frac{\alpha_s^2}{4\pi} b. \quad (7.4.2)$$

Here the coefficient b and higher order coefficients are determined by group theoretical numbers.

For $SU(N_c)$ with N_f flavors,

$$b = \frac{11}{3} N_c - \frac{2}{3} N_f, \quad (7.4.3)$$

and the running coupling is given by

Running coupling for SU(N_c) with N_f flavors

$$\frac{1}{g_s^2(\mu)} = \frac{1}{g_s^2(\Lambda)} - \frac{1}{(4\pi)^2} \left[\frac{11}{3} N_c - \frac{2}{3} N_f \right] \log \left(\frac{\Lambda^2}{\mu^2} \right) \quad (7.4.4)$$

where Λ is some reference scale.

In comparison, the running of the QED coupling is given by

$$\frac{1}{e^2(\mu)} = \frac{1}{e^2(\Lambda)} + \frac{1}{12\pi^2} \log \left(\frac{\Lambda^2}{\mu^2} \right) . \quad (7.4.5)$$

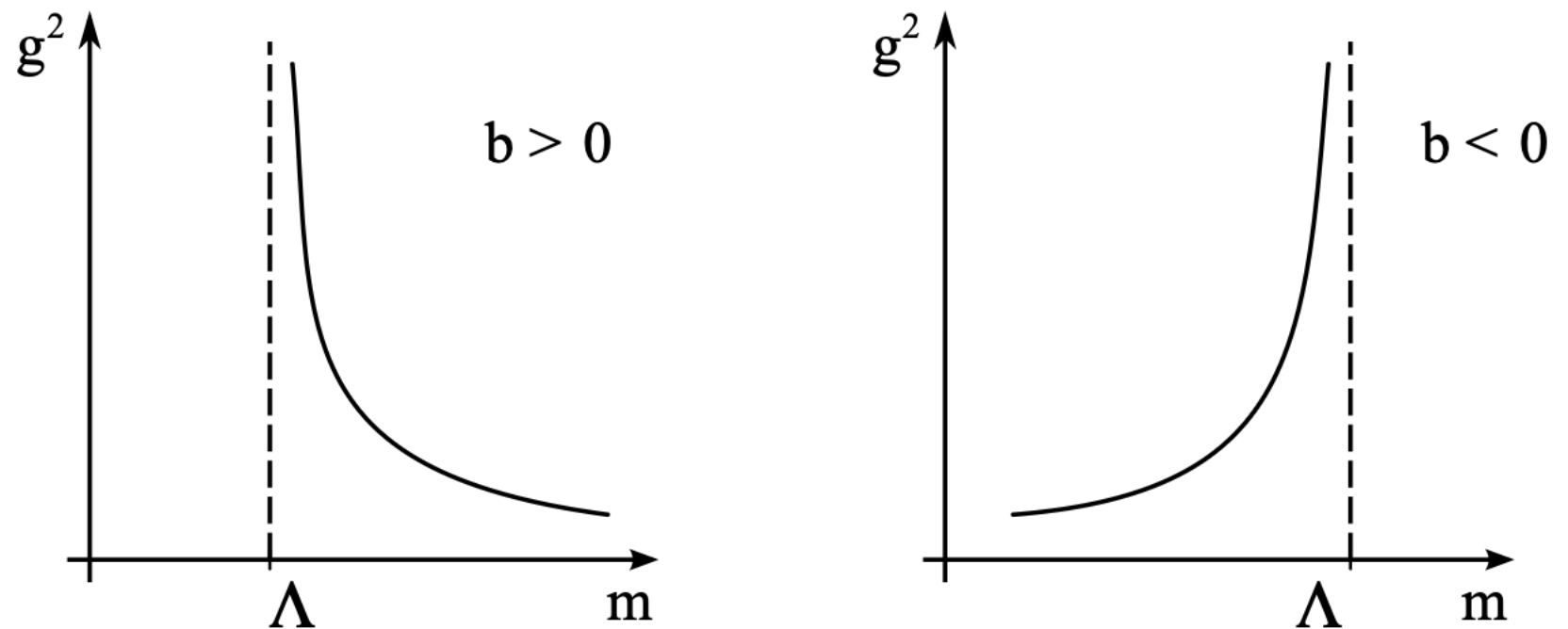
It leads to a Landau pole at $\mu \simeq 10^{286} \text{eV}$ at which point the theory ceases to be valid.

However, this energy is so high that it is clear the theory will need modifications at much smaller energies, knowing for instance that gravity cannot be neglected at the Planck scale.

Yang-Mills is much richer than QED.

In QCD, we need to distinguish the different behaviors depending on whether N_c is greater or smaller than 2N_f/11, cf. Fig. 7.4.

Figure 7.4 The running of the coupling constants for asymptotically free ($b > 0$) and for asymptotically slave ($b < 0$) theories.



For QCD, we have $N_c = 3$ and $N_f = 6$ which implies that the theory is asymptotically free!

This means that, despite the coupling being relatively strong at low energies, it decreases logarithmically with energy and the theory is well behaved in the ultra-violet.

In this sense it is better behaved than QED.

The main difference is the factor $-11N_c/3$ due to the self interactions of gauge bosons.

This contribution has a definite sign opposite from that due to matter fields.

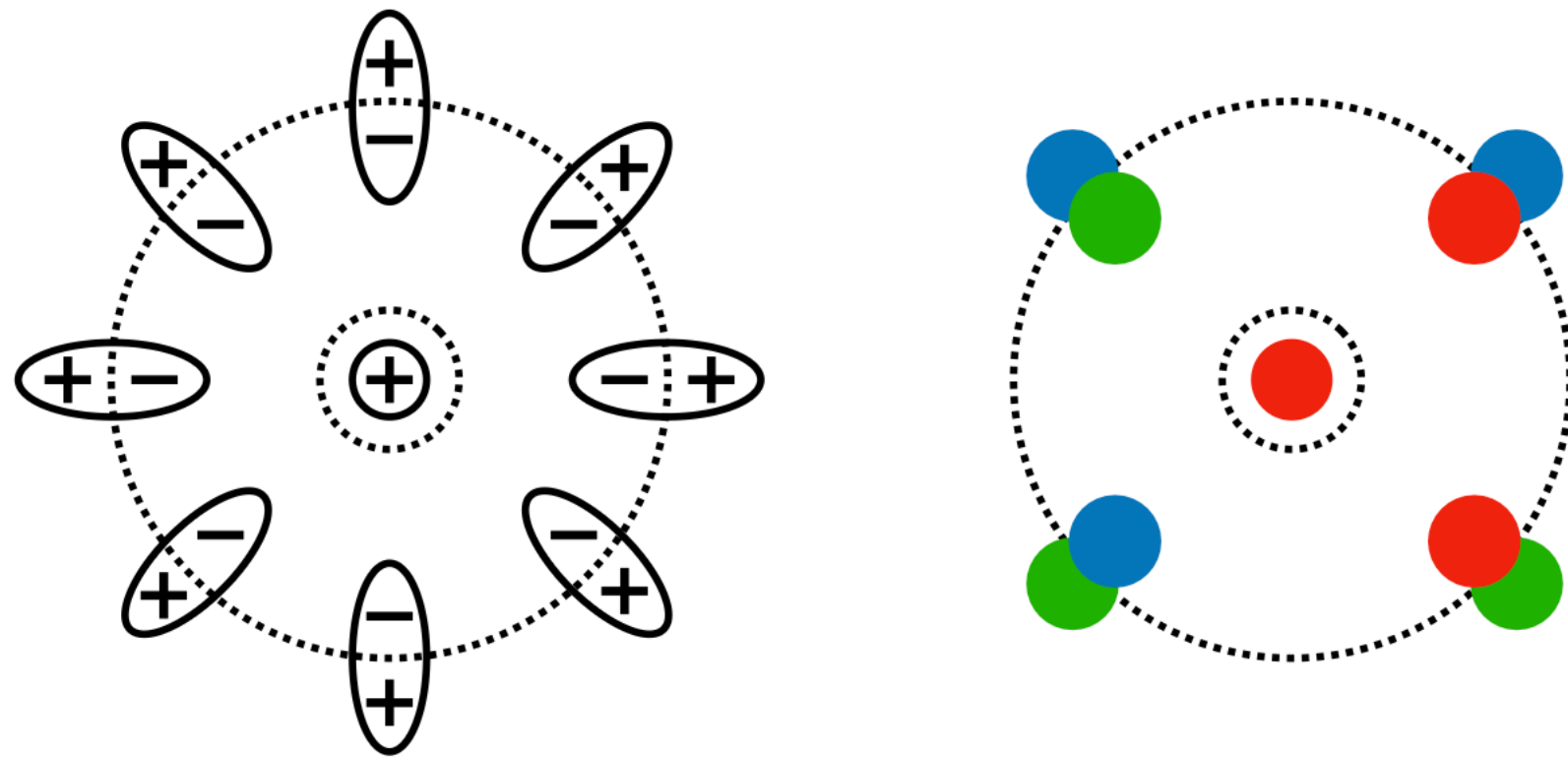
If we insist that a given gauge theory is well defined in the UV, this puts a bound on the number of matter fields, in this case on $N_f < 11N_c/2$ which is well satisfied by QCD.

This difference in sign between QED and QCD is crucial.

In QED it leads to what is known as the screening effect in which the contributions to the vacuum polarization screen the value of the bare electric charge.

We can imagine a vacuum with pairs of particle and anti-particles being produced in the vicinity of an electric charge in which the electromagnetic attraction somehow screens the value of the bare charge in such a way that the effective charge decreases with distance (see Fig. 7.5).

Figure 7.5: Screening vs Anti-screening. A cartoon representation of the screening effect in QED ($b < 0$) for which the effective electric charge increases closer to the bare charge and the anti-screening effect for asymptotically free theories ($b > 0$) such as QCD in which the net color (red in this case) decreases closer to the bare particle.



For QCD the opposite happens, i.e., there is an anti-screening effect in which the effective color decreases closer to the colored particle.

Since QCD is asymptotically free, we can see that it is a weakly interacting theory at high energies, explaining the observational fact that in deep inelastic scattering (DIS) the components of the proton behaved as free particles.

It would also imply that, in the early universe at high temperatures, quarks and gluons were essentially free, interacting in a quark-gluon plasma soup together with leptons and photons.

Once the universe cools down and particles interacted less violently, the coupling becomes stronger and at some point hadronization happens, that means the observable spectrum becomes hadrons (and leptons) and no longer quarks and gluons.

Knowing how the coupling changes with energies, we may estimate the value of this scale.

The scale Λ_{QCD} is called the *confining scale* corresponding to the QCD Landau pole.

It is usually estimated by computing the value of μ for which $1/g_s^2 \rightarrow 0$ and is given by

$$\Lambda_{QCD} = \Lambda_{UV} \exp \left(-\frac{8\pi^2}{bg_s^2} \right) . \quad (7.4.6)$$

Notice the non-perturbative nature of this expression (the function e^{-1/x^2} is such that the function and all its derivatives vanish at $x = 0$ and therefore it does not have a proper Taylor expansion).

The above relation is known as *dimensional transmutation*: the dimensionless coupling in the action g_s can be traded for an energy scale at which perturbation theory breaks down Λ_{QCD} .

In the case of QCD, it is the natural way of getting a smaller scale, namely $\Lambda_{\text{QCD}} \approx 200\text{MeV}$, from high energy scales.

The changing of the strong coupling with energy has been checked experimentally with the best fit giving $\alpha_s = 0.1184 \pm 0.0007$ at the Z mass scale $\mu = M_Z$.

This can be compared with the value of the QED fine structure 'constant' $\alpha_{EM} \approx 1/129 \approx 0.0077$ at the same scale.

This explains why we observe QED as a weakly coupled theory and QCD as strong, but the relation changes at higher energies since QCD gets weaker and QED gets stronger.

The values of α_s and Λ_{QCD} have been computed using corrections up to 4-loops to the beta function with excellent agreement with experiment.

Quark Confinement and Hadron Masses*

We argued in the preceding section that only color singlets are allowed as bound states: mesons and baryons.

Now, asymptotic freedom allows us to have an intuitive understanding of why we cannot see quarks and gluons in isolation.

At low energies the interaction is strong enough as to keep quarks confined into the singlet bound states.

This is equivalent to have electrons and protons bound in a Hydrogen atom.

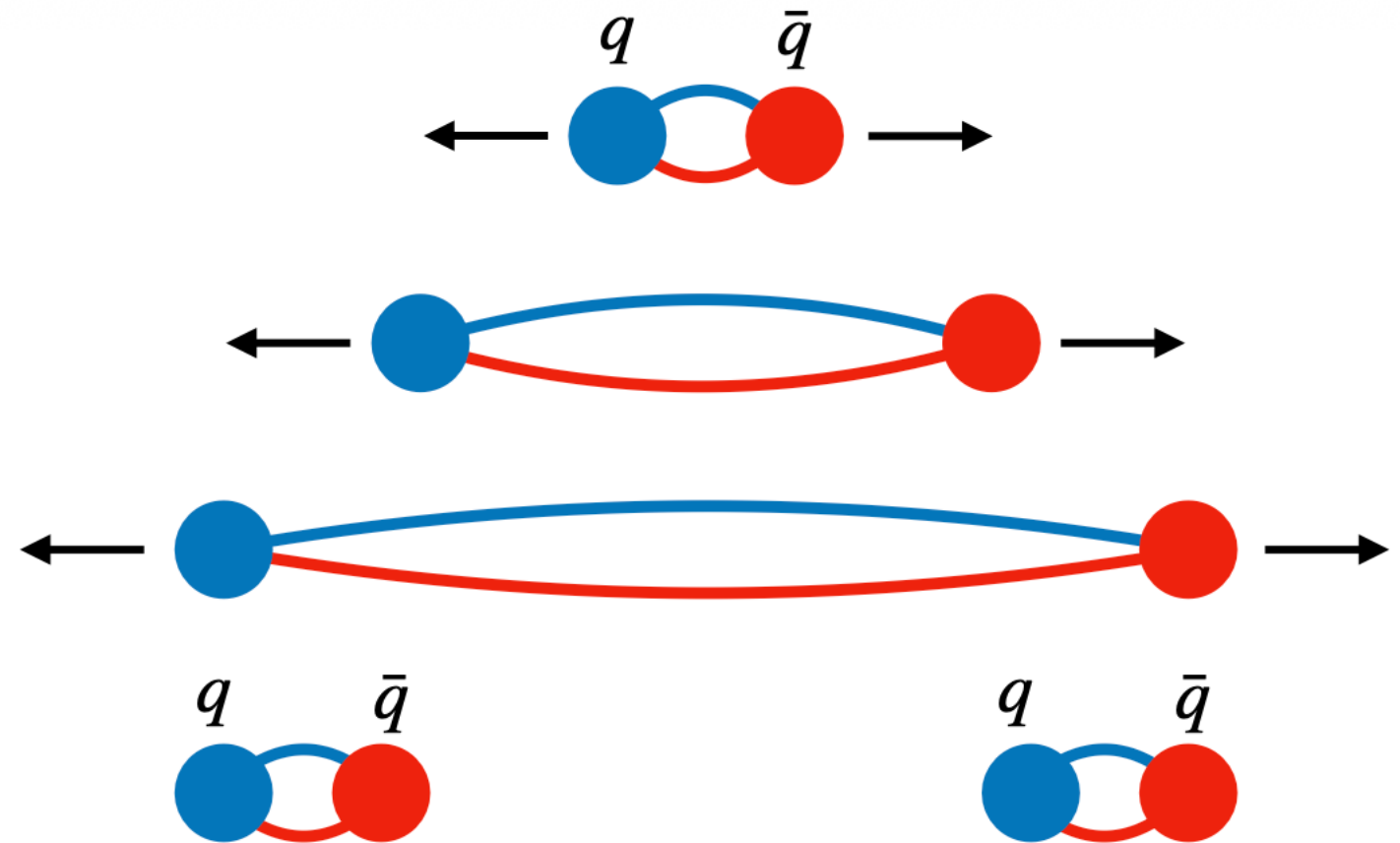
But for atoms, if we apply enough energy we can eject the electron from the atom, which is possible since the attractive interaction decreases with the separation ($V(r) \propto -1/r$).

However, due to the anti-screening implied by asymptotic freedom, applying energy to try to separate a quark from a hadron, the farther apart they are the stronger the attractive interaction.

At some point it is energetically preferred to create a quark-anti-quark pair than to break-up the hadron and create an isolated quark.

Experimentally this would appear as jets of mesons being produced. (See Fig. 7.6)

Figure 7.6: Trying to split the quark and anti-quark from a meson. The gluons act as providing a constant force. Flux tubes are formed and at some point energetics favored the creation of new quark-anti-quark pair, thereby preventing the generation of a single quark. This is a potential visualization of quark and gluons confinement.



An effective linear interaction ($V(r) \propto r$) can describe this behavior. Even though this is a compelling picture, due to our technical limitations to explicitly compute observable quantities at strong coupling, a proper proof of confinement is still an open question.

Over several decades a technique known as lattice QCD has been developed in which quarks and gluons are described in a lattice discretization of 3-dimensional space.

With the help of huge computer facilities, this approach has partially succeeded in computing some amplitudes and quark masses.

There are several techniques to extract information about the physics of quarks inside hadrons.

In general since $m_u, m_d, m_s < \Lambda_{\text{QCD}} < m_c, m_b, m_t$, we can see that hadrons made-up of u, d, s quarks may have different properties than those made-up of the c, b, t quarks.

For instance let us roughly estimate the values of hadron masses.

Since Λ_{QCD} determines the energy of confinement, we may roughly say, using the uncertainty principle, that the momentum of quarks inside a hadron is of order $p \sim 1/r_c \sim \Lambda_{\text{QCD}}$ with r_c the confinement radius.

The mass of the hadron is determined by the total energy $E = \sqrt{p^2 + m_q^2}$ with m_q the mass of the constituent quarks.

For u, d, s we know $m_q < \Lambda_{\text{QCD}}$ and therefore the corresponding hadrons would have a mass of order $m_{\text{hadron}} \simeq E \sim \Lambda_{\text{QCD}}$, whereas for c, b, t the total energy is dominated by the quark mass and so $m_{\text{hadron}} \simeq E \simeq m$.

This essentially agrees with the pattern of the hundreds of hadrons known so far.

With very few notable exceptions, such as pions, which are hierarchically lighter than Λ_{QCD} .

This will be explained more thoroughly in the next section.

Before we finish this section, let us go one step further and use these simple techniques to make a more precise numerical estimate of the size of the proton.

We know that in a Hydrogen atom the total energy is actually $E = m_p + m_e + E_b$ where $E_b \sim -13.6\text{eV}$ the binding energy which is the work that needs to be done to eject the electron from the atom.

For QCD, we know that a proton is a bound state of the form uud.

The total energy is then $E = E_{\text{quarks}} + E_b$ where now $E_{\text{quarks}} \sim 3p \sim 3/r$ and the binding energy is of the form $E_b = kr$ with k a constant known as the string tension.

The total energy is then

$$E \sim \frac{3}{r} + kr . \quad (7.4.7)$$

This function has an extremum at $k = 3/r_m^2$ with r_m the minimum distance between quarks.

Plugging this back into the expression for the energy gives

$$E(r_m) = m_{\text{proton}} = \frac{3}{r_m} + \frac{3}{r_m} = \frac{6}{r_m} \sim 1 \text{ GeV} \quad (7.4.8)$$

where we have used the known value of the proton mass ($\sim 1 \text{ GeV}$).

This implies that the minimum quark separation is $r_m \sim 6 \text{ GeV}^{-1} \sim 10^{-15}\text{cm} \sim 1.2 \text{ Fermi}$.

The string tension is then $k = 3r_m^2 = 1/12 \text{ GeV}^{-2}$.

Finally the proton radius is determined by assuming the three quarks form an equilateral triangle of side r_m and the radius is the radius of the circle embedding the corresponding triangle which by trigonometry we know it is $r_m/\sqrt{3}$.

Therefore the proton radius is $R_{\text{proton}} \sim r_m/\sqrt{3} \sim 0.7$ Fermi which fits well with experiments which suggest that $R^{\text{experiment}}_{\text{proton}} \sim 0.84 \sim 0.87$ Fermi.

7.5 Effective Chiral Lagrangian

Asymptotic freedom is somehow a present from nature to physicists that can now use the perturbative techniques based on Feynman diagrams and loop expansions for QCD despite being a strongly coupled theory at low energies, as long as the calculations are done for processes at high enough energies to justify the weak coupling approximation.

It also shows that QCD is by itself UV complete which is a major achievement.

However, we still need to be able to describe the physics at scales lower than Λ_{QCD} in which it should describe the confinement of quarks and gluons into hadrons.

It is important to emphasize that a formal proof of confinement is not yet available and it is considered as one of the top challenges in mathematics.

Physicists are only convinced it is true given all the evidence from experiment and asymptotic freedom.

The challenge is then to perform reliable calculations in the strong coupling regime of QCD.

For this the most powerful tool available at the moment is Lattice QCD in which for more than 40 years a large community has been able to make progress by putting QCD in a discretized space, a lattice, with the lattice size as an inverse cut-off scale and together with heavy computational power concrete calculations such as hadron masses and couplings have been obtained.

Other approaches have been proposed over the years to extract some information in the strong coupling regime.

Here we will concentrate on a general approach known as *chiral perturbation theory*.

This is a way of treating QCD in terms of an effective field theory.

Recall that EFT's are good descriptions at given energies.

We can organize physics by energies by integrating out high energy modes.

For this, notice the hierarchy in the quark masses in Table 7.1 where the confining scale Λ_{QCD} is located between strange- and charm-quark mass

$$m_s < \Lambda_{\text{QCD}} < m_c . \tag{7.5.1}$$

quark	u	d	s	c	b	t
mass	1.7 – 3.3MeV	4.1 – 5.8MeV	104MeV	1270MeV	4GeV	173GeV

Table 7.1: List of all the quark masses.

It is clear from the table that the u and d quarks are substantially lighter than the rest.

We also learn that the s quark may still be considered in an effective theory below Λ_{QCD} , but the c, b, t quarks are clearly in another regime heavier than Λ_{QCD} with the top quark far heavier than the rest.

Let us then consider Λ_{QCD} with only u and d quarks as an effective low-energy theory so that

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} \left(G_{\mu\nu}^A \right)^2 + i\bar{u}^L \not{D} u^L + i\bar{u}^R \not{D} u^R + i\bar{d}^L \not{D} d^L + i\bar{d}^R \not{D} d^R \\ & - m_u (\bar{u}_L u_R + \bar{u}_R u_L) - m_d (\bar{d}_L d_R + \bar{d}_R d_L) . \end{aligned} \tag{7.5.2}$$

The quark masses are generated originally by the Yukawa couplings to the Higgs, but for our purposes here they are just free small parameters.

In the limit $m_u, m_d \rightarrow 0$, the Lagrangian clearly has the following global symmetry

$$G = \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_V \times \text{U}(1)_A \tag{7.5.3}$$

where $\text{SU}(2)_L$ acts on $(u)_L, (d)_L$ and $\text{SU}(2)_R$ on $(u)_R, (d)_R$ building together the chiral symmetry.

$\text{U}(1)_V$ is associated with baryon number, whereas $\text{U}(1)_A$ is anomalous as we discussed in part 5.

Since $\text{U}(1)_V$ is already a symmetry even in the massive case and $\text{U}(1)_A$ is clearly anomalous we will concentrate on the approximate $\text{SU}(2)_L \times \text{SU}(2)_R$ chiral symmetry.

Let us stress here that the symmetry is approximate in the sense that it is only a symmetry if the quark masses were identically zero.

Nonetheless, its existence will help us understand a lot about the spectrum of hadrons as we now explain.

Let us define the generators as follows

$$T_V^a = T_L^a + T_R^a, \quad T_A^a = T_L^a - T_R^a, \quad a = 1, 2, 3 \quad (7.5.4)$$

generating $SU(2)_V$ and $SU(2)_A$ respectively.

Then

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \exp(i(\theta_V^a T^a + \gamma_5 \theta_A^a T^a)) \begin{pmatrix} u \\ d \end{pmatrix}. \quad (7.5.5)$$

Notice that as the γ_5 presence indicates, $SU(2)_A$ can map a hadron $|h\rangle$ to another hadron $|h'\rangle$ of opposite chirality, but all other quantum numbers the same.

If this symmetry was not spontaneously broken, it would imply that both hadrons would be degenerate in mass.

Since such a pair of degenerate hadrons does not exist, this means that this symmetry should be spontaneously broken.

The natural order parameter would be

$$\langle \bar{u}_L u_R \rangle = \langle \bar{d}_L d_R \rangle \neq 0 \quad (7.5.6)$$

breaking partially the chiral symmetry to the diagonal (or vector) $SU(2)_V$

$$SU(2)_V \otimes SU(2)_A \rightarrow SU(2)_V . \quad (7.5.7)$$

The unbroken vector symmetry acts equally on u_L and u_R quarks as

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow g \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \quad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \rightarrow g \begin{pmatrix} u_R \\ d_R \end{pmatrix} \quad (7.5.8)$$

and so

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow g \begin{pmatrix} u \\ d \end{pmatrix} . \quad (7.5.9)$$

The associated quantum number is isospin.

This is the same isospin introduced by Heisenberg when he proposed that strong interactions would make no difference between protons and neutrons and that they would be related as different spin states of the rotation group given by an internal $SU(2)$.

Notice $p = uud$ and $n = udd$ have the same isospin and therefore

$$\begin{pmatrix} p \\ n \end{pmatrix} \rightarrow g \begin{pmatrix} p \\ n \end{pmatrix} \quad (7.5.10)$$

under $g \in \text{SU}(2)_V$.

This is remarkable since now we can explain this isospin symmetry as just an approximate symmetry derived from QCD and the symmetry is only approximate as long as the u and d quarks are considered massless.

Once their mass is taken into account the proton and neutron do not have to be degenerate in mass.

Note that the mass of proton and neutron are of order 1 GeV, but their mass difference is only of order 1 MeV which is of the same order as the mass of u and d quarks.

From the early universe perspective, we may say that in the early universe the $\text{SU}(2)_L \times \text{SU}(2)_R$ symmetry was manifest, but once the universe cools down to temperatures of order $T < T_c \simeq \Lambda_{\text{QCD}}$ the binding energy among quarks is strong enough to confine them (and the gluons) into hadrons and the quark-anti-quark pair can condense (get a VEV) to break the symmetry.

We may now rightfully ask: How can we study the quark condensate phase?

Recall that a quark condensate is similar to the EFT of superconductivity.

In that case, the abelian Higgs model was an appropriate EFT to describe the main aspects of superconductors, cf. Section 5.5.1.

Even though the corresponding scalar field is not a physical field in the spectrum, it captures the physics of Cooper pairs of electrons in the medium.

Let follow here a similar logic and introduce scalar fields $\Sigma_{ij}(x)$ transforming under $G = \text{SU}(2)_L \times \text{SU}(2)_R$ via

$$\Sigma \rightarrow g_L \Sigma g_R^\dagger, \quad \Sigma^\dagger \rightarrow g_R \Sigma^\dagger g_L. \quad (7.5.11)$$

We then consider an effective Lagrangian with the same global symmetries as the original Lagrangian, namely $\text{SU}(2)_L \times \text{SU}(2)_R$.

The most general such a renormalizable Lagrangian is

$$\mathcal{L} = \text{Tr} \left[\partial_\mu \Sigma (\partial^\mu \Sigma)^\dagger \right] + m^2 \text{Tr}(\Sigma \Sigma^\dagger) - \frac{\lambda}{4} \text{Tr} \left[\Sigma \Sigma^\dagger \Sigma \Sigma^\dagger \right]. \quad (7.5.12)$$

For positive m^2 and λ this gives rise to spontaneous symmetry breaking as we have seen already several times.

The VEV of Σ that keeps the diagonal $\text{SU}(2)$ unbroken is

$$\langle \Sigma_{ij} \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v = \frac{2m}{\sqrt{\lambda}}. \quad (7.5.13)$$

Then $\text{SU}(2)_L \times \text{SU}(2)_R \rightarrow \text{SU}(2)_V$ as required.

We can make contact with quark condensate as follows

$$v \sim \Lambda_{QCD} \sim \langle \bar{u}u \rangle^{\frac{1}{3}}. \quad (7.5.14)$$

Notice the power 1/3 is due to dimensional analysis and $[u] = [d] = 3/2$.

Now we can expand around the vacuum

$$\Sigma(x) = \frac{v + \sigma(x)}{\sqrt{2}} \exp \left(\frac{2i T^a \pi^a(x)}{(F_\pi)} \right) = \frac{v + \sigma(x)}{\sqrt{2}} U(x) \quad (7.5.15)$$

where the constant $F_\pi = v$ is introduced to keep standard notation in the literature.

Here, as usual, $\sigma(x)$ is a massive Higgs field, invariant under $SU(2)_V$ and $\pi^a(x)$ are the massless Goldstone modes that transform in the adjoint of the unbroken group.

More generally, one finds

$$\Sigma \rightarrow g_L \Sigma g_R^\dagger \implies \delta \pi^a = \frac{F_\pi}{2} (\theta_L^a - \theta_R^a) - \frac{1}{2} f^{abc} (\theta_L^b + \theta_R^b) \pi^c + \dots \quad (7.5.16)$$

Concentrating only on the low-energy theory, we can integrate out the massive $\sigma(x)$ field (mass of order Λ_{QCD}) and work only with the effective theory for the low energy Goldstone modes π^a in terms of the field $U(x)$ defined as

$$U(x) = \exp \left(\frac{2i T^a \pi^a(x)}{F_\pi} \right) = \exp \left(\frac{i}{F_\pi} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix} \right) \quad (7.5.17)$$

with $\pi^0 = \pi^3$ and $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$.

The invariant Lagrangian for Goldstone modes can then be written as an expansion in derivatives known as the Chiral Lagrangian

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4} \text{Tr} \left[(D^\mu U)(D_\mu U)^\dagger \right] + \lambda_1 \text{Tr} \left[(D^\mu U)(D_\mu U)^\dagger \right]^2 + \dots \quad (7.5.18)$$

Expanding the exponentials

$$\mathcal{L}_\chi \rightarrow \frac{1}{2}(\partial_\mu \pi^0)(\partial^\mu \pi^0) + (D^\mu \pi^+)(D_\mu \pi^-)^\dagger + \frac{1}{F_\pi^2} \left[-\frac{1}{3} \pi^0 \pi^0 D_\mu \pi^+ D^\mu \pi^- + \dots \right] + \dots \quad (7.5.19)$$

where in general D_μ is the electroweak covariant derivative (but without including the QCD gauge fields since the field $U(x)$ is a color singlet).

Notice that the normalization factor $F_\pi^2/4$ implies that the pion fields have canonical kinetic terms.

The higher derivative terms have arbitrary coefficients λ_i that as usual should be bound by experiment.

This is an expansion in powers of E/F_π , since an expansion in derivatives corresponds to an expansion in momenta and then energy E , so the above is only predictive for energies $E \ll F_\pi$.

This formalism is called *chiral-perturbation theory* (χ^{PT}).

It can easily be seen that $\pi^0, \pi^\pm$ have the same quantum numbers as the well known pion fields!

So these fields can be seen as both

- hadrons made out of quarks, or as
- (pseudo-)Goldstone bosons of the approximate $SU(2)_L \times SU(2)_R$ chiral symmetry breaking.

This is remarkable.

We have now an effective field theory obtained from QCD, but not in terms of the fundamental degrees of freedom (quarks and gluons), but in terms of the lightest hadrons, the pions.

Furthermore, this makes contact with the original proposal of Yukawa treating pions as mediators of strong interactions.

However, Yukawa's concept of strong interactions was based on direct interactions among hadrons, whereas we know that this is only a low-energy behavior in the hadronic phase of QCD and not the fundamental degrees of freedom.

Therefore, χ^{PT} , though not fundamental, is an appropriate effective field theory to describe interactions among hadrons.

The Lagrangian above is only based on derivatives of the $U(x)$ field.

We may also add 'mass terms' of the form

$$\delta\mathcal{L} = C\text{Tr} \left(MU + M^\dagger U^\dagger \right) \quad (7.5.20)$$

where M is the u, d mass matrix $M = \text{diag} (m_u, m_d)$ and C a dimension 3 constant of order $C \simeq \Lambda_{\text{QCD}}^3$.

Expanding the exponential in U we can get the quadratic terms in the π fields proportional to

$$\delta\mathcal{L}_{\text{quadratic}} = \frac{C}{F_\pi^2} (m_u + m_d) (\pi_0^2 + \pi_1^2 + \pi_2^2) \quad (7.5.21)$$

implying that the mass² of the pions is of order

$$m_\pi^2 = \frac{C}{F_\pi^2} (m_u + m_d) \sim \Lambda_{\text{QCD}} (m_u + m_d) \quad (7.5.22)$$

fitting well with the experimental results with $m_{\pi^0} \sim 135 \text{ GeV}$ and $m_{\pi^\pm} \sim 139.6 \text{ GeV}$.

Note however that we considered only the lightest quarks u and d and the chiral symmetry is only approximate due to the lightness of u, d quarks.

We may move on and extend this formalism to include the other quark whose mass is lighter than Λ_{QCD} , namely the s quark.

In this case the approximate chiral symmetry and its breaking would be extended to

$$\text{SU}(3)_L \times \text{SU}(3)_R \rightarrow \text{SU}(3)_V \quad (7.5.23)$$

with $16 - 8 = 8$ Goldstone bosons,

$$\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix} \quad (7.5.24)$$

that can be identified with the known mesons $\eta, K, \tilde{K}$ besides the pions.

The remaining unbroken symmetry $\text{SU}(3)_V$ is nothing else but the flavor global symmetry proposed to classify hadrons as a consequence of an approximate symmetry coming from the more fundamental theory that is QCD.

Furthermore, the approximate nature of this symmetry is under less solid grounds since the mass of the s quark is smaller than Λ_{QCD} , but not much smaller and it is substantially heavier than u and d.

Therefore, the extra Goldstone modes are hierarchically heavier than the pions, as observed in nature.

As in the case of $\text{SU}(2)$ an effective Lagrangian can be written describing the interactions of all these mesons.

Baryons can also be introduced as

$$B = \varepsilon_{ijk} q^i q^j q^k \quad (7.5.25)$$

using the well known $SU(3)_V$ product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}. \quad (7.5.26)$$

We stress that these are representations of $SU(3)_V = SU(3)_f$ which is not $SU(3)_c$, in particular the indices i, j, k above are flavor indices corresponding to u, d, s .

Therefore, for example, the $SU(2)$ isospin doublet containing the proton and the neutron can be embedded into a full $SU(3)$ multiplet, the following octet of baryons

$$\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\Lambda \end{pmatrix}. \quad (7.5.27)$$

Couplings of B to U can be introduced to have then a low-energy theory of interacting mesons and baryons.

Once again, we stress that this approach can really only be used for the u, d, s quarks and their corresponding hadrons.

For hadrons made out of c, b, t this is not suitable since these quarks are heavier than Λ_{QCD} and there would be no justification for a low-energy effective action.

Other approaches (such as heavy quark effective theory) should be used in this case.

Note that somehow we closed a circle.

We started with the historical introduction of isospin and the eightfold way that lead to the proposal of quarks as fundamental degrees of freedom and then to color as the appropriate gauge symmetry to treat strong interactions.

Now we started with QCD as the fundamental theory and ended up explaining why there are these approximate symmetries of isospin and the eightfold way.

The existence of these symmetries is explained by the fact that there is a hierarchy of quark masses and 2 or 3 of them can be taken to be approximately zero compared to Λ_{QCD} and the other heavier quarks.

Why this hierarchy of masses exists, is still an open question.

PART 8 THE STANDARD MODEL

Confusion before clarity

I think it is important to understand how confusing these things seemed back then, and no one knows better than I do how confused I was.

Steven Weinberg

The journey toward the complete Standard Model has been a fascinating one, full of theoretical challenges, experimental breakthroughs, and conceptual shifts.

The theory as we know it today is the culmination of decades of collaborative effort and ingenuity, unifying the fundamental forces – except gravity – under a single quantum framework.

The Standard Model represents a major triumph of modern physics, explaining not only the electromagnetic and weak forces but also the strong interactions.

In this final part of these lectures, we will collect and synthesize all the key concepts developed in the previous discussions of electroweak and strong interactions to arrive at the full picture.

In the preceding parts, we introduced the two major components that together form the backbone of the Standard Model.

The electroweak interactions, described by a spontaneously broken gauge theory, have shown us how the electromagnetic and weak forces can be unified.

Meanwhile, the strong force, governed by quantum chromodynamics, demonstrates the phenomenon of confinement and explains the behavior of hadrons.

These two branches of fundamental interactions display very different characteristics, but their unification under the Standard Model reflects the depth and consistency of modern theoretical physics.

Together, they illustrate two distinct phases of gauge theories, which account for the existence of short-range interactions – either through the Higgs mechanism or quark confinement.

In this part, we bring together all the elements we have developed throughout these lectures.

However, the result is more than just a straightforward combination of the two Lagrangians.

We will present the full Standard Model as a gauge theory for the group $SU(3)_c \times SU(2)_L \times U(1)_Y$, which is spontaneously broken to $SU(3)_c \times U(1)_{em}$, and provide the following key insights:

- Count the total number of free parameters that can be determined through experiments.
- Explain why all gauge symmetries are anomaly-free.
- Identify the global symmetries present, including approximate, accidental, and anomalous ones.
- Highlight a non-trivial parameter, Θ , which requires the consideration of both weak and strong interactions since it is related to the phases in the CKM matrix $(6.4.81)$. This parameter is central to one of the most significant puzzles in the Standard Model: the *strong CP problem*.

Though this part of the lectures consolidates the material discussed in earlier sections, it also introduces new perspectives and presents the completion of our understanding of fundamental interactions.

The structure of the Standard Model, with its symmetry-breaking mechanisms, anomalies, and free parameters, serves as the best-tested theory in physics, and yet it leaves us with profound open questions.

While straightforward in some aspects, the content of this part reflects the culmination of the deepest theoretical insights of 20th-century physics

8.1 The Standard Model – all at once

In this section, we synthesize the results from previous discussions on electroweak and strong interactions to present the complete Standard Model.

Both interactions, formulated as gauge theories, are naturally unified into a single Lagrangian invariant under local transformations from the group

$$G_{\text{SM}} = \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y . \quad (8.1.1)$$

As before, we denote the fundamental gauge fields as G^A_μ , W^a_μ , B_μ , while after SSB we write G^A_μ , $W^\pm_\mu$, Z_μ , A_μ .

The Lagrangian for the kinetic and self-interaction terms of the gauge bosons is

$$\mathcal{L}^{\text{gauge}} = -\frac{1}{4} \left(G^A_{\mu\nu} \right)^2 - \frac{1}{4} \left(W^a_{\mu\nu} \right)^2 - \frac{1}{4} \left(B_{\mu\nu} \right)^2 - \Theta_G G^A_{\mu\nu} \tilde{G}^A_{\mu\nu} - \Theta_W W^a_{\mu\nu} \tilde{W}^a_{\mu\nu} - \Theta_B B_{\mu\nu} \tilde{B}_{\mu\nu} . \quad (8.1.2)$$

In principle, the three topological θ -terms coming with different angles Θ_G , Θ_W and Θ_B are allowed.

Even though classically and in perturbation theory they correspond to a total derivative that does not affect the equations of motion, quantum mechanically they can play a role once non-perturbative effects are included.

We will discuss later their importance in QCD.

The fermionic matter content can be summarized as follows

$$\begin{aligned}
Q_L^i &= \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}, & (\mathbf{3}, \mathbf{2}, \frac{1}{6}) \\
u_R^i &= \{u_R, c_R, t_R\}, & (\bar{\mathbf{3}}, \mathbf{1}, \frac{2}{3}) \\
d_R^i &= \{d_R, s_R, b_R\}, & (\bar{\mathbf{3}}, \mathbf{1}, -\frac{1}{3}) \\
L_L^i &= \left\{ \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix} \right\}, & (\mathbf{1}, \mathbf{2}, -\frac{1}{2}) \\
e_R^i &= \{e_R, \mu_R, \tau_R\}, & (\mathbf{1}, \mathbf{1}, -1) \\
\left(\nu_R^i = \{\nu_{e,R}, \nu_{\mu,R}, \nu_{\tau,R}\}, \right. & (\mathbf{1}, \mathbf{1}, 0) \Big)^* & (8.1.3)
\end{aligned}$$

The Lagrangian for the fermions can be split into two pieces so that

$$\mathcal{L}_F = \mathcal{L}_F^{\text{kinetic}} + \mathcal{L}_F^{\text{Yukawa}} \quad (8.1.4)$$

where the kinetic term for the fermions is of the form

$$\begin{aligned}
\mathcal{L}_F^{\text{kinetic}} &= i\bar{L}_L^i \not{D} L_L^i + i\bar{Q}_L^i \not{D} Q_L^i + i\bar{e}_R^i \not{D} e_R^i \\
&\quad + i\bar{\nu}_R^i \not{D} \nu_R^i + i\bar{u}_R^i \not{D} u_R^i + i\bar{d}_R^i \not{D} d_R^i.
\end{aligned} \quad (8.1.5)$$

The covariant derivative can be written as

$$D_\mu = \partial_\mu - ig_s G_\mu^A T^A - ig W_\mu^a T^a - ig' Y B_\mu. \quad (8.1.6)$$

The Yukawa couplings are defined as

$$\mathcal{L}_F^{\text{Yukawa}} = -y_{ij}^d \bar{Q}_L^i H d_R^j - y_{ij}^u \bar{Q}_L^i \tilde{H} u_R^j + y_{ij}^e \bar{L}_L^i H e_R^j (+y_{ij}^\nu \bar{L}_L^i H \nu_R^j)^* + \text{h.c.} \quad (8.1.7)$$

The Higgs field is a complex scalar doublet

$$H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}, \quad \left(\mathbf{1}, \mathbf{2}, \frac{1}{2} \right) \quad (8.1.8)$$

with Lagrangian

$$\mathcal{L}^{\text{Higgs}} = D_\mu H (D^\mu H)^\dagger + V(H), \quad V(H) = m^2 |H|^2 - \lambda |H|^4. \quad (8.1.9)$$

Altogether, the Standard Model Lagrangian is given by

Table 8.1: Parameter count for the Standard Model

Sector	Parameters	Physical	Number
Gauge	$g_s, g, g', \theta_G, \theta_W, \theta_B$	$g_s, e, \cos(\theta_W), \bar{\theta}$	4
Higgs	m^2, λ	m_h, m_W	2
Yukawa	$y_{ij}^u, y_{ij}^d, y_{ij}^e, (y_{ij}^\nu)^*$	m_i^u, m_i^d, m_i^e	9
		V_{CKM}	4
		V_{PMNS}	6?
		$m_i^\nu, M_i^{\nu R}$	3+?
Total			25+

Standard Model Lagrangian

$$\mathcal{L}_{SM} = \mathcal{L}^{\text{gauge}} + \mathcal{L}_F^{\text{kinetic}} + \mathcal{L}_F^{\text{Yukawa}} + \mathcal{L}^{\text{Higgs}} . \quad (8.1.10)$$

Let us perform a parameter count for the Standard Model summarised in Table 8.1.

We have more than 25 free parameters in the Standard Model.

We do not specify the number of parameters coming from the right-handed neutrinos since it is yet not known how right-handed neutrinos will appear and couple to the remaining fields in the Standard Model.

In particular, their number does not have to be restricted to the number of families as for the other fields since right-handed neutrinos are simply fermions that do not couple to any of the gauge fields of the Standard Model.

The Standard Model Lagrangian (8.1.10) is renormalizable and can be expanded in terms of operators of different dimensions.

Let us write

$$\mathcal{L}_{SM} = \sum_i c_i \mathcal{O}_i , \quad [c_i] + [\mathcal{O}_i] = 4 . \quad (8.1.11)$$

The dimensions of the individual operators $\mathcal{O}_i$ are

- $[\mathcal{O}_i] = 0$: $c_0 = \Lambda$ is the constant term in the scalar potential ($\lambda v^4/2$ in the Higgs potential). Once coupled to gravity, this term would correspond to the *cosmological constant*.
- $[\mathcal{O}_i] = 2$: $c_2 = m^2$ with m^2 the coefficient in the quadratic term $m^2 H^2$ of the Higgs potential.
- $[\mathcal{O}_i] = 3$: there is no dimension 3 operator in the Standard Model Lagrangian. But if right-handed neutrinos are involved then the corresponding Majorana mass $c_3 = M^\nu$ multiplying $\nu_R \nu_R = \mathcal{O}_3$.
- $[\mathcal{O}_i] = 4$: all the other terms implying the coefficients (gauge couplings, Yukawa couplings, θ terms are dimensionless).

Notice that

- no mass terms are allowed for gauge fields because of gauge invariance, and
- no mass terms are allowed for fermions, again from (chiral) gauge symmetry. The fermion masses arise from Yukawa couplings and $\langle H \rangle = v \neq 0$.

Then, unlike the case of the Higgs, fermion and gauge field masses are only generated after symmetry breaking and quantum corrections to the Lagrangian cannot induce masses for gauge fields and fermions.

The only missing piece in the Standard Model Lagrangian is gravity.

To include gravity, we have to introduce the metric degrees of freedom and make the Lagrangian invariant under general coordinate transformations.

We then arrive at an non-renormalizable EFT with Lagrangian

$$\mathcal{L}_{SM} \rightarrow \sqrt{-g} \left(\mathcal{L}'_{SM} + \Lambda + M_P^2 R + \dots \right) \quad (8.1.12)$$

where g is the determinant of the metric, Λ the cosmological constant and R the Ricci scalar.

We also defined

$$\mathcal{L}'_{SM} = \mathcal{L}_{SM}[D_\mu \rightarrow \mathcal{D}_\mu] \quad (8.1.13)$$

in terms of the covariant derivative D_μ suitably adjusted for gravity.

Here, we wrote the gravity component as an expansion in powers of the curvature (which is a derivative expansion) with leading order term the cosmological constant Λ , next order the Einstein-Hilbert action in terms of the Ricci scalar and then higher powers of curvature terms $\mathcal{O}(R^2)$.

This is the model (or theory or framework) that explains and describes the world we observe.

It is

- Mathematically consistent in the sense that the Lagrangian is consistent with all spacetime and internal symmetries, it is the most general renormalisable Lagrangian consistent with these symmetries and all gauge symmetries are free from anomalies.
- Explains *all* the experiments done before the formulation of the model (all interactions, decay rates, etc.).

- Made predictions that were spectacularly confirmed over the years (neutral currents, $W^\pm$, Z^0 , Higgs).
- Precision tests: theory $\leftrightarrow$ experiment with agreeing precision at many decimal figures.
- Explains the observed “approximate” and accidental symmetries such as
 - Baryon number (accidental)
 - Lepton number (accidental)
 - Isospin (approximate)
 - $SU(3)_f$ (eightfold way) (approximate)
- Consistent coupling to gravity as an EFT at energies

$$E \ll M_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19} \text{ GeV} . \quad (8.1.14)$$

It is important to compare the magnitude of this scale compared to the relevant scales in the Standard Model namely the VEV of the Higgs $v = \langle H \rangle = 246 \text{ GeV}$ and the QCD scale: $\Lambda_{\text{QCD}} \sim 250 \text{ MeV}$.

We summarize the massive particle content of the Standard Model together with the relevant scales in Fig. 8.1

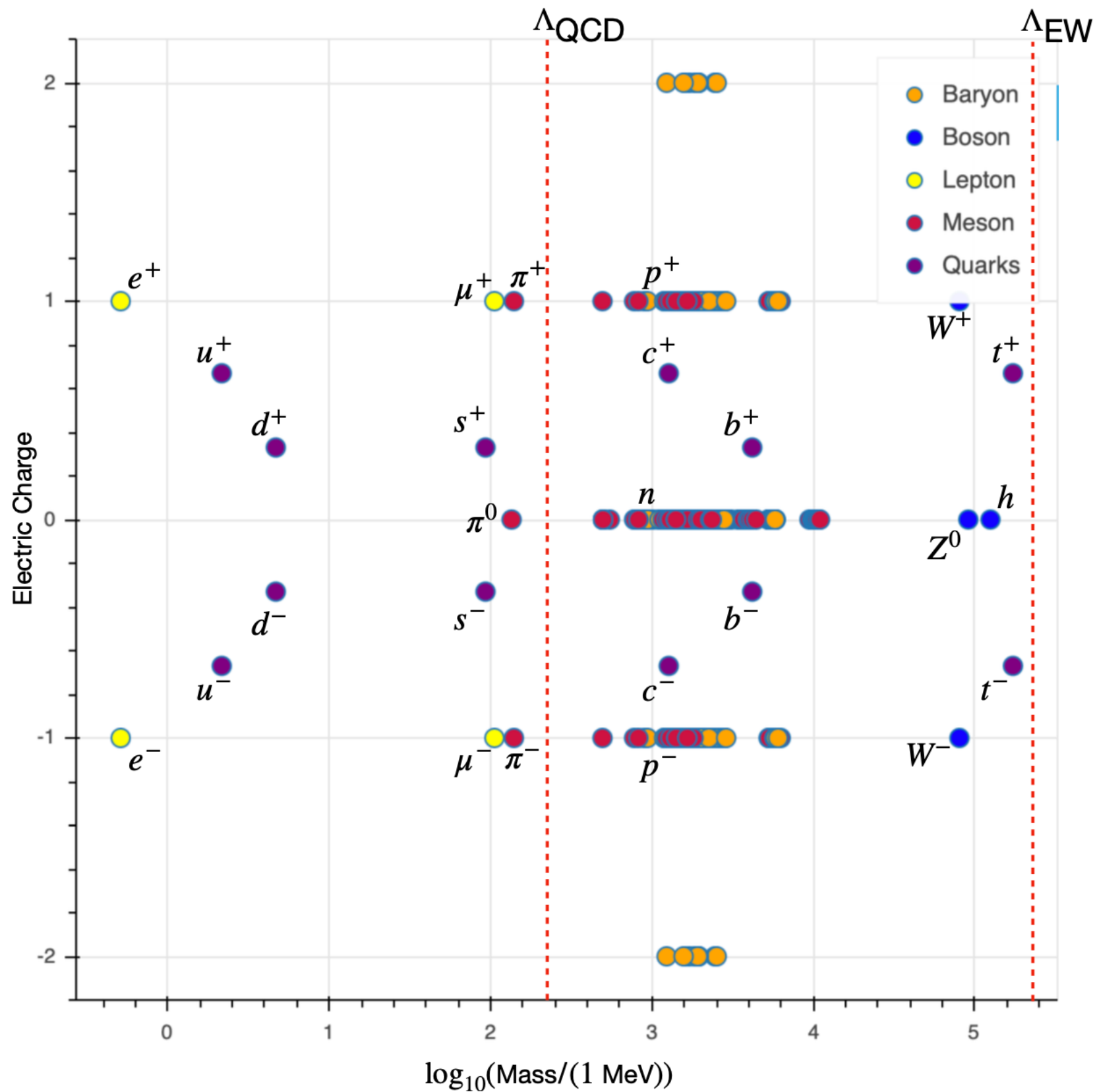


Figure 8.1: Summary of massive particles in the Standard Model. We highlight the types of particles with different colors and assign labels to the most prominent ones. The QCD scale $\Lambda_{\text{QCD}} \sim 250 \text{ MeV}$ and electroweak scale $\Lambda_{\text{EW}} = v = 246 \text{ GeV}$ are also indicated as red lines.

8.2 Global symmetries in the Standard Model

Throughout these lectures, we mentioned various types of global symmetries of the Standard Model.

Having completed the full structure of the Standard Model in the previous section, we are in the right position to actually collect all of the global symmetries in the Standard Model.

Let us start with the Lagrangian in the absence of any interaction (including self-interactions of the gauge fields) or mass terms.

This determines the maximum possible symmetry group of the theory.

It can be determined by counting the number of real degrees of freedom for each spin:

- the Higgs H as a complex doublet corresponds to 4 real degrees of freedom,
- there are three generations of fermions and 15 different species, namely one for e_R^i , two from L_L^i , three from each u_R^i and d_R^i and six from Q_L^i . Altogether, this leads to 45 complex degrees of freedom, and
- the number of independent helicity-1 fields is obtained by adding the ranks of all gauge groups. In total, we have $8 + 3 + 1 = 12$ real degrees of freedom.

We can use orthogonal rotations to transform the various bosonic fields of the same spin into each other and similarly unitary transformations for the fermions.

This allows us to define the maximum possible global symmetry group as

$$G_{\max} = G_0 \times G_{1/2} \times G_1 = \mathrm{O}(4) \times \mathrm{U}(45) \times \mathrm{O}(12) .$$

The next step is to consider individual interactions in the Standard Model Lagrangian in order to determine the subgroup of $G_{\max}$ that is preserved.

Clearly, the gauge group of the Standard Model

$$G_{SM} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset G_{\max}$$

must be preserved by construction.

We distinguish the following couplings:

- *gauge self-interactions*: the only transformations that leave the structure constants invariant by virtue of Jacobi's identity are the transformations associated with G_{SM} . As a result, there are no additional global symmetries in this sector. In fact, this is a general statement about Lie groups meaning that there will never arise accidental symmetries in this sector.
- *scalar self-couplings*: these terms leave the full $G_0 = \mathrm{O}(4)$ intact since the scalar potential $V(H)$ in the Higgs Lagrangian (8.1.9) is invariant under rotations (considering the Higgs as a real 4-vector).
- *scalar-gauge couplings*: the Standard Model only includes a single irreducible representation of scalar fields and there is no other subgroup than $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ that commutes with $\mathrm{O}(4)$. So only the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ subgroup survives and no additional global symmetries arise. However, for small gauge coupling g' associated with $\mathrm{U}(1)_Y$, the $\mathrm{O}(4)$ symmetry is a good *approximate* symmetry.

- *fermion-gauge couplings*: if we pick a basis for the fermions in the corresponding representations of G_{SM} , then there are no larger subgroup than G_{SM} transforming particles into each other in a single generation. However, there are 3 generations and we can perform unitary rotations relating a fermion in one to another in a different generation. Thus, the subgroup of $G_{1/2}$ giving rise to an accidental global symmetry is

$$G_f = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R} \times U(3)_{L_L} \times U(3)_{e_R} \subset G_{1/2}. \quad (8.2.3)$$

Each factor corresponds to a unitary transformation in the space of generations for the individual representations of G_{SM} .

- *Yukawa interactions*: It is convenient to work in a basis in which the mass matrix and therefore also the Yukawa couplings are diagonal. In this basis, the couplings are diagonal and real. Using the experimental observations that the eigenvalues of the Yukawa coupling matrix are all non-zero and non-degenerate, we can restrict the five choices of $U(3)$ matrices from (8.2.3) to be also diagonal with pure phases along the diagonal. This breaks each factor to an a priori independent $U(1)$. However, since the Yukawa couplings couple left- and right-handed fields to each other, the $U(1)$'s associated to the RH fields equal the LH $U(1)$'s. So for the leptons there are three independent $U(1)$'s for each generation so that $U(3)_{L_L} \times U(3)_{e_R} \rightarrow U(1)_e \times U(1)_\mu \times U(1)_\tau$. For the quarks, we also need to guarantee that the transformations leave the charge current interactions invariant. As we have seen in (6.4.80) these currents involve the CKM-matrix in the basis of mass eigenstates. For a generic such unitary CKM-matrix, the only way of guaranteeing the invariance is by having a diagonal matrix which is why $U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R} \rightarrow U(1)_B$ breaks to a single $U(1)$ corresponding to the choice of one phase on the diagonal of the 3×3 -matrix.

Interestingly, the observed fermion masses and, hence, the measured Yukawa couplings are small in comparison to, e.g., m_W and m_Z . Ignoring the Yukawa couplings results in a larger *approximate* flavour symmetry

$$\text{leptons: } \text{U}(3)_{L_L} \times \text{U}(3)_{e_R}, \quad \text{quarks: } \text{U}(3)_{Q_L} \times \text{U}(3)_{u_R} \times \text{U}(3)_{d_R}. \quad (8.2.4)$$

For more details on approximate symmetries in the Standard Model

- *chiral symmetry*: in the absence of electroweak interactions, we find the approximate chiral symmetry

$$G_{\text{chiral}} = \text{U}(3)_L \times \text{U}(3)_R. \quad (8.2.5)$$

It becomes relevant when studying the low energy behaviour of strongly interacting quarks, especially the lightest particles as we discussed in section 7.5

6.3

All in all, we find that the accidental global symmetry group of the Standard Model is given by

$$G_{ac} = \text{U}(1)_e \times \text{U}(1)_\mu \times \text{U}(1)_\tau \times \text{U}(1)_B. \quad (8.2.6)$$

Each generator of these groups is associated with a quantum number that is confirmed to be conserved experimentally.

The associated quantum numbers are

- As discussed in Sect. 6.3, the *electron number* L_e can be defined in terms of the generators Q_L and Q_R so that

$$L_e = 2Q_L + Q_R \quad (8.2.7)$$

and hence

$$L_e L_L^1 = L_L^1, \quad L_e e_R^1 = e_R^1, \quad L_e \nu_R^1 = \nu_R^1. \quad (8.2.8)$$

Acting on any of the other fields, the charges are zero. Similarly, one can define *muon number* L_μ as

$$L_\mu L_L^2 = L_L^2, \quad L_\mu e_R^2 = e_R^2, \quad L_\mu \nu_R^2 = \nu_R^2 \quad (8.2.9)$$

and *tau number* L_τ as

$$L_\tau L_L^3 = L_L^3, \quad L_\tau e_R^3 = e_R^3, \quad L_\tau \nu_R^3 = \nu_R^3. \quad (8.2.10)$$

It is again understood that L_μ, L_τ acting on all other fields vanishes. The number

$$L_{\text{tot}} = L_e + L_\mu + L_\tau \quad (8.2.11)$$

is called *lepton number*.

b) *baryon number*: Baryon number B is defined as

$$B Q_L^i = \frac{1}{3} Q_L^i, \quad B u_R^i = \frac{1}{3} u_R^i, \quad B d_R^i = \frac{1}{3} d_R^i. \quad (8.2.12)$$

The conservation of these quantum numbers has direct physical implications.

First, it ensures the stability of the lightest particles carrying a non-zero charge in each generation.

Thus, all neutrinos, the electron and the lightest baryon (corresponding to the proton!) are predicted to be absolutely stable. .

The neutron in an isolated environment is unstable with a lifetime of approximately 880 second, albeit being stable when bound in a nucleus

Otherwise, life as we know it would not be possible.

Moreover, the above observations suggest that processes like $\mu \rightarrow e\gamma$ are forbidden by conservation of L_e and L_μ .

Due to the fact that neutrinos are not perfectly massless, there is evidence that the separate lepton numbers are not conserved, but the violation is so tiny that they are hardly detectable in particle experiments.

Hence, the last statement is true only approximately and these *so called charged lepton flavor violations* are expected to appear in nature.

Baryon and lepton number are symmetries of nature not because of an ad-hoc assumption, but simply because the Lagrangian written as the most general renormalizable Lagrangian consistent with the gauge and spacetime symmetries of the theory happens to be also invariant under baryon and lepton number.

We have to bear in mind though that these symmetries are anomalous, as we show below, and hence would be broken in the quantum theory, which for global symmetries is not a sign of inconsistency.

8.3 Anomalies in the Standard Model

Let us recall our discussion about anomalies from section 5.6.

The general idea is to understand the potential consequences of breaking classical symmetries in quantum theories.

Noether's theorem states that every global continuous symmetry has an associated conserved current.

The conservation is violated in the presence of anomalies.

We then distinguish between anomalies of local and global symmetries.

a) The former constitute a violation of Ward identities invalidating the quantum theory.

This is because a massless helicity-1 particle coupling to a non-conserved current gives rise to an unphysical longitudinal polarization violating unitarity.

In general, the absence of such gauge anomalies is a very strong consistency requirement on any QFT.

b) In contrast, anomalies of global symmetries are not dangerous for the existence of a theory.

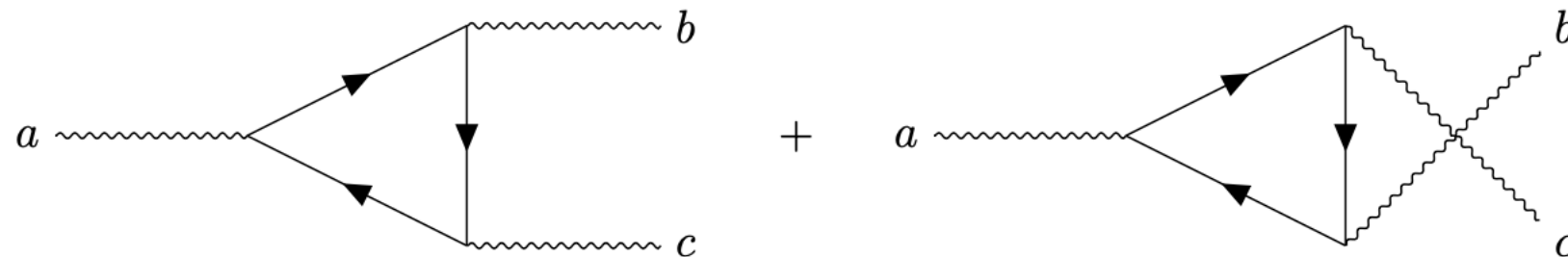
They are nonetheless essential to single out conserved quantities.

The prime example is baryon number that we introduced in section 6.4.4 which assigns a quantum number $B = 1/3$ to all quarks and $B = 0$ to all leptons.

As we show below, the associated Noether current $J^\mu_B = \sum_i \bar{q}_i \gamma^\mu q_i$ is anomalous.

The violation of baryon number is a crucial ingredient to explain the matter/anti-matter asymmetry in the observed Universe.

In section 5.6, we argued that anomalies can be derived from triangle diagrams of the form



which leads to a contribution

$$A^{abc} = \text{Tr}(T^a \{T^b, T^c\}) = A(R) d^{abc} \quad (8.3.1)$$

for some representation R of the fermions running in the loop.

These are the so-called *anomaly-coefficients* which need to vanish to guarantee the absence of anomalies.

The trace implies that we need to sum over all types of fermions that can contribute to these processes which in the case of the Standard Model typically means summing over every color, flavor and generation.

Overall, the result of summing over the two triangle diagrams leads to (recall (5.6.32))

$$\partial^\alpha J_\alpha^a = \left(\sum_{\text{left}} A(R_l) - \sum_{\text{right}} A(R_r) \right) \frac{g^2}{128\pi^2} d^{abc} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^b F_{\alpha\beta}^c \quad (8.3.2)$$

with a sum over all left-handed and over all right-handed particles.

If the right hand side is non-vanishing, the current is clearly not conserved.

As we argued already section 5.6, a non-chiral theory, i.e., a theory with an equal amount of left-handed and right-handed fields is automatically free of anomalies which is obvious from the expression (8.3.2).

Since the SM is a chiral gauge theory with several gauge group factors, the cancellation of all gauge anomalies is not obvious at all.

8.3.1 Local anomaly cancellation in the Standard Model

Let us get started and show that Standard Model is free of gauge anomalies.

Let us denote the anomaly coefficients $A(n_1, n_2, n_3)$ for $n_i \in \{1, 2, 3\}$ denoting a choice of gauge group factor in $G = \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)$.

We use the $\text{SU}(3)$ generators

$$T^\alpha = \frac{\lambda^\alpha}{2}, \quad \alpha = 1, \dots, 8 \quad (8.3.3)$$

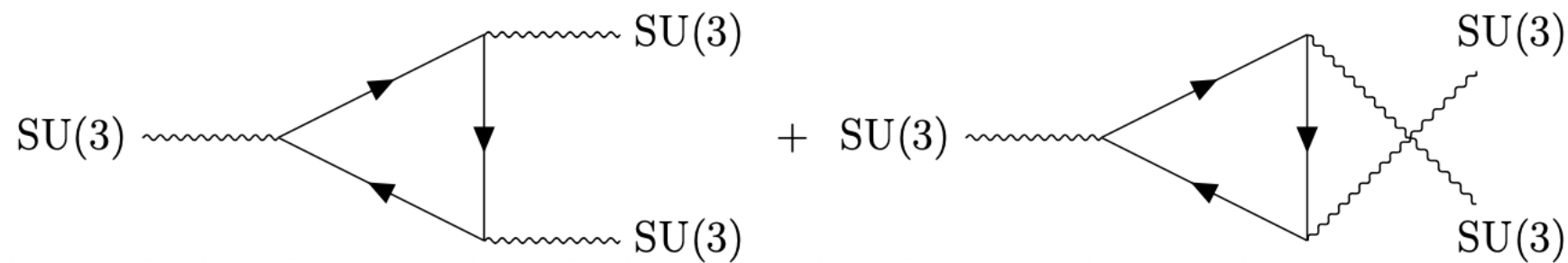
in terms of the 3×3 -matrices λ_α called the *Gellmann-matrices* and similarly the SU(2) generators

$$T^a = \frac{\sigma^a}{2}, \quad a = 1, 2, 3 \quad (8.3.4)$$

in terms of the Pauli-matrices σ^a .

By virtue of (5.6.32), the coefficient of the anomaly will be proportional to $\text{Tr} [\{T_i, T_j\}, T_k]_L - \text{Tr} [\{T_i, T_j\}, T_k]_R$ with T_i the generators of the corresponding groups and L, R meaning left- and right-handed fermions respectively.

For example, the diagrams



correspond to the coefficients $A(3, 3, 3)$ given by

$$A(3, 3, 3)_{\alpha\beta\gamma} = \text{Tr}(T^\alpha \{T^\beta, T^\gamma\}) . \quad (8.3.5)$$

Before we start, we can make a few simplifications.

Notice that anomaly coefficients associated with e.g. $U(1)^2\text{SU}(2)$ are

$$A(1, 1, 2)_a = \text{Tr} (Y^2 T_a) = \text{Tr}(Y^2) \text{Tr} \left(\frac{\sigma_a}{2} \right) = 0 \quad (8.3.6)$$

because the Pauli-matrices are traceless.

In general, the generators of SU(N) are all traceless which is why we only need to consider the anomaly coefficients associated with SU(N)³ or SU(N)²U(1).

This means that the only non-trivial anomalies could arise from

$$U(1)^3, \quad U(1)SU(2)^2, \quad U(1)SU(3)^2, \quad SU(2)^3, \quad SU(3)^3. \quad (8.3.7)$$

Let us now go through all these possibilities and show that each anomaly vanishes individually:

- $A(1, 1, 1)$: we need to sum over all fermions so that

$$\begin{aligned} A(1, 1, 1) &= \sum_{\text{fermions}} Y^3 \\ &= 3 \cdot \left(3 \cdot 2 \cdot Y_Q^3 + 2 \cdot Y_L^3 - (Y_{e_R}^3 + 3 \cdot Y_{u_R}^3 + 3 \cdot Y_{d_R}^3) \right) \\ &= 3 \cdot \left(3 \cdot 2 \cdot \left(\frac{1}{6} \right)^3 + 2 \cdot \left(-\frac{1}{2} \right)^3 - (-1)^3 - 3 \cdot \left(\frac{2}{3} \right)^3 - 3 \cdot \left(-\frac{1}{3} \right)^3 \right) \\ &= 3 \left(\frac{1}{36} - \frac{1}{4} + 1 - \frac{8}{9} + \frac{1}{9} \right) \\ &= 0. \end{aligned} \quad (8.3.8)$$

- $A(1, 2, 2)$: Using the fact that

$$\{\sigma^a, \sigma^b\} = 2\delta^{ab}\mathbb{1}_2, \quad (8.3.9)$$

we find by summing over all fermions in non-trivial representations of $SU(2)$ (effectively summing over all doublets)

$$\begin{aligned} A(1, 2, 2)^{ab} &= \text{Tr} \left(Y \left\{ \frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right\} \right) \\ &= \delta^{ab} \sum_{\text{doublets}} Y \\ &= \delta^{ab} 3 (Y_L + 3Y_Q) \\ &= \delta^{ab} 3 \left(-\frac{1}{2} + 3 \left(\frac{1}{6} \right) \right) \\ &= 0. \end{aligned} \quad (8.3.10)$$

- $A(1, 3, 3)$: Using the fact that

$$\{\lambda^\alpha, \lambda^\beta\} = \frac{4}{3}\delta^{\alpha\beta}\mathbb{1}_3 + 2d^{\alpha\beta\gamma}\lambda^\gamma, \quad (8.3.11)$$

we find by summing over all fermions in non-trivial representations of $SU(3)$ (effectively summing over all triplets)

$$\begin{aligned} A(1, 3, 3)^{\alpha\beta} &= \text{Tr} \left(Y \left\{ \frac{\lambda^\alpha}{2}, \frac{\lambda^\beta}{2} \right\} \right) \\ &= \delta^{\alpha\beta} \sum_{\text{triplets}} Y \end{aligned}$$

$$\begin{aligned}
&= \delta^{\alpha\beta} 3 (-Y_{u_R} - Y_{d_R} + 2Y_Q) \\
&= \delta^{\alpha\beta} 3 \left(-\frac{2}{3} + \frac{1}{3} + 2 \left(\frac{1}{6} \right) \right) \\
&= 0.
\end{aligned} \tag{8.3.12}$$

- $A(2, 2, 2)$: Here, we have

$$\begin{aligned}
A(2, 2, 2)^{cab} &= \text{Tr} \left(\frac{\sigma^c}{2} \left\{ \frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right\} \right) \\
&= \frac{1}{4} \text{Tr} \left(\sigma^c \delta^{ab} \mathbb{1}_2 \right) \\
&= 0.
\end{aligned} \tag{8.3.13}$$

We could also have used the fact that

$$\sigma_a^* = -\sigma_2 \sigma_a \sigma_2 \tag{8.3.14}$$

which is why the corresponding representations are pseudo-real. This means that the anomaly-coefficients vanish identically.

- $A(3, 3, 3)$: Here, we can use that the $\text{SU}(3)$ gauge theory is non-chiral and the corresponding coefficients must vanish by the the fact that in general pseudo-real representations are anomaly free.

In addition to these constraints, we also need to discuss potential gravitational anomalies.

Although we always think about the Standard Model in Minkowski spacetime, we would like to be able to couple it to gravity.

In this case, the Poincaré group should not be viewed as a global, but as a local group since we work with a dynamical, massless spin-2 field.

Notice that there is nothing wrong with quantizing gravity perturbatively and treating it as an effective field theory below the Planck scale.

We can compute all kinds of observables in analogy to Yang-Mills theory with only a few modifications.

Now, what are gravitational anomalies?

Computing anomalies for one gauge boson and two gravitons leads to

$$\partial^\alpha J_\alpha^a \sim \text{Tr}(T^a) \varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu}{}^{\gamma\delta} R_{\alpha\beta\gamma\delta} . \quad (8.3.15)$$

SU(N) generators are traceless, the anomalies of $\text{grav}^2\text{SU}(N)$ are automatically zero.

The only non-vanishing coefficient therefore is $A(1, J, J)$ for some Lorentz generators J.

This coefficient corresponds to the sum over all fermions

$$\begin{aligned}
A(1, J, J) &= 3 (2Y_L - Y_{e_R} + 6Y_Q - 3Y_{u_R} - 3Y_{d_R}) \\
&= 3 \left(2 \left(-\frac{1}{2} \right) - (-1) + 6 \left(\frac{1}{6} \right) - 3 \left(\frac{2}{3} \right) - 3 \left(-\frac{1}{3} \right) \right) \\
&= 0 .
\end{aligned} \tag{8.3.16}$$

All in all, this implies the the SM as a chiral gauge theory is free of gauge and gravitational anomalies and, as such, is well defined.

Although historically anomaly cancellation was not an initial assumption (simply because they were discovered quite late in the development of the SM, namely around 1969 by Adler as well as Bell, Jackiw), the final theory turns out to be anomaly free.

Furthermore, anomaly cancellation conditions give strict guidelines on possible extensions of the SM as will be exemplified with right-handed neutrinos below.

In total, we have the following four anomaly cancellation conditions

$$2Y_L^3 - Y_{e_R}^3 + 6Y_Q^3 - 3Y_{u_R}^3 - 3Y_{d_R}^3 = 0 , \tag{8.3.17}$$

$$Y_L + 3Y_Q = 0 , \tag{8.3.18}$$

$$-Y_{u_R} - Y_{d_R} + 2Y_Q = 0 , \tag{8.3.19}$$

$$2Y_L - Y_{e_R} + 6Y_Q - 3Y_{u_R} - 3Y_{d_R} = 0 . \tag{8.3.20}$$

Since we have in total 5 independent hypercharges for the fermions, we can fix them uniquely by choosing the value for one of them.

So choosing a certain value of the hypercharge for, say, the electron leads to a unique assignment of hypercharges for all matter fields by requiring the absence of gauge anomalies.

Charge of electron and proton

From the anomaly cancellation conditions derived above, the most interesting one is almost certainly (8.3.18), i.e.,

$$Y_L + 3Y_Q = 0 . \tag{8.3.21}$$

This conditions arises due to the chirality of the electroweak sector relating (left-handed) leptons and quarks.

There is no contribution from the RH fields and the multiplicities are determined from the associated representation with respect to $SU(3)_c$.

The electric charge of the proton is

$$Q_P = 2Q_u + Q_d \tag{8.3.22}$$

where the electric charges of up- and down-quark are determined from

$$QQ_L^i = (T^3 + Y_Q \mathbb{1}_2) Q_L^i = \begin{pmatrix} \frac{1}{2} + Y_Q & 0 \\ 0 & -\frac{1}{2} + Y_Q \end{pmatrix} Q_L^i = \begin{pmatrix} Q_u & 0 \\ 0 & Q_d \end{pmatrix} Q_L^i. \quad (8.3.23)$$

Thus, the electric charge is given by (using (8.3.18))

$$Q_P = 2 \left(\frac{1}{2} + Y_Q \right) - \frac{1}{2} + Y_Q = \frac{1}{2} - Y_L. \quad (8.3.24)$$

On the other hand, the electric charge of the leptons is determined from

$$QL_L^i = \begin{pmatrix} \frac{1}{2} + Y_L & 0 \\ 0 & -\frac{1}{2} + Y_L \end{pmatrix} L_L^i = \begin{pmatrix} Q_\nu & 0 \\ 0 & Q_e \end{pmatrix} L_L^i \quad (8.3.25)$$

and, in particular, of the electron

$$Q_e = -\frac{1}{2} + Y_L. \quad (8.3.26)$$

This immediately implies that

$$Q_P = -Q_e. \quad (8.3.27)$$

This is remarkable since this allows us to understand a very basic property of matter, that is that the magnitude of the electric charge of protons is exactly the same as that of the electrons with opposite sign.

On the Lie algebra level, there is no argument of why hypercharge or electric charge should be quantized in the SM.

However, the anomaly cancellation condition (8.3.21) implies that, whatever the actual value of the electric charges is, being it ± 1 or $\pm \pi$, the electron e^- and the proton p have exactly opposite charges and not just approximately.

This is crucial to build stable atoms and hence any type of matter!

For the neutron, we obtain

$$Q_n = Q_u + 2Q_d = -\frac{1}{2} - Y_L = -Q_\nu. \quad (8.3.28)$$

Hence, (8.3.21) on its own implies

$$Q_P = -Q_e, \quad Q_n = -Q_\nu \quad (8.3.29)$$

whatever values we assign to the hypercharges.

Only after imposing $Y_L = -1/2$, we recover

$$Q_P = -Q_e = 1, \quad Q_n = Q_\nu = 0. \quad (8.3.30)$$

We conclude that charges must always be quantized given that these are the lightest particles in nature out of which matter is being formed.

While the lightest mesons such as the pions $\phi^0, \phi^\pm$ decay into electrons, neutrinos and photons, the proton is in fact believed to be stable or at least to have an huge lifetime of more than 10^{34} years (compare this to the age of the Universe which is $\sim 10^{10}$ years).

Gauge anomalies including right-handed neutrinos

Finally, let us add the right-handed neutrinos to the Standard Model.

The new terms in the Lagrangian are

$$\mathcal{L}^{\text{RH } \nu} = \bar{\nu}_R^i i\gamma^\mu D_\mu \nu_R^i + (-y_{ij}^\nu \bar{L}_L^i \tilde{H} \nu_R^j + \text{h.c.}) . \quad (8.3.31)$$

The anomaly cancellation conditions become

$$2Y_L^3 - Y_{e_R}^3 - Y_{\nu_R}^3 + 6Y_Q^3 - 3Y_{u_R}^3 - 3Y_{d_R}^3 = 0 , \quad (8.3.32)$$

$$Y_L + 3Y_Q = 0 , \quad (8.3.33)$$

$$-Y_{u_R} - Y_{d_R} + 2Y_Q = 0 , \quad (8.3.34)$$

$$2Y_L - Y_{e_R} - Y_{\nu_R} + 6Y_Q - 3Y_{u_R} - 3Y_{d_R} = 0 . \quad (8.3.35)$$

These are still 4 equations, but now in 6 variables.

Our previous argument about charge quantization remains true.

However, there are now more than one solution.

Clearly, nothing changes when setting $Y_\nu = 0$ which leads to well known hypercharges discussed before.

The most general solutions for the above equations are

$$\begin{aligned} Y_L &= -\frac{a}{2} - b, & Y_{e_R} &= -a - b, & Y_{\nu_R} &= -b, \\ Y_Q &= \frac{a}{6} + \frac{b}{3}, & Y_{u_R} &= \frac{2a}{3} + \frac{b}{3}, & Y_{d_R} &= -\frac{a}{3} + \frac{b}{3} \end{aligned} \quad (8.3.36)$$

for any a, b and

$$Y_Q = Y_L = 0, \quad Y_{u_R} = c, \quad Y_{d_R} = -c, \quad Y_{e_R} = d, \quad Y_{\nu_R} = -d, \quad (8.3.37)$$

for any c, d which are the only two solutions up to $u_R \leftrightarrow e_R$ and $e_R \leftrightarrow \nu_R$.

The Standard Model is obtained from the first solutions with the assignments $a = 1$ and $b = 0$.

Again, we only need to impose for instance $Y_{\nu_R} = 0$ to obtain $b = 0$.

Then $a = 1$ can be obtained by rescaling the U(1) coupling constant so that the Standard Model hypercharge is again uniquely determined from the first.

The second solution spoils this argument, but it does not match with observations.

Taking into account ν_R^i , we find another solution B - L (difference between baryon and lepton number).

Hypercharge of the Higgs

One way to constrain the hypercharge for the Higgs is by demanding a non-trivial Yukawa coupling with the electron giving rise to the mass of the electron.

We would like the Yukawa couplings to involve only hypercharge neutral terms in order to guarantee gauge invariance.

Recall that for the leptons

$$\mathcal{L}^{\text{Yukawa, leptons}} = -y_{ij}^e \bar{L}_L^i H e_R^j + \text{h.c.} \quad (8.3.38)$$

To ensure the vanishing of the net hypercharge, we require

$$-Y_L + Y_H + Y_{e_R} = 0 \quad (8.3.39)$$

which results in

$$Y_H = \frac{1}{2} \quad (8.3.40)$$

as expected.

The Lagrangian for all Yukawa couplings is given by

$$\mathcal{L}^{\text{Yukawa}} = -y_{ij}^e \bar{L}_L^i H e_R^j - y_{ij}^d \bar{Q}_L^i H d_R^j - y_{ij}^u \bar{Q}_L^i \tilde{H} u_R^j + \text{h.c.} \quad (8.3.41)$$

which gives rise to the following three conditions

$$-Y_Q + Y_H + Y_{d_R} = 0, \quad (8.3.42)$$

$$-Y_Q - Y_H + Y_{u_R} = 0, \quad (8.3.43)$$

$$-Y_L + Y_H + Y_{e_R} = 0. \quad (8.3.44)$$

It is a remarkable outcome that, with this assignment in combination with anomaly cancellation, not only does the electron get a mass, but also all left- and right-handed quarks due to the presence of non-vanishing Yukawa couplings

8.3.2 Anomalies of global symmetries in the Standard Model

In this section, we discuss the possible anomalies associated with the global symmetries G_{ac} found in the previous section.

We begin with the baryon number for which

- $A(B, B, B)$: This coefficient is **non-anomalous**

$$A(B, B, B) = \sum_{\text{all}} 2B^3 = 2 \frac{36 - 18 - 18}{27} = 0. \quad (8.3.45)$$

- $A(B, B, 1)$: This coefficient is **non-anomalous**

$$\begin{aligned} A(B, B, 1) &= \sum_{\text{all}} 2Y B^2 \\ &= 36 \left(\frac{1}{6}\right) \left(\frac{1}{3}\right)^2 - 18 \left(\frac{2}{3}\right) \left(-\frac{1}{3}\right)^2 + 18 \left(\frac{1}{3}\right) \left(-\frac{1}{3}\right)^2 \\ &= 0. \end{aligned} \quad (8.3.46)$$

- $A(B, 1, 1)$: This coefficient is **anomalous**

$$\begin{aligned}
 A(B, 1, 1) &= \sum_{\text{all}} 2Y^2 B \\
 &= 36 \left(\frac{1}{6}\right)^2 \frac{1}{3} + 18 \left(\frac{2}{3}\right)^2 \left(-\frac{1}{3}\right) + 18 \left(\frac{1}{3}\right)^2 \left(-\frac{1}{3}\right) \\
 &= -3.
 \end{aligned} \tag{8.3.47}$$

- $A(B, 2, 2)$: This coefficient is **anomalous**

$$A(B, 2, 2) = \sum_{\text{doublets}} B = 9 \left(\frac{1}{3}\right) = 3. \tag{8.3.48}$$

- $A(B, 3, 3)$: This coefficient is **non-anomalous**

$$\begin{aligned}
 A(B, 3, 3) &= \sum_{\text{quarks}} B \\
 &= 6 \left(\frac{1}{3}\right) + 3 \left(-\frac{1}{3}\right) + 3 \left(-\frac{1}{3}\right) \\
 &= 0.
 \end{aligned} \tag{8.3.49}$$

- $A(B, J, J)$: This coefficient is **non-anomalous**

$$A(B, J, J) = \sum_{\text{all}} B = 12 - 6 - 6 = 0. \tag{8.3.50}$$

For the individual lepton numbers, we find with $L \in \{L_e, L_\mu, L_\tau\}$

- $A(L, L, L)$: This coefficient is **anomalous**

$$\begin{aligned} A(L, L, L) &= \sum_{\text{all}} 2L^3 \\ &= 4(1) + 2(-1)^3 \\ &= 2. \end{aligned} \tag{8.3.51}$$

- $A(L, L, 1)$: This coefficient is **non-anomalous**

$$\begin{aligned} A(L, L, 1) &= \sum_{\text{all}} 2YL^2 \\ &= 4\left(-\frac{1}{2}\right)(1)^2 + 2(1)(-1)^2 \\ &= 0. \end{aligned} \tag{8.3.52}$$

- $A(L, 1, 1)$: This coefficient is **anomalous**

$$\begin{aligned} A(L, 1, 1) &= \sum_{\text{all}} 2Y^2L \\ &= 4\left(-\frac{1}{2}\right)^2(1) + 2(1)^2(-1) \\ &= -1. \end{aligned} \tag{8.3.53}$$

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- $A(L, 2, 2)$: This coefficient is **anomalous**

$$\begin{aligned} A(L, 2, 2) &= \sum_{\text{doublets}} L \\ &= 1 . \end{aligned} \tag{8.3.54}$$

- $A(L, J, J)$: This coefficient is **anomalous**

$$\begin{aligned} A(L, J, J) &= \sum_{\text{all}} L \\ &= 2 (1) + 1 (-1) \\ &= 1 . \end{aligned} \tag{8.3.55}$$

Some of the anomaly coefficients above are non-zero which implies that the Standard Model suffers from global anomalies.

Thus, these symmetries do not survive in the quantum theory.

However, this should be understood as a scale dependent observation.

At energies well below m_W , the violation of the associated conservation laws due to quantum effects is exponentially suppressed.

Hence, most of these symmetries are approximately realized in the Standard model, although those involving $SU(3)_c$ are strongly broken.

The only anomaly free global symmetries of the Standard Model are obtained by taking appropriate linear combinations, namely

$$L_e - L_\mu, \quad L_e - L_\tau \quad (8.3.56)$$

with the third $L_\mu - L_\tau$ being linearly dependent on the first two.

To see this, let us consider any linear combination $L = L_1 - L_2$ with $L_1, L_2 \in \{L_e, L_\mu, L_\tau\}$.

Then for example

$$A(L, 1, 1) = A(L_1, 1, 1) - A(L_2, 1, 1) = 0. \quad (8.3.57)$$

The only non-trivial term is

$$\begin{aligned} A(L, L, L) &= \sum_{\text{all}} 2(L_1 - L_2)^3 \\ &= \sum_{\text{all}} 2(L_1^3 - 3L_1^2L_2 + 3L_1L_2^2 - L_2^3) \\ &= A(L_1, L_1, L_1) - 3A(L_1, L_2, L_2) + 3A(L_1, L_2, L_2) - A(L_2, L_2, L_2) \\ &= 0 \end{aligned} \quad (8.3.58)$$

which is zero because $A(L_1, L_2, L_2) = A(L_1, L_2, L_2)$ by the equality of the associated charges.

One can show that the total lepton number

$$L_{\text{tot}} = L_e + L_\mu + L_\tau \quad (8.3.59)$$

has the same gauge anomalies as B, only the anomaly coefficients for B^3 , L_{tot}^3 and the gravitational anomaly disagree.

In the presence of right handed neutrinos, all anomaly coefficients agree so that the combination

$$B - L_{\text{tot}} \tag{8.3.60}$$

is anomaly free.

This makes this symmetry suitable for a potential extension of the Standard Model in which $U(1)_{B-L}$ could be promoted to a full gauge symmetry with a massive Z' gauge field.

8.4 The θ parameter and quark masses

Most of the structure of the Standard Model Lagrangian is simply adding the electroweak terms of part 6 to the strong interactions Lagrangian of part 7 and just taking care of the full covariant derivatives.

However, by putting them together, there is a term that becomes more relevant once the strong and electroweak interactions are combined in the full Standard Model Lagrangian.

This is the θ_3 term for the strong interactions

$$\frac{g_s^2 \theta_3}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^A G_{\rho\sigma}^A, \quad \theta_3 = \frac{64\pi^2}{g_s^2} \Theta_G. \tag{8.4.1}$$

First, let us recall three important properties of this term:

- *Total derivative.* This term is a total derivative since $\epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^A G_{\rho\sigma}^A = \partial^\mu K_\mu$ with K_μ the Chern-Simons current defined as

$$K^\mu = \epsilon^{\mu\nu\rho\sigma} \left(G_\nu^A G_{\rho\sigma}^A - \frac{g_s}{3} f^{ABC} G_\nu^A G_\rho^B G_\sigma^C \right). \quad (8.4.2)$$

This means that this term is not relevant in perturbation theory. However it is important once non-perturbative effects are included ¹

- *CP violation.* This term violates CP (see example sheet for full details). The easiest way to see it is that this term has the same structure as the QED case (abelian) for which it is just $\mathbf{E} \cdot \mathbf{B}$ which changes sign under time reversal (and parity) since $\mathbf{E}$ being a polar vector does not change sign under time reversal, but $\mathbf{B}$ does. By the CPT theorem, if it violates T, it violates CP.
- *Quantisation.* In fact, it needs to be taken into account to ensure a proper quantum description.

It is important to keep in mind that it is not some mathematical obscurity, but quintessential for the quantization process itself.

For SU(2) and U(1) the corresponding ω parameter can be redefined away as we will explain at the end of this section.

We will show now why for QCD the θ_3 parameter is physical.

Let us consider the mass term for the six quark flavors (u, d, s, c, b, t) in the mass basis

$$\mathcal{L}_{mass} = m_j \bar{\psi}_L^j \psi_R^j, \quad j = 1, \dots, 6 \quad (8.4.3)$$

Then, we consider a chiral field redefinition for all quark fields

$$\psi_L^j \rightarrow e^{i\alpha_j} \psi_L^j, \quad \psi_R^j \rightarrow e^{-i\alpha_j} \psi_R^j. \quad (8.4.4)$$

If the quark masses m_j were real, these rotations can be compensated by transforming the masses as

$$m_j \rightarrow e^{2i\alpha_j} m_j, \quad j = 1, 2, \dots, 6 \quad (8.4.5)$$

at the price of making the quark masses complex.

Also, as we discussed in part 5, each of these U(1) axial transformations is anomalous and would induce a θ term for the corresponding gauge field

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \rightarrow \exp \left[-\frac{ig_s^2 \sum_j \alpha_j}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^A G_{\rho\sigma}^A \right] \mathcal{D}\psi \mathcal{D}\bar{\psi}. \quad (8.4.6)$$

Therefore, the above transformations shift the θ_3 parameter a

$$\theta_3 \rightarrow \theta_3 + 2 \sum_{j=1}^6 \alpha_j. \quad (8.4.7)$$

Thus, by a suitable choice of the parameters α_j , we can rotate away the θ_3 setting it to zero.

However, this will be done at the price of including a phase in the mass matrix and an extra parameter beyond those of the CKM matrix.

This is important because this parameter is a phase that breaks CP.

This means that we only moved the CP violating parameter θ_3 into another CP violating parameter, but now in the quark masses.

There is one physically relevant parameter which is the invariant combination

$$\bar{\theta} \equiv \theta_3 - \sum_{j=1}^6 \arg m_j = \theta_3 - \arg \prod m_j = \theta_3 - \arg \det(M) \quad (8.4.8)$$

where M denotes the quarks mass matrix, recall the discussion in Sect. 6.4.3.

This is the effective θ -parameter that cannot be rotated away: the chiral anomaly shifts θ_3 and $\arg \det(M)$ by exactly the same constant making $\bar{\theta}$ invariant.

Theta vacua*

This parameter ω illustrates a particular property of the QCD vacuum in the following sense.

We know from the previous part of these lectures that in the effective chiral Lagrangian for the Goldstone modes π^a we can have a 'mass term'

$$\delta\mathcal{L} = C\text{Tr} \left(MU + M^\dagger U^\dagger \right) \quad (8.4.9)$$

With M the u, d mass matrix $M = \text{diag} (m_u, m_d)$, assumed to be real, and C a dimension 3 constant of order $C \sim \Lambda^3_{\text{QCD}}$.

Expanding the exponential in U we can get the quadratic terms in the π fields proportional to

$$\delta\mathcal{L} = \frac{C}{F_\pi^2} (m_u + m_d) \left(\pi_0^2 + \pi_1^2 + \pi_2^2 \right) + \dots$$

implying that the mass² of the pions is of order

$$m_\pi^2 = \frac{C}{F_\pi^2} (m_u + m_d) \sim \Lambda_{\text{QCD}} (m_u + m_d) .$$

as we have seen above.

Now, if the quark mass matrices include the $\bar{\theta}$ phase

$$M = \text{diag}(m_u, m_d) e^{i\bar{\theta}} ,$$

then $\delta\mathcal{L}$ acquires a dependence on $\bar{\theta}$.

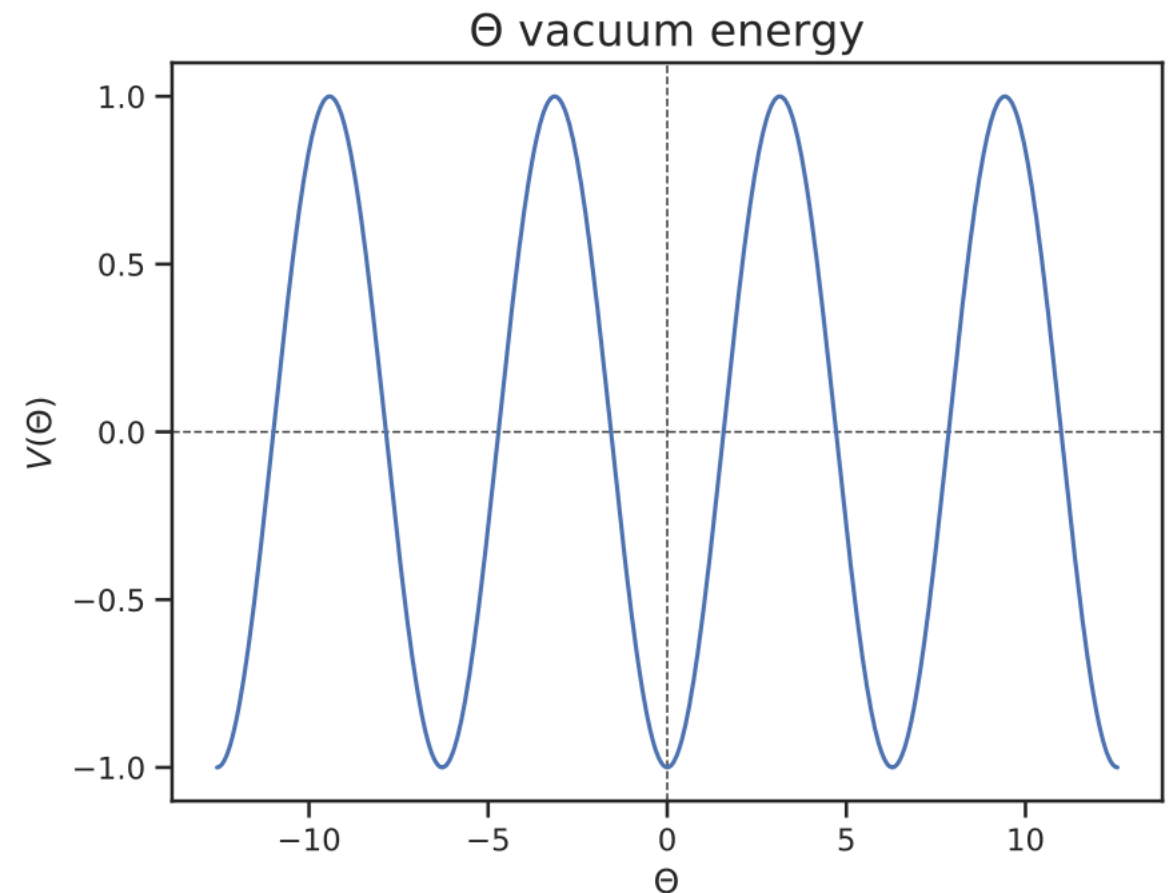
Taking the leading term in the expansion of U (the identity) implies

$$E(\bar{\theta}) = -2C(m_u + m_d) \cos \bar{\theta} = -F_\pi^2 m_\pi^2 \cos \bar{\theta}.$$

This illustrates the fact that the vacuum structure of QCD depends on $\bar{\theta}$ and each minimum of this energy function is a vacuum with degenerate energies.

These are known as theta vacua or θ -vacua, see Fig. 8.2.

Figure 8.2: The vacuum in QCD has a degeneracy in the sense that it depends on $\bar{\theta}$ as a periodic function on $\bar{\theta}$. Each of the minima of this function is an allowed vacuum.



What about θ for electroweak interactions?*

We have seen that a chiral transformation on fermions, being anomalous, generates an effective θ term.

In the case of QCD, we can perform a quark transformation to rotate away the θ parameter, but this introduces an additional phase in the mass matrix.

The physical parameter is $\bar{\theta} = \theta - \arg \det M$.

For weak interactions, once the corresponding θ parameter is shifted to the mass matrix, for instance through a chiral left-handed transformation, we can rotate only the right-handed fermions to remove the phase from the mass matrix.

Since right-handed fields are invariant under $SU(2)_L$, they do not reintroduce a θ term for $SU(2)_L$.

Therefore, unlike in QCD, the weak interaction θ term has no physical significance.

A similar argument applies to electromagnetism, where neutrinos can be used to rotate away the corresponding ω parameter, as they are neutral with respect to electromagnetic interactions.

However, in condensed matter systems, electromagnetic interactions involving boundary effects can render the associated θ term relevant.

This is an area of active research and falls beyond the scope of these lectures.

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PART 9 FINAL REMARKS

Good ideas take time

I am emphasizing here that it took a long time before we realized what these ideas were good for partly because I want to encourage today's string theorists, who I think also have good ideas that are taking a long time to mature.

Steven Weinberg

This concludes the lectures.

The primary goal has been to convey the conceptual foundations of the Standard Model, allowing for a deeper understanding and appreciation of the remarkable achievements that have culminated in this comprehensive framework of how nature operates.

The structure of these lectures was designed to provide a modern perspective on the logical principles that define the Standard Model.

We aimed to illustrate how both inevitable and compelling this model is for describing the world, based on first principles and basic experimental input.

Given the constraints a limited number of lectures and the vast range of material related to the Standard Model, we had to be selective in our approach.

The focus was on presenting an overarching view, starting with a historical context, followed by a first-principles approach, now that the subject is mature and well-established.

A wide range of earlier classes was necessary to ensure coverage of topics vital to the Standard Model, while trying to avoid unnecessary repetition.

For instance, important aspects of QED, including the calculation of scattering amplitudes and decay rates, were covered in the RQM and QFT classes.

Anomalies were discussed again since they are so important, though restricted our discussion to basic concepts and tools due to the subject's breadth.

The quantization of Yang-Mills theory was covered in the QFT class, and the discussion of running couplings was also covered in the QFT class.

By bringing together this material, we hope to provide a broad understanding of the subject, equipping you guys with the knowledge to engage with advanced textbooks and eventually reproduce more detailed calculations of decay rates and scattering amplitudes if you so desire

Let us finally wrap these lectures up by summarizing the main lines of arguments followed in this class:

1. The fundamental theories we take as basic postulates are just Special Relativity and Quantum Mechanics.
2. The symmetries of Special Relativity given by the Poincaré group include translations and Lorentz transformations. The basic representations of the Lorentz group correspond to left-handed and right-handed Weyl spinors obtained from the 2 to 1 homomorphism between the $SL(2, \mathbb{C})$ and $SO(3, 1)$ groups (indicating that rotations by 4π rather than 2π are the ones that are identical to the identity). Out of these two independent representations all other representations of the Lorentz group (vectors, tensors, etc.) can be obtained. This also includes the more standard Dirac spinors which are a reducible representation composed of left- and right-handed Weyl spinors.

3. Physical states correspond to unitary irreducible representations of the Poincaré group. They have very different properties for massive and massless cases.
 - For the massive case, they are labelled by the eigenvalues of the Casimir operators $C_1 = P^\mu P_\mu$ and $C_2 = W^\mu W_\mu$ with P^μ momentum generators and W_μ the Pauli-Ljubanski vector. Each state within a representation is labelled by the eigenvalues of momenta, chosen in a frame as $p^\mu = (m, 0, 0, 0)$, and the corresponding Little group associated with the generators that leave invariant p^μ . In the massive case, the Little group is $SO(3)$ and so the states are $|m, j; p^\mu, j_3\rangle$ with m the mass, $j = 0, 1/2, 1, \dots$ the spin and $j_3 = -j, \dots, j$ the component of the spin in the direction of motion. These states describe massive particles.
 - For massless states the same procedure leads to $C_1 = C_2 = 0$ and the Little group, after imposing finite dimensional representations, reduces to $SO(2)$ so the states are only labelled by momenta that in a particular frame is $p^\mu = (E, 0, 0, E)$ and helicity $\lambda = 0, \pm 1/2, \pm 1, \dots$ as $|p^\mu, \lambda\rangle$.

These are the basic quantities to consider to be the building blocks for all matter.

4. To study interactions among the elementary particle states described above, we impose extra criteria of locality and unitarity which require us to assign a field to every particle state. Then interactions are described by the interaction Hamiltonian and captured by a Lagrangian. Further conditions of stability guarantee the existence of a vacuum state and renormalisability to guarantee predictability, thereby restricting the possible Lagrangians. This condition is understood as only an approximation to a more general Lagrangian written in an Effective Field Theory (EFT) for which the Lagrangian is expanded as $\mathcal{L} = c_i \mathcal{O}_i$ with coefficients c_i of higher dimensional operators $\mathcal{O}_i$ suppressed with respect to a energy scale M by $c_i = k_i/M^{i-4}$ and k_i dimensionless constants.

5. Field theories for particles of spin/helicity $0, 1/2$ (for which both massless and massive states have the same number of degrees of freedom) do not offer particular challenges, even though they are very constrained to have Lagrangians with only a few terms. However, for higher values of spin/helicity constraints become much more severe:

- Massless states of helicity $\lambda = \pm 1$ have only two polarisation states, whereas the massive particles of spin $j = 1$ have $2j + 1 = 3$ degrees of freedom. The polarisation vectors ϵ_μ , with originally 4-components, in both cases are constrained by $p^\mu \epsilon_\mu = 0$ reducing the number of degrees of freedom to 3. This is already sufficient for the massive case, but not for the massless case. In this case, since the on-shell condition reads $p^\mu p_\mu = 0$, we have to take into account that there is a redundancy in the polarisation vectors. That is, $\epsilon_\mu + \alpha(p)p_\mu$ is equivalent to the polarisation ϵ_μ for any arbitrary function $\alpha(p)$. Moving to position space from this momentum space condition leads to $A_\mu(x) + \partial_\mu \alpha(x)$ being equivalent to A_μ . This manifests gauge invariance for a Lagrangian built to describe the interactions of a field $A_\mu(x)$. Given this redundancy, the polarisation vectors for a helicity 1 field do not transform as vectors under Lorentz transformations, but they transform as vectors up to $\epsilon_\mu \rightarrow \epsilon_\mu + \alpha(p)p_\mu$. In particular an amplitude of the form $\mathcal{M} = \mathcal{M}_\mu \epsilon^\mu$ would not be Lorentz invariant unless $p^\mu \mathcal{M}_\mu = 0$. This is the famous Ward identity.

* Using the Ward identity and considering a general scattering process with arbitrary incoming and outgoing particles and attaching a "soft photon" to each line, we proved that the charges of the particles (defined as the interaction of the photon to the corresponding particle) are conserved, that is: $\sum_{in} Q_i = \sum_{out} Q_i$.

- * Using a process equivalent to Compton scattering with arbitrary couplings at the vertex between the a helicity 1 particle and the matter fields i and j as T_{ij}^a . We proved that contributions from the s and t channels satisfy the Ward identity if the couplings satisfy $[T^a, T^b] = 0$ (like charges satisfy in QED) unless there is a self-interaction among the helicity 1 particles with vertex f_{abc} for a cubic (three particle) interaction. In this case the couplings satisfy $[T^a, T^b] = f^{abc}T^c$ which is Yang-Mills theory.

This means that only using Lorentz invariance already implies the Ward identity and this in turn implies abelian and non-abelian gauge theories. The corresponding gauge symmetries are not an input but an output of any theory involving helicity ± 1 particles. So they are unavoidable.

- For massive spin $j = 1$ particles, in order to construct a properly defined theory we proved that amplitudes will increase with energy. Since amplitudes are after all probability amplitudes, having an arbitrary large amplitude would break unitarity. Therefore theories of massive spin 1 particles are not well defined at high energies and need a UV completion.
- For massless particles of helicity $\lambda = \pm 2$, following the same steps as for charge conservation for helicity ± 1 particles, we established that the equivalent condition to charge conservation is of the form $\sum_{in} \kappa_i p_i^\mu = \sum_{out} \kappa_i p_i^\mu$ with κ_i the couplings and p_i^μ the momenta of the matter particles. This linear condition on momenta is on top of the standard momentum conservation that reads $\sum_{in} p_i^\mu = \sum_{out} p_i^\mu$. Having an extra linear constraint on momenta would not be possible (unless restricting motion to lower dimensional surfaces) and therefore these two conditions should be equivalent implying all the κ_i are the same. That means that the interaction mediated by the helicity 2 particle is the same for all particles. This is the principle of equivalence allowing us to identify this interaction with gravity.

- Doing the same for higher helicities, say $\lambda = 3$, the condition would extend to $\sum_{in} \gamma_i p_i^\mu p_i^\nu = \sum_{out} \gamma_i p_i^\mu p_i^\nu$ for which there are no solutions (keeping in mind momentum conservation) unless the couplings vanish $\gamma_i = 0$. This implies that there are no interacting theories for helicity $\lambda > 2$ massless particles. This is a very powerful result limiting the possible interactions to be mediated by helicity 1 particles with abelian or non-abelian gauge symmetries or helicity ± 2 corresponding to gravity and no more. We may add interactions mediated by scalar particles if we prefer to interpret the Higgs as a mediator of interactions rather than a matter particle but this is just a question of semantics. It is then no surprise that the interactions we have observed are precisely of this type. All this is derived only from Lorentz invariance and quantum mechanics.
 - For massive particles of spin $j \geq 2$ as for spin $j = 1$, they do not have perturbative unitarity and need an UV completion which is not known. Notice that there are known composite particles of high spin, but not massive elementary particles of spin $j \geq 2$.
6. Having singled out the theories to those with spin/helicity $0, 1/2, 1, 3/2, 2$, we then concentrated on the Yang-Mills case which also includes QED as the abelian case. Yang-Mills theories are very rich and the number of gauge groups is infinite. First we argued that in order to have positive kinetic terms, we are limited to gauge groups which are compact and simple or semi-simple, that means the groups classified by Cartan. Specifically, the relevant groups are $SO(N), SU(N), Sp(N), G_2, F_4, E_6, E_8$ which eliminates the infinite number of non-compact groups, but still leaves a large number of options for gauge symmetries, including all representations for matter fields. Fortunately most of the results are independent of which symmetry group we worked with. We wrote the most general renormalisable Lagrangian coupling gauge fields to matter fields through covariant derivatives and found general properties of these theories.

- *Spontaneous symmetry breaking.* Once coupled to scalar fields, the potential for the scalars is of two types depending on the sign of the quadratic term $m^2|H|^2$. If positive, the minimum of the potential is at $\langle H \rangle = 0$ and the symmetry is manifest. If negative, the minimum is at $\langle H \rangle \neq 0$ implying spontaneous symmetry breaking.
 - * We proved Goldstone's theorem stating that, once a continuous symmetry is broken, there are $\dim(G/H)$ massless particles, the Goldstone bosons.
 - * We described the Higgs mechanism in which gauge symmetry is spontaneously broken and the originally massless gauge boson acquires a mass by absorbing the degrees of freedom of the original Goldstone bosons. This solves several problems at once: no massless Goldstone bosons are seen, no massless Yang-Mills fields have been seen but together they become massive spin-1 fields with the extra bonus that the couplings to the massive Higgs field restores perturbative unitarity. Therefore, we ended up with a UV complete theory describing interactions of massive spin $j = 1$ particles with matter fields.
- *Asymptotic freedom.* Yang-Mills theories are unique in the sense that the corresponding gauge couplings evolve with energy in a way that the strength of the interactions tends to decrease with increasing energy (asymptotic freedom), unless the theory is abelian (QED) or has a substantial number of matter fields. This allows for the possibility of having fundamental degrees of freedom confined at lower energies. But at high energies the theories are UV complete.

7. *Electroweak theory.* Finally we considered concrete examples of Yang-Mills theory. First an example of spontaneous symmetry breaking describing the weak and electromagnetic interactions. We justified, based on the critical experimental evidence for chirality that the natural group for weak interactions is $SU(2)_L \times U(1)_Y$ with $SU(2)_L$ acting only on left-handed fermions. The scalar Higgs field breaks the symmetry to $U(1)_{EM}$ and so includes electromagnetism automatically within the model. We described this model in detail including all couplings of gauge fields among themselves, their couplings to fermions (quarks and leptons), to the Higgs particle and the couplings of the Higgs to matter.
8. *Quantum Chromodynamics.* Strong interactions are an example of asymptotically free Yang-Mills theories explaining the interactions among quarks and gluons and why at low energies it is natural to see only the composite states, namely hadrons. We also proved that well known symmetries such as baryon number are only accidental symmetries in the sense that the most general renormalisable Lagrangian for QCD is automatically symmetric under a global symmetry corresponding to the conservation of baryon number and higher order corrections to the Standard Model Lagrangian would break these symmetries. Furthermore, in describing QCD at energies below Λ_{QCD} , we uncover the well known approximate symmetries corresponding to isospin (relating protons and neutrons) as well as the eightfold way (the flavour $SU(3)$ symmetry that historically gave rise to the proposal of quarks as the basic building blocks of matter and colours as the symmetry behind the strong interactions). Explaining in this way the origin of these symmetries and their approximate nature due to the fact that there is a hierarchy of masses for the quarks.

It is crucial to understand the fundamental principles of the Standard Model in order to gain insight into how we might extend it, particularly in these times when the optimal path forward remains uncertain.

Notably, we have made significant progress without direct experimental input, allowing us to conclude that the basic constituents of matter must transform as scalars, fermions, vectors, or symmetric tensors – and nothing more.

We know that spin-1 theories require an underlying symmetry, whether abelian or non-abelian, that such theories are typically asymptotically free and lead to confinement, and that when coupled to scalar fields (whether elementary or composite), they result in symmetry breaking via the Higgs mechanism.

Remarkably, all of this follows from the principles of relativity and quantum mechanics alone, making the core features of the Standard Model largely predictable.

The primary choice we face is selecting the appropriate symmetry group and representations.

However, not all options are viable, as consistency conditions – such as the requirement anomaly cancellation – impose significant constraints (leading, for example, to charge quantization).

Thus, although the development of the Standard Model involved many confusing paths, we can now confidently assert that its structure is highly robust.

It is likely that the gauge symmetries and matter content will be modified at higher energies, with the current framework representing a minimal case.

Fortunately, significant challenges remain.

A final note on symmetries: the only true symmetry we have assumed is Poincaré invariance from special relativity and CPT symmetry, which is a consequence of relativistic quantum field theories.

All gauge symmetries, by contrast, are simply redundancies rather than real symmetries, and global symmetries are only approximate.

Additionally, all spacetime symmetries that can be broken, such as P and CP (or T), are broken.

Therefore, although symmetries offer a powerful tool for systematically constructing theories through invariant Lagrangians, they may not, in the end, form part of the fundamental principles underlying these theories – aside from Poincaré or more general spacetime symmetries.

We hope that the techniques introduced in this class will help you fully grasp the core principles of the Standard Model and guide you in your future readings, including, maybe, those that explore theories beyond the Standard Model.

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