Ordinary Differential Equations(ODE)

ODEs are equations involving derivatives in various ways. The highest derivative specifies the order of the equation. Single variable equations are ODEs and multi-variable equations are Partial Differential Equations(PDE)s.

An ODE

$$L(y(x)) = R(x)$$

is linear if the differential operator L is a linear operator satisfying the linearity property

$$L(a_1 y_1(x) + a_2 y_2(x)) = a_1 L(y_1(x)) + a_2 L(y_2(x))$$

where a_1 and a_2 are constants.

All physical systems not far from **EQUILIBRIUM** (small displacements)are linear systems.

Linear ODEs are simpler to solve than non-linear ODEs.

This is a consequence of the following two superposition principles which apply only to linear ODEs:

(1) If $y_1(x)$ and $y_2(x)$ are **any** two solutions of a **homogeneous linear ODE**

$$L(y(x)) = 0$$

then

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \qquad c_1, c_2 = \text{ constants}$$

is also a solution, i.e.,

$$Ly_h(x) = L(c_1 y_1(x) + c_2 y_2(x)) = c_1 L(y_1(x)) + c_2 L(y_2(x)) = 0$$

(2) If $y_h(x)$ is a (i.e.,any) solution of the homogeneous linear ODE, $L(y_h(x)) = 0$ and $y_p(x)$ is a solution of the inhomogeneous linear ODE

$$L(y_p(x)) = R(x)$$

then the linear combination

$$y(x) = y_p(x) + Ay_h(x) \qquad A = \text{ constant}$$

is a solution of the inhomogeneous linear ODE

$$L(y(x)) = R(x)$$

i,e,

$$L(y(x)) = L(y_p(x) + Ay_h(x)) = L(y_p(x)) + AL(y_h(x)) = R(x)$$

2nd-Order Homogeneous ODEs with Constant Coefficients

Consider the ODE

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$$

If we let

$$D = \frac{d}{dx}$$

then we have the equation

$$(D^2 + 5D + 4)y = 0$$

These equations can be solved by **substitution(guessing)** and converting the ODE into an **algebraic (quadratic)** equation.

We choose(guess) the solution

$$y = ce^{ax}$$

Substitution gives the allowed values of a (corresponding to possible solutions)

$$(a^2 + 5a + 4)ce^{ax} = 0 \rightarrow a^2 + 5a + 4 = 0 \rightarrow a = -1 \text{ and } a = -4$$

The most general solution is a superposition of all possible solutions with arbitrary multiplicative constants (number of possible solutions = order of ODE) so

$$y = c_1 e^{-x} + c_2 e^{-4x}$$

The arbitrary constants c_1 and c_2 are determined by boundary conditions where we specify

$$y(x_0) \text{ and } \frac{dy(x_0)}{dx} \quad , \quad x_0 = \text{ arbitrary point}$$

For example suppose,

$$x_0 = 0 \text{ and } y(0) = 2 \text{ and } \frac{dy(0)}{dx} = -3$$

then we have

$$y(0) = 2 = c_1 + c_2$$

$$\frac{dy(0)}{dx} = -3 = -c_1 - 4c_2$$

$$\rightarrow c_1 = \frac{5}{3} \text{ and } c_2 = \frac{1}{3}$$

and the general solution with these boundary conditions is

$$y(x) = \frac{5}{3}e^{-x} + \frac{1}{3}e^{-4x}$$

The solutions for a can be imaginary numbers (trigonometric solutions) or complex numbers (mixed trigonometric and exponential solutions).

Second-Order Homogeneous ODEs in General

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation with boundary conditions

$$y_1(a) = 1 \ , \quad y_1'(a) = 0 \ , \ \ y_2(a) = 0 \ , \ \ y_2'(a) = 1$$

It is clear that $y_1(x)$ and $y_2(x)$ can never be proportional to each other, since they already differ at the boundary $x = a$. This means that they are linearly independent.

They are also the only linearly independent solutions because 2nd-order ODEs have only two independent boundary conditions $y(a)$ and $y'(a)$.

A general solution satisfying the boundary conditions

$$y(a) = \ c_1 \quad , \ \ y'(a) = c_2$$

is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

The 2nd Homogeneous Solution

Now, if one solution $y_1(x)$ of a 2nd order homogeneous linear ODE is known, a second solution $y_2(x)$, linearly independent of the first, can be constructed with the help of the Wronskian:

$$\frac{d}{dx}\left[\frac{y_2}{y_1}\right] = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W(x)}{y_1^2(x)}$$

A simple integration then gives

$$\frac{y_2(x)}{y_1(x)} - \frac{y_2(b)}{y_1(b)} = \int_b^x \frac{W(x')}{y_1^2(x')} dx' = g(x)$$

$$y_2(x) = g(x) y_1(x) + \frac{y_2(b)}{y_1(b)} y_1(x)$$

Now we can drop the last term since it is proportional to $y_1(x)$ and we already have a term proportional to $y_1(x)$ in the general solution

$$y(x) = a y_1(x) + b y_2(x)$$

Therefore, we have

$$y_2(x) = g(x)y_1(x)$$

Since

$$W(x) = W(b)\exp\left(-\int_b^x p(x')dx'\right)$$

we get

$$g(x) = \int_b^x \frac{W(x')}{y_1^2(x')}dx' = W(b)\int_b^x \frac{\exp\left(-\int_b^{x'} p(x'')dx''\right)}{y_1^2(x')}dx'$$

Inhomogeneous Solutions

The solution of the inhomogeneous 2nd order linear ODE

$$Ly(x) = R(x)$$

has the general form

$$y(x) = y_p(x) + [c_1y_1(x) + c_2y_2(x)]$$

where

$$y_p(x) = \text{ any particular solution}$$

$$[c_1 y_1(x) + c_2 y_2(x)] = \text{ complementary solution}$$

The complementary solution is a combination of 2 linearly independent solutions of the homogeneous equation.

Series Solution of Homogeneous 2nd Order Linear ODEs: Method of Frobenius

We have seen

(1) given one solution of a homogeneous 2nd order ODE we can generate a second linearly independent solution integration

(2) if we are given both solutions, then a particular solution of the inhomogeneous ODE also can be calculated by integration

Thus, it remains for us to obtain at least 1 solution for the homogeneous equation.

We do not want to spend time looking for a particular solution with JUST the right boundary conditions because they are very hard to obtain in general. Anyway we are unlikely to find one with exactly the correct boundary conditions.

The complementary function now comes to our rescue.

A method, with which it is usually possible to do so, is to obtain the solution $y(x)$ of the homogeneous ODE

$$Ly(x) = 0$$

using a **power series** in x.

We choose

$$y(x) = x^s \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda \qquad a_0 \neq 0$$

We then have

$$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (\lambda + s) x^{\lambda+s-1}$$

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (\lambda + s)(\lambda + s - 1) x^{\lambda+s-2}$$

Substitution back into the homogeneous ODE produces a powers series which sums to zero.

In this case, because each term in the series is linearly independent, each coefficient must separately $= 0$.

This allows us to determine the values of s and the a-coefficients and hence the solution to the problem.

Consider the equation

$$\frac{d^2u}{dx^2} + x^2u = 0$$

Let us first use the simpler method which is valid for certain equations where $s =$ integer.

$$y(x) = x^s \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda \qquad a_0 \neq 0$$

In this case, we can choose

$$u(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda \qquad a_0 \neq 0$$

i.e., we can use the simpler series when $s =$ integer.

We then have

$$\frac{d^2u}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda(\lambda)(\lambda-1)x^{\lambda-2}$$

Substitution gives

$$\sum_{\lambda=0}^{\infty} a_\lambda(\lambda)(\lambda-1)x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+2} = 0$$

or

$$a_2(1)(2)x^0 + a_3(2)(3)x^3 + [a_4(3)(4) + a_0]x^2$$
$$+ .... + [a_{k-2} + a_{k+2}(k+2)(k+1)]x^k = ... = 0$$

We then have

$$a_0 \neq 0$$

$$a_1 \neq 0$$

$$a_2 = 0$$

$$a_3 = 0$$

$$a_k + a_{k+4}(k+3)(k+4) = 0$$

$$a_{k+4} = -\frac{a_k}{(k+3)(k+4)}$$

$$a_2 = 0 \rightarrow a_6 = a_{10} = a_{14} = ... = 0$$

$$a_3 = 0 \rightarrow a_7 = a_{11} = a_{15} = ... = 0$$

Starting with a_0 we get

$$a_4 = -\frac{a_0}{(3)(4)}$$

$$a_8 = -\frac{a_4}{(7)(8)} = \frac{a_0}{(3)(4)(7)(8)}$$

Starting with a_1 we get

$$a_5 = -\frac{a_1}{(4)(5)}$$

$$a_9 = -\frac{a_5}{(8)(9)} = \frac{a_1}{(4)(5)(8)(9)}$$

and the general solution is

$$u(x) = c_1\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - ....\right) + c_2\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - .....\right)$$

Let us now do this same equation with s left in (more general).

We choose

$$u(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+s} \qquad a_0 \neq 0$$

We then have

$$\frac{d^2u}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (s+\lambda)(s+\lambda-1)x^{\lambda-2+s}$$

Substitution gives

$$\sum_{\lambda=0}^{\infty} a_\lambda (s+\lambda)(s+\lambda-1)x^{\lambda-2+s} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+s+2} = 0$$

or

$$a_0 s(s-1)x^{s-2} + a_1 s(s+1)x^{s-1} + a_2(s+1)(s+2)x^s$$
$$+ a_3(s+2)(s+3)x^{s+1}$$
$$+ [a_4(s+3)(s+4) + a_0]x^{s+2} + ...$$
$$+ [a_{k-2} + a_{k+2}(s+k+2)(s+k+1)]x^{s+k} + ... = 0$$

We then have

$$a_0 s(s-1) = 0 \rightarrow a_0 \neq 0, s = 0, 1$$

$$a_1 s(s+1) = 0 \rightarrow a_1 \neq 0, s = 09 \quad \text{and} \quad a_1 = 0, s = 1$$

$$a_2(s+1)(s+2) = 0 \rightarrow a_2 = 0$$

$$a_3(s+2)(s+3) = 0 \rightarrow a_3 = 0$$

$$[a_{k-2} + a_{k+2}(s+k+2)(s+k+1) = 0$$

$$a_{k+4} = -\frac{a_k}{(s+k+3)(s+k+4)}$$

$$a_2 = 0 \rightarrow a_6 = a_{10} = a_{14} = ... = 0$$

$$a_3 = 0 \rightarrow a_7 = a_{11} = a_{15} = ... = 0$$

Starting with $s = 0$ and choosing $a_0 = a_1 = 1$ we get

$$a_4 = -\frac{a_0}{(3)(4)}\ ,\ a_8 = -\frac{a_4}{(7)(8)} = \frac{a_0}{(3)(4)(7)(8)}\ ,\ .....$$

$$a_5 = -\frac{a_1}{(4)(5)}\ ,\ a_9 = -\frac{a_5}{(8)(9)} = \frac{a_1}{(4)(5)(8)(9)}\ ,\ .....$$

The case $s = 1$ does not generate any new series (always happens if $s =$ integer).

The general solution is the same as earlier

$$u(x) = c_1\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - ....\right) + c_2\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - .....\right)$$

We now consider the equation

$$8x^2\frac{d^2u}{dx^2} + 6x\frac{du}{dx} + (x-1)u = 0$$

Assume

$$u(x) = x^s\sum_{\lambda=0}^{\infty} a_\lambda x^\lambda \qquad a_0 \neq 0$$

Substitution gives

$$8\sum_{\lambda=0}^{\infty} a_\lambda(\lambda+s)(\lambda+s-1)x^{\lambda+s} + 6\sum_{\lambda=0}^{\infty} a_\lambda(\lambda+s)x^{\lambda+s} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+s+1}$$
$$- \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+s} = 0$$

$$[8a_0 s(s-1) + 6a_0 s - a_0]x^s + [8a_1 s(s+1) + 6a_1(s+1) - a_1 + a_0]x^{s+1}$$
$$+ [8a_2(s+1)(s+2) + 6a_2(s+2) - a_2 + a_1]x^{s+2} + ...$$
$$+ [8a_n(s+n-1)(s+n) + 6a_n(s+n) - a_n + a_{n-1}]x^{s+n} + ... = 0$$

The coefficient of x^s is called the **indicial equation**.

It determines the allowed values of s.

We choose $a_0 = 1$ for simplicity.

We get

$$8s(s-1)+6s-1 = 0 = 8s^2 - 2s - 1 \rightarrow s = \frac{2 \pm \sqrt{4+32}}{16} = \frac{1}{2}, -\frac{1}{4}$$

These two allowed s values will generate the two linearly independent homogeneous solutions.

We have from the other coefficients

$$8a_n(s+n-1)(s+n)+6a_n(s+n)=a_n+a_{n-1}=0$$

$$a_n=-\frac{a_{n-1}}{8(s+n-1)(s+n)+6(s+n)-1}$$

This last relation is a recursion relation and determines all higher order coefficients(for each allowed s value) in terms of a_0.

We get

$$s=\frac{1}{2}$$

$$\begin{aligned}u_1(x)&=x^{1/2}-\frac{1}{8(1/2)(3/2)+6(3/2)-1}x^{3/2}\\&+\frac{1}{(8(3/2)(5/2)+6(5/2)-1)(8(1/2)(3/2)+6(3/2)-1)}x^{5/2}-\ldots\\&=x^{1/2}-\quad\frac{1}{14}x^{3/2}+\frac{1}{616}x^{5/2}-\ldots.\end{aligned}$$

$$s = -\frac{1}{4}$$

$$u_2(x) = x^{-1/4} - \frac{1}{2}x^{3/4} + \frac{1}{40}x^{7/4} - \ldots\ldots\ldots$$

and the most general solution is

$$u(x) = au_1(x) + bu_2(x)$$

Legendre Equation This ODE arises in many physical systems that we shall investigate later

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \ell(\ell+1)u = 0$$

$$P_\ell(x) = \sum_{n=0}^{N}(-1)^n \frac{(2\ell-2n)!}{2^\ell n!(\ell-n)!(\ell-2n)!}x^{\ell-2n} \qquad N = \begin{cases} \frac{\ell}{2} & \ell \text{ even} \\ \frac{\ell-1}{2} & \ell \text{ odd} \end{cases}$$

Hermite Equation

Now we turn our attention to Hermite's equation

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2\alpha u = 0 \quad , \quad \alpha = \text{ nonnegative integer}$$

As we shall see, the solutions are polynomials called **Hermite polynomials**.

Legendre and Hermite polynomials are the first two examples of a large class of **special functions** that play a major role in the solutions of partial differential equations representing physical systems.

$$H_n(x) = n! \sum_{k=0}^{N} \frac{(-1)^k}{k!(n-2k)!}(2x)^{n-2k}$$

$$N = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

The Schrödinger equation for the 1-dimensional harmonic oscillator with a potential function

$$V(x) = \frac{1}{2}kx^2$$

is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x)$$

The standard ODE solution of this equations goes as follows.

Let

$$\xi = \alpha x \quad , \quad \alpha^4 = \frac{mk}{\hbar^2}$$

$$\lambda = \frac{2E}{\hbar\omega_0} \quad , \quad \omega_0 = \left(\frac{k}{m}\right)^{1/2}$$

This gives the new equation

$$\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0$$

If we consider the behavior of the equation as $\xi \to \pm\infty$ we can neglect λ and the equation becomes

$$\frac{d^2\psi}{d\xi^2} - \xi^2\psi = 0$$

which says that the large ξ behavior of the solution is

$$\psi = e^{-\xi^2/2}$$

This suggests that we try a solution of the form

$$\psi = H(\xi)e^{-\xi^2/2}$$

which should remove the large ξ behavior from the equation.

We get the equation

$$H'' - 2\xi H' + (\lambda - 1)H = 0$$

This is Hermite's equation.

That is why we are studying all these special equations - we will then recognize them when we solve the equations for real physical systems and can just quote our earlier results to get solutions!

Laguerre Equation

The Laguerre equation takes the form

$$x\frac{d^2u}{dx^2} + (1-x)\frac{du}{dx} + Nu = 0$$

These are the Laguerre polynomials:

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = 1 - 2x + \frac{x^2}{2!}$$

They will appear in the hydrogen atom solution of the Schrödinger equation.

Bessel's Equation

We now consider the one of the most important equation in physics.

$$L(x)u(x) = \left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + (x^2 - \mu^2)\right)u(x) = 0$$

$$L(x) = L(-x) \rightarrow \text{ exists even and odd solutions}$$

$$J_\mu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)^{2n+\mu} n!\Gamma(\mu + n + 1)} s^{2n+\mu}$$

which is the Bessel function of the first kind of order μ.

Partial Differential Equations (PDEs) In Physics, there are differential equations of motion that describe the response of systems to external disturbances.

These are called ordinary differential equations (ODEs).

There are also differential equations of states, or field equations, whose solutions give the space-time dependence of physical properties.

These are called partial differential equations (PDEs) in the 4 variables x, y, z, t.

In general, the PDE's we will discuss describe three-dimensional situations. The independent variables are the position vector $\vec{r}$ and the time t.

The actual variables used to specify $\vec{r}$ are dictated by the coordinate system in use, i.e., (x, y, z), (ρ, ϕ, z), (r, θ, ϕ), etc.

The most important PDEs are:

(1) **The wave equation**

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

This equation describes as a function of position and time the displacement from equilibrium, $u(\vec{r}, t)$, of a vibrating string or membrane, or a vibrating solid, gas or liquid.

The equation also occurs in electromagnetism, where $u(\vec{r}, t)$ may be a component of the electric or magnetic field in an electromagnetic wave, or the current or voltage along a transmission line.

The quantity c is the speed of propagation of the waves.

(2) **The diffusion equation**

$$\nabla^2 u = \frac{1}{D}\frac{\partial u}{\partial t}$$

This equation describes the temperature $u(\vec{r}, t)$ in a region containing no heat sources or sinks.

It also applies to the diffusion of a chemical that has concentration $u(\vec{r}, t)$.

The constant D is called the diffusivity.

(3) **Laplace equation**

$$\nabla^2 u = 0$$

This equation is obtained by setting

$$\frac{\partial u}{\partial t} = 0$$

in the diffusion equation and describes (for example) the **steady-state** temperature of a solid in which there are no heat sources - i.e., the temperature after a long time has elapsed.

Laplace's equation also describes the gravitational potential in a region containing no matter, or the electrostatic potential in a charge-free region.

It also applies to the flow of an incompressible fluid with no sources, sinks or vortices - in this case $u(\vec{r}, t)$ is the velocity potential, from which the velocity is given by $\vec{v} = \nabla u$.

(4) **Poisson equation**

$$\nabla^2 u = \rho(\vec{r})$$

This equation describes the same physical situations as Laplace's equation, but in regions containing matter, charges, or sources of heat or fluid. The function $\rho(\vec{r})$ is called the source density, and in physical applications usually contains some multiplicative physical constants.

(5) **Schrödinger equation**

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\vec{r})u = i\hbar\frac{\partial u}{\partial t}$$

This equation describes the quantum mechanical wave function $u(\vec{r}, t)$ of a non-relativistic particle of mass m; $\hbar$ is Planck's constant divided by 2π. All of these equations are **linear**, They are all 2nd-order in the space variables and of 1st- or 2nd-order in time.

There are many general methods for studying and solving for general aspects/properties of PDEs and their solutions.

We will concentrate in this course on one solution method, namely, **separation of variables.**

Separation of Variables and Eigenfunction Expansions

Under certain circumstances the solution of a PDE may be written as a sum of terms, each of which is the product of functions of only one of the variable.

This is called solution by **separation of variables (SOV)**.

Let us illustrate the procedure by an example:

Consider a 1-dimensional wave equation describing the transverse vibrations of a string

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial x^2}$$

In the SOV method we simply look for a solution of the form

$$u(x,t) = X(x)T(t)$$

Direct substitution then gives

$$\frac{\partial^2}{\partial t^2}u(x,t) = T(t)\frac{d^2X(x)}{dx^2} = \frac{1}{c^2}X(x)\frac{d^2T(t)}{dt^2}$$

or

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}$$

It is clear that we have **separated the variables**.

Since the LHS is a function of x only and the RHS is a function of t only, both sides of this equation must be equal for all x and t.

The only way this is possible is for both of them to be equal to the same constant, say λ.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}$$

which gives

$$\frac{d^2X(x)}{dx^2} - \lambda X(x) = 0 \quad , \quad \frac{d^2T(t)}{dt^2} - \lambda c^2 T(t) = 0$$

These are two separated ODEs (single variable).

They are not completely independent of each other since the same separation constant λ must appear in both.

They are both eigenvalue/eigenfunction equations.

The general solutions of these equations are:

$$X(x) = A\cos\left(\sqrt{-\lambda}x\right) + B\sin\left(\sqrt{-\lambda}x\right)$$

$$T(t) = D\cos\left(\sqrt{-\lambda}ct\right) + E\sin\left(\sqrt{-\lambda}ct\right)$$

As a rule, **all possible values** of the separation constant λ are allowed unless **explicitly forbidden** by the physics of the system. i.e., certain values of λ can be forbidden when the corresponding solution $X(x)$, which depends on λ, does not have the correct properties.

The properties in question, are the **boundary conditions** imposed by the physics of the system.

It may happen that one or more of these boundary conditions can be satisfied only when the separation constant takes on a set of special values. This set then contains the only permissible values, or **eigenvalues**, for the problem.

The corresponding solutions are called **eigenfunctions**.

Dealing with the Laplacian

To deal with other equations we must work with the full Laplacian operator. The first step is always to separate out the t dependence using the SOV procedure.

So we always assume

$$u(\vec{r}, t) = u(\vec{r})\, T(t)$$

Most of the PDEs of interest then reduce to the form

$$(\nabla^2 + k^2) u(\vec{r}) = 0$$

where

$$k^2 = 0 \quad \text{for the Laplace equation}$$

$$k^2 > 0 \quad \text{for the helmholtz equation}$$

$$k^2 < 0 \quad \text{for the diffusion equation}$$

The Schrödinger equation is an exception to the rule because of the appearance of an extra term due to the potential function

$$\frac{2m}{\hbar^2}V(\vec{r})$$

The Laplacian in Different Coordinate Systems

Cartesian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical Polar:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical Polar:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Cartesian SOV Solution: assuming

$$u(\vec{r}) = X(x)y(y)Z(z)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0$$

We then choose

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m_y^2 \rightarrow \frac{d^2Y}{dy^2} + m_y^2 Y = 0$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -m_z^2 \rightarrow \frac{d^2Z}{dz^2} + m_z^2 Z = 0$$

which gives

$$\frac{d^2X}{dx^2} + (k^2 - k_y^2 - k_z^2)X = 0 = \frac{d^2X}{dx^2} + m_x^2 X$$

so that we end up with **three separate ODEs**.

Cylindrical Polar SOV Solution: assuming

$$u(\vec{r}) = R(\rho)\Phi(\phi)Z(z)$$

$$\frac{1}{\rho R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \frac{1}{\rho^2 \Phi}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z}\frac{\partial^2}{\partial z^2} + k^2 = 0$$

We then choose

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \rightarrow \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{Z}\frac{d^2}{dz^2} = \alpha^2 \rightarrow \frac{d^2}{dz^2} - \alpha^2 Z = 0$$

which gives

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (\beta^2\rho^2 - m^2)R = 0 \quad , \quad \beta^2 = k^2 + \alpha^2$$

Again, we end up with **three separate ODEs**.

Spherical Polar SOV Solution: assuming

$$u(\vec{r}) = u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\sin\theta\,\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{r^2\sin\theta\,\Phi}\frac{\partial^2\Phi}{\partial\phi^2}+k^2=0$$

We then choose

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \rightarrow \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{\sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta = -c$$

$$\rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + c\Theta = 0$$

which gives

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{c}{r^2}R + k^2R = 0$$

Again, we end up with **three separate ODEs**.

Solutions

Cartesian

The three ODEs for X, Y, and Z are all solved either by trigonometric functions or exponentials.

Cylindrical Polar The Z and Φ equations are solved either by trigonometric functions or exponentials.

The R or **radial equation** is

$$\left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+(z^2-m^2)\right]J_m(z)=0 \quad , \quad z=\beta\rho$$

where $J_m(z)$ is the cylindrical Bessel function of order m.

It is the solution of this equation that behaves like

$$\left(\frac{z}{2}\right)^m$$

as $z \to 0$.

Spherical Polar

The Φ equation is solved either by trigonometric functions or exponentials.

The equation for θ is familiar.

Putting $x = \cos\theta$ we get

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2z\frac{d}{dx} - \left(c - \frac{m^2}{1-x^2}\right)\right] P(x) = 0$$

which is the Associated Legendre equation.

The ODE Legendre equation we studied is a special case of this equation with $m = 0$.

This equation can be solved by a power series using the standard Frobenius method.

The infinite Frobenius series is only convergent for $|x| < 1$.

The solution at $x = 1$ is infinite(not good) unless

$$c = \ell(\ell+1) \quad , \quad \ell = 0, 1, 2, 3, ...$$

With this choice of c the Frobenius infinite series terminates as a polynomial, which is called the associated Legendre polynomial $P_\ell^m(x)$.

Since physical solutions must be bounded, we choose always to terminate the series and get associated Legendre polynomials as the solutions with ℓ non-negative.

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2z\frac{d}{dx} - \ell(\ell+1) + \frac{m^2}{1-x^2})\right] P_\ell^m(x) = 0$$

The r or radial equation is then

$$\left[z^2\frac{d^2}{dz^2} + 2z\frac{d}{dz} + (z^2 - \ell(\ell+1))\right] j_\ell(z) = 0$$

where $j_\ell(z)$ is the spherical Bessel function of order ℓ and

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+1/2}(z)$$

These Bessel functions are among the most useful of the so-called **"higher transcendental functions"**.

A transcendental function is one that is not **"algebraic"**.

An algebraic function of x is one that can be generated from x by a finite number of algebraic operations like $+, -, x, /$.

For example, a polynomial is algebraic but $\log(x)$ is not.

The elementary transcendental functions are exp, log, sin, cos, sinh, cosh, ...

Steady State Temperature in a Sphere

We have a sphere of radius $r = 1$.

The upper hemisphere is kept at $T = 100$ and the lower hemisphere is kept at $T = 0$.

We now know the process so we will just write down all of the steps.

Equations:

$$\nabla^2 T = 0$$

$$T(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\sin\theta\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

We choose

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2 \rightarrow \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{\sin\theta\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = -k$$

$$\rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + k\Theta = 0$$

which gives

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - kR = 0$$

Solutions:

First,

$$\Phi(\phi) = a\sin m\phi + b\cos m\phi$$

where $m =$ integer so that

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

We then choose $k = \ell(\ell + 1)$ to get convergent polynomials, i.e.,

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

We neglect the Legendre functions of the second kind because they are singular at $\cos\theta = 1$.

Finally, we have

$$R(r) = cr^\ell + dr^{-(\ell+1)}$$

We must, however, choose $d = 0$ or the solution diverges at $r = 0$.

General Solution

$$T(r, \theta, \phi) = r^\ell P_\ell^m(\cos\theta)(a \sin m\phi + b \cos m\phi)$$

We have boundary condition that are independent of ϕ.

Therefore we can have $m = 0$ only and our general solution becomes

$$T(r, \theta, \phi) = \sum_{\ell=0}^{\infty} b_\ell r^\ell P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta) =$ Legendre polynomial.

We determine the coefficients by using the boundary condition

$$T(1, \theta, \phi) = \sum_{\ell=0}^{\infty} b_\ell P_\ell(\cos\theta) = f(\theta) = \begin{cases} 100 & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 \le \theta \le \pi \end{cases}$$

This is always possible to do because the Legendre functions are a orthonormal basis on the interval $[0, 1]$.

We finally get

$$\int_{-1}^{1} dx P_n(x) \sum_{\ell=0}^{\infty} b_\ell P_\ell(\cos\theta) = \int_{-1}^{1} dx P_n(x) f(x) = 100 \int_{0}^{1} dx P_n(x)$$

$$\sum_{\ell=0}^{\infty} b_\ell \int_{-1}^{1} dx P_n(x) P_m(x) = 100 \int_{0}^{1} dx P_n(x)$$

$$\sum_{\ell=0}^{\infty} b_\ell \delta_{n\ell} \frac{2}{2n+1} = 100 \int_{0}^{1} dx P_n(x) = \frac{2}{2n+1} b_n \to b_n = \frac{50(2n+1)}{\int_0^1 dx P_n(x)}$$

Therefore,

$$T(r,\theta,\phi) = 50 \sum_{\ell=0}^{\infty} \frac{50(2\ell+1)}{\int_0^1 dx P_\ell(x)} r^\ell P_\ell(\cos\theta)$$

Now,

$$\int_0^1 dx P_0(x) = \int_0^1 dx = 1 \quad , \quad \int_0^1 dx P_1(x) = \int_0^1 x dx = \frac{1}{2}$$

$$\int_0^1 dx P_2(x) = \frac{1}{2}\int_0^1 dx(3x^2 - 1) = 0$$

$$\int_0^1 dx P_3(x) = \frac{1}{8} \quad , \quad \int_0^1 dx P_4(x) = 0 \quad , \quad \int_0^1 dx P_5(x) = \frac{1}{16}$$

so that we get the final answer

$$T(r,\theta,\phi)$$
$$= 100\left[\frac{1}{2}P_0(\cos\theta) + \frac{3}{4}P_1(\cos\theta) - \frac{7}{16}P_3(\cos\theta) + \frac{11}{32}P_5(\cos\theta) + ...\right]$$