Group Theory and Elementary Particles

Introduction to Group Theory

The theory of finite groups and continuous groups is a very useful tool for studying symmetry and invariance.

In order to use group theory we need to introduce some definitions and concepts.

A **group** G is defined as a set of objects or operations (called **elements**) that may be combined or **multiplied** to form a well-defined product and that satisfy the following four conditions.

If we label the elements $a, b, c, ...$, then the conditions are:

1. If a and b are any two elements, then the product ab is an element.
2. The defined multiplication is associative, $(ab)c = (a(bc)$
3. There is a unit element I, with $Ia = aI = a$ for all elements a.
4. Each element has an inverse $b = a^{-1}$, with $a^{-1} = a^{-1}a = I$ for all elements a.

In physics, these abstract conditions will take on direct physical meaning in terms of transformations of vectors and tensors.

As a very simple, but not trivial, example of a group, consider the set $\{I, a, b, c\}$ that combine according to the group multiplication table below.

	I	a	b	c
I	I	a	b	c
a	a	b	c	I
b	b	c	I	a
c	c	I	a	b

Clearly, the four conditions of the definition of group are satisfied.

The elements $\{I, a, b, c\}$ are abstract mathematical entities, completely unrestricted except for the above multiplication table.

A **representation** of the group is a set of particular objects $\{I, a, b, c\}$ that satisfy the multiplication table.

Some examples are:

$$\{I = 1, a = i, b = -1, c = -i\}$$

where the combination rules$=$ordinary multiplication.

This group representation is labelled C_4.

Since multiplication of the group elements is "commutative", i.e., $ab = ba$ for any pair of elements, the group is labelled "commutative or abelian".

This group is also a "cyclic" group, since all of the elements can be written as successive powers of one element. In this case,

$$\{I = i^4, a = i, b = i^2, c = i^3\}$$

Another representation is given by the successive 90° rotations in a plane.

Remember the matrix that represents a rotation through angle ϕ in 2-dimensions is given by

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

With $\phi = 0, \pi/2, \pi, 3\pi/2$, we have

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus, we have seen two explicit representations.

One with numbers(complex) using ordinary multiplication and the other with matrices using matrix multiplication.

There maybe a correspondence between the elements of two groups.

The correspondence can be one-to-one , two-to-one or, many-to-one.

If the correspondence satisfies the same group multiplication table, then it is said to be **homomorphic**.

If it is also one-to-one, then it is said to be **isomorphic**.

The two representations that we discussed above are one-to-one and preserve the multiplication table and hence, are **isomorphic**.

The representation of group elements by matrices is a very powerful technique and has been almost universally adopted among physicists.

It can be proven that all the elements of finite groups and continuous groups of the type important in physics can be represented by unitary matrices.

For unitary matrices we have

$$A^{\dagger} = A^{-1}$$

or, in words, the **complex conjugate/transposed** is equal to the **inverse**.

If there exists a transformation that will transform our original representation matrices into diagonal or block-diagonal form, for example,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{44} & A_{44} \end{pmatrix} \Rightarrow \begin{pmatrix} P_{11} & P_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} \end{pmatrix}$$

such that the smaller portions or submatrices are no longer coupled together, then the original representation is said to be **reducible**.

Equivalently, we have

$$R = SAS^{-1} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

which is a “similarity” transformation.

We write

$$R = P \oplus Q$$

and say that R has been **decomposed** into the representations P and Q.

The **irreducible** representations play a role in group theory that is roughly analogous to the unit vectors of vector analysis.

They are the simplest representations → all others may be built up from them.

If a group element x is transformed into another group element y by means of a similarity transformation involving a third group element g_i

$$g_i x g_i^{-1} = y$$

then y is said to be **conjugate** to x.

A **class** is a set of mutually conjugate group elements.

A class is generally not a group.

The **trace** of each group element (each matrix of our representation) is invariant under unitary transformations. We define

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii} = \mathrm{Trace}(A)$$

Then we have

$$
\begin{aligned}
\mathrm{Tr}(SAS^{-1}) &= \sum_{i=1}^{n}(SAS^{-1})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} s_{ik}a_{kj}s^{-1}_{ji} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}\sum_{i=1}^{n} s^{-1}_{ji}s_{ik} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}(S^{-1}S)_{jk} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}(I)_{jk} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}\delta_{jk} \\
&= \sum_{j=1}^{n} a_{jj} = \mathrm{Tr}(A)
\end{aligned}
$$

We now relabel the trace as the **character**.

Then we have that all members of a given class (in a given representation) have the same character.

This follows directly from the invariance of the trace under similarity or unitary transformations.

If a subset of the group elements (including the identity element I) satisfies the four group requirements (means that the subset is a group also), then we call the subset a **subgroup**.

Every group has two trivial subgroups: the identity element alone and the group itself.

The group C_4 we discussed earlier has a nontrivial subgroup comprised of the elements $\{I, b\}$.

The **order** of a group is equal to the number of group elements.

Consider a subgroup H with elements $\{h_i\}$ and a group element x not in H.

Then xh_i and h_ix are not in the subgroup H.

The sets generated by

$$xh_i \ , \ i = 1, 2, ... \ \text{ and } \ h_i x \ , \ 1 = 1, 2....$$

are called the left and right **cosets**, respectively, of the subgroup H with respect to x.

The coset of a subgroup has the same number of distinct elements as the subgroup.

This means that the original group G can be written as the sum of H and its cosets:

$$G = H + x_1 H + x_2 H + ...$$

Since H and all of its cosets have the same order(same number of elements), the order(number of elements) of any subgroup is a divisor of the order(number of elements) of the group.

The similarity transformation of a subgroup H by a fixed group element x **not** in H

$$xHx^{-1}$$

yields a subgroup.

If this subgroup is identical to H for all x, i.e., if

$$xHx^{-1} = H$$

then H is called an **invariant, normal or self-conjugate** subgroup.

In physics, groups usually appear as a set of operations that leave a system unchanged or invariant.

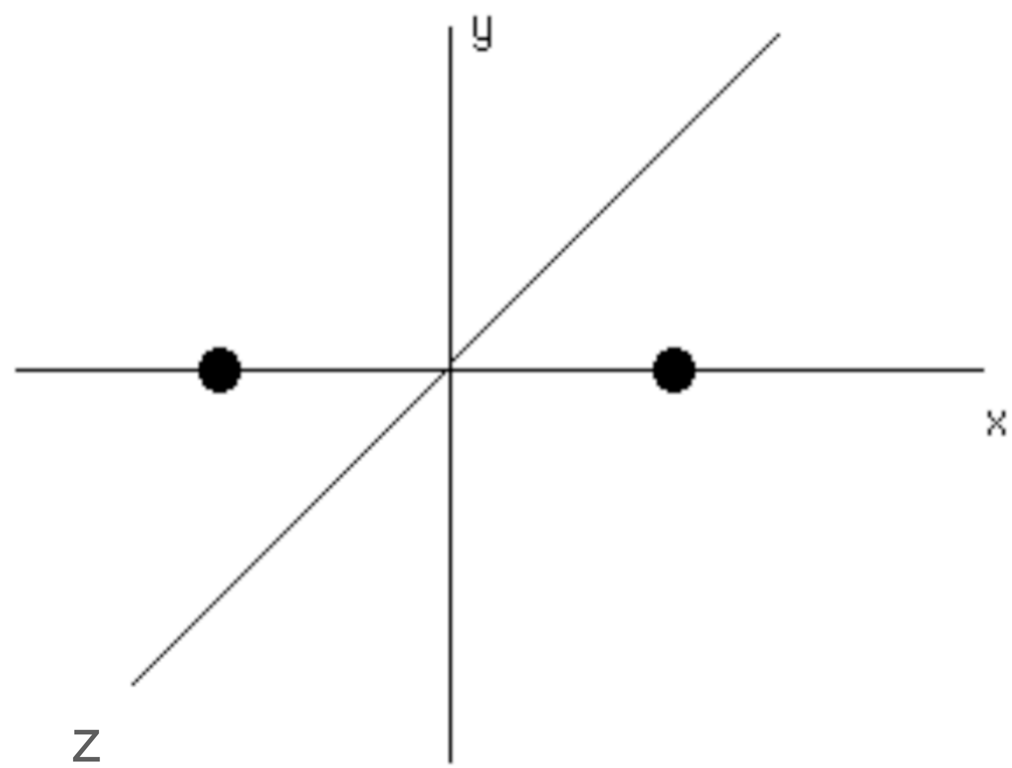
This is an expression of symmetry.

A symmetry is usually defined as the invariance of the Hamiltonian of a system under a group of transformations.

We now investigate the symmetry properties of sets of objects such as the atoms in a molecule or a crystal.

Two Objects - Twofold Symmetry Axis

Consider first the two-dimensional system of two identical atoms in the xy-plane located at $(1, 0)$ and $(-1, 0)$ as shown below.



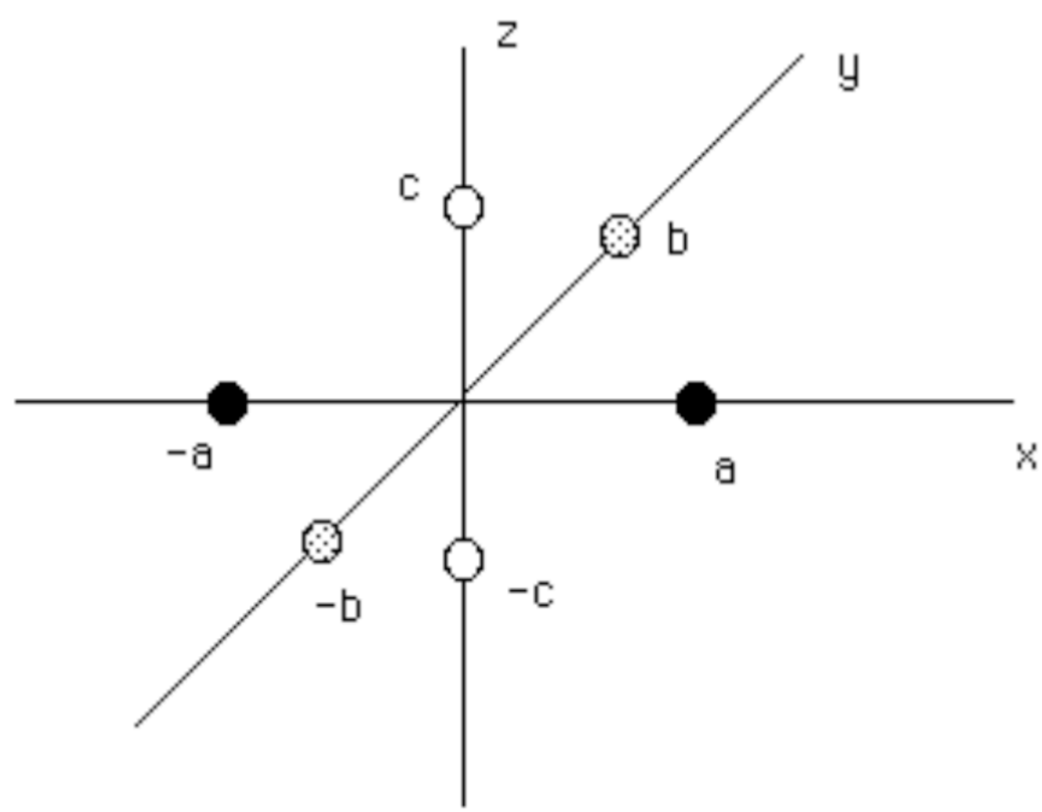
What rotations can be carried out (keeping both atoms in the xy-plane) that will leave the system invariant?

The only ones are I (always works) and a rotation by π about the z-axis.

So we have a rather uninteresting group of two elements.

The z-axis is labelled a **twofold** symmetry axis, corresponding to the two rotation angles 0 and π that leave the system invariant.

Our system becomes more interesting in 3-dimensions. Now imagine a molecule (or part of a crystal) with atoms of element X at $\pm a$ on the x-axis, atoms of element Y at $\pm b$ on the y-axis, and atoms of element Z at $\pm c$ on the z-axis as shown below:



Clearly, each axis is now a twofold symmetry axis.

The matrix representation of the corresponding rotations are:

$$R_x(\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_y(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R_z(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These four elements $\{I, R_x(\pi), R_y(\pi), R_z(\pi)\}$ form an abelian group with a group multiplication table:

	I	R_x	R_y	R_z
I	I	R_x	R_y	R_z
R_x	R_x	I	R_z	R_y
R_y	R_y	R_z	I	R_x
R_z	R_z	R_y	R_x	I

The products in this table can be obtained in either of two distinct ways:

(1) We may analyze the operations themselves - a rotation of π about the x-axis followed by a rotation of π about the y-axis is equivalent to a rotation of π about the z-axis

$$R_y(\pi)R_x(\pi) = R_z(\pi)$$

and so on.

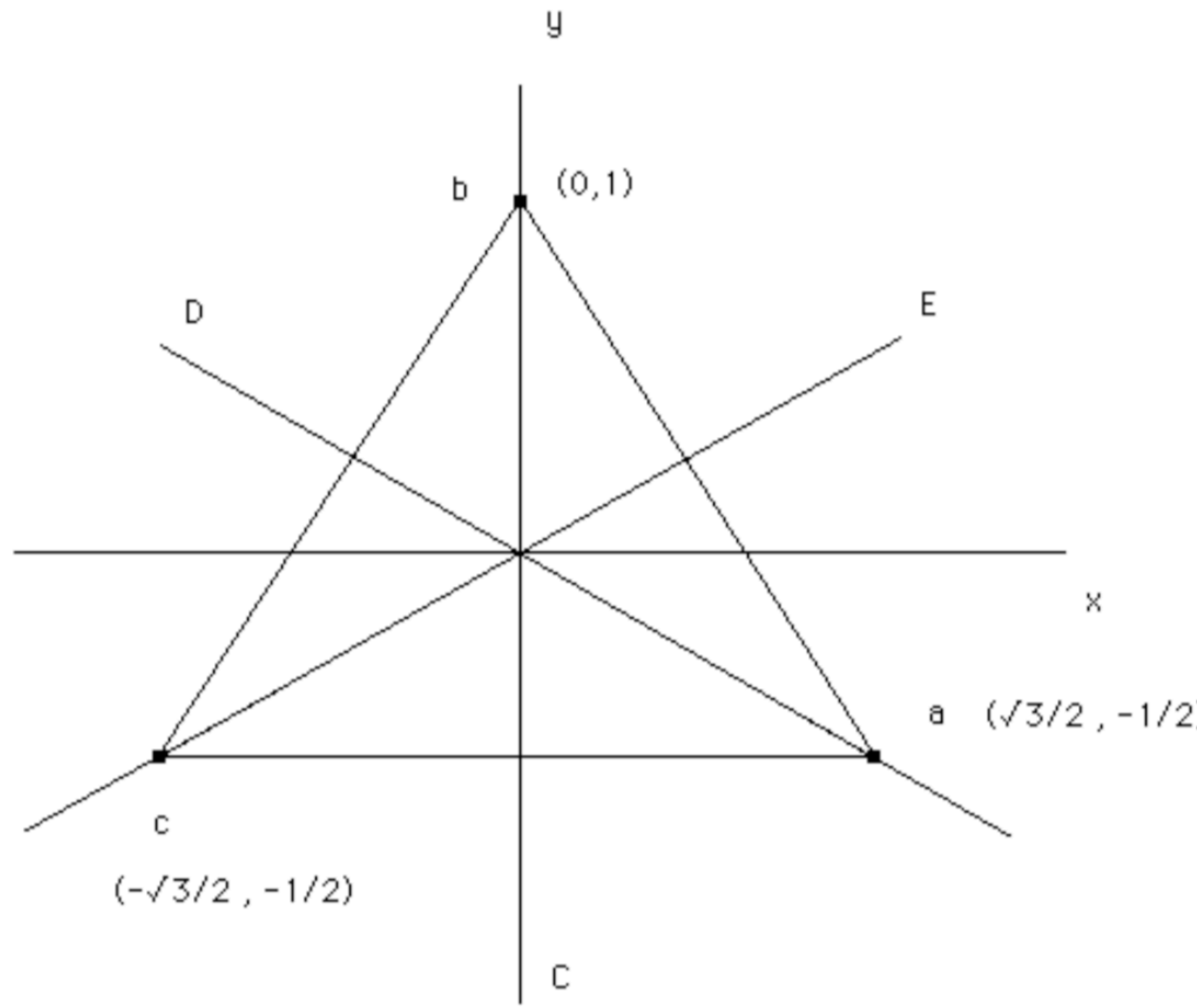
(2) Alternatively, once the matrix representation is established, we can obtain the products by direct matrix multiplication.

This latter method is especially important when the system is too complex for direct physical interpretation.

This symmetry group is often labelled D_2, where the D signifies a **dihedral** group and the subscript 2 signifies a twofold symmetry axis (and that no higher symmetry axis exists).

Three Objects – Threefold Symmetry Axis

Consider now three identical atoms at the vertices of an equilateral triangle as shown below:



Rotations (in this case we assume counterclockwise) of the triangle of $0, 2\pi/3, and 4\pi/3$ leave the triangle invariant.

In matrix form, we have

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = R_z(2\pi/3) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$B = R_z(4\pi/3) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

The z-axis is a threefold symmetry axis.

$\{I, A, B\}$ form a cyclic group, which is a subgroup of the complete 6-element group that we now discuss.

In the $x - y$ plane there are three additional axes of symmetry - each atom (vertex) and the origin define the twofold symmetry axes C, D, and E.

We note that twofold rotation axes can also be described via reflections through a plane containing the twofold axis.

In this case, we have for rotation of π about the C-axis.

$$C = R_C(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The rotation of π about the D-axis can be replaced by a rotation of $4\pi/3$ about the z-axis followed by a reflection in the $y - z$ plane $(x \to -x)$ or

$$D = R_D(\pi) = CB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

Similarly the rotation of π about the E-axis can be replaced by a rotation of $2\pi/3$ about the z-axis followed by a reflection in the $y - z$ plane $(x \to -x)$ or

$$E = R_E(\pi) = CA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

The complete group multiplication table is:

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	D	E	C
B	B	I	A	E	C	D
C	C	E	D	I	B	A
D	D	C	E	A	I	B
E	E	D	C	B	A	I

Notice that each element of the group appears only once in each row and in each column.It is clear from the multiplication table that the group is not abelian.

We have explicitly constructed a 6-element group and a 2×2 irreducible matrix representation of it.

The only other distinct 6-element group is the cyclic group $\{I, R, R^2, R^3, R^4, R^5\}$ with

$$R = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

The group $\{I, A \;\; , \;\; B, C, D, E\}$ is labeled D_3 in crystallography, the dihedral group with a threefold axis of symmetry.

The three axes (C, D, E) in the $x - y$ plane are twofold symmetry axes and, as a result, (I, C), (I, D), and (I, E) all form two-element subgroups.

None of these two-element subgroups are invariant.

There are two other irreducible representations of the symmetry group of the equilateral triangle.

They are ,

1. the trivial $\{1, 1, 1, 1, 1, 1\}$
2. the almost as trivial $\{1, 1, 1, -1, -1, -1\}$

Both of these representations are homomorphic to D_3.

A general and most important result for finite groups of h elements is that

$$\sum_i n_i^2 = h$$

where n_i is the dimension of the matrices of the i^{th} irreducible representation. equality is called the **dimensionality theorem** and is very useful in establishing the irreducible representations of a group. Here for D_3 we have

$$1^2 + 1^1 + 2^2 = 6$$

for our three representations.

This means that no other representations of the symmetry group of three objects exists.

We note from the examples above that for the transformations involving rotations and reflections, the transformations involving only pure rotations have determinant $= 1$ and the those involving a rotation and a reflection have determinant $= -1$.

Continuous Groups

The groups that we have been discussing all have a finite number of elements.

If we have a group element that contains one or more parameters that vary continuously over some range, then variation of the parameter will produce a continuum of group elements (an infinite number).

These groups are called continuous groups.

If continuous group has a rule for determining combination of elements or the transformation of an element or elements into a different element which is an analytic function of the parameters, where analytic means having derivatives of all orders, then it is called a **Lie** group.

For example, suppose we have the transformation rule

$$x_i' = f_i(x_1, x_2, x_3, \theta)$$

then for this transformation group to be a Lie group the functions f_i must be analytic functions of the parameter θ.

This analytic property will allow us to define infinitesimal transformations thereby reducing the study of the whole group to a study of the group elements when they are only slightly different than the identity element I.

Orthogonal Groups

We begin our study by looking at the orthogonal group O_3.

In particular, we consider the set of $n \times n$ real, orthogonal matrices with determinant $= +1$ (no reflections).

The defining property of a real orthogonal matrix is

$$A^T = A^{-1}$$

where T represents the **transpose** operation.

In terms of matrix elements this gives

$$\sum_i (A^T)_{ji} A_{ik} - \sum_i (A^{-1})_{ji} A_{ik} = I_{jk} - \delta_{jk} = 0$$

The ordinary rotation matrices are an example of real 3×3 orthogonal matrices:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices follow the convention that the rotation is a counterclockwise rotation of the coordinate system to a new orientation.

Each $n \times n$ real orthogonal matrix with determinant $= +1$ has $n(n-1)/2$ independent parameters.

For example, 2-dimensional rotations are described by 2×2 real orthogonal matrices and they need only one independent parameter $(2(2-1)/2 = 1)$, namely the rotation angle.

On the other hand, 3-dimensional rotations require three independent angles $(3(3-1)/2 = 3)$.

Let us see this explicitly for the 2×2 case: Suppose we have the general 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all the elements are real.

Imposing the condition determinant $= +1$ gives

$$ad - bc = 1$$

Imposing the condition that the transpose is the inverse gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These equations imply the solution

$$\begin{pmatrix} a & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a \end{pmatrix}$$

which shows that there is only one independent parameter.

Special Unitary Groups, $SU(2)$

The set of complex $n \times n$ unitary matrices also forms a group.

This group is labeled $U(n)$.

If we impose an additional restriction that the determinant of the matrices be $+1$, then we get the special unitary group, labeled $SU(n)$.

The defining relation for complex unitary matrices is

$$A^\dagger = A^{-1}$$

where $\dagger$ represents the complex-conjugate-transpose or adjoint operation.

In this case we have $n^2 - 1$ independent parameters.

This can be seen explicitly below: Suppose we have the general 2×2 matrix representing the transformation U

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all the elements are complex.

Imposing the condition determinant $= +1$ gives

$$ad - bc = 1$$

Imposing the condition that the adjoint is the inverse gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ a^*c + b^*d & |c|^2 + |d|^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These equations imply the solution

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where

$$|a|^2 + |b|^2 = aa^* + bb^* = 1$$

Since each complex number has two independent components, this shows that there are three $(2^2 - 1)$ independent parameters.

For $n = 3$, there will be eight $(3^2 - 1)$ independent parameters.

This will become the famous **eightfold way** of elementary particle physics(later).

An alternative general form is for the 2×2 case is

$$U(\xi, \eta, \zeta) = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix}$$

Now we will determine the irreducible representations of $SU(2)$.

Since the transformations are 2×2 matrices, the objects being transformed (the vectors or states) will be a two-component complex column vector (called a spinor):

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

or

$$u' = au + bv \quad , \quad v' = -b^* u + a^* v$$

From the form of this result, if we were to start with a homogeneous polynomial of the n^{th} degree in u and v and carry out the unitary transformation given by the last two equations, we would still have a homogeneous n^{th} degree polynomial.

To illustrate what happens we use the function (due to Wigner):

$$f_m(u,v) = \frac{u^{j+m}v^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

The index m will range from $-j$ to j, covering all terms of the form $u^p v^q$ with $p+q=2j$.

The denominator makes sure that our representation will be unitary.

Now we can show that

$$Uf_m(u,v) = f_m(u',v')$$

which becomes

$$Uf_m(u,v) = f_m(au+bv, -b^*u+a^*v') = \frac{(au+bv)^{j+m}(-b^*u+a^*v)^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

We now express this last equation as a linear combination of terms of the form $f_m(u, v)$.

The coefficients in the linear combination will give the matrix representation in the standard way.

We expand the two binomials using the binomial theorem:

$$(au + bv)^{j+m} = \sum_{k=0}^{j+m} \frac{(j+m)!}{k!(j+m-k)!} a^{j+m-k} u^{j+m-k} b^k v^k$$

$$(-b^* u + a^* v)^{j-m} = \sum_{n=0}^{j-m} (-1)^{j-m-n} \frac{(j-m)!}{n!(j-m-n)!} b^{*(j-m-n)} u^{j-m-n} a^{*n} v^n$$

Upon rearranging and changing some dummy indices we get

$$Uf_m(u, v) = f_m(u', v') = \sum_{m'=-j}^{j} U_{mm'} f_{m'}(u, v)$$

where the matrix element $U_{mm'}$ is given by

$$U_{mm'}$$
$$= \sum_{k=0}^{j+m} (-1)^{m'-m+k} \frac{\sqrt{((j+m)!)^2((j-m)!)^2}}{k!\sqrt{(j-m'-k)!(j+m-k)!(m'-m+k)!}} a^{j+m-k} a^{*(j-m'-k)} b^k b^{*(m'-m+k)}$$

The index k starts with zero and runs up to $j+m$, but the factorials in the denominator guarantee that the coefficient will vanish if any exponent goes negative (since $(-n)! = \pm\infty$ for $n = 1, 2, 3, \ldots$).

Since m and m' each range from $-j$ to j in unit steps, the matrices $U_{mm'}$ representing $SU(2)$ have dimensions $(2j+1) \times (2j+1)$.

If $j = 1/2$, then we get

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} U_{m=1/2,m'=1/2} & U_{m=1/2,m'=-1/2} \\ U_{m=-1/2,m'=1/2} & U_{m=-1/2,m'=-1/2} \end{pmatrix}$$

which is identical to the earlier equation.

$SU(2) - O_3$ Homomorphism (for the mathematically inclined)

The operation of $SU(2)$ on a matrix is given by a unitary transformation

$$M' = UMU^\dagger$$

Now any 2×2 matrix M can be written in terms of I and the three Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

that is, these four matrices span the 2×2 world; they are a basis.

Let M be a zero-trace matrix(then I does not contribute) so that in general

$$M = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

Since the trace is invariant under a unitary transformation, M' must have the same form

$$M = x'\sigma_1 + y'\sigma_2 + z'\sigma_3 = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix}$$

The determinant is also invariant under a unitary transformation.

Therefore

$$-(x^2 + y^2 + z^2) = -(x'^2 + y'^2 + z'^2)$$

or $x^2 + y^2 + z^2$ is invariant under this operation of $SU(2)$ (same as with O_3, the rotation group). $SU(2)$ must, therefore, describe some kind of rotation.

In fact, it is easy to check that the square of the length of a spinor (2-element column vector)

$$s^\dagger s = \begin{pmatrix} u^* & v^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = u * u + v^* v$$

is invariant under the transformation U.

This suggests that $SU(2)$ and O_3 may be isomorphic or homomorphic.

We can figure out what rotation $SU(2)$ describes by considering some special cases.

Consider

$$U_z(\alpha/2) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

Now we carry out the transformation of each of the three Pauli matrices.

For example,

$$U_z \sigma_1 U_z^\dagger = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}$$

or

$$U_z x\sigma_1 U_z^\dagger = (x\cos\alpha)\sigma_1 - (x\sin\alpha)\sigma_2$$

Similarly,

$$U_z x\sigma_2 U_z^\dagger = (y\sin\alpha)\sigma_1 + (y\cos\alpha)\sigma_2$$

$$U_z x\sigma_3 U_z^\dagger = (z)\sigma_3$$

Our earlier expression for M' then gives

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

$$z' = z$$

Thus, the 2×2 unitary transformation using $U_z(\alpha/2)$ is equivalent to the rotation operator $R_z(\alpha)$.

The same correspondence can be shown for

$$U_y(\beta/2) = \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix}$$

$$U_x(\phi/2) = \begin{pmatrix} \cos(\phi/2) & i\sin(\phi/2) \\ i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}$$

In general, we have

$$U_k(\gamma/2) = I\cos(\gamma/2) + i\sigma_k \sin(\gamma/2)$$

The correspondence

$$U_z(\alpha/2) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \Leftrightarrow R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha 0 & \\ 0 & 0 & 1 \end{pmatrix}$$

is not a simple one-to-one correspondence.

Specifically, as α in $R_z(\alpha)$ ranges from 0 to 2π the parameter in $U_z(\alpha/2)$, $\alpha/2$, goes from 0 to π.

We find

$$R_z(\alpha + 2\pi) = R_z(\alpha)$$

$$U_z(\alpha/ + \pi) = \begin{pmatrix} -e^{i\alpha/2} & 0 \\ 0 & -e^{-i\alpha/2} \end{pmatrix} = -U_z(\alpha/2)$$

Therefore both $U_z(\alpha/2)$ and $U_z(\alpha/ + \pi) = -U_z(\alpha/2)$ correspond to $R_z(\alpha)$.

The correspondence is 2 to 1, or $SU(2)$ and O_3 are homomorphic.

The establishment of the correspondence between representations of $SU(2)$ and those of O_3 means that the known representations of $SU(2)$ automatically provide us with representations of O_3.

Combining the various rotations, we find that a unitary transformation using

$$U(\alpha, \beta, \gamma) = U_z(\gamma/2) U_y(\beta/2) U_z(\alpha/2)$$

corresponds to the general Euler rotation $R_z(\alpha)R_y(\beta)R_z((\gamma)$ (studied Advanced Mechanics and Quantum Mechanics courses) .

By direct multiplication we get

$$U(\alpha, \beta, \gamma) = U_z(\gamma/2) U_y(\beta/2) U_z(\alpha/2)$$

From our alternative original definition of U, using

$$\xi = (\gamma + \alpha)/2 \ , \ \eta = \beta/2 \ , \ \psi = (\gamma - \alpha)/2$$

we identify the parameters a and b as

$$a = e^{i(\gamma+\alpha)/2} \cos(\beta/2) \ \ , \ \ b = e^{i(\gamma-\alpha)/2} \sin(\beta/2)$$

Finally the $SU(2)$ representation of $U_{mm'}$ becomes

$$U_{mm'}(\alpha,\beta,\gamma)$$
$$= \sum_{k=0}^{j+m} (-1)^k \frac{\sqrt{((j+m)!)^2((j-m)!)^2}}{k!\sqrt{(j-m'-k)!(j+m-k)!(m'-m+k)!}}$$
$$\times\, e^{im\gamma}(\cos(\beta/2)^{2j+m-m'-2k}(-\sin(\beta/2)^{m-m'+2k} e^{im'\alpha}$$

Here the irreducible representations are in terms of the Euler angles.

This result allows us to calculate the $(2j+1)\times(2j+1)$ irreducible representations of $SU(2)$ for all j $(j = 0, 1/2, 1, 3/2, ...)$ and the irreducible representations of $)_3$ for integral orbital angular momentum j $(j = 0, 1, 2, 3, ...)$ (in Quantum Mechanics course).

Generators

In all cases, rotations about a **common axis** combine as

$$R_z(\phi_2)R_z(\phi_1) = R_z(\phi_1 + \phi_2)$$

that is, multiplication of these matrices is equivalent to addition of the arguments.

This suggests that we look for an **exponential** representation of the rotations, i.e.,

$$\exp(\phi_2)\exp(\phi_1) = \exp(\phi_1 + \phi_2)$$

Suppose we define the exponential function of a matrix by the following relation:

$$U = e^{iaH} = I + iaH + \frac{(iaH)^2}{2!} + ...$$

where a is a real parameter independent of matrix(operator) H.

It is easy to prove that if H is Hermitian, then U is unitary and if U is unitary, the H is Hermitian.

In group theory, H is labeled a **generator**, the generator of U.

The following matrix describes a finite rotation of the coordinates through an angle ϕ counterclockwise about the z-axis:

$$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi 0 & \\ 0 & 0 & 1 \end{pmatrix}$$

Now letR_z be an infinitesimal rotation through an angle $\delta\phi$.

We can then write

$$R_z(\delta\phi) = I + i\delta\phi M_z$$

where

$$M_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, we find

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad M_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

A finite rotation ϕ can be compounded out of successive infinitesimal rotations $\delta\phi$

$$R_z(\delta\phi_2 + \delta\phi_1) = (I + i\delta\phi_2 M_z)(I + i\delta\phi_1 M_z)$$

Letting $\delta\phi = \phi/N$ for N rotations, with $N \to \infty$, we get

$$R_z(\phi) = \lim_{N\to\infty} (I + (i\phi/N)M_z))^N = \exp(i\phi M_z)$$

This implies that M_z is the generator of the group $R_z(\phi)$, a subgroup of O_3.

Before proving this result, we note:

(a) M_z is Hermitian and thus, $R_z(\phi)$ is unitary

(b) $\mathrm{Tr}(M_z) = 0$ and $\det(R_z(\phi)) = +1$

In direct analogy with M_z, M_x may be identified as the generator of R_x, the subgroup of rotations about the x-axis, and M_y as the generator of R_y .

$$R_z(\phi) = \exp(i\phi M_z) = I + i\phi M_z + \frac{(i\phi M_z)^2}{2!} + \frac{(i\phi M_z)^3}{3!} + \ldots\ldots$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ i\phi M_z + \frac{(i\phi)^2}{2!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{(i\phi)^3}{3!} M_z + \ldots\ldots$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (1 - \frac{\phi^2}{2!} + ...) + i(\phi - \frac{\phi^3}{3!} + ...) M_z + ...$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\phi + i \sin\phi M_z + ...$$

$$= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where we have used the relations

$$M_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad M_z^3 = M_z \quad , \quad , \quad \text{etc}$$

and we recognized the two series as $\cos\phi$ and $\sin\phi$.

Thus, we get the same matrix $R_z(\phi)$ as earlier.

Other relations we can write are:

1. all infinitesimal rotations commute

$$[R_i(\delta\phi_i), R_j(\delta\phi_j)] = 0$$

2. an infinitesimal rotation about an axis defined by unit vector $\hat{n}$ is

$$R(\delta\phi) = I + i\delta\phi\ \hat{n}M$$

3. the generators satisfy the commutation rules

$$[M_i, M_j] = \sum_k i\epsilon_{ijk} M_k$$

   This will turn out to be the commutation relations for angular momentum operators in quantum mechanics.

In a similar way the elements (U_1, U_2, U_3) of the two-dimensional unitary group, $SU(2)$, may be generated by

$$\exp\left(a\sigma_1/2\right) \quad , \quad \exp\left(a\sigma_2/2\right) \quad , \quad \exp\left(a\sigma_3/2\right)$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices and a, b, and c are real parameters.

We note that the σ's are Hermitian and have zero trace and thus, the elements of $SU(2)$ are unitary and have determinant $= +1$.

We also note that the generators in diagonal form such as σ_3 will lead to conserved quantum numbers.

Finally, if we define $s_i = \frac{1}{2}\sigma_i$ then we have

$$[s_i, s_j] = \sum_k i\epsilon_{ijk} s_k$$

which are the angular momentum commutation rules implying that the $s_i$are angular momentum operators.(Quantum mechanics course)

SU(3) and the Eightfold Way of Elementary Particles

The basic elements of a group are unitary operators (matrices of determinant $= +1$ or unimodular matrices) that transform the basis vectors $|q_i\rangle$ (we switch to Dirac notation which you learned in Quantum class for convenience) among themselves as

$$|q_j'\rangle = \sum_{i=1}^{n} u_{ji}\,|q_i\rangle$$

We **assume** that the **observed** elementary particle states or vectors, as linear combinations (or composites) of these basis vectors also transform into one another under the unitary operations of the group. It is most convenient to work with infinitesimal unitary transformations where

$$U = \exp\left(\frac{i}{2}\sum_j \epsilon_j \lambda_j\right) = I + \frac{i}{2}\sum_j \epsilon_j \lambda_j$$

where the ϵ_j are infinitesimal.

Now

$$\det(U) = 1 = \det\left(I + \frac{i}{2}\sum_j \epsilon_j \lambda_j\right) = \det(I) + \frac{i}{2}\sum_j \epsilon_j \mathrm{Tr}(\lambda_j) + O(\epsilon^2)$$

Therefore we have, in the general case,

$$\sum_j \epsilon_j \mathrm{Tr}(\lambda_j) = 0$$

We can satisfy this condition by choosing matrices λ_j such that

$$\mathrm{Tr}(\lambda_j) = 0$$

for all j.

For $n \times n$ matrices, there are $n^2 - 1$ linearly independent traceless matrices.

Since these matrices now have all of the properties of the generators of a group as discussed in the earlier parts of these notes, we will make that designation. The λ_j,$j = 1, 2, 3, ..., n^2 - 1$ are the generators of the group.

We need, therefore, only study the commutation relations among the generators in order to understand all of the group properties.

To see how this all works and to connect it to earlier ideas we will now use $SU(2)$ as an illustration.

As we stated earlier, the algebra is defined by the commutation relations among a certain group of operators(the generators).

In order to find

1. all the operators, and
2. the defining commutation relations of the algebra we start with the simplest physical realization of this algebra.

We assume the existence some entity with 2 states (we end up with the 2-dimensional representation) and represent it by a 2-element column vector, i.e.,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \text{amount of ``upness''} \\ \text{amount of ``downness''} \end{pmatrix}$$

The example we will use is a quantity called **isospin** (I_1, I_2, I_3) and the "upness" and "downness" will refer to the values of $I_3 = I_z$, i.e.,

$$|proton\rangle = |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |neutron\rangle = |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The operator (2×2 matrix) which converts a neutron into a proton (a **raising** operator) is given by

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where

$$\tau_+ \, |n\rangle = |p\rangle \quad , \quad \tau_= \, |p\rangle = 0$$

Similarly, its Hermitian adjoint (a **lowering** operator) is

$$\tau_- = \tau_+^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where

$$\tau_- \, |p\rangle = |n\rangle \quad , \quad \tau_- \, |n\rangle = 0$$

Note that τ_+ "annihilates" protons and τ_- "annihilates" neutrons.

Forming their commutator we find that

$$[\tau_+, \tau_-] = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\tau_3 \, |p\rangle = |p\rangle \quad , \quad \tau_3 \, |n\rangle = - \, |n\rangle$$

We also find that

$$[\tau_3, \tau_\pm] = \pm 2\tau_\pm$$

We then set

$$\mathsf{I} \rightarrow (I_1, I_2, I_3) \rightarrow \text{ Isospin}$$

$$\boldsymbol{\tau} \rightarrow (\tau_1, \tau_2, \tau_3)$$

and then define

$$\mathsf{I} = \frac{1}{2}\boldsymbol{\tau} \Rightarrow (I_1, I_2, I_3) = \frac{1}{2}(\tau_1, \tau_2, \tau_3)$$

so that the "upness" values are given by $\pm\frac{1}{2}$. i.e.,

$$I_3 \left| p \right\rangle = \frac{1}{2} \left| p \right\rangle \quad , \quad I_3 \left| n \right\rangle = -\frac{1}{2} \left| n \right\rangle$$

That completes the discussion of the 2-dimensional representation.

The same commutation relations are also used to derive higher order representations.

Note that these three commutators only involve the same three operators.

This means that these three commutators complete the commutation relations (all further commutators can be expressed in terms of these three operators or we have a **closed commutator algebra**).

The three operators τ_+, τ_-, τ_3 are the basic set for $SU(2)$.

An equivalent set is

$$\tau_1 = i(\tau_+ - \tau_-) \ , \ \tau_2 = i(\tau_- + \tau_+) \ , \ \tau_3$$

This also correlates with the earlier statements that we would have $n^2 - 1 = 3$ generators for $SU(2)$.

To obtain the other representations of the $SU(2)$ algebra, we must only use these defining commutation relations.

It is clear from the commutators that only one of the Hermitian operators τ_1, τ_2, τ_3 can be diagonalized in a given representation of the space(using a given set of basis vectors).

We will choose a representation in which τ_3 is diagonal (this was the choice in the 2-dimensional representation above also).

Remember that these diagonal elements are the eigenvalues of τ_3.

We label the state by the eigenvalue m of τ_3, i.e.,

$$\tau_3 \left|m\right\rangle = m \left|m\right\rangle$$

Now let us use the commutation relations.

$$\tau_3 \tau_+ \left|m\right\rangle = (\tau_+ \tau_3 + 2\tau_+) \left|m\right\rangle = (m+2)\tau_+ \left|m\right\rangle$$

This implies that

$$\tau_+ \left|m\right\rangle = c \left|m+2\right\rangle$$

In a similar manner, we can show that

$$\tau_- |m\rangle = c' |m-2\rangle$$

Now, if we start with any state in a given representation, we can generate states with higher eigenvalues, until we reach the state with maximum eigenvalue M (which we assume exists).

Since we cannot go any higher, this state must have the property that

$$\tau_+ |M\rangle = 0$$

This was the case for the 2-dimensional representation where

$$\tau_+ |p\rangle = 0$$

We now start with this highest weight state, which it turns out is unique for an irreducible representation.

We then have

$$\tau_- |M\rangle = \lambda_1 |M-2\rangle$$

which implies that (assuming λ_1 is real)

$$\langle M| \tau_+ = \lambda_1 \langle M-2|$$

You can make sense of this by thinking in terms of matrices.

As usual, we choose to normalize our basis states to 1, i.e., $\langle m \,|\, m\rangle = 1$ and we then obtain

$$\begin{aligned}
\langle M| \tau_+\tau_- |M\rangle &= \langle M-2| \lambda_1\lambda_1 |M-2\rangle = \lambda_1^2 \langle M-2 \,|\, M-2\rangle = \lambda_1^2 \\
&= \langle M| \left([\tau_+, \tau_-] + \tau_-\tau_+\right) |M\rangle \\
&= \langle M| [\tau_+, \tau_-] |M\rangle + \langle M| \tau_-\tau_+ |M\rangle \\
&= \langle M| \tau_3 |M\rangle = M
\end{aligned}$$

where we have used

$$\tau_+ |M\rangle = 0 \;\;,\;\; [\tau_+, \tau_-] = \tau_3 \;\;,\;\; \tau_3 |M\rangle = M |M\rangle$$

So we get

$$\lambda_1^2 = M$$

We also note that

$$\langle M-2|\,\tau_-\,|M\rangle = \lambda_1 = \langle M|\,\tau_+\,|M-2\rangle$$

which implies that

$$\tau_+\,|M-2\rangle = \lambda_1\,|M\rangle$$

Now consider

$$\tau_-\,|M-2\rangle = \lambda^2\,|M-4\rangle$$

which gives (same algebra as before)

$$\begin{aligned}
\lambda_2^2 &= \langle M-2|\,\tau_+\tau_-\,|M-2\rangle = \langle M-2|\,([\tau_+,\tau_-]+\tau_-\tau_+)\,|M-2\rangle \\
&= \langle M-2|\,[\tau_+,\tau_-]\,|M-2\rangle + \langle M-2|\,\tau_-\tau_+\,|M-2\rangle \\
&= \langle M-2|\,\tau_3\,|M-2\rangle + \langle M|\,\lambda_1^2\,|M\rangle = \lambda_1^2 + M - 2
\end{aligned}$$

In general, we can state..... if

$$\tau_- \left| M - 2(p-1) \right\rangle = \lambda_p \left| M - 2p \right\rangle$$

we get

$$\lambda_p^2 = \lambda_{p-1}^2 + M - 2(p-1)$$

Now $\lambda_0^2 = 0$ (from definition of maximum eigenvalue and also see below).

Thus, we have

$$\lambda_1^2 = \lambda_0^2 + 1(M - 1 + 1) = M \quad \text{as before!}$$
$$\lambda_2^2 = \lambda_1^2 + m - 2(2-1) = \lambda_1^2 + M - 2 = 2(M - 2 + 1)$$
$$\lambda_3^2 = \lambda_2^2 + M - 2(3-1) = 3(M - 3 + 1)$$

or generalizing

$$\lambda_p^2 = p(M - p + 1)$$

When $\lambda_p^2 = 0 \, (p \neq 0)$ we reach the minimum eigenvalue.

This gives

$$p = M + 1$$

which implies the minimum eigenvalue is $-M$, that is,

$$\tau_- \left| M - 2(p-1) \right\rangle = \tau_- \left| M - 2(M+1-1) \right\rangle$$

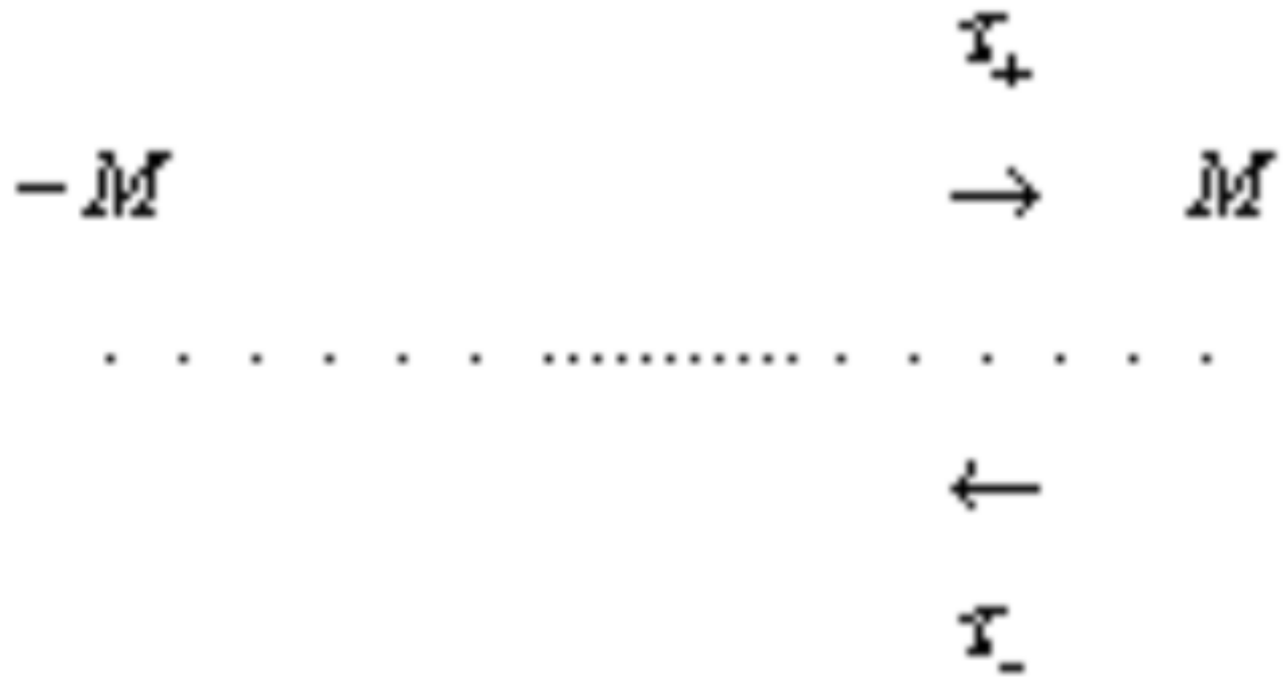
$$= \tau_- \left| -M \right\rangle = 0 \quad (\lambda_{M+1} = 0)$$

$$\rightarrow \tau_- \left| -M \right\rangle = 0$$

The total number of states contained in this set is

$$p = M + 1 = \quad \text{multiplicity}$$

Let us represent the states by a linear array



The operators $\tau_\pm$ represent steps between equally spaced points.

There is only **one** state for a given value of m and only **one** set of states (an irreducible representation) is generated from a given maximum state $|M\rangle$.

This result implies that there exists one independent operator that can be constructed from $\tau_\pm$ and τ_3 which commutes with $\tau_\pm$ and τ_3 (has simultaneous eigenstates with τ_3 and thus can simultaneously label the states) and which serves to distinguish representations.

We consider the operator (the square of the isospin)

$$\Gamma = \frac{1}{2}(\tau_+\tau_- + \tau_-\tau_+) + \frac{1}{4}\tau_3^2 = \frac{1}{4}(\tau_1^2 + \tau_2^2 + \tau_3^2) = \frac{1}{4}\tau^2$$

We have

$$[\Gamma, \tau_i] = 0$$

which implies that

$$\Gamma \left|m\right\rangle = c \left|m\right\rangle \quad \text{for all} \;\; m$$

or

$$\left\langle m\right| \Gamma \left|m\right\rangle = \left\langle M\right| \Gamma \left|M\right\rangle = c$$

Using the commutation relations we have

$$\Gamma = \frac{1}{4}\tau_3^2 + \frac{1}{2}\tau_3 + \tau_- \tau_+$$

which gives

$$c = \langle M | \, \Gamma \, | M \rangle = \frac{1}{4}M^2 + \frac{1}{2}M = \frac{M}{2}\left(\frac{M}{2} + 1\right)$$

If we let $t = M/2$ then we have $c = t(t+1)$.

This means that

$$p = M + 1 = 2t + 1 = \quad \text{multiplicity}$$

where $2t + 1$ is an integer.

We note the connection to ordinary Isospin I:

$$I_3 = \frac{1}{2}\tau_3 \quad , \quad I^2 = \Gamma = \frac{1}{4}\tau^2$$

Now to delineate the representations.

First, the 2-dimensional representation (so that we can see the general stuff give back the specific case that we started with).

This is the smallest or lowest representation.

We choose $M = 1$ corresponding to

$$t = 1 \quad \text{and } p = \quad \text{multiplicity } = 2$$

We are most interested in finding the basis states or the simultaneous eigenvectors of I^2 and I_3 since these will turn out to represent elementary particles.

We have (as before)

$$|proton\rangle = |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |neutron\rangle = |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$I = \frac{1}{2}$$

$$I_3|p\rangle = \frac{1}{2}|p\rangle \quad , \quad I_3|n\rangle = -\frac{1}{2}|n\rangle$$

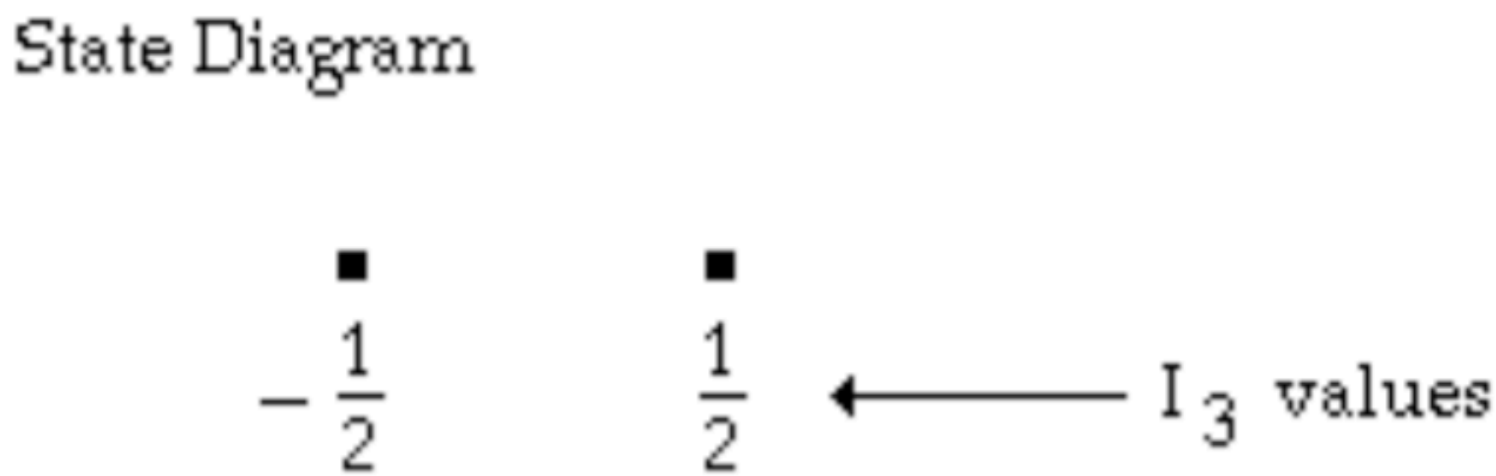
$$I^2|p\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)|p\rangle = \frac{3}{4}|p\rangle \quad , \quad I^2|n\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)|n\rangle = \frac{3}{4}|n\rangle$$

$$I_+|n\rangle = |p\rangle \quad , \quad I_-|p\rangle = |n\rangle$$

where we have used

$$I_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \;\; , \;\; I_+ = \begin{pmatrix} 0 & 1 \\ 0 & -0 \end{pmatrix} \;\; , \;\; I_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

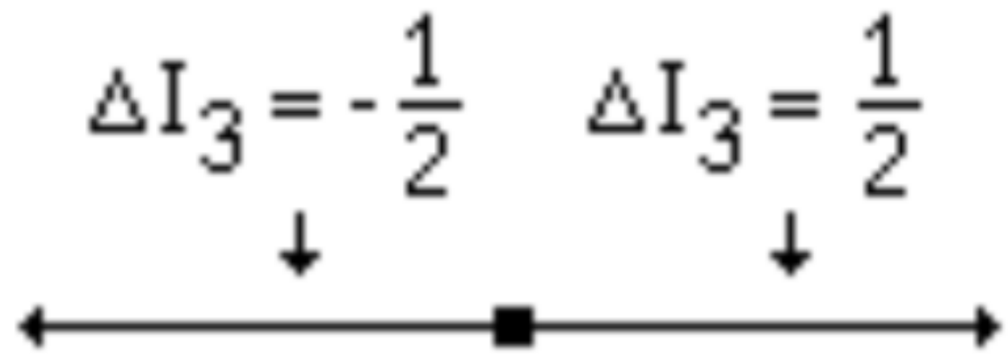
Therefore the state diagram for the $\frac{1}{2}$-representation (representations are labeled by I value) is given by



where

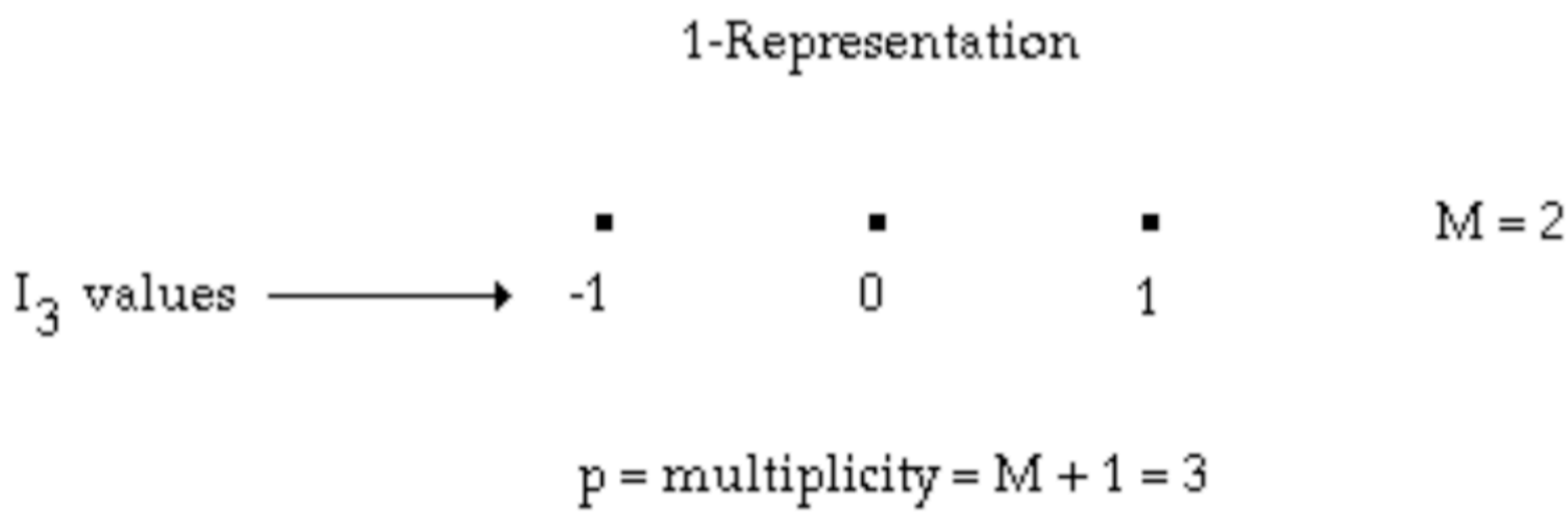
$$M = 2I_3(\text{maximum}) = 1 \quad , \quad p = \quad \text{multiplicity} \quad M + 1 = 2$$

We now define a weight diagram for SU(2) by the following symbol:



SU(2) Weight Diagram

Generating the 1-representation is just as easy and leads to a general graphical method for generating higher representations via the weight diagrams.



Generalizing, the I-representation ($M = 2I$) has a multiplicity $2I + 1$ or $2I + 1$ equally spaced points ($\Delta I_3 =$ constant) representing states in the representation.

We now consider the higher representation generated by combining two $\frac{1}{2}$-representations.

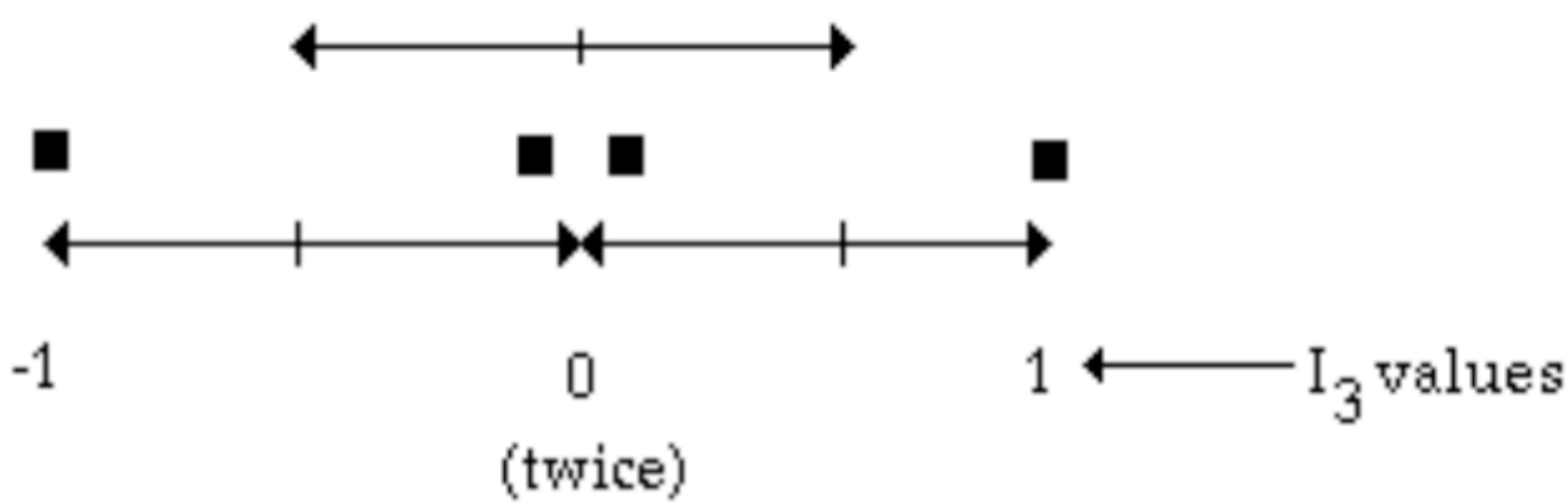
The formal name for this product is a **direct product** and it is written

$$\frac{1}{2} \otimes \frac{1}{2}$$

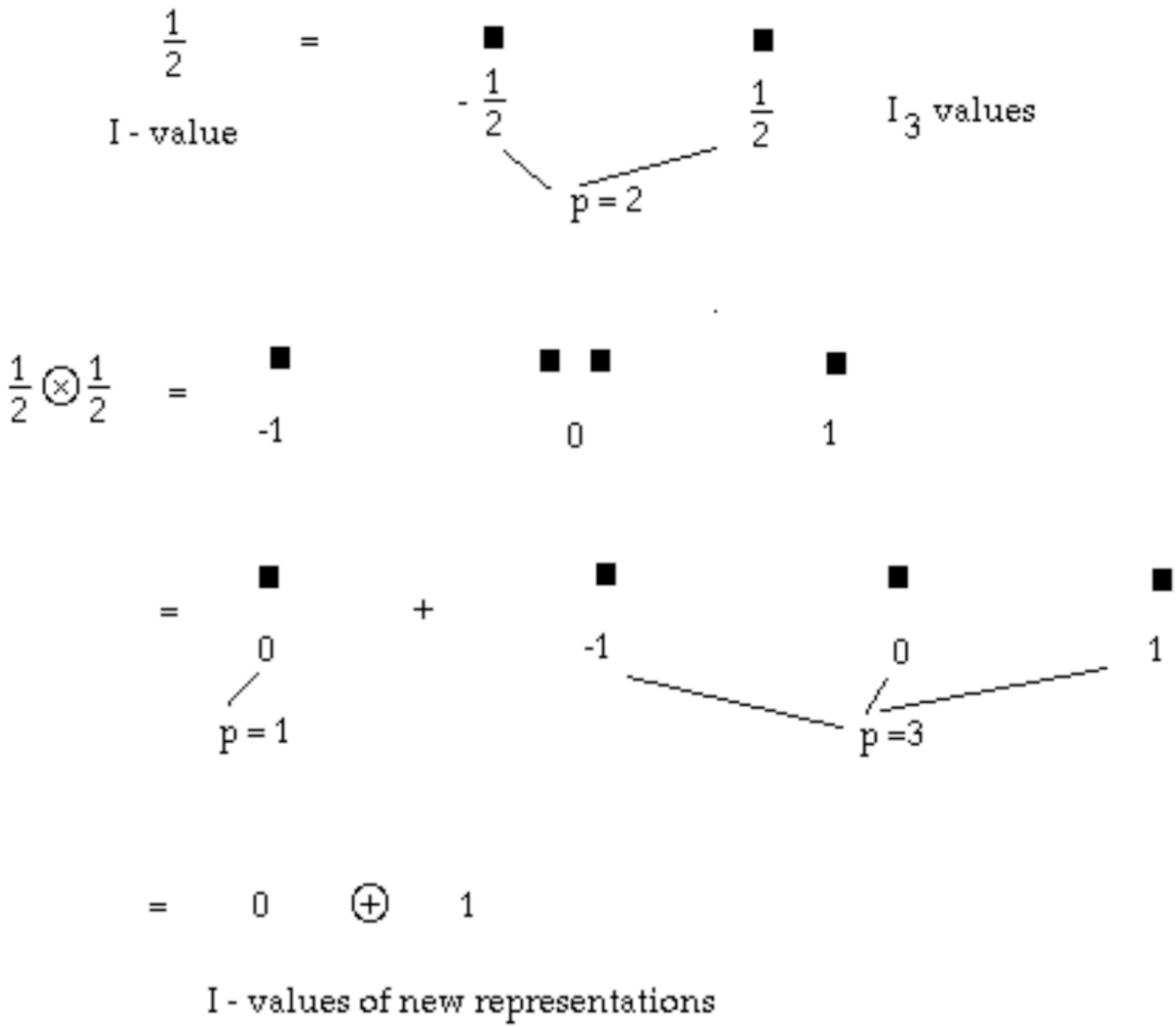
The **rules** for using the weight diagrams are as follows:

1. **draw** the first weight diagram
2. **superpose** a set of weight diagrams **centered** at the arrow tips of the first weight diagram
3. **allowed** states are at the final arrow tips

For the case we are considering this looks like:

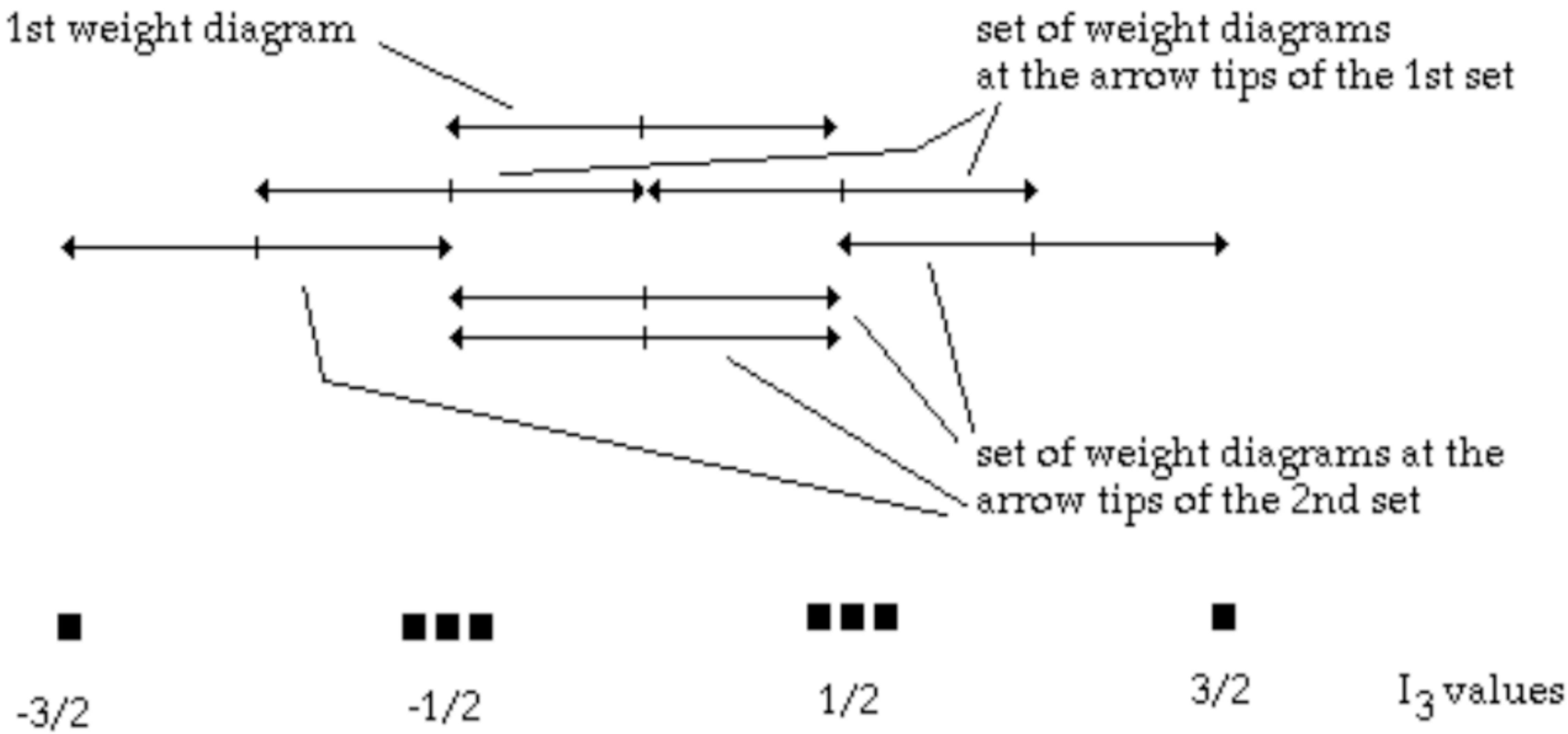


We observe that



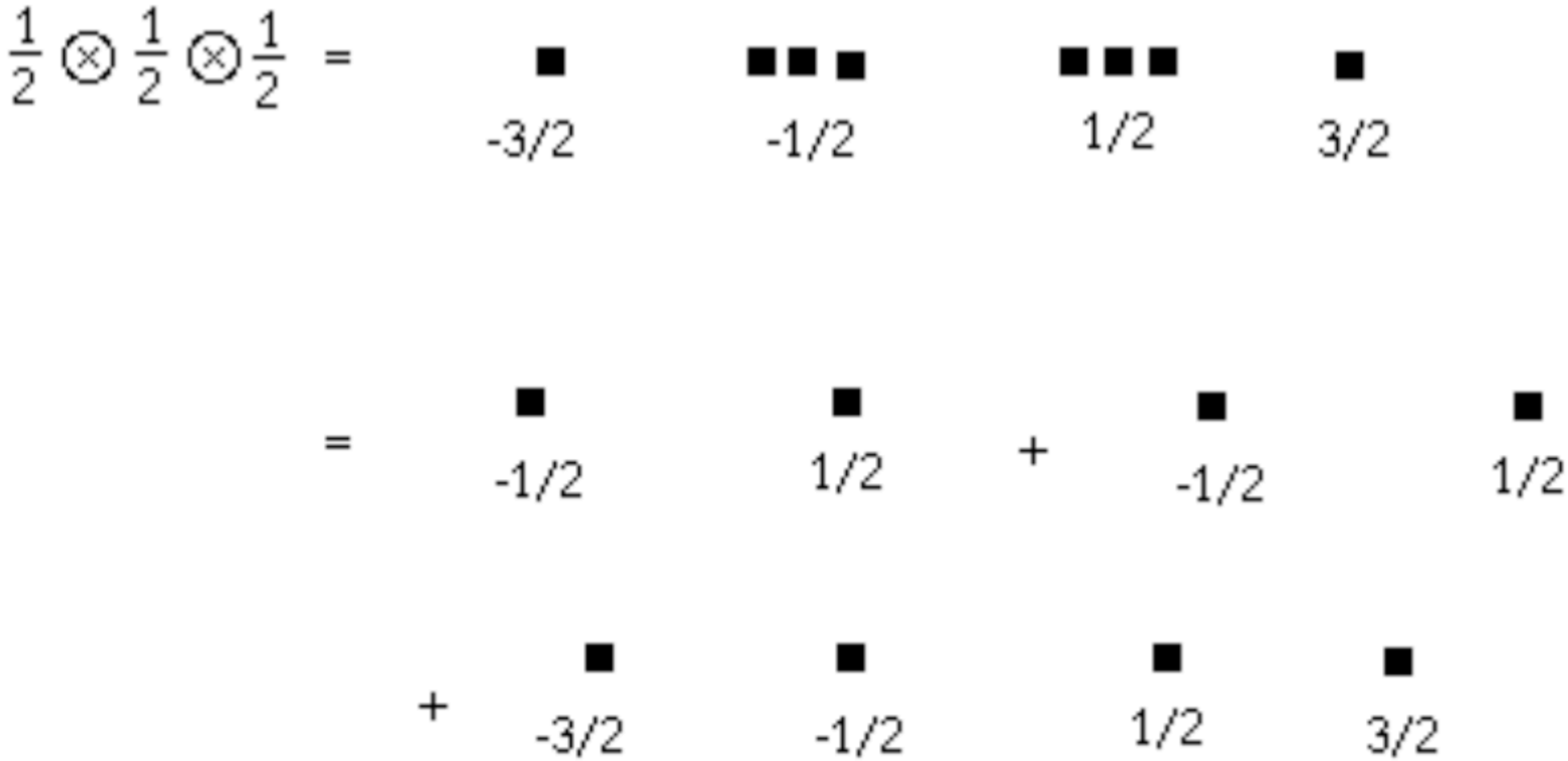
The superposed weight diagram picture looks like:

$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$



Note how multiple arrow tips require multiple weight diagrams in the next stage and multiple arrow tips at the end imply multiple states with that value of I_3.

Rearranging we have



which gives the result

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

Another way to think about it is to write

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = 0 \otimes \frac{1}{2} \oplus 1 \otimes \frac{1}{2}$$

$$0 \otimes \frac{1}{2} = \frac{1}{2} \quad , \quad 1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

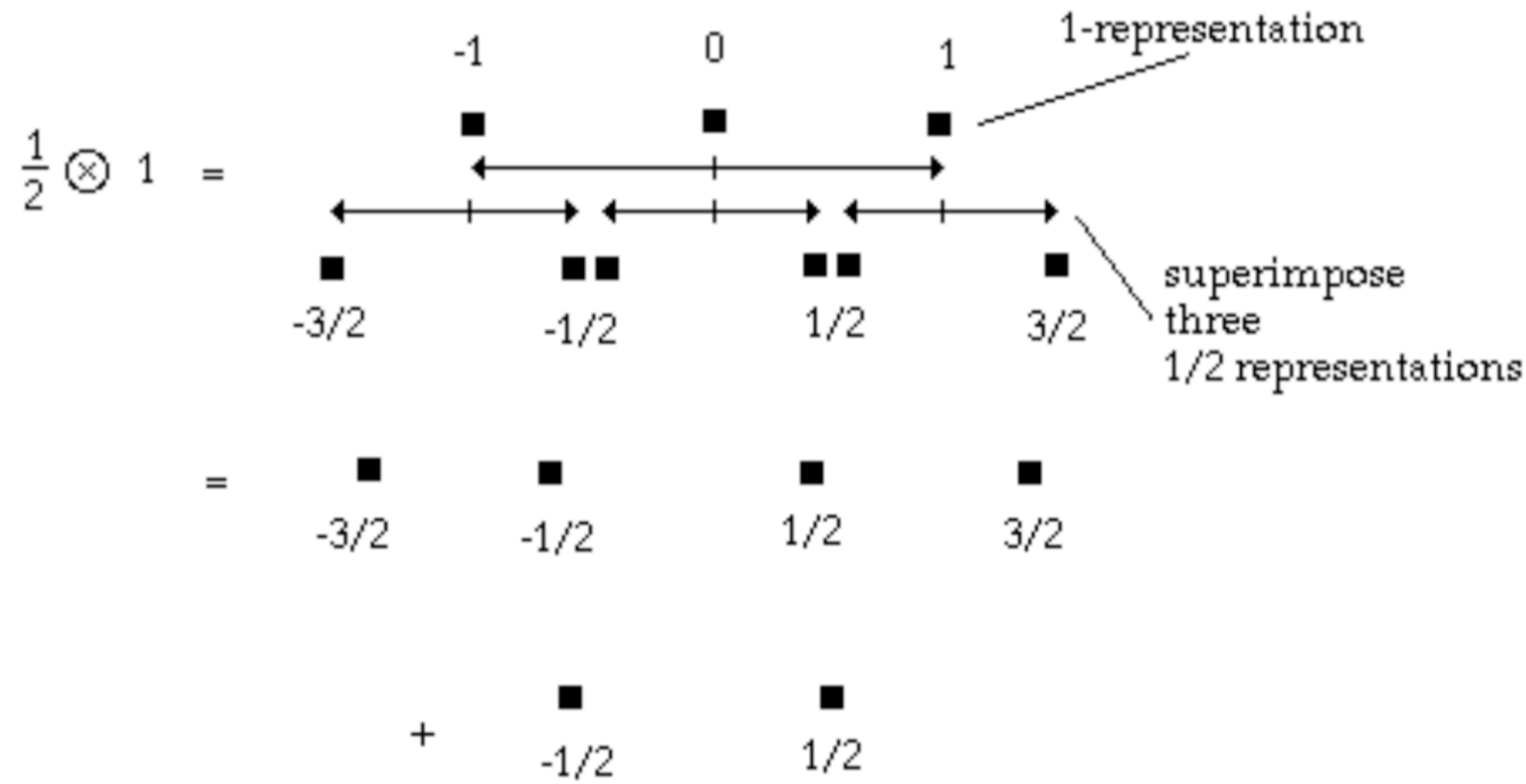
where we have used the fact that the 0-representation represents no additional states

$$0 \otimes \frac{1}{2} = \frac{1}{2}$$

and

$$1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$$

as can be seen from the weight diagrams below:



This will all reappear in a Quantum mechanics course because it is exactly the same as the angular momentum coupling rule (remember the homomorphism).

We have the general rule:

$$l_1 \otimes l_2 = |l_1 - l_2| \oplus |l_1 - l_2| + 1 \oplus \ldots\ldots \oplus l_1 + l_2$$

Now if $SU(2)$ were the correct group for describing the real world, then all possible states (all known particles) should be generated from the $\frac{1}{2}$-representation (most fundamental building block), i.e., from $|p\rangle$ and $|n\rangle$ as the fundamental building blocks.

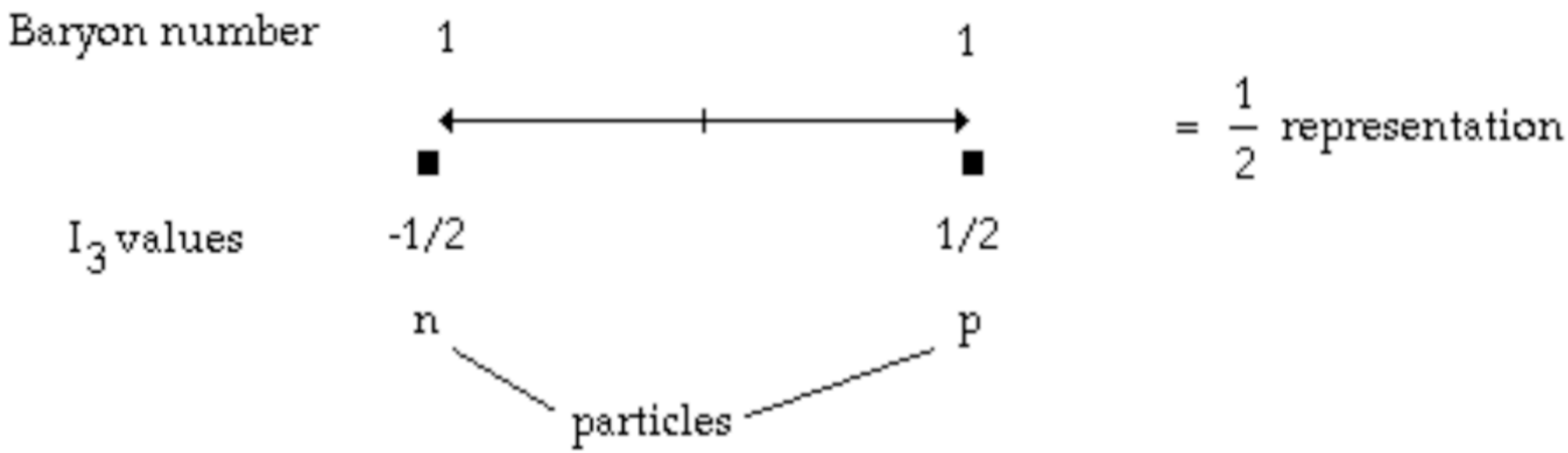
We know this is not possible, however, because:

1. we have not included any anti-baryons, which implies that we cannot have any states (particles) with baryon number $B = 0$.
2. we have not introduced strangeness (or hypercharge) so our states cannot represent the **real** world

Let us correct (1) first, within the context of $SU(2)$, and then attack (2), which will require that we look at $SU(3)$.

Let us now add particle labels to our weight diagrams.

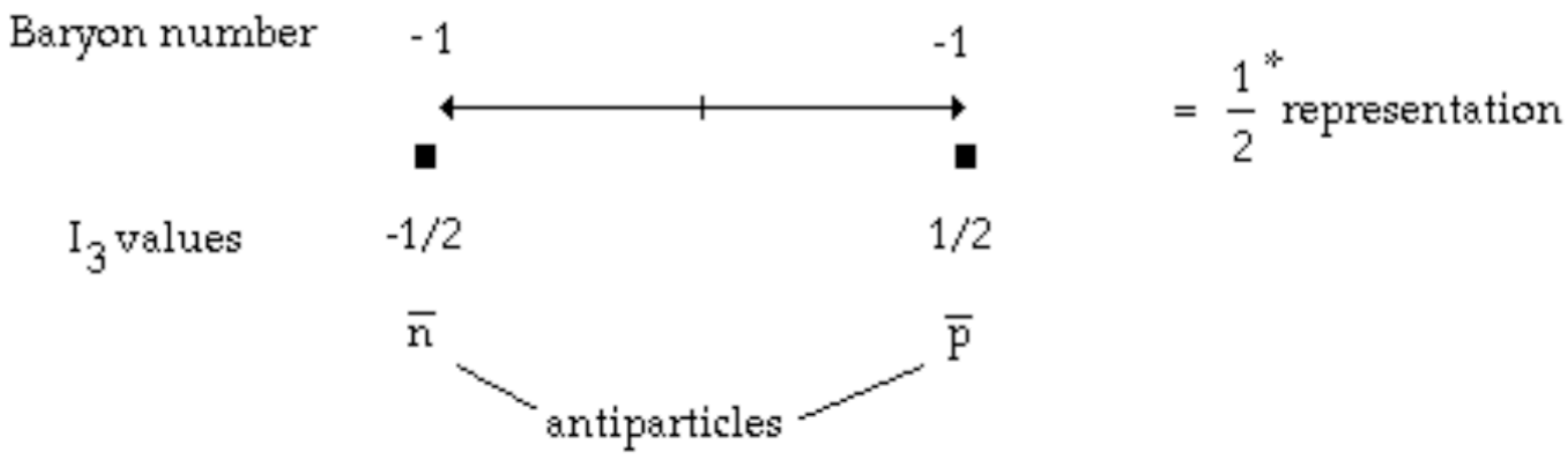
The weight diagram for the $\frac{1}{2}$-representation or as we now rename it, the **pn weight diagram** is given by



where we assumed Baryon number $= B = 1$ for each particle.

We also assume that Baryon number is additive.

We then define the $\frac{1}{2}^*$-representation by the weight diagram



where we assumed Baryon number $= B = -1$ for each antiparticle.

Now let us see the effect of this assumption on higher representations.

Remember

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

Now we would have

$\frac{1}{2} \otimes \frac{1}{2}$	$=$	$0 \oplus 1$	$=$	$\blacksquare$	$+$	$\blacksquare \quad \blacksquare \quad \blacksquare$	
		I_3	$=$	0		-1	0	1
		B	$=$	2		2	2	2
		state	$=$	$\lvert np-pn\rangle$		$\lvert nn\rangle$	$\lvert np+pn\rangle$	$\lvert pp\rangle$

Now consider

$$\frac{1}{2}^* \otimes \frac{1}{2} = 0 \oplus 1 = \blacksquare + \blacksquare \quad \blacksquare \quad \blacksquare$$

I_3	=	0	-1	0	1
B	=	0	0	0	0
state	=	$\lvert p\bar{p}+n\bar{n}\rangle$	$\lvert n\bar{p}\rangle$	$\lvert p\bar{p}-n\bar{n}\rangle$	$\lvert p\bar{n}\rangle$

The isospin structure is the same as

$$\frac{1}{2} \otimes \frac{1}{2}$$

but $B = 0$ (these could be mesons!).

The original identifications were a “triplet” of mesons (π^+, π^0, π^-) and a “singlet” meson η^0.

The mesons would somehow be bound states made from the baryon-antibaryon pairs! It was the correct idea but the wrong baryons!

In order to include hypercharge Y or strangeness S into our structure, we must start with three basic states (no identification with any known particles will be attempted at this point).

We choose:

$$|q_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |q_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |q_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Since we will be working with vectors containing three elements and our operators will be 3×3 matrices, there will be $3^2 - 1 = 8$ independent matrices representing the generators of the transformations (this is the group $SU(3)$).

These 8 operators will :

(1) represent I_3

(2) represent Y

where we will want $[I_3, Y] = 0$ so that our states can be simultaneous eigenstates of both operators and hence our particles can be characterized by values of both quantum numbers.

(3) represent $|q_1\rangle \rightarrow |q_2\rangle$

(4) represent $|q_1\rangle \rightarrow |q_3\rangle$

(5) represent $|q_2\rangle \rightarrow |q_1\rangle$

(6) represent $|q_2\rangle \rightarrow |q_3\rangle$

(7) represent $|q_3\rangle \rightarrow |q_1\rangle$

(8) represent $|q_3\rangle \rightarrow |q_2\rangle$

which correspond to raising and lowering operators. I will now write down(just straightforward algebra) the shift operators, the isospin and hypercharge operators and a whole mess of commutators and state equations.

After that we will discuss their meanings.

Shift Operators

$$E_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_{-1}^{+} \qquad E_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_{-2}^{+} \qquad E_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = E_{-3}^{+}$$

where

$$E_1|q_2\rangle = |q_1\rangle \quad , \quad E_2|q_3\rangle = |q_1\rangle \quad , \quad E_3|q_3\rangle = |q_2\rangle$$
$$E_{-1}|q_1\rangle = |q_2\rangle \quad , \quad E_{-2}|q_1\rangle = |q_3\rangle \quad , \quad E_{-3}|q_2\rangle = |q_3\rangle$$

Commutation Relations

$$[E_1, E_2] = [E_1, E_{-3}] = [E_2, E_3] = [E_{-1}, E_{-2}] = [E_{-1}, E_{-3}] = [E_{-2}, E_{-3}] = \mathbf{0}$$
$$[E_1, E_{-2}] = -\frac{1}{\sqrt{6}}E_{-3} \quad , \quad [E_1, E_3] = \frac{1}{\sqrt{6}}E_2 \quad , \quad [E_2, E_{-3}] = \frac{1}{\sqrt{6}}E_1$$
$$[E_{-1}, E_2] = \frac{1}{\sqrt{6}}E_3 \quad , \quad [E_{-1}, E_{-3}] = -\frac{1}{\sqrt{6}}E_{-2} \quad , \quad [E_{-2}, E_3] = -\frac{1}{\sqrt{6}}E_{-1}$$

and

$$[E_1, E_{-1}] = \frac{1}{\sqrt{3}}H_1 \quad , \quad [E_2, E_{-2}] = \frac{1}{2\sqrt{3}}H_1 + \frac{1}{2}H_2$$
$$[E_3, E_{-3}] = -\frac{1}{2\sqrt{3}}H_1 + \frac{1}{2}H_2$$

where

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad H_2 = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and finally

$$[H_1, E_1] = \frac{1}{\sqrt{3}} E_1 \quad , \quad [H_1, E_2] = \frac{1}{2\sqrt{3}} E_2 \quad , \quad [H_1, E_3] = -\frac{1}{2\sqrt{3}} E_3$$

$$[H_2, E_1] = 0 \quad , \quad [H_2, E_2] = \frac{1}{2} E_2 \quad , \quad [H_2, E_3] = \frac{1}{2} E_3 \quad , \quad [H_1, H_2] = 0$$

It is clear from the commutators that we have a closed algebra, i.e., all of the commutators among the 8 operators only involve the same 8 operators.

We note that $|q_1\rangle, |q_2\rangle, |q_3\rangle$ are eigenstates of both H_1 and H_2 simultaneously, which suggests that we will be able to use H_1 and H_2 to represent I_3 and Y.

Now we have

$$H_1 |q_1\rangle = \frac{1}{2\sqrt{3}} |q_1\rangle \ , \ H_1 |q_2\rangle = -\frac{1}{2\sqrt{3}} |q_2\rangle \ , \ H_1 |q_3\rangle = 0$$

$$H_2 |q_1\rangle = \frac{1}{6} |q_1\rangle \ , \ H_2 |q_2\rangle = \frac{1}{6} |q_2\rangle \ , \ H_2 |q_3\rangle = -\frac{1}{3} |q_3\rangle$$

Thus, if we define $I_3 = \sqrt{3} H_1$, $Y = 2H_2$ we get

	I_3	Y
q_1	1/2	1/3
q_2	-1/2	1/3
q_3	0	-2/3

Construction of Representations

We define eigenstates $|m_1, m_2\rangle$ where we assume

$$H_1 |m_1, m_2\rangle = m_1 |m_1, m_2\rangle \quad , \quad H_2 |m_1, m_2\rangle = m_2 |m_1, m_2\rangle$$

Now $\sqrt{6}E_1, \sqrt{6}E_{-1}, 2\sqrt{3}H_1$ obey the same commutation relations as τ_+, τ_-, τ_3 – in fact, they form a sub-algebra of the closed algebra of the 8 operators.

Therefore, we construct the operator

$$I^2 = 3(E_1 E_{-1} + E_{-1} E_1) + 3H_1^2$$

in analogy to

$$\Gamma = \frac{1}{2}(\tau_+\tau_- + \tau_-\tau_+) + \frac{1}{4}\tau_3^2$$

Since $[I^2, H_1] = [I^2, H_2] = 0$ the states $|m_1, m_2\rangle$ are also eigenstates of I^2 or

$$I^2 |m_1, m_2, i\rangle = i(i+1) |m_1, m_2, i\rangle$$

We note that

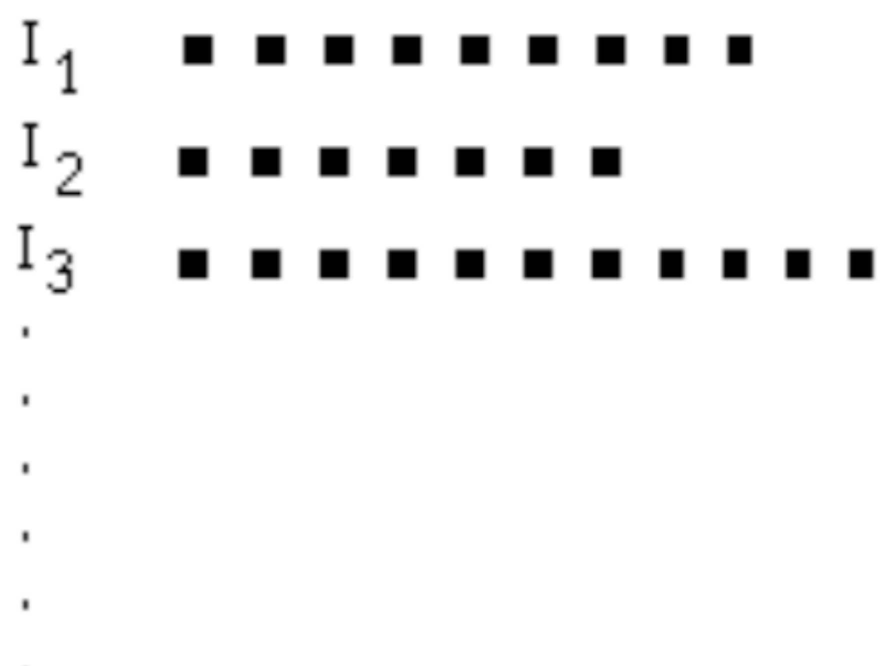
$$[I^2, E_{\pm 1}] = 0$$

but

$$[I^2, E_{\pm 2}] \neq 0 \ \ , \ \ [I^2, E_{\pm 3}] \neq 0$$

Which implies that I^2 is **not** an **invariant** operator or that **it can change** within a representation.

This suggests the following possible picture:



where each line is the same as in our earlier discussion ($I^2 =$ constant), but there are many different lines (values of I^2) in the representation.

It is not difficult to deduce the following table:

operator/effect	Δm_1	Δm_2
E_1	$1/\sqrt{3}$	0
E_{-1}	$-1/\sqrt{3}$	0
E_2	$1/2\sqrt{3}$	1/2
E_{-2}	$-1/2\sqrt{3}$	-1/2
E_3	$-1/2\sqrt{3}$	1/2
E_{-3}	$1/2\sqrt{3}$	-1/2

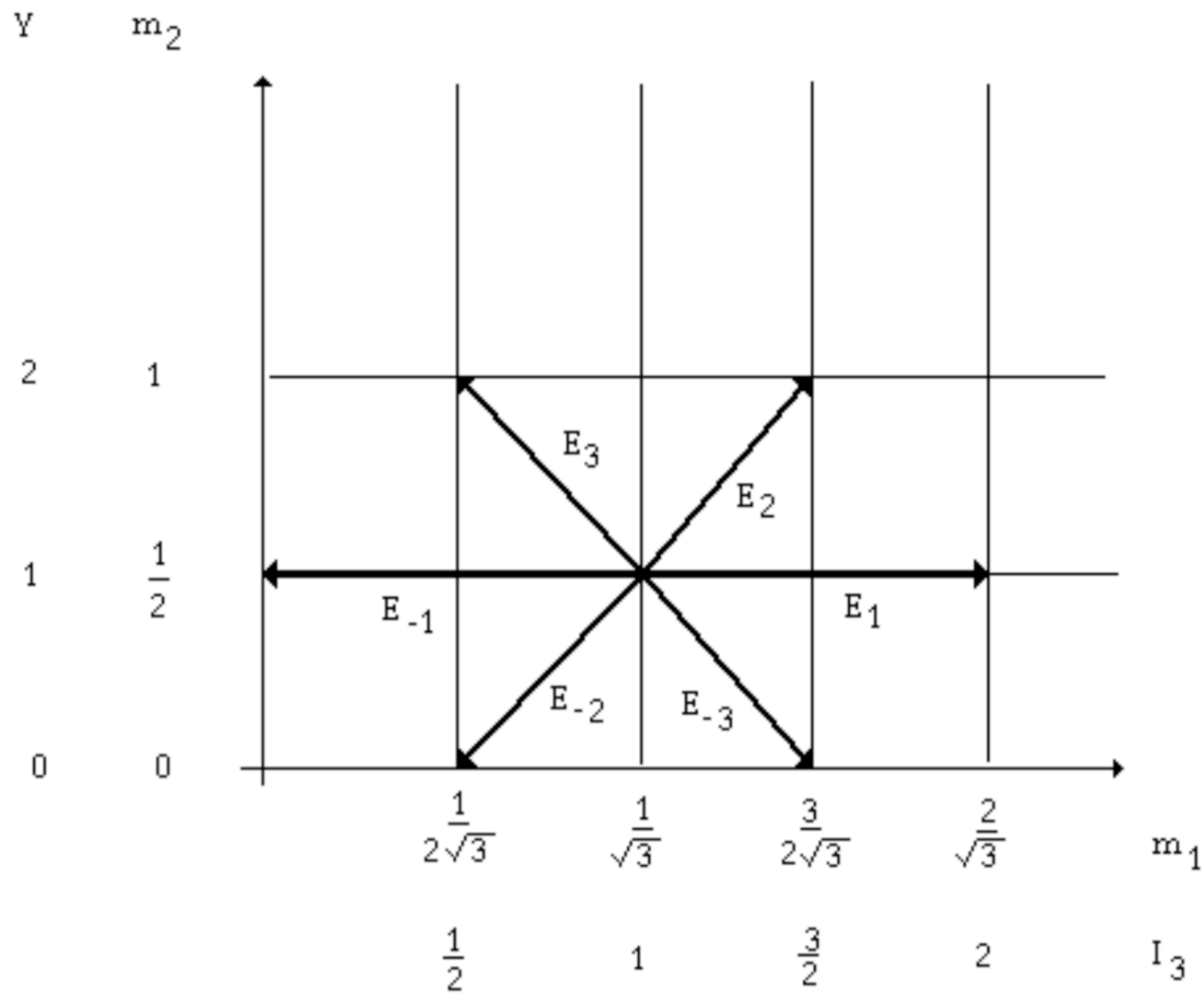
This table implies relation like:

$$H_1 E_1 \left|m_1, m_2\right\rangle = E_1 \left(H_1 + \frac{1}{\sqrt{3}}\right) \left|m_1, m_2\right\rangle = \left(m_1 + \frac{1}{\sqrt{3}}\right) E_1 \left|m_1, m_2\right\rangle$$

or

$$E_1 \left|m_1, m_2\right\rangle = \text{constant} \left|m_1 + \frac{1}{\sqrt{3}}, m_2\right\rangle$$

Graphically, this table is represented by the diagram below.



Let us now represent our states on a $Y - I_3$ diagram and indicate the effect of the operators

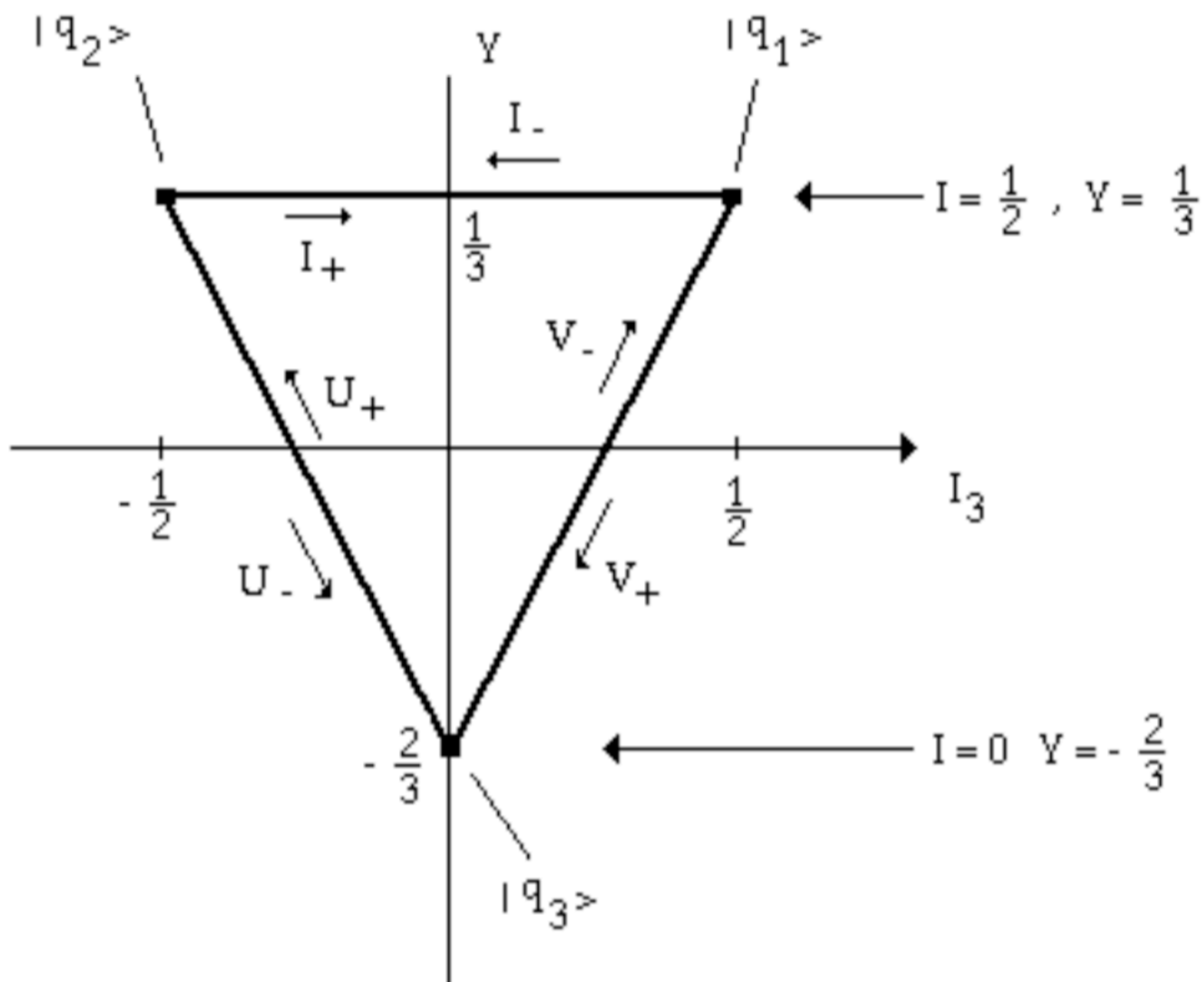
$$I_\pm = E_{\pm 1} \qquad (\Delta I_3 = \pm 1, \Delta Y = 0)$$

$$V_\pm = E_{\pm 2} \qquad (\Delta I_3 = \pm 1/2, \Delta Y = \pm 1)$$

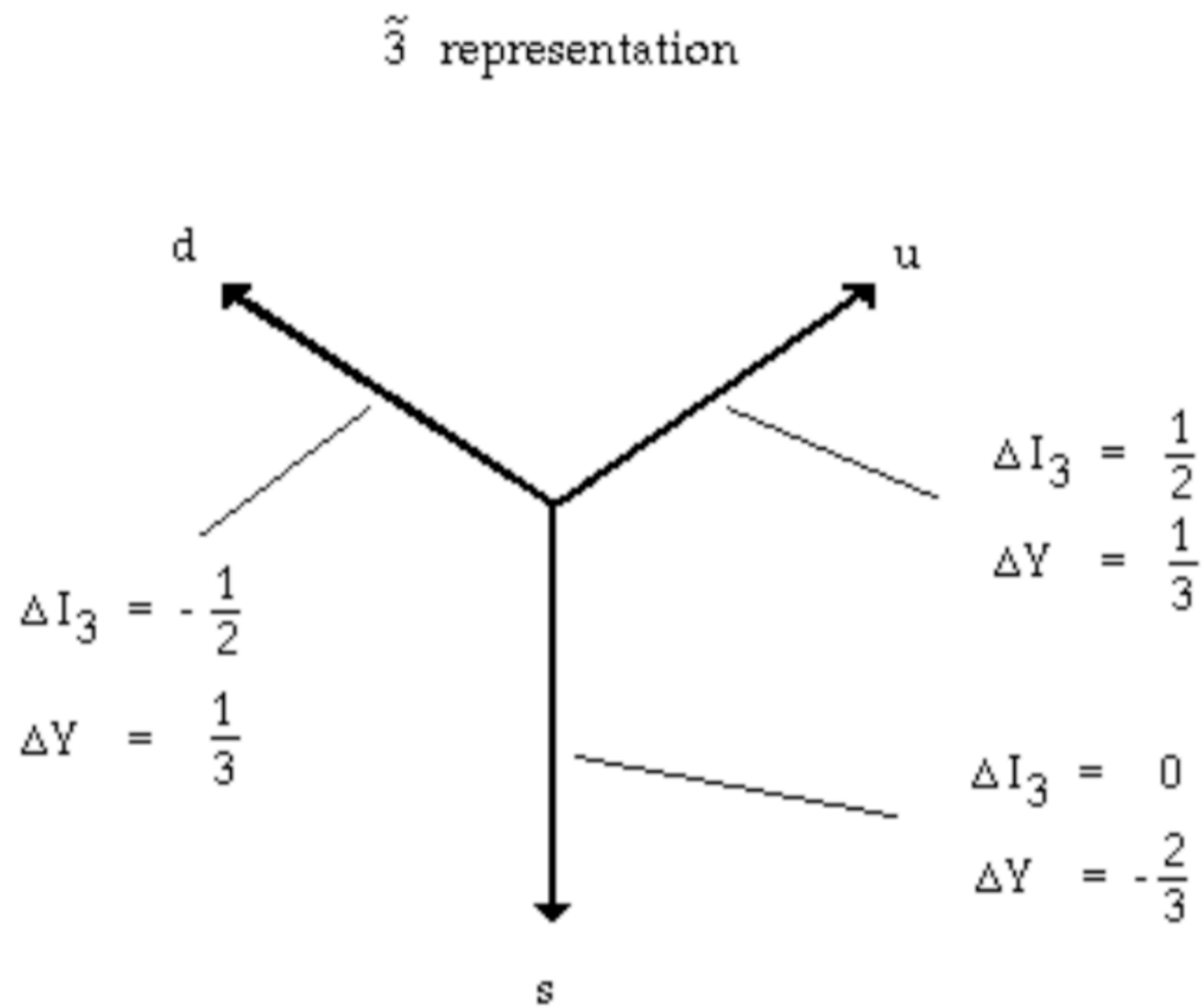
$$U_\pm = E_{\pm 3} \qquad (\Delta I_3 = \pm 1/2, \Delta Y = \pm 1)$$

Remember that $I_3 = \sqrt{3}H_1$ and $Y = 2H_2$.

This gives the simplest representation of SU(3):



and suggests a weight diagram as shown below:



We have other related quantum numbers:

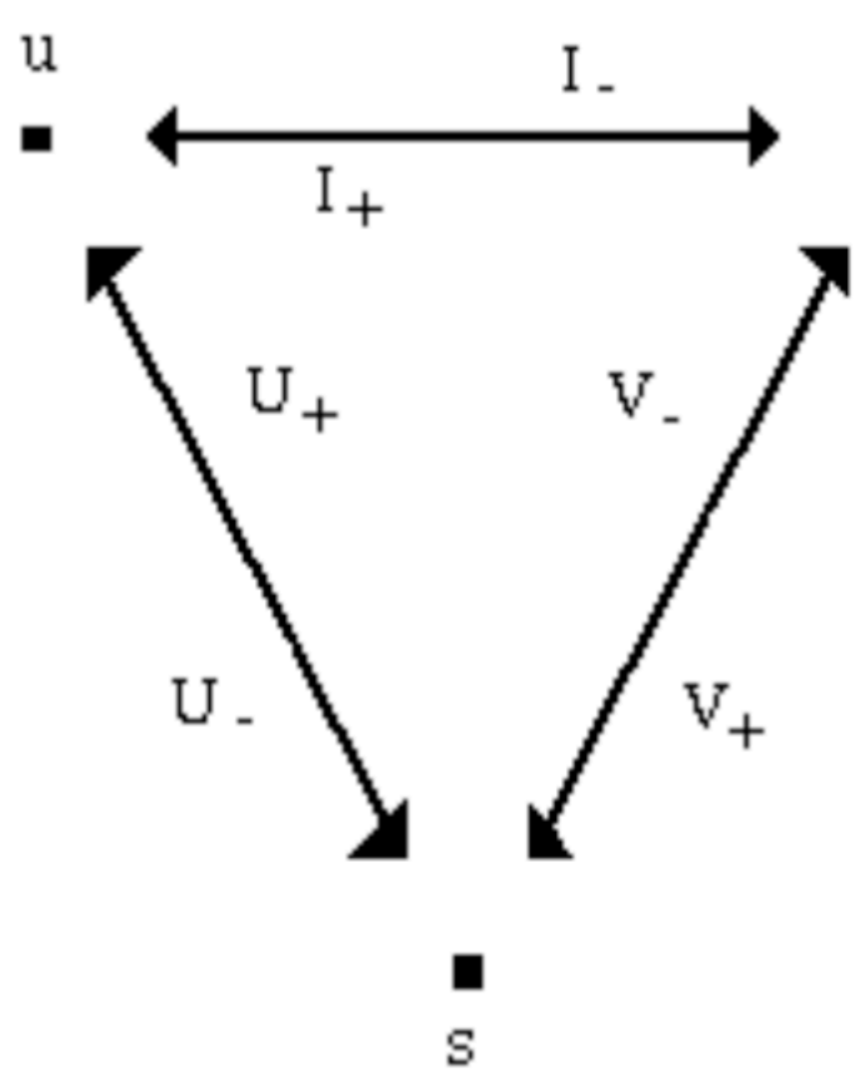
$$Q(\text{charge}) = I_3 + \frac{Y}{2} \quad , \quad Y = B + S$$

This leads to the table below:

Quark	I	I_3	Y	Q	B	S
u	1/2	1/2	1/3	2/3	1/3	0
d	1/2	-1/2	1/3	-1/3	1/3	0
s	0	0	-2/3	-1/3	1/3	-1

The baryon number of each of these states or particles is $B = 1/3$ and they represent the original three quarks.....**up, down and strange**.

Another way of visualizing the shifts (shift operators) is the diagram below:

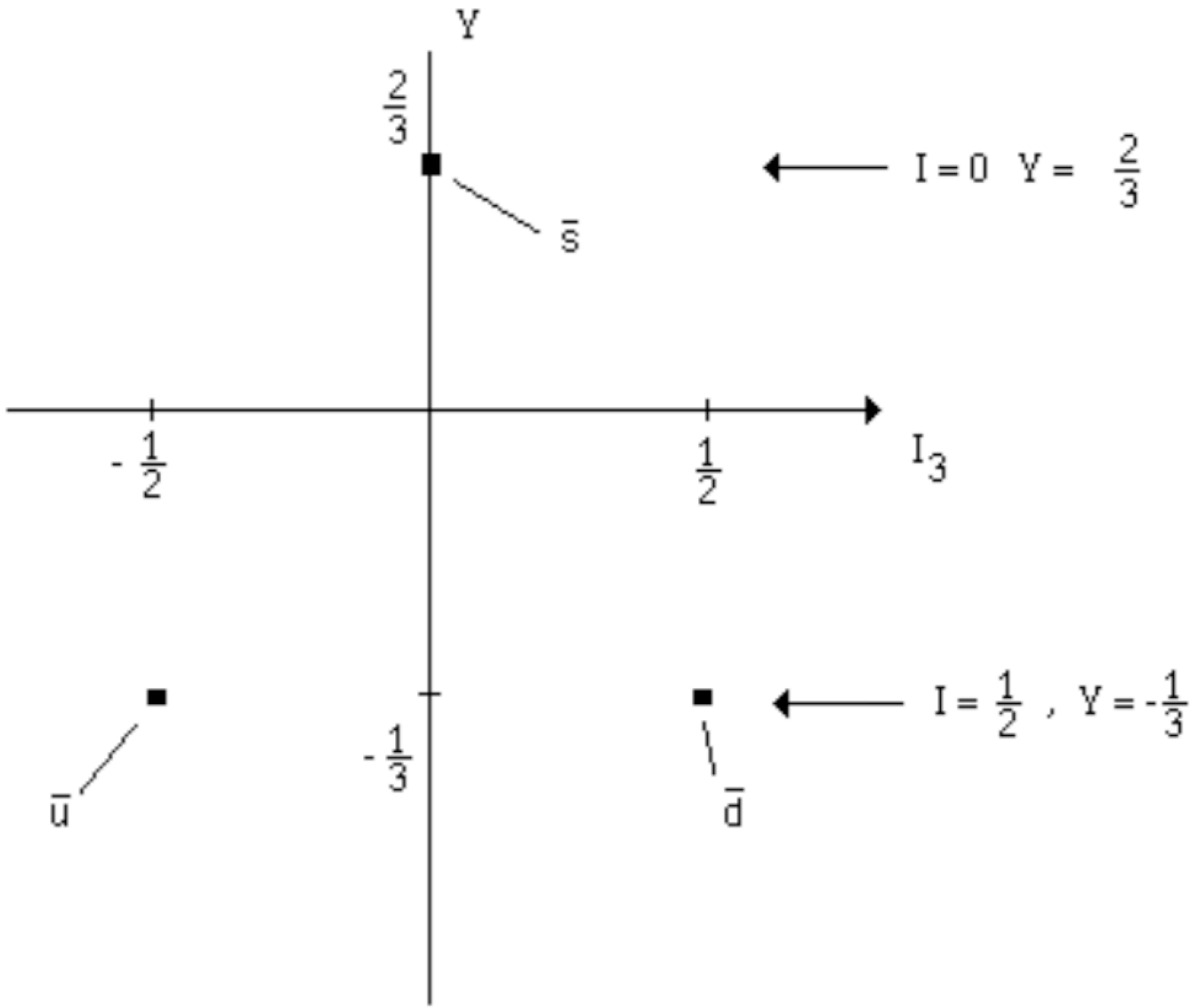


These shift operator transform quarks into each other.

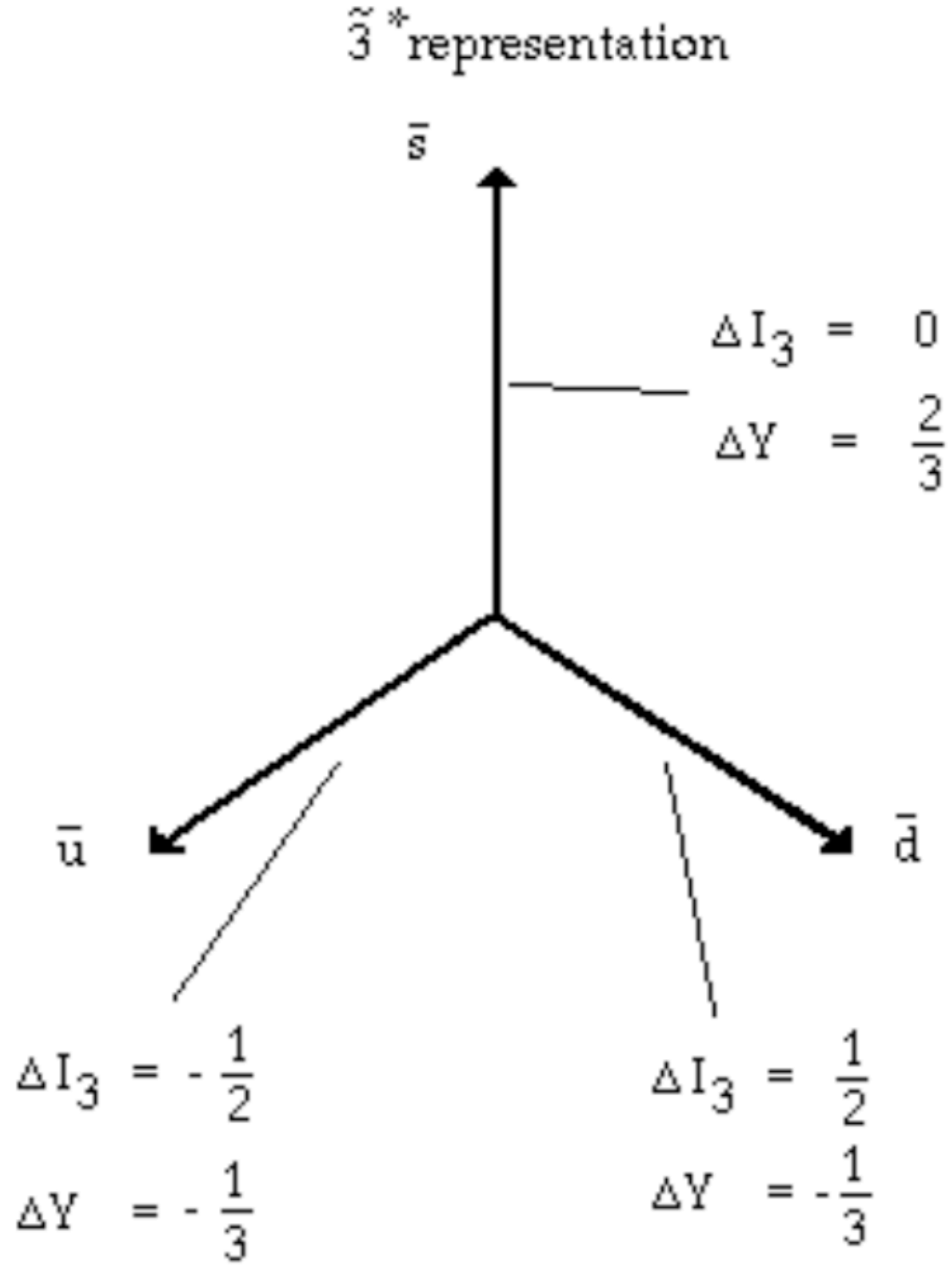
They will turn out to represent **gluons** later on.

It turns out that there exists an equivalent simple representation.

It is shown below:



It corresponds to the weight diagram below:



It is the $\tilde{3}^*$ representation and the particles or states are the anti-quarks.

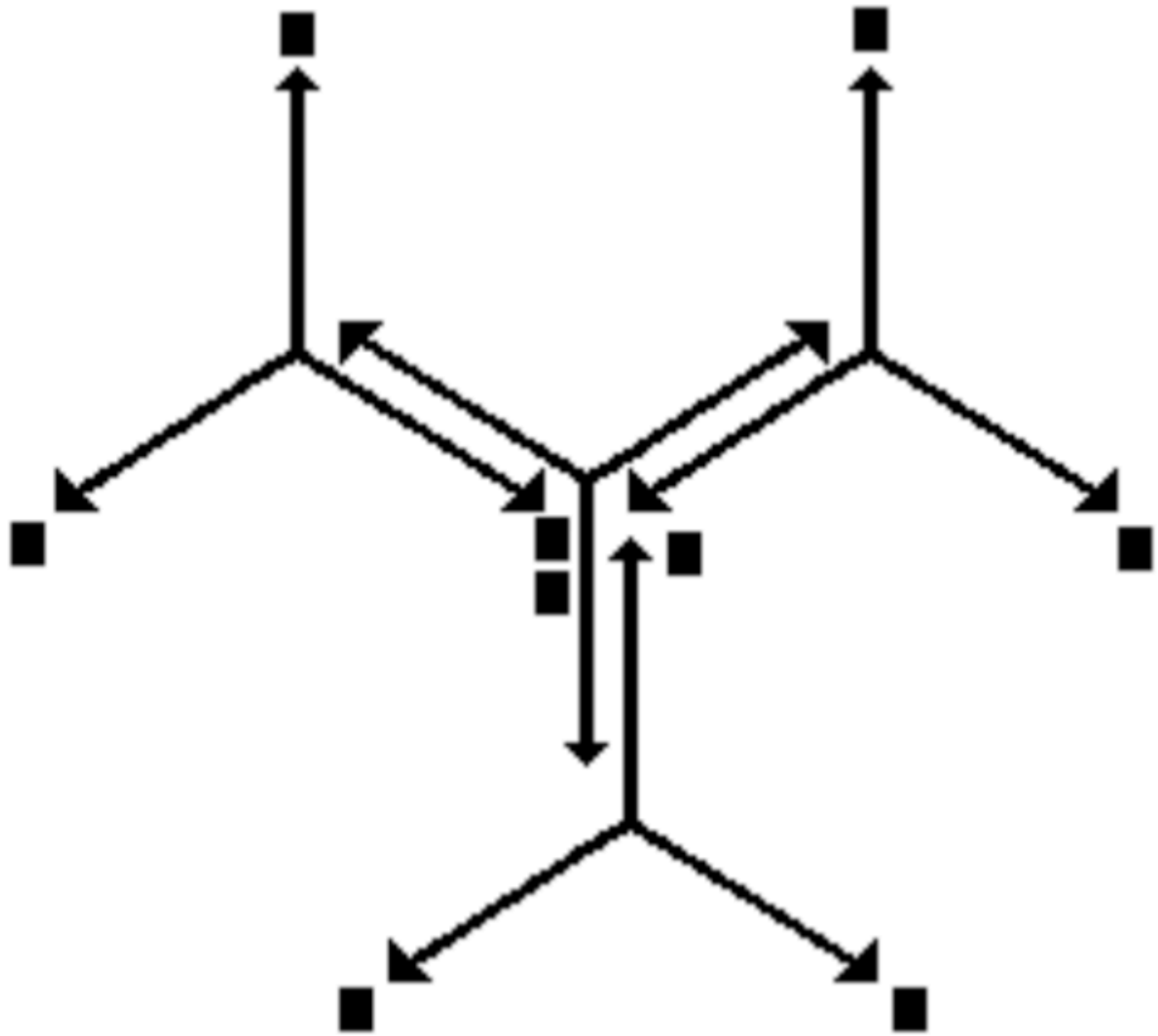
The corresponding table looks like:

Quark	I	I_3	Y	Q	B	S
anti-u=ū	1/2	-1/2	-1/3	-2/3	-1/3	0
anti-d=đ	1/2	1/2	-1/3	1/3	-1/3	0
anti- s=ŝ	0	0	2/3	1/3	-1/3	1

We now use these simplest representations or simplest weight diagrams to build higher representations.

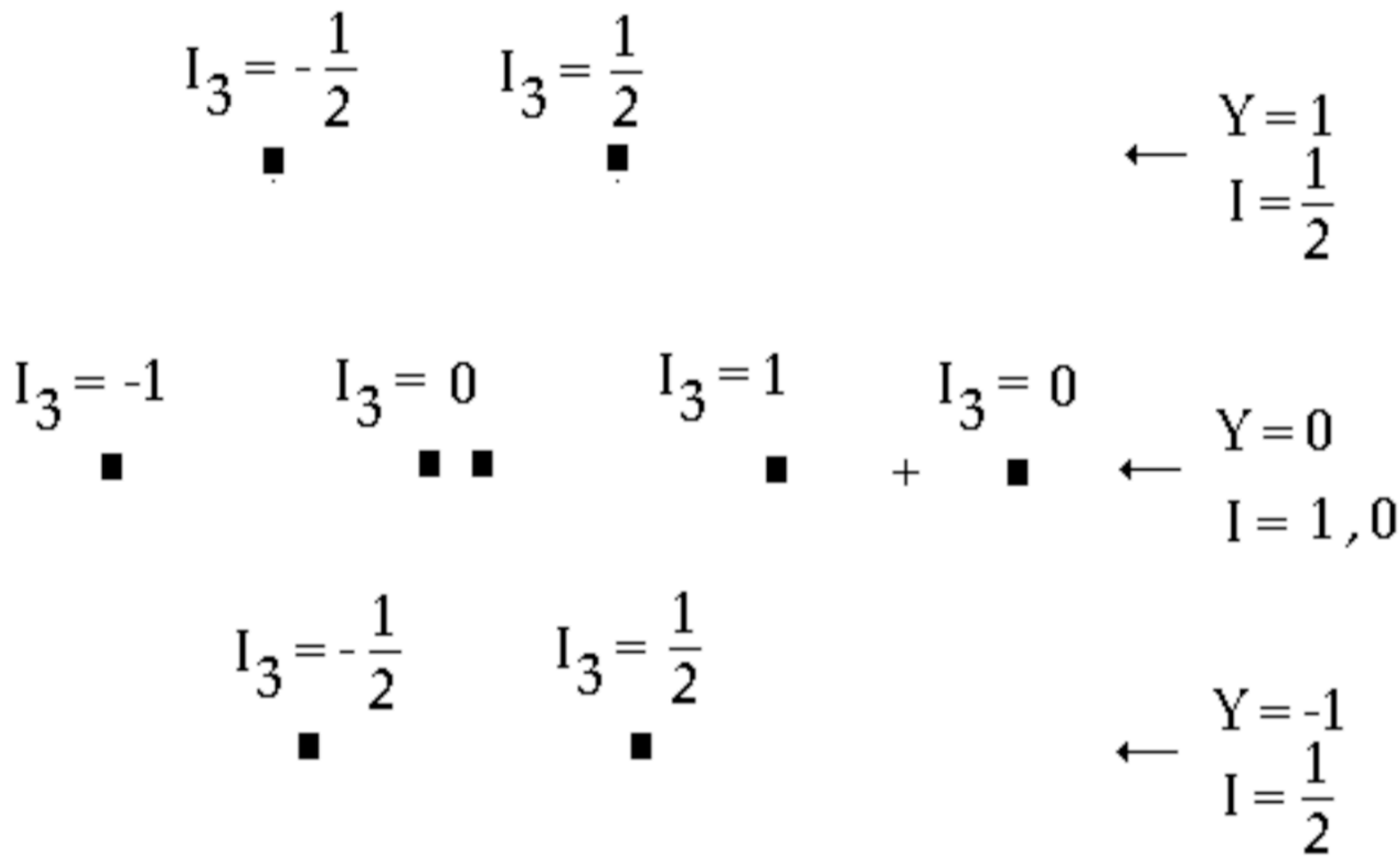
We note that $\tilde{3} \otimes \tilde{3}^*$ will have $B = 0$ (quark + anti-quark) and should represent **mesons** and $\tilde{3} \otimes \tilde{3} \otimes \tilde{3}$ will have $B = 1$ and should represent **baryons**.

So we now consider $\tilde{3} \otimes \tilde{3}^*$ using the weight diagrams as shown below:



where the black dots represent the final states of the new (higher) representations.

Looking only the states we have



The I-values associated with a line are determined by the multiplicity of a line (remember each line is a different I-value).

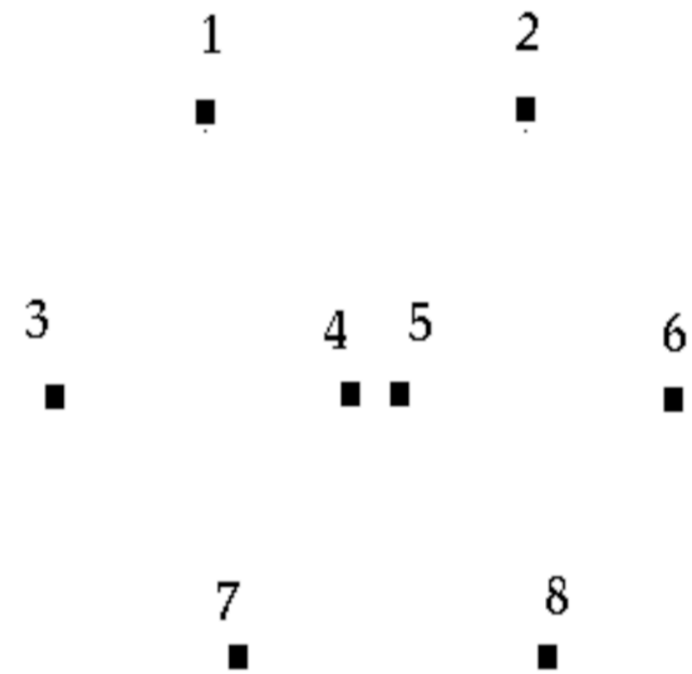
Note that there were three(3) final states at the same point.

Symbolically we have

$$\tilde{3} \otimes \tilde{3}^* = \tilde{8} \oplus \tilde{1} = \text{ octet + singlet}$$

We now look more carefully at the OCTET representation.

If we label the states as



We get the table

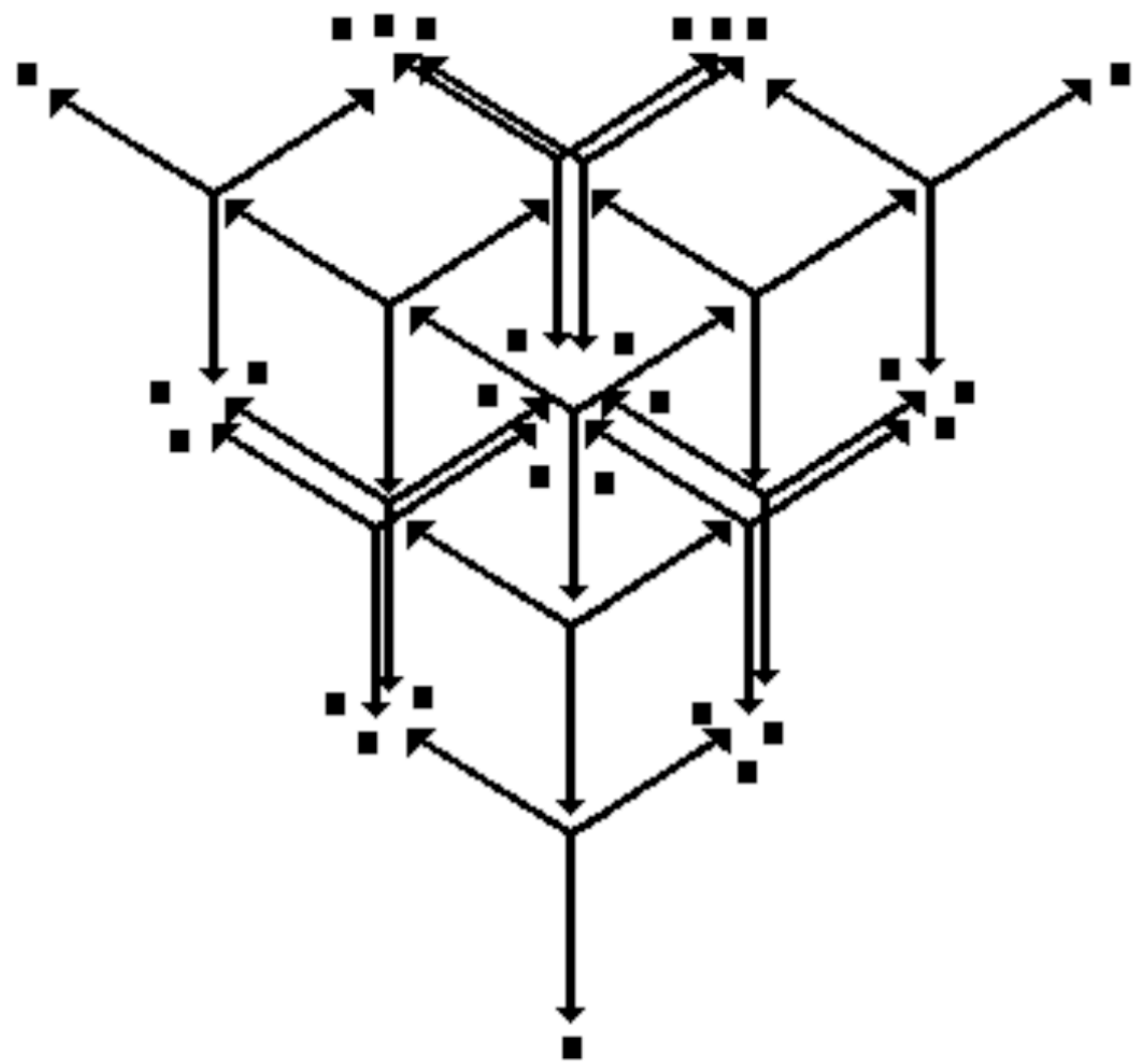
#	quarks	I	I3	Y	Q	B	S	meson
1	dŝ	1/2	-1/2	1	0	0	1	K^0
2	uŝ	1/2	1/2	1	1	0	1	K^+
3	dū	1	-1	0	-1	0	0	π^-
4	d**d̄**-uū	1	0	0	0	0	0	π^0
5	d**d̄**+uū-2sŝ	0	0	0	0	0	0	η^0
6	-u**d̄**	1	1	0	1	0	0	π^+
7	sū	1/2	-1/2	-1	-1	0	-1	K^-
8	s**d̄**	1/2	1/2	-1	0	0	-1	K^0-bar

Our $SU(3)$ scheme has produced the known mesons (including the strange particles) as linear combination of quarks and anti-quarks.

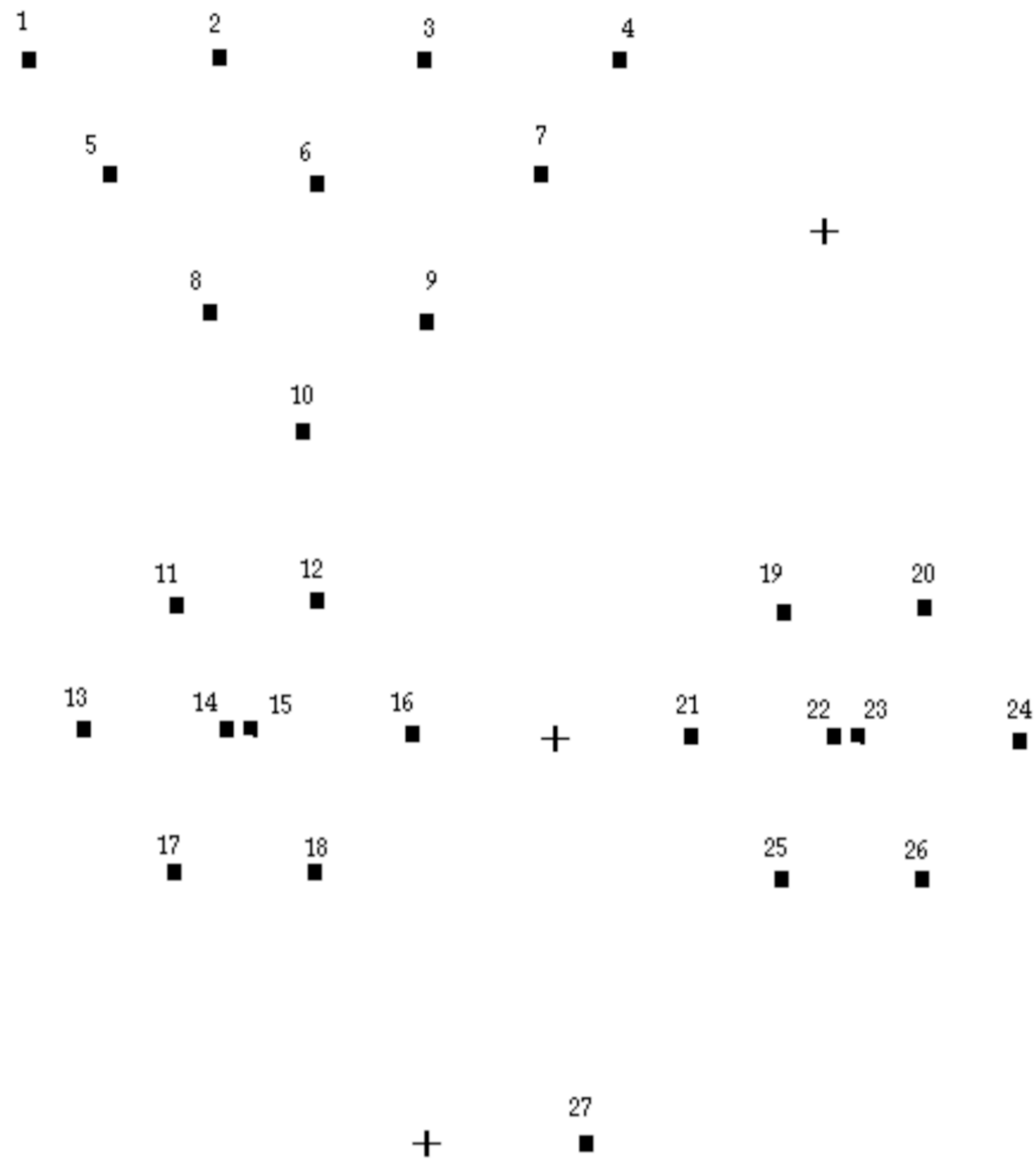
Let us push it even further and consider the baryon states.

This means we look at the $\tilde{3} \otimes \tilde{3} \otimes \tilde{3}$ representation.

Using weight diagrams we get



These states separate as follows:



The particle identifications are given in the table below (only partly filled in):

#	States	I	I3	Y	Q	B	S	Particle
1	ddd	3/2	-3/2	1	-1	1	0	Δ-
2	udd+dud+ddu	3/2	-1/2	1	0	1	0	Δ^0
3	duu+udu+uud	3/2	1/2	1	1	1	0	Δ^+
4	uuu	3/2	3/2	1	2	1	0	Δ^{++}
5	sdd+dsd+dds	1	-1	0	-1	1	-1	Σ*-
6	sdu+dsu+dus +sud+usd+uds	1	0	0	0	1	-1	Σ^{*0}
7	suu+usu+uus	1	1	0	1	1	-1	Σ^{*+}
8	dss_sds+ssd	1/2	-1/2	-1	-1	1	-2	Ξ*-
9	uss+sus+ssu	1/2	1/2	-1	0	1	-2	Ξ^{*0}
10	sss	0	0	-2	-1	1	-3	Ω-
11	d[ud]	----	----	----	----	----	----	----
12	u[ud]	----	----	----	----	----	----	----
13	d[ds]	----	----	----	----	----	----	----
14	u[ds]+d[su]-2s[ud]	----	----	----	----	----	----	----
15	u[ds]-d[su]	----	----	----	----	----	----	----
16	u[su]	----	----	----	----	----	----	----
17	s[ds]	----	----	----	----	----	----	----
18	s[su]	----	----	----	----	----	----	----
19	----	----	----	----	----	----	----	----
20	uud+udu-2duu	----	----	----	----	----	----	----
----	----	----	----	----	----	----	----	----
27	u[ds]+d[su]+s[ud]	----	----	----	----	----	----	----

Consider the DECUPLET representation.

All of the particles had been previously observed except for the Ω^-, which we will discuss further later.

We end up with two OCTET representations.The first is antisymmetric in the second and third quarks and the second is symmetric in the second and third quarks. They are completely equivalent representations within $SU(3)$ and represent the same physical particles.

The Mass Formula

We will simply state and discuss the mass formula here. The most general form for the mass formula is given by (from group theory arguments):

$$M = A + BY + C\left(I(I+1) - \frac{1}{4}Y^2\right) + DI_3$$

where the A, B, and C terms represent the average mass (within a row) and the D term represents electromagnetic splittings.

Some OCTET experimental data implies that:

$$M_8 = 1109.80 - 189.83Y + 41.49\left(I(I+1) - \frac{1}{4}Y^2\right) + DI_3$$

which gave the following table(decades ago - it is even better agreement now) :

particle	experiment	formula
n	939.5	943.2
p	938.2	938.3
Λ^0	1115.4	1109.8
Σ^-	1189.4	1190.3
Σ^0	1193.2	1192.8
Σ^+	1197.6	1195.2
Ξ^-	1316	1317.9
Ξ^0	1321	1322.8

This is an average error of only 0.19%.

The experimental data in the table below was also available.

particle	experiment
Δ	1238
Σ*	1385
Ξ*	1530

As we said above, the Ω^- particle had not yet been discovered.

This data gives the DECUPLET mass formula below:

$$M_{10} = 1501.8 - 58.5Y - 58.4\left(I(I+1) - \frac{1}{4}Y^2\right)$$

The mass formula then allows us to predict the mass of the Ω^- particle ($SU(3)$ also predicts all of the other quantum numbers). The prediction is $M_{\Omega^-} = 1677$ (1685 was observed or about a 0.5% error (even better agreement now)).

Introduction of Color We now know there are 6 quarks.

u(up),d(down),s(strange),c(charm),b(bottom),t(top)

Let me discuss a few of the reasons why the other quarks had to exist.

A basic problem with the quark model of hadrons is with the wave function antisymmetry.

Consider the Ω^- state.

It is strangeness $S = -3$ baryon with isospin $I = 3/2$.

In the 3 quark model we have been discussing

$$\left|\Omega^-\right\rangle = |s\rangle\,|s\rangle\,|s\rangle$$

Each of these quarks has the following states

$$|s\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle_{spin}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{isospin}\left|B = \frac{1}{3}, S = -1\right\rangle$$

and, thus, the final state is **symmetric** in all of its quantum numbers (the orbital angular momentum states are also symmetric when one solves the bound state problems), which violates the Pauli Principle for fermions - they must have **antisymmetric** states.

The solution to the problem was to generalize the definition of the quarks such that each quarks can have one of three colors (**r=red, b=blue, g=green**) or

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} u_r & u_b & u_g \\ d_r & d_b & d_g \\ s_r & s_b & s_g \end{pmatrix}$$

The associated observable is called **flavor**.

Quarks now are assumed to come in three flavors, namely, red, blue and green.

The problem with the Ω^- state is fixed by assuming that it is completely antisymmetric in the flavor space or we write it as

$$\left|\Omega^-\right\rangle = \left|s_r\right\rangle \left|s_b\right\rangle \left|s_g\right\rangle - \left|s_r\right\rangle \left|s_g\right\rangle \left|s_b\right\rangle - \left|s_b\right\rangle \left|s_r\right\rangle \left|s_g\right\rangle + \left|s_b\right\rangle \left|s_g\right\rangle \left|s_r\right\rangle + \left|s_g\right\rangle \left|s_r\right\rangle \left|s_b\right\rangle - \left|s_g\right\rangle \left|s_b\right\rangle \left|s_r\right\rangle$$

It is equal parts red, green and blue!

Another type of problem was that certain reactions were not occurring when SU(3) said they were allowed. This usually signals that an extra (new) quantum number is needed to prevent the reaction.

The new quantum number would have a new conservation law that would be violated if the reaction occurred and hence it is disallowed.

The new quantum number introduced was **charm**. In the quark model this is treated as the appearance of a new quark.

I will delineate only the starting point of this discussion below. It leads to the formalism of $SU(4)$ if we carried out the same steps as we did with $SU(3)$.

Quark states:

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |c\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We now have

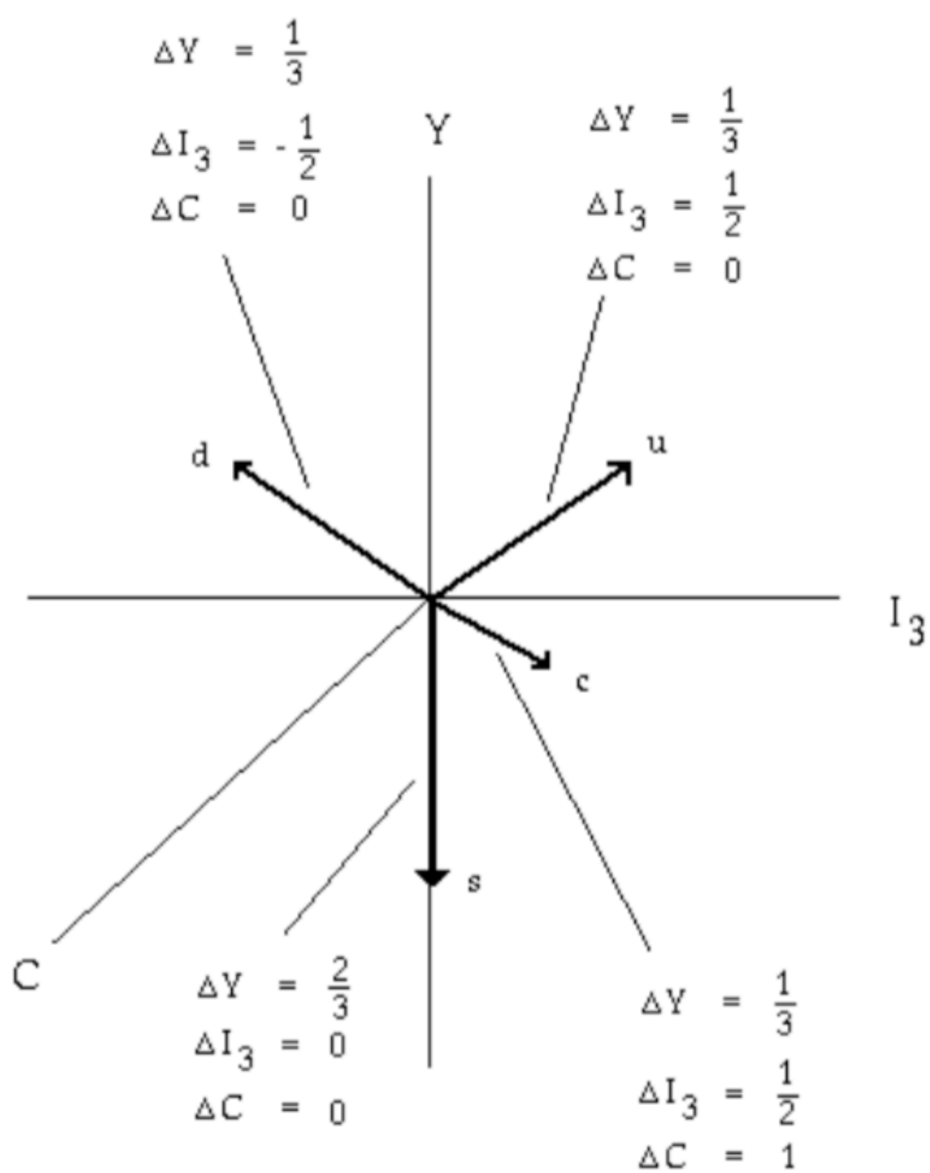
$$Q = I_3 + \frac{B+S+C}{2} \quad , \quad C = \text{ charm}$$

The four quarks then have quantum numbers

Quark	Y	Q	B	S	C
u	1/3	2/3	1/3	0	0
d	1/3	-1/3	1/3	0	0
s	-2/3	-1/3	1/3	-1	0
c	1/3	2/3	1/3	0	1

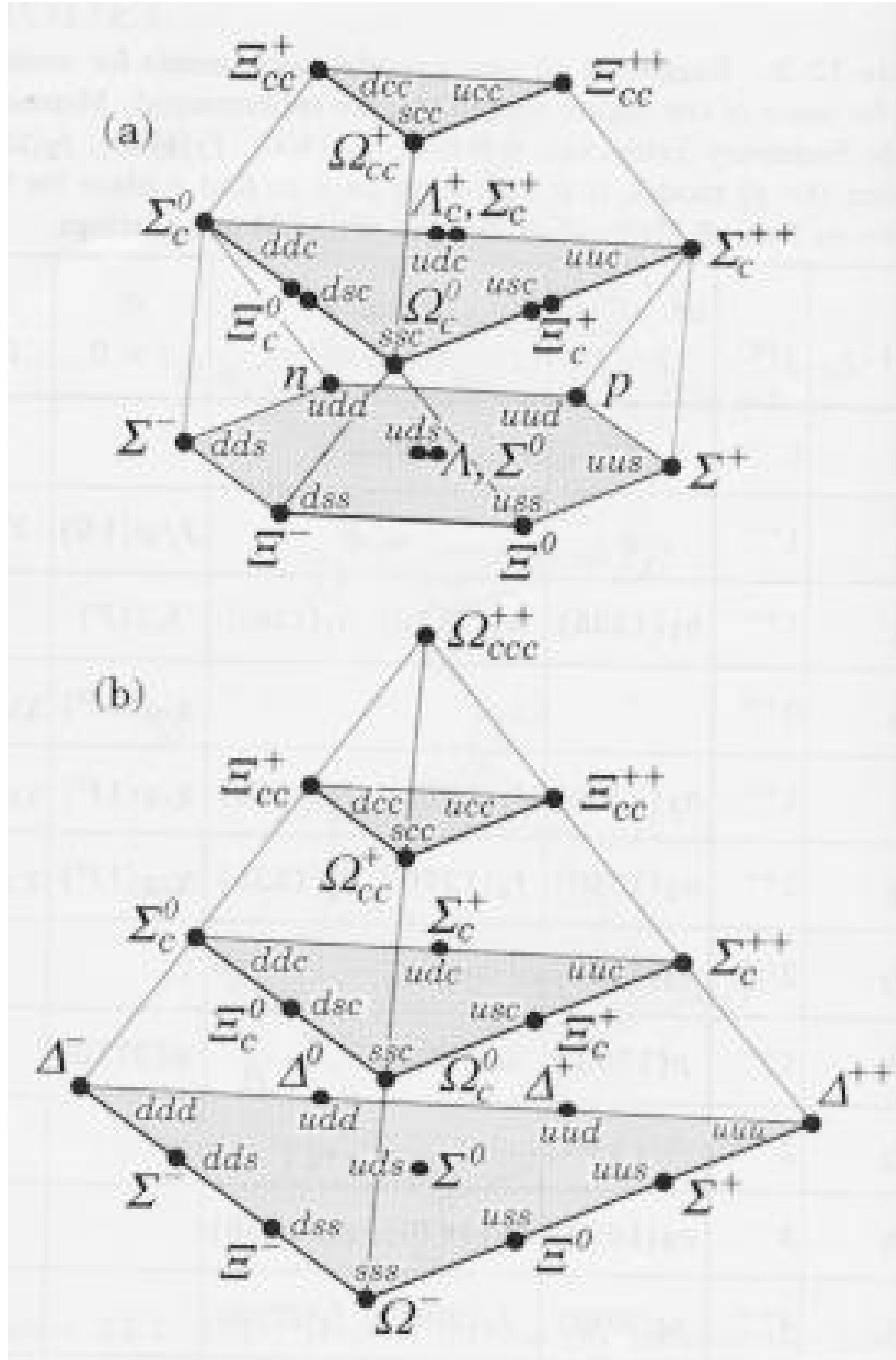
The charm quark is the same as the up quark except for the charm value.

The corresponding lowest representation weight diagram is



This is the $\tilde{4}$ representation.

The typical particle diagrams (now in 3 dimensions) look like



That concludes the discussion of the Eightfold Way and we end the course.