

Problems and Solutions Introductory Mechanics

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September 6, 2020

Assignment 1 - Problems

Problem 1 - Given two vectors, $\vec{A} = 2\hat{i} - 3\hat{j} + 7\hat{k}$ and $\vec{B} = -2\hat{i} - 3\hat{j} - \hat{k}$, find:

(a) $\vec{A} + \vec{B}$

(b) $\vec{A} - \vec{B}$

(c) $\vec{A} \cdot \vec{B}$

(d) $\vec{A} \times \vec{B}$

Problem 2 - Find the cosine of the angle between $\vec{A} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = 5\hat{i} + \hat{j} + 2\hat{k}$

Problem 3 - Show that if $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$, then $\vec{A}$ is perpendicular to $\vec{B}$.

Problem 4 - Prove that the diagonals of an equilateral parallelogram are perpendicular.

Problem 5 - Prove the law of sines using the cross product. It should only take a couple of lines. (HINT: Consider the area of a triangle by $\vec{A}$, $\vec{B}$, $\vec{C}$, where $\vec{A} + \vec{B} + \vec{C} = 0$).

Problem 6 - Let $\hat{a}$ and $\hat{b}$ be unit vectors in the xy plane making angles θ and ϕ with the x -axis, respectively. Show that $\hat{a} = \cos \theta \hat{i} + \sin \theta \hat{j}$, $\hat{b} = \cos \phi \hat{i} + \sin \phi \hat{j}$ and using vector algebra prove that

$$\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$$

Problem 7 - Find a unit vector perpendicular to $\vec{A} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 3\hat{k}$.

Problem 8 - Let $\vec{A}$ be an arbitrary vector and let $\hat{n}$ be a unit vector in some fixed direction. Show that $\vec{A} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$.

Problem 9 - Given the vectors

$$\vec{A} = 5\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{B} = -3\hat{i} + \hat{j} - 7\hat{k}$$

$$\vec{C} = 4\hat{i} + 4\hat{j} + 4\hat{k}$$

obtain the magnitude of the sum of the three vectors and determine the direction it makes with respect to the x -, y -, and z -axes.

Problem 10 -Find the unit vector that is orthogonal to the plane containing the vectors

$$\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$$

$$\vec{B} = -\hat{i} + \hat{j} + 2\hat{k}$$

Problem 11 - Given the vectors

$$\vec{A} = 5\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{B} = -3\hat{i} + \hat{j} - 7\hat{k}$$

$$\vec{C} = 4\hat{i} + 4\hat{j} + 4\hat{k}$$

compute

$$\vec{A} \cdot (\vec{B} \times \vec{C}) \quad \text{and} \quad \vec{A} \times (\vec{B} \times \vec{C})$$

and show that

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

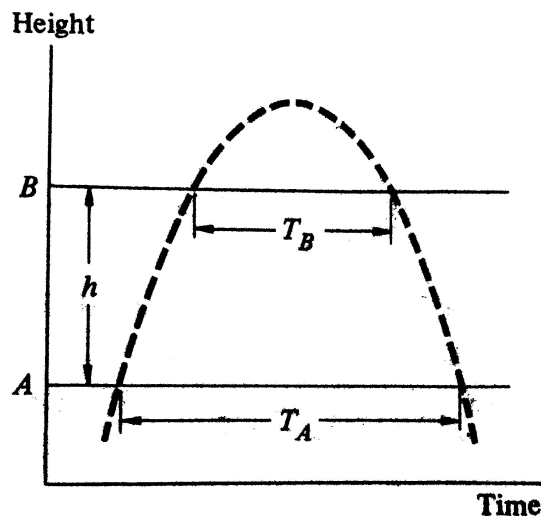
Problem 12 - A particle moves according to $\vec{r}(t) = 2t^2\hat{i} + 5t\hat{j} - t^3\hat{k}$, where t is in seconds and r is in meters.

(a) Find the speed of the particle at $t = 2$ s.

(b) Calculate the particle's acceleration at $t = 2$ s.

Assignment 2 - Problems

Problem 1 - The acceleration of gravity can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions. See figure below.

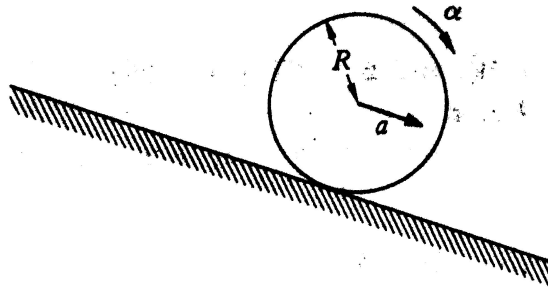


Show that if the time the body takes to pass a horizontal line A in both directions is T_A , and the time to go by a second line B in both directions is T_B , then, assuming that the acceleration is constant, its magnitude is

$$g = \frac{8h}{T_A^2 - T_B^2}$$

Problem 2 - An elevator ascends from the ground with uniform speed. At time T_1 a boy drops a marble through the floor. The marble falls with uniform acceleration $g = 9.8 \text{ m/s}^2$, and hits the ground T_2 seconds later. Find the height of the elevator at time T_1 .

Problem 3 - A drum of radius R rolls down a slope without slipping. See figure below. Its axis has acceleration a parallel to the slope. What is the drum's angular acceleration α ?

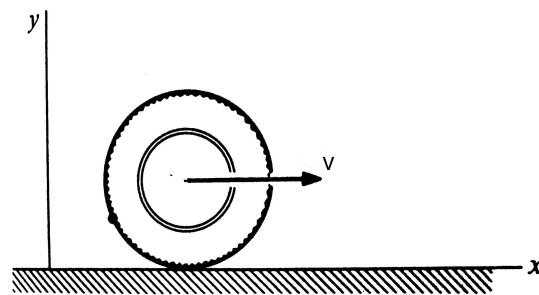


Problem 4 - A sportscar, Fiasco 1, can accelerate uniformly to 120 mi/h in 30 s. Its *maximum* braking rate cannot exceed $0.7g$. What is the minimum time required to go $\frac{1}{2}$ mi, assuming it begins and end at rest. HINT: A graph of velocity versus time would be helpful.

Problem 5 - A particle moves in a plane with constant radial velocity $\dot{r} = 4\text{ m/s}$. The angular velocity is constant and has magnitude $\dot{\theta} = 2\text{ rad/s}$. When the particle is 3 m from the origin, find the magnitude of

- (a) the velocity
- (b) the acceleration

Problem 6 - A tire rolls in a straight line without slipping. See figure below.

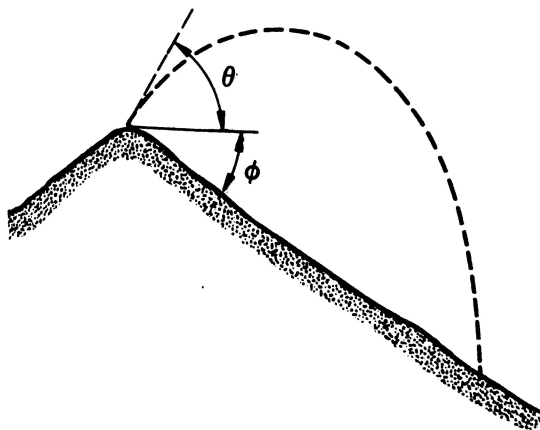


Its center moves with constant speed V . A small pebble lodged in the tread of the tire touches the road at $t = 0$. Find the pebble's position, velocity and acceleration as functions of time.

Problem 7 - A particle moves along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant. $A = (1/\pi) m/rad$. θ increases in times according to $\theta = \alpha t^2/2$, where α is a constant.

- (a) Sketch the motion, and indicate the approximate velocity and acceleration at a few points.
- (b) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2} rad$.
- (c) At what angles do the radial and tangential accelerations have equal magnitude?

Problem 8 - A boy stands at the peak of a hill which slopes downward uniformly at angle ϕ . See figure below.



At what angle θ from the horizontal should he throw a rock so that it has the greatest range?

Problem 9 - You are traveling in the fabled Mines of Moria when you come across an open water well in the stone floor of the tunnel. To find out how deep the well is, you drop a stone into the well and measure the time until you hear the splash. If you **hear** the splash 3.0 s after you drop the stone, how deep is the well?

Problem 10 - A spaceship is approaching Starbase beta at an initial velocity of 130 km/s in the $+x$ direction. The ensign sets its computer to initiate a braking program starting at time $t = 0$. After this time, the computer controls

the ship until the ship docks at the time $t = T$ so that its x -acceleration is $a_x = b(t - T)$, where $b = 0.26 \text{ m/s}^2$. Note that $a_x < 0$ for $t < T$, but gets smaller as t approaches the docking time T .

- (a) What must T be so that the ship's x -velocity is also zero at the time of docking?
- (b) How far is the ship from the station at $t = 0$?

Problem 11 - You are a police officer. Your squad car is at rest on the shoulder of an interstate highway when you notice a car fitting the description given in an all-points bulletin passing you at its top speed of 85 mi/h. You jump in your car, start the engine, and find a break in the traffic, a process which takes 25 s. You know from the squad car's manual that when it starts from rest with its accelerator pressed to the floor, the magnitude of its acceleration is $a = a_0 - bt^2$, where $a_0 = 2.5 \text{ m/s}^2$ and $b = 0.0028 \text{ m/s}^4$, until $a_0 = bt^2$ and then remains zero thereafter. Can you catch the car before it reaches the next exit 5.3 mi away?

Make a plot of distance versus time for **both** the police officer and the speeder. Indicate on your plot (using arrows and labels) the important points in your calculation.

Problem 12 - A body initially at rest ($\theta = 0$, $\dot{\theta} = \omega = 0$, $t = 0$) is accelerated in a circular path of radius 1.3 m according to the equation

$$\ddot{\theta} = \alpha(t) = 120t^2 - 48t + 16$$

- (a) Find the angular position and angular velocity of the body as functions of time.
- (b) Determine the r and θ components of the acceleration.

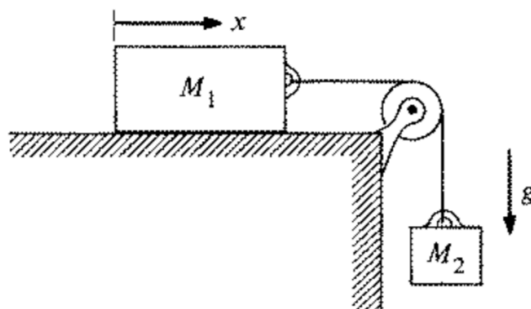
Problem 13 - A point is moving on a circle according to the law

$$s = t^3 + 2t^2$$

where s is measured in feet along the circle and t is measured in seconds. If the total acceleration of the point is $16\sqrt{2} \text{ ft/s}^2$ when $t = 2$ s, calculate the radius of the circle.

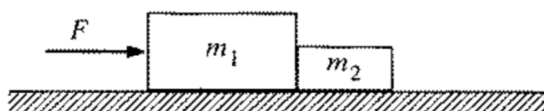
Assignment 3 - Problems

Problem 1 - The two blocks shown in the sketch are connected by a string of negligible mass.



If the system is released from rest, find how far block M_1 slides in time t . Neglect friction.)

Problem 2 - Two blocks are in contact on a horizontal table.

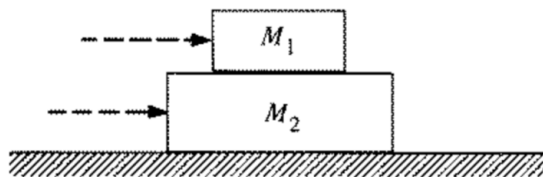


A horizontal force is applied to one of the blocks, as shown in the drawing. If $m_1 = 2\text{ kg}$, $m_2 = 1\text{ kg}$, and $F = 3\text{ N}$, find the force of contact between the two blocks.

Problem 3 - In a concrete mixer, cement, gravel, and water are mixed by tumbling action in a slowly rotating drum. If the drum spins too fast the ingredients stick to the drum wall instead of mixing.

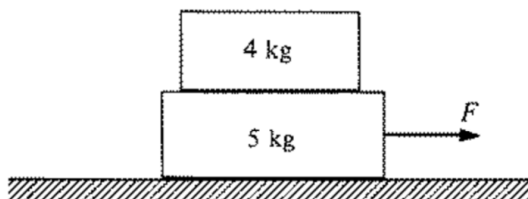
Assume that the drum of a mixer has radius R and that it is mounted with its axle horizontal. What is the fastest the drum can rotate without the ingredients sticking to the wall all the time? Assume $g = 32\text{ ft/s}^2$.

Problem 4 - A block of mass M_1 rests on a block of mass M_2 which lies on a frictionless table.



The coefficient of friction between the blocks is μ . What is the maximum horizontal force which can be applied to the blocks for them to accelerate without slipping on one another if the force is applied to (a) block 1 and (b) block 2?

Problem 5 - A 4-kg block rests on top of a 5-kg block, which rests on a frictionless table.

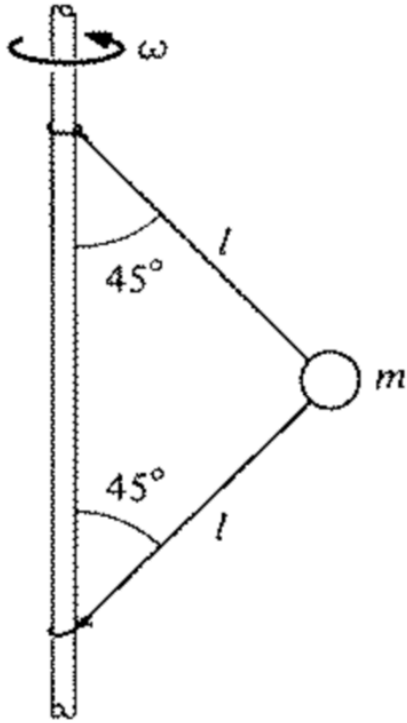


The coefficient of friction between the two blocks is such that the blocks start to slip when the horizontal force F applied to the lower block is 27 N. Suppose that a horizontal force is now applied only to the upper block. What is the maximum value for the blocks to slide without slipping relative to each other?

Problem 6 - A mass m is connected to a vertical revolving axle by two strings of length l each making an angle of 45° with the axle, as shown. Both the axle and mass are revolving with angular velocity ω . Gravity is directed downward.

- (a) Draw a clear force diagram for m .
- (b) Find the tension in the upper string, T_{up} , and in the lower string, T_{low} .

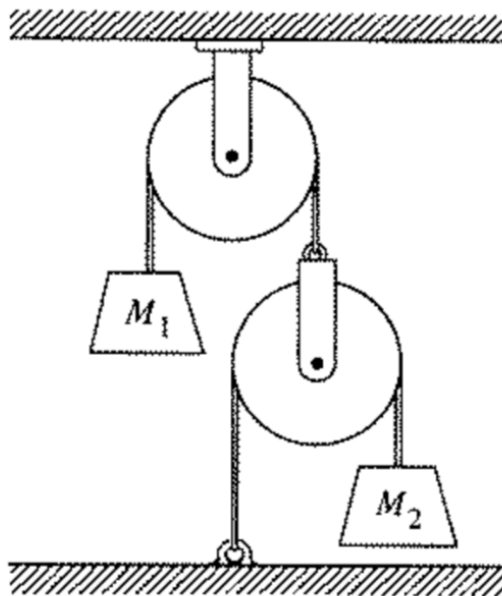
Problem 7 - If you have courage and a tight grip, you can yank a tablecloth out from under the dishes on a table. What is the longest time in which



the cloth can be pulled out so that a glass 6 in. from the edge comes to rest before falling off the table? Assume that the coefficient of friction of the glass sliding on the tablecloth or sliding on the tabletop is 0.5. (For the trick to be effective the cloth should be pulled out so rapidly that the glass does not move appreciably.)

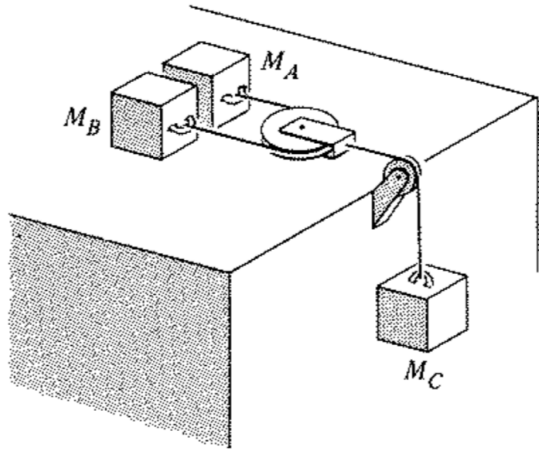
Assignment 4 - Problems

Problem 1 - Masses M_1 and M_2 are connected to a system of strings and pulleys as shown.



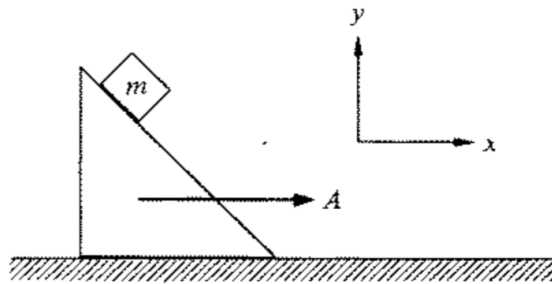
The strings are massless and inextensible, and the pulleys are massless and frictionless. Find the acceleration of M_1 .

Problem 2 - Two masses, A and B , lie on a frictionless table (see below).



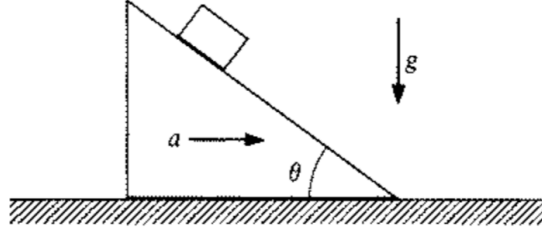
They are attached to either end of a light rope of length l which passes around a pulley of negligible mass. The pulley is attached to a rope connected to a hanging mass, C . Find the acceleration of each mass.

Problem 3 - A 45° wedge is pushed along a table with constant acceleration A . A block of mass m slides without friction on the wedge. (Gravity is directed down.)



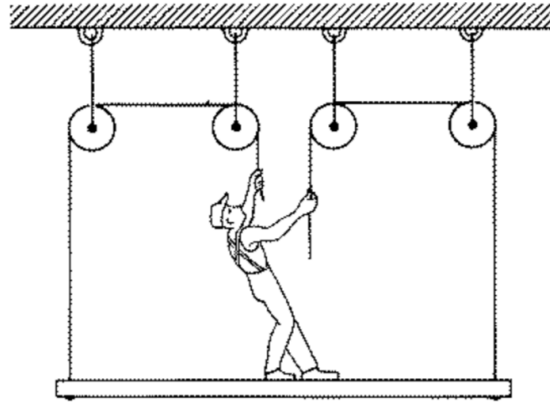
Find the acceleration.

Problem 4 - A block rests on a wedge inclined at angle θ . The coefficient of friction between the block and plane is μ .



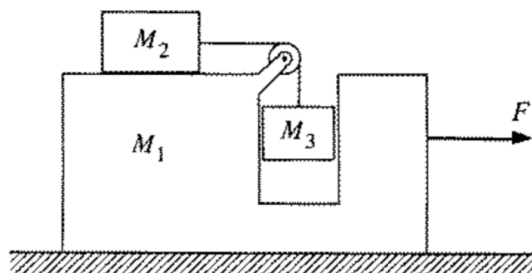
- (a) Find the maximum value of θ for the block to remain motionless on the wedge when the wedge is fixed in position.
- (b) The wedge is given horizontal acceleration a , as shown. Assuming that $\tan \theta < \mu$, find the minimum acceleration for the block to remain on the wedge without sliding.
- (c) Repeat part b, but find the maximum value of the acceleration.

Problem 5 - A painter of mass M stands on a platform of mass m and pulls himself up by two ropes which hang over pulleys, as shown.



He pulls each rope with force F and accelerates upward with a uniform acceleration a . Find a - neglecting the fact that no one could do this for long.

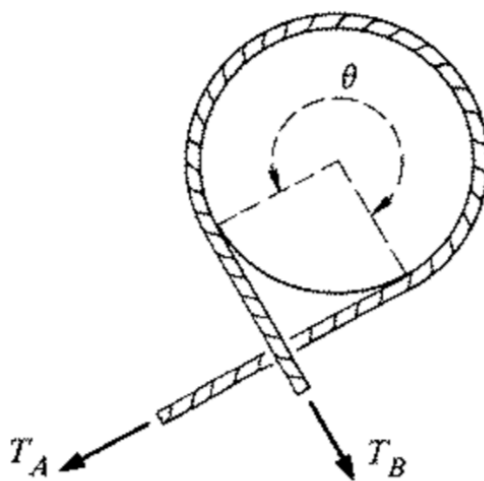
Problem 6 - A “Pedagogical Machine” is illustrated in the sketch below.



All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

Problem 7 - A uniform rope of mass m and length l is attached to a block of mass M . The rope is pulled with force F . Find the tension at distance x from the end of the rope. Neglect gravity.

Problem 8 - A device called a capstan is used aboard ships in order to control a rope which is under great tension.



The rope is wrapped around a fixed drum, usually for several turns (the

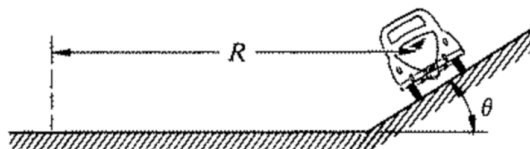
drawing shows about three-fourths turn). The load on the rope pulls it with a force T_A , and the sailor holds it with a much smaller force T_B . Can you show that

$$T_B = T_A e^{-\mu\theta}$$

where μ is the coefficient of friction and θ is the *total* angle subtended by the rope on the drum?

Assignment 5 - Problems

Problem 1 - An automobile enters a turn whose radius is R . The road is banked at angle θ , and the coefficient of friction between wheels and road is μ .

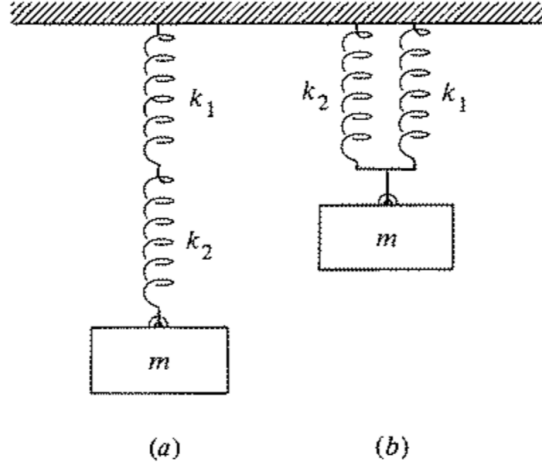


Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

Problem 2 - A car is driven on a large revolving platform which rotates with constant angular speed ω . At $t = 0$ a driver leaves the origin and follows a line painted radially outward on the platform with constant speed v_0 . The total weight of the car is W , and the coefficient of friction between the car and stage is μ .

- (a) Find the acceleration of the car as a function of time using polar coordinates. Draw a clear vector diagram showing the components of acceleration at some time $t > 0$.
- (b) Find the time at which the car just starts to skid.
- (c) Find the direction of the friction force with respect to the instantaneous position vector $\vec{r}$ just before the car starts to skid. Show your result on a clear diagram.)

Problem 3 - Find the frequency of oscillation of mass m suspended by two springs having constants k_1 and k_2 , in each of the configurations shown.

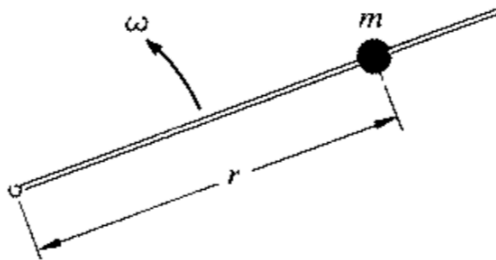


Problem 4 - A wheel of radius R rolls along the ground with velocity V . A pebble is carefully released on top of the wheel so that it is instantaneously at rest on the wheel.

- (a) Show that the pebble will immediately fly off the wheel if $V > \sqrt{Rg}$.
- (b) Show that in the case where $V < \sqrt{Rg}$, and the coefficient of friction is $\mu = 1$, the pebble starts to slide when it has rotated through an angle given by

$$\theta = \cos^{-1} \left[\frac{1}{\sqrt{2}} \frac{V^2}{Rg} \right] - \frac{\pi}{4}$$

Problem 5 - A particle of mass m is free to slide on a thin rod.



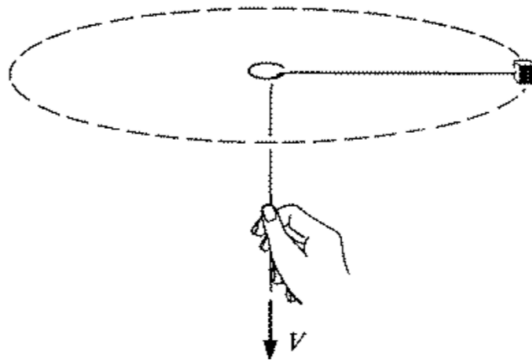
The rod rotates in a plane about one end at constant angular velocity ω . Show that the motion is given by

$$r = Ae^{-\gamma t} + Be^{\gamma t}$$

where γ is a constant which you must find and A and B are arbitrary constants. Neglect gravity.

Show that for a particular choice of initial conditions [that is, $r(t = 0)$ and $v(t = 0)$], it is possible to obtain a solution such that r decreases continually in time, but that for any other choice r will eventually increase (Exclude cases where the bead hits the origin).

Problem 6 - A mass m whirls around on a string which passes through a ring, as shown.



Neglect gravity. Initially the mass is a distance r_0 from the center and is revolving at angular velocity ω_0 . The string is pulled with constant velocity V starting at $t = 0$ so that the radial distance to the mass decreases. Draw a force diagram and obtain a differential equation for ω . This equation is quite simple and can be solved either by inspection or by formal integration. Find

- (a) $\omega(t)$
- (b) The force needed to pull the string.

Problem 7 - his problem involves a simple differential equation. You should be able to integrate it after a little “playing around”.

A particle of mass m moving along a straight line is acted on by a retarding force (one always directed against the motion)

$$F = be^{\alpha v}$$

where b and α are constants and v is the velocity. At $t = 0$ it is moving along with velocity v_0 . Find the velocity at later times.

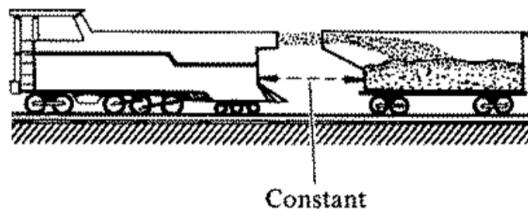
Assignment 6 - Problems

Problem 1 - Find the center of mass of a thin uniform plate in the shape of an equilateral triangle with sides a .

Problem 2 - An instrument-carrying projectile accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L . The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the earth's curvature.

Problem 3 - A light plane weighing 2,500 lb makes an emergency landing on a short runway. With its engine off, it lands on the runway at 120 ft/s. A hook on the plane snags a cable attached to a 250-lb sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is 0.4, and if the plane's brakes give an additional retarding force of 300 lb, how far does the plane go before it comes to a stop?

Problem 4 - A sand-spraying locomotive sprays sand horizontally into a freight car as shown in the sketch.

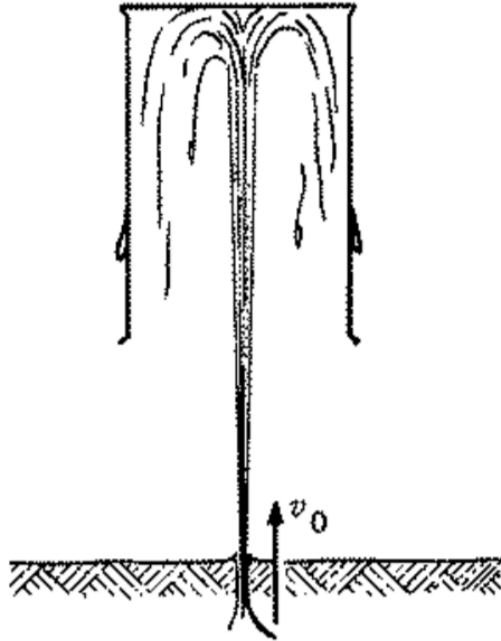


The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate $dm/dt = 10 \text{ kg/s}$ with a velocity of 5 m/s relative to the locomotive. The car starts from rest with an initial mass of 2,000 kg. Find its speed after 100 s.

Problem 5 - A rope of mass M and length l lies on a frictionless table, with a short portion, l_0 , hanging through a hole. Initially the rope is at rest.

- (a) Find a general equation for $x(t)$, the length of rope through the hole.
- (b) Evaluate all unknown constants so that the initial conditions are satisfied.

Problem 6 - An inverted garbage can of weight TfV is suspended in air by water from a geyser.



The water shoots up from the ground with a speed v_0 , at a constant rate dm/dt . The problem is to find the maximum height at which the garbage can rides. What assumption must be fulfilled for the maximum height to be reached?

Problem 7 - A raindrop of initial mass M_0 starts falling from rest under the influence of gravity. Assume that the drop gains mass from the cloud at a rate proportional to the product of its instantaneous mass and its instantaneous velocity:

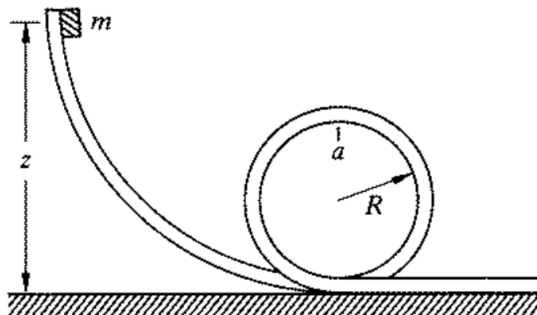
$$\frac{dM}{dt} = kMV$$

where k is a constant.

Show that the speed of the drop eventually becomes effectively constant, and give an expression for the terminal speed. Neglect air resistance.

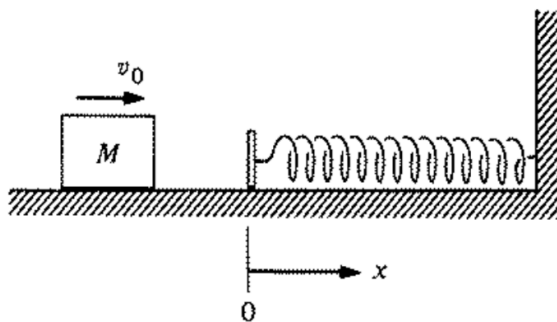
Assignment 7 - Problems

Problem 1 - A small block of mass m starts from rest and slides along a frictionless loop-the-loop as shown in the figure.



What should be the initial height z , so that m pushes against the top of the track (at a) with a force equal to its weight?

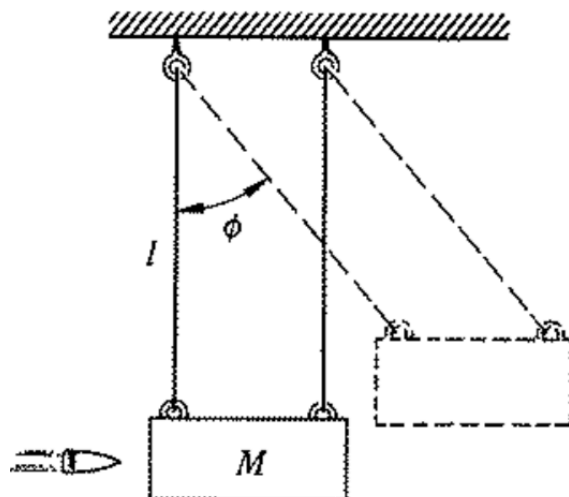
Problem 2 - A block of mass M slides along a horizontal table with speed v_0 . At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force (see figure).



The coefficient of friction is variable and is given by $\mu = bx$, where b is a constant. Find the loss in mechanical energy when the block has first come momentarily to rest.

Problem 3 - A simple way to measure the speed of a bullet is with a *ballistic pendulum*. As illustrated, this consists of a wooden block of mass M into

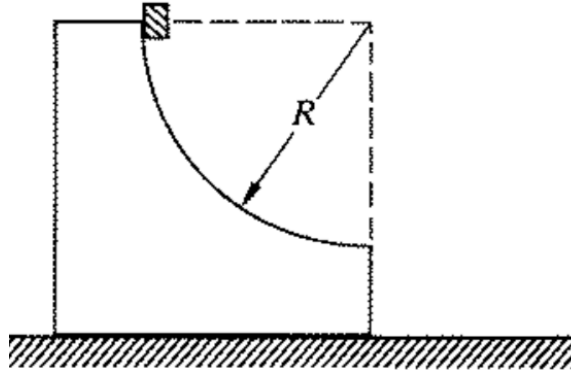
which the bullet is shot.



The block is suspended from cables of length l , and the impact of the bullet causes it to swing through a maximum angle ϕ , as shown. The initial speed of the bullet is v , and its mass is m .

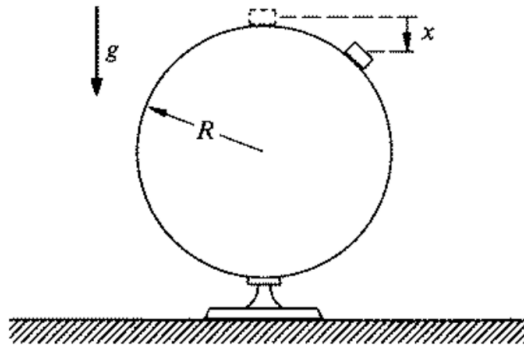
- (a) How fast is the block moving immediately after the bullet comes to rest? (Assume that this happens quickly.)
- (b) Show how to find the velocity of the bullet by measuring m , M , l , and ϕ .

Problem 4 - A small cube of mass m slides down a circular path of radius R cut into a large block of mass M . If, as shown below.



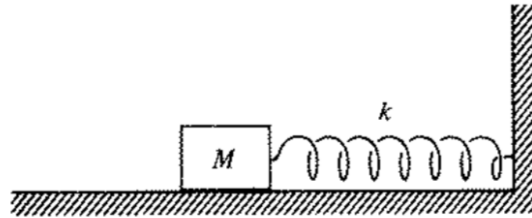
M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path. Find the velocity v of the cube as it leaves the block.

Problem 5 - A small block slides from rest from the top of a frictionless sphere of radius R (see below).



How far below the top x does it lose contact with the sphere? The sphere does not move.

Problem 6 - A block of mass M on a horizontal frictionless table is connected to a spring (spring constant k), as shown.



The block is set in motion so that it oscillates about its equilibrium point with a certain amplitude A_0 . The period of motion is $T_0 = 2\pi\sqrt{M/k}$.

- (a) A lump of sticky putty of mass m is dropped onto the block. The putty sticks without bouncing. The putty hits M at the instant when the velocity of M is zero. Find
- (1) The new period
 - (2) The new amplitude
 - (3) The change in the mechanical energy of the system
- (b) Repeat part (a), but this time assume that the sticky putty hits M at the instant when M has its maximum velocity.

Problem 7 - During the Second World War the Russians, lacking sufficient chutes for airborne operations, occasionally dropped soldiers inside bales of hay onto snow. The human body can survive an average pressure on impact of 30 lb/in^2 .

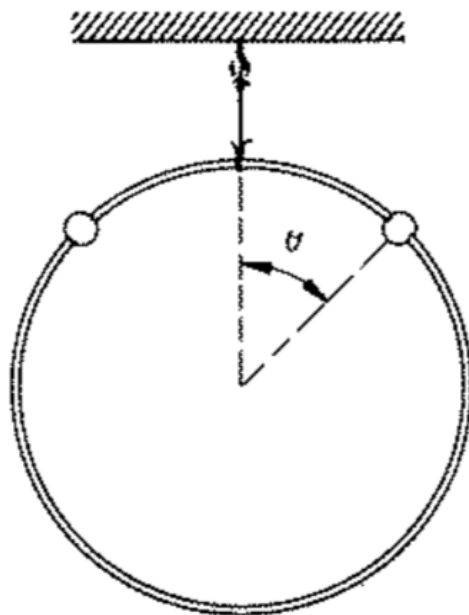
Suppose that the lead plane drops a dummy bale equal in weight to a loaded one from an altitude of 150 ft, and that the pilot observes that it sinks about 2 ft into the snow. If the weight of an average soldier is 144lb and his effective area is 5 ft^2 , is it safe to drop the men?

Problem 8 - A snowmobile climbs a hill at 15 mi/hr. The hill has a grade

of 1 ft rise for every 40 ft. The resistive force due to the snow is 5% of the vehicle's weight. How fast will the snowmobile move downhill, assuming its engine delivers the same power?

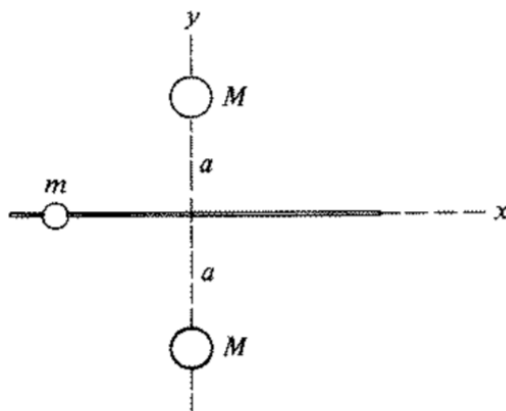
Assignment 8 - Problems

Problem 1 - A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction (see figure).



The beads are released simultaneously from the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > 3M/2$, and find the angle at which this occurs.

Problem 2 - A bead of mass m slides without friction on a smooth rod along the x axis. The rod is equidistant between two spheres of mass M . The spheres are located at $x = 0, y = \pm a$ as shown, and attract the bead gravitationally.



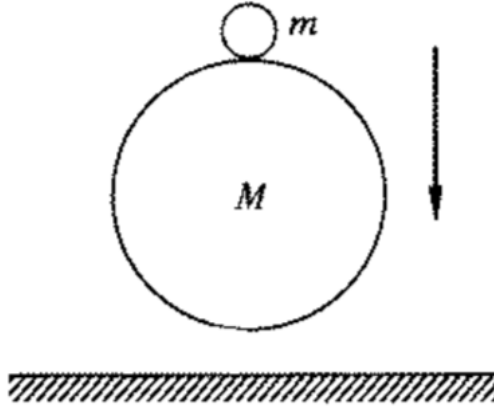
- (a) Find the potential energy of the bead.
- (b) The bead is released at $x = 3a$ with velocity v_0 toward the origin. Find the speed as it passes the origin.
- (c) Find the frequency of small oscillations of the bead about the origin.

Problem 3 - A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed toward the origin with magnitude B , and an inverse square law repulsive force with magnitude A/x^2 .

- (a) Find the potential energy function $U(x)$.
- (b) Sketch the energy diagram for the system when the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.
- (c) Find the equilibrium position, x_0
- (d) What is the frequency of small oscillations about x_0 ?

Problem 4 - A ball drops to the floor and bounces, eventually coming to rest. Collisions between the ball and floor are inelastic; the speed after each collision is e times the speed before the collision where $e < 1$, (e is called the *coefficient of restitution*). If the speed just before the first bounce is v_0 , find the time to come to rest.

Problem 5 - A small ball of mass m is placed on top of a “superball” of mass M , and the two balls are dropped to the floor from height h .



How high does the small ball rise after the collision? Assume that collisions with the superball are elastic, and that $m \ll M$. To help visualize the problem, assume that the balls are slightly separated when the superball hits the floor. (If you are surprised at the result, try demonstrating the problem with a marble and a superball).

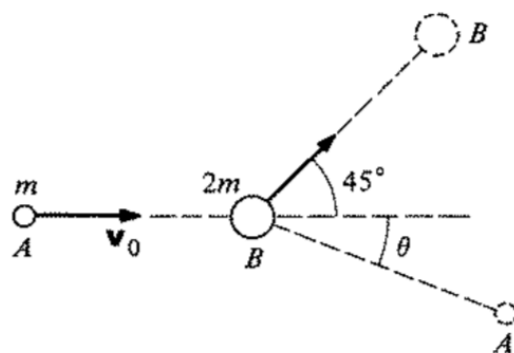
Problem 6 - Cars B and C are at rest with their brakes off.



Car A plows into B at high speed, pushing B into C . If the collisions are completely inelastic, what fraction of the initial energy is dissipated in car C ? Initially the cars are identical.

Problem 7 - A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with $\frac{4}{9}$ of its initial kinetic energy. Find the ratio of the mass of the unknown particle to the mass of the proton, assuming the collision is elastic.

Problem 8 - Particle A of mass m has initial velocity v_0 . After colliding with particle B of mass $2m$ initially at rest, the particles follow the paths shown in the figure.

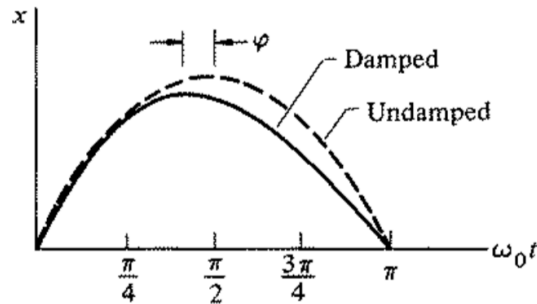


Find the angle θ .

Assignment 9 - Problems

Problem 1 - A 0.3 kg mass is attached to a spring and oscillates at 2 Hz with a Q of 60. Find the spring constant and the damping constant.

Problem 2 - In an undamped free harmonic oscillator the motion is given by $x = A \sin \omega_0 t$. The displacement is maximum exactly midway between the zero crossings.



In a damped oscillator the motion is no longer sinusoidal, and the maximum is advanced before the midpoint of the zero crossings. Show that the maximum is advanced by a phase angle ϕ given approximately by

$$\phi = \frac{1}{2Q}$$

where we assume that Q is large.

Problem 3 - The *logarithmic decrement* δ is defined to be the natural logarithm of the ratio of successive maximum displacements (in the same direction) of a free damped oscillator. Show that $\delta = \pi/Q$.

Find the spring constant k and damping constant b of a damped oscillator having a mass of 5 kg, frequency of oscillation 0.5 Hz, and logarithmic decrement of 0.02.

Problem 4 - If the damping constant of a free oscillator is given by $\gamma = 2\omega_0$, the system is said to be critically damped. Show by direct substitution that

in this case the motion is given by

$$x = (A + Bt)e^{-(\gamma/2)t}$$

where A and B are constants

A critically damped oscillator is at rest at equilibrium. At $t = 0$ it is given a blow of total impulse I . Sketch the motion, and find the time at which the displacement starts to decrease.

Problem 5

- (a) A mass of 10 kg falls 50 cm onto the platform of a spring scale, and sticks. The platform eventually comes to rest 10 cm below its initial position. The mass of the platform is 2 kg. Find the spring constant.
- (b) It is desired to put in a damping system so that the scale comes to rest in minimum time without overshoot. This means that the scale must be critically damped (see text). Find the necessary damping constant and the equation for the motion of the platform after the mass hits.

Problem 6 - Show that for a lightly damped forced oscillator near resonance

$$\frac{\text{average energy stored in the oscillator}}{\text{average energy dissipated per radian}} \approx \frac{\omega_0}{\gamma} = Q$$

Problem 7 - For a particle undergoing simple harmonic motion, determine whether the maximum value of each of the following occurs at the equilibrium point, or at one of the endpoints:

- (a) the force on the particle
- (b) its speed
- (c) its acceleration

What changes could you make to a simple harmonic oscillator to double its

- (d) maximum speed
- (e) maximum acceleration

Problem 8 - A spring of spring constant k hangs unstretched. A mass m is attached to the free end and released. Show that even though the mass is acted on by gravity as well as the spring force, it still executes simple harmonic motion about a new equilibrium point and that the frequency is independent of g .

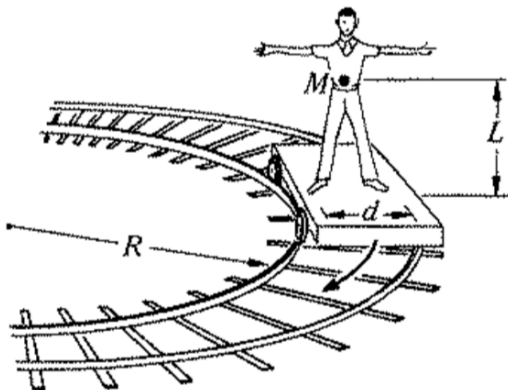
Problem 9 - An elephant of mass 75 kg stands on a platform that goes up and down with simple harmonic motion of amplitude 0.5 meter. Determine the smallest value for the period of the motion so that the elephant will not lose contact with the platform.

Problem 10 - A horizontal board is moving horizontally with simple harmonic motion with an amplitude of 1.5 m. If the board is oscillating at a rate of 15 oscillations per minute, calculate the minimum value of the coefficient of friction in order that a body placed on the board will not slide when the board is moving.

Assignment 10 - Problems

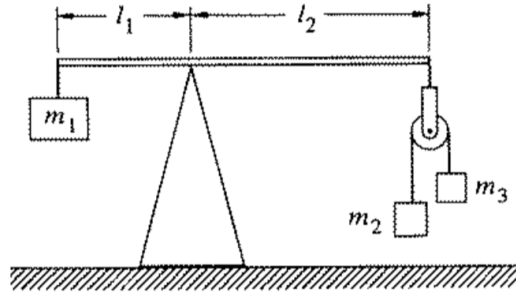
Problem1 - A ring of mass M and radius R lies on its side on a frictionless table. It is pivoted to the table at its rim. A bug of mass m walks around the ring with speed v , starting at the pivot. What is the rotational velocity of the ring when the bug is (a) halfway around and (b) back at the pivot.

Problem 2 - A man of mass M stands on a railroad car which is rounding an unbanked turn of radius R at speed v . His center of mass is height L above the car, and his feet are distance d apart. The man is facing the direction of motion.



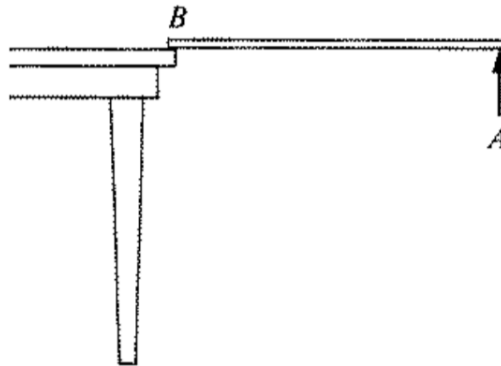
How much weight is on each of his feet?

Problem 3 - A pivoted beam has a mass m_1 suspended from one end and an Atwood's machine suspended from the other (see figure below).



The frictionless pulley has negligible mass and dimension. Gravity is directed downward, and $m_2 > m_3$. Find a relation between m_1 , m_2 , m_3 , l_1 , and l_2 which will ensure that the beam has no tendency to rotate just after the masses are released.

Problem 4 - A uniform stick of mass M and length l is suspended horizontally with end B on the edge of a table, and the other end, A is held by hand.



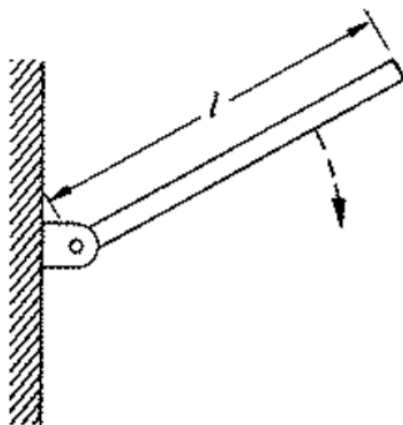
Point A is suddenly released. At the instant after release:

- What is the torque about B ?
- What is the angular acceleration about B ?
- What is the vertical acceleration of the center release:

(d) From part (c), find by inspection the vertical force at B .

Problem 5 - A physical pendulum is made of a uniform disk of mass M and radius R suspended from a rod of negligible mass. The distance from the pivot to the center of the disk is l . What value of l makes the period a minimum?

Problem 6 - A thin plank of mass M and length l is pivoted at one end (see figure below).

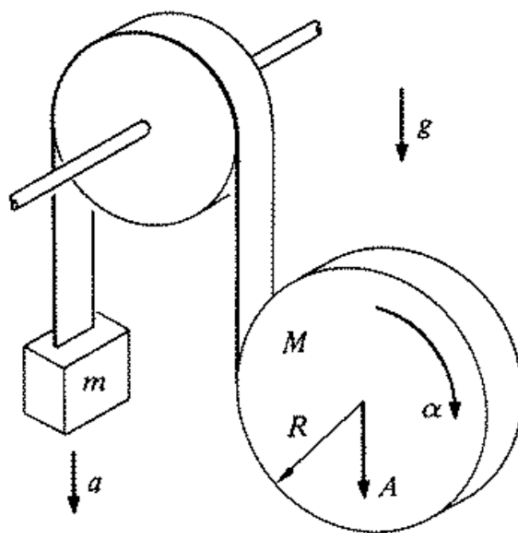


The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?

Assignment 11 - Problems

Problem 1 - A cylinder of radius R and mass M rolls without slipping down a plane inclined at angle θ . The coefficient of friction is μ . What is the maximum value of θ for the cylinder to roll without slipping?

Problem 2 - A disk of mass M and radius R unwinds from a tape wrapped around it (see figure below).



The tape passes over a frictionless pulley, and a mass m is suspended from the other end. Assume that the disk drops vertically.

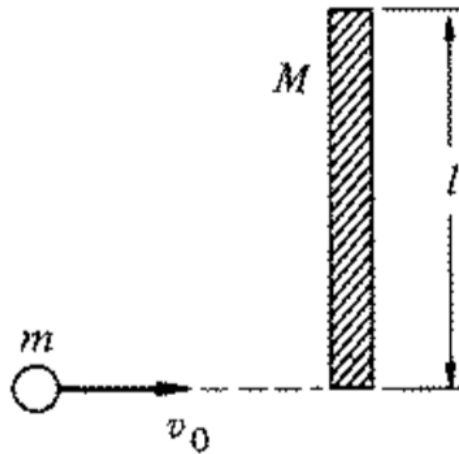
- (a) Relate the accelerations of m and the disk, a and A , respectively, to the angular acceleration of the disk.
- (b) Find a , A , and α .

Problem 3 - A marble of mass M and radius R is rolled up a plane of angle θ . If the initial velocity of the marble is v_0 , what is the distance l it travels up the plane before it begins to roll back down?

Problem 4 - A bowling ball is thrown down the alley with speed v_0 . Initially it slides without rolling, but due to friction it begins to roll. Show that its

speed when it rolls without sliding is $\frac{5}{7}v_0$.

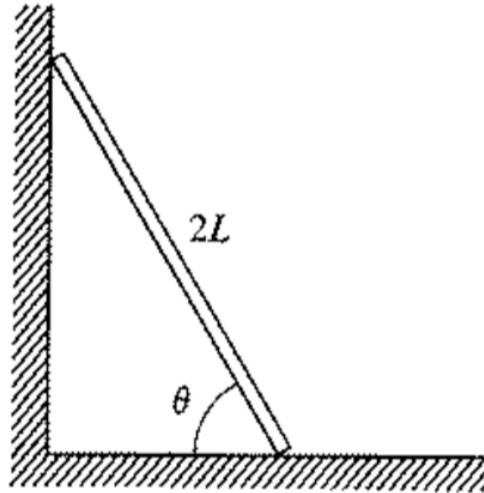
Problem 5 - A boy of mass m runs on ice with velocity v_0 and steps on the end of a of l and mass M which is perpendicular to his path.



- (a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.
- (b) One point on the plank is at rest immediately after the collision. Where is it?

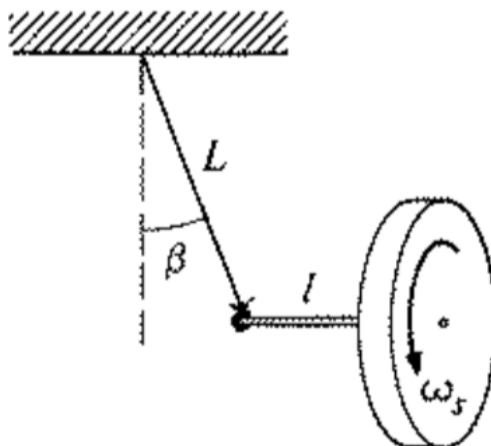
Problem 6 - This problem utilizes most of the important laws introduced worth a substantial effort. However, the problem is tricky (although not really complicated), so don't be alarmed if the solution eludes you.

A plank of length $2L$ leans against a wall.



Show that the top of the plank loses contact with the wall when it is at two-thirds of its initial height. Hint: Only a single variable is needed to describe the system. Note the motion of the center of mass.

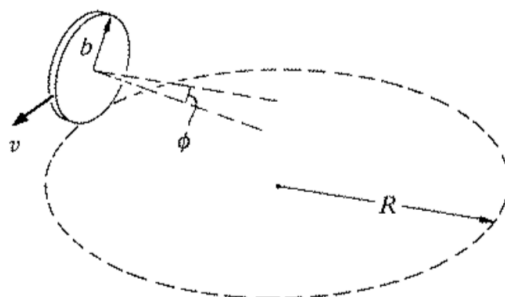
Problem 7 - A gyroscope wheel is at one end of an axle of length l . The other end of the axle is suspended from a string of length L .



The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of inertia about its center of mass I_0 . Its spin angular velocity is ω_s . Neglect the mass of the shaft and of the string.

Find the angle β that the string makes with the vertical. Assume that β is so small that approximations like $\sin \beta \approx \beta$ are justified.

Problem 8 - If you start a coin rolling on a table with care, you can make it roll in a circle. As you can see from the figure, the coin “leans” inward, with its axis tilted.



The radius of the coin is b , the radius of the circle it follows on the table is R , and its velocity is v . Assume that there is no slipping. Find the angle ϕ that the axis makes with the horizontal.

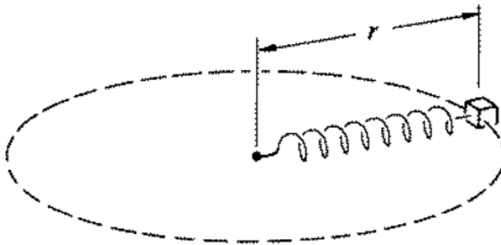
Assignment 12 - Problems

Problem 1 - A particle of mass 50 g moves under an attractive central force of magnitude $4r^3$ dynes. The angular momentum is equal to $1,000 \text{ g} \cdot \text{cm}^2/\text{s}$.

- (a) Find the effective potential energy.
- (b) Indicate on a sketch of the effective potential, the total energy for circular motion.
- (c) The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0 .

Problem 2 - A particle moves in a circle under the influence of an inverse cube law force. Show that the particle can also move with uniform radial velocity, either in or out. (This is an example of unstable motion. Any slight perturbation to the circular orbit will start the particle moving radially, and it will continue to do so.) Find θ as a function of r for motion with uniform radial velocity. **Problem 3** - For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$, where $A > 0$.

Problem 4 - A 2-kg mass on a frictionless table is attached to one end of a massless spring. The other end of the spring is held by a frictionless pivot.



The spring produces a force of magnitude $3r$ newtons on the mass, where r is the distance in meters from the pivot to the mass. The mass moves in a circle and has a total energy of 12 J.

- (a) Find the radius of the orbit and the velocity of the mass.

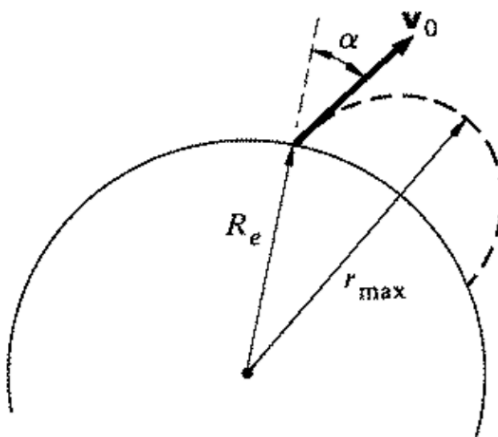
- (b) The mass is struck by a sudden sharp blow, giving it instantaneous velocity of 1 m/s radially outward. Show that the state of the system before and after the blow on a sketch of the energy diagram.
- (c) For the new orbit, find the maximum and minimum values of r .

Problem 5 - A particle of mass m moves under an attractive central force Kr^4 with angular momentum l . For what energy will the motion be circular, and what is the radius of the circle? Find the frequency of radial oscillations if the particle is given a small radial impulse.

Problem 6 - A rocket is in elliptic orbit around the earth. To put it into an escape orbit, its engine is fired briefly, changing the rocket's velocity by $\Delta\vec{V}$. Where in the orbit, and in what direction, should the firing occur to attain escape with a minimum value of $\Delta\vec{V}$?

Assignment 13 - Problems

Problem 1 - A projectile of mass m is fired from the surface of the earth at an angle α from the vertical.



The initial speed v_0 is equal to $\sqrt{GM_e/R_e}$. How high does the projectile rise? Neglect air resistance and the earth's rotation. (Hint: It is probably easier to apply the conservation laws directly instead of using the orbit equations.)

Problem 2 - Halley's comet is in an elliptic orbit about the sun. The eccentricity of the orbit is 0.967 and the period is 76 years. The mass of the sun is 2×10^{30} kg, and $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$.

- (a) Using these data, determine the distance the sun at perihelion and at aphelion.
- (b) What is the speed of Halley's comet when it is closest to the sun?

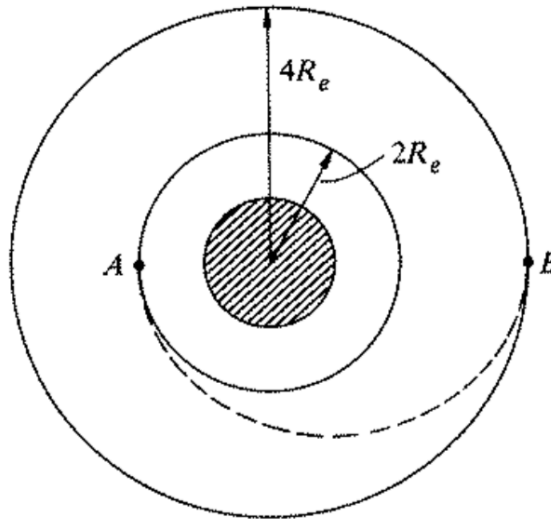
Problem 3

- (a) A satellite of mass m is in circular orbit about the earth. The radius of the orbit is r_0 and the mass of the earth is M_e . Find the total mechanical energy of the satellite.

- (b) Now suppose that the satellite moves in the extreme upper atmosphere of the earth where it is retarded by a constant feeble friction force f . The satellite will slowly spiral toward the earth. Since the friction force is weak, the change in radius will be very slow. We can therefore assume that at any instant the satellite is effectively of average radius r . Find the approximate change in in a circular orbit radius per revolution of the satellite, Δr .
- (c) Find the approximate change in kinetic energy of the satellite per revolution, ΔK .

Problem 4 - Before landing men on the moon, the Apollo 11 space vehicle was put into orbit about the moon. The mass of the vehicle was 9,979 kg and the period of the orbit was 119 min. The maximum and minimum distances from the center of the moon were 1,861 km and 1,838 km. Assuming the moon to be a uniform spherical body, what is the mass of the moon according to these data? $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$

Problem 5 - A space vehicle is in circular orbit about the earth. The mass of the vehicle is 3,000kg and the radius of the orbit is $2R_e = 12,800 \text{ km}$. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$.



- (a) What is the minimum energy expenditure required for the transfer?
- (b) An efficient way to accomplish the transfer is to use a semielliptical orbit (known as a Hohmann transfer orbit), as shown. What velocity changes are required at the points of intersection, A and B ?

Problem 6 - Black Hole Physics

Although a Black Hole is really a complicated solution to Einstein's equations of general relativity, we can use those equations to derive a potential that can be used with the methods we have developed to study the motion of a small point mass in the vicinity of a Black Hole. The result is:

$$V_{eff}(r) = -\frac{GM}{r} + \frac{L^2}{2mr^2} - \frac{1}{2} \left(\frac{L}{mc} \right)^2 \frac{GM}{r^3}$$

where

M = mass of Black Hole, c = speed of light

L = angular momentum of small mass, G = Gravitational constant

m = small mass (choose = 1)

- (1) For general L , find the location of maxima, minima, and points of inflection of the effective potential energy function.
- (2) Plot the effective potential for different values of L and determine the corresponding motions qualitatively. Look at the range

$$\frac{2GM}{c} \leq L \leq \frac{5GM}{c}$$

Why might we conclude that this potential represents a "black hole"?

Assignment 14 - Problems

Problem 1 - An event occurs in S at $x = 6 \times 10^8$ m, and in S' at $x' = 6 \times 10^8$ m, $t' = 4$ s. Find the relative velocity of the systems.

Problem 2 - An observer sees two spaceships flying apart with speed $0.99c$. What is the speed of one spaceship as viewed by the other?

Problem 3 - Two particles of rest mass m_0 approach each other with equal and opposite velocity v , in the laboratory frame. What is the total energy of one particle as measured in the rest frame of the other?

Problem 4 - A particle of rest mass m_0 and kinetic energy xm_0c^2 , where x is some number, strikes and sticks to an identical particle at rest. What is the rest mass of the resultant particle?)

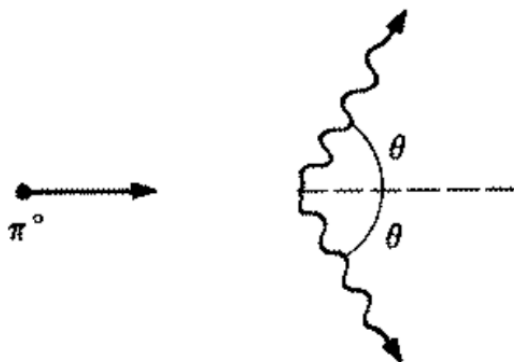
Problem 5 - In the laboratory frame a particle of rest mass m_0 and speed v is moving toward a particle of mass m_0 at rest. What is the speed of the inertial of the system is zero?

Problem 6 - A 1-kW light beam from a laser is used to levitate a solid aluminum sphere by focusing it on the sphere from below.



What is the diameter of the sphere, assuming that it floats freely in the light beam? The density of aluminum is 2.7 g/cm^3 .

Problem 7 - A neutral pi meson, rest mass 135 MeV, decays symmetrically into two photons while moving at high speed. The energy of each photon in the laboratory system is 100 MeV.



- (a) Find the meson's speed V . Express your answer as a ratio V/c .
- (b) Find the angle θ in the laboratory system between the momentum of each photon and the initial line of motion.

Problem 8 - A high energy photon (γ ray) collides with a proton at rest. A neutral pi meson (π^0) is produced according to the reaction

$$\gamma + p \rightarrow p + \pi^0$$

What is the minimum energy the γ ray must have for this reaction to occur? The rest mass of a proton is 938 MeV, and the rest mass of a π^0 is 135 MeV.

Assignment 1 - Solutions

Problem 1

$$1-1 \quad \vec{A} = 2\hat{i} - 3\hat{j} + 7\hat{k}, \quad \vec{B} = 5\hat{i} + \hat{j} + 2\hat{k}$$

$$(a) \quad \vec{A} + \vec{B} = (2+5)\hat{i} + (-3+1)\hat{j} + (7+2)\hat{k} \\ = 7\hat{i} - 2\hat{j} + 9\hat{k}$$

$$(b) \quad \vec{A} - \vec{B} = (2-5)\hat{i} + (-3-1)\hat{j} + (7-2)\hat{k} \\ = -3\hat{i} - 4\hat{j} + 5\hat{k}$$

$$(c) \quad \vec{A} \cdot \vec{B} = \overset{A_x}{(2)}\overset{B_x}{(5)} + \overset{A_y}{(-3)}\overset{B_y}{(1)} + \overset{A_z}{(7)}\overset{B_z}{(2)} \\ = 10 - 3 + 14 = 21$$

$$(d) \quad \vec{A} \times \vec{B} = \begin{pmatrix} \overset{A_1}{(-3)}\overset{B_2}{(2)} - \overset{A_2}{(7)}\overset{B_1}{(1)} \\ \overset{A_2}{(7)}\overset{B_3}{(5)} - \overset{A_3}{(2)}\overset{B_2}{(2)} \\ \overset{A_3}{(2)}\overset{B_1}{(1)} - \overset{A_1}{(-3)}\overset{B_3}{(5)} \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \\ = (-6 - 7)\hat{i} + (35 - 4)\hat{j} + (2 + 15)\hat{k} \\ = -13\hat{i} + 31\hat{j} + 17\hat{k}$$

Problem 2

$$1-2 \quad \vec{A} = 3\hat{i} + \hat{j} + \hat{k}, \quad \vec{B} = -2\hat{i} - 3\hat{j} - \hat{k}$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB}$$

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z = (3)(-2) + (1)(-3) + (1)(-1) \\ &= -6 - 3 - 1 = -10 \end{aligned}$$

$$A^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 = 9 + 1 + 1 = 11 \rightarrow A = \sqrt{11}$$

$$B^2 = \vec{B} \cdot \vec{B} = B_x^2 + B_y^2 + B_z^2 = 4 + 9 + 1 = 14 \rightarrow B = \sqrt{14}$$

$$\rightarrow \cos \theta = \frac{-10}{\sqrt{154}} = -0.805$$

$$\theta = 143.6^\circ$$

Problem 3

1-4 If $|A-B| = |A+B|$, then

$$|A-B|^2 = |A+B|^2$$

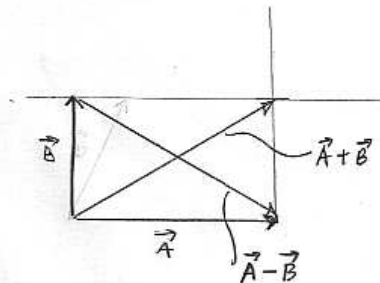
$$\Rightarrow (\vec{A}-\vec{B}) \cdot (\vec{A}-\vec{B}) = (\vec{A}+\vec{B}) \cdot (\vec{A}+\vec{B})$$

$$A^2 - 2\vec{A} \cdot \vec{B} + B^2 = A^2 + 2\vec{A} \cdot \vec{B} + B^2$$

$$\rightarrow 4\vec{A} \cdot \vec{B} = 0$$

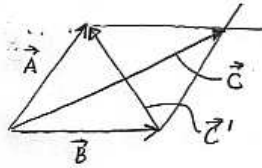
$$\rightarrow \vec{A} \cdot \vec{B} = 0$$

$\rightarrow \vec{A}$ is perpendicular (orthogonal) to $\vec{B}$



Problem 4

1-5



$$\vec{A} = \vec{B} \rightarrow A^2 = B^2$$

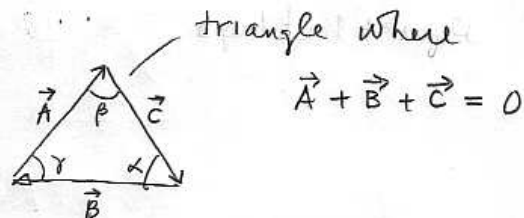
$$\left. \begin{aligned} \vec{C} &= \vec{A} + \vec{B} \neq 0 \\ \vec{C}' &= \vec{A} - \vec{B} \neq 0 \end{aligned} \right\} \rightarrow \text{diagonals}$$

$$\begin{aligned} \vec{C} \cdot \vec{C}' &= (\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) \\ &= \vec{A}^2 + \vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{B} - \vec{B}^2 \\ &= A^2 - B^2 = 0 \end{aligned}$$

$\rightarrow \vec{C}$ is orthogonal to $\vec{C}'$

Problem 5

1-6



$$\begin{aligned} \text{Therefore } 0 &= \vec{A} \times (\underbrace{\vec{A} + \vec{B} + \vec{C}}_{=0}) \\ &= \underbrace{\vec{A} \times \vec{A}}_{=0} + \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \end{aligned}$$

$$\therefore 0 = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$\rightarrow \vec{A} \times \vec{B} = -\vec{A} \times \vec{C}$$

$$\rightarrow |\vec{A} \times \vec{B}| = |\vec{A} \times \vec{C}|$$

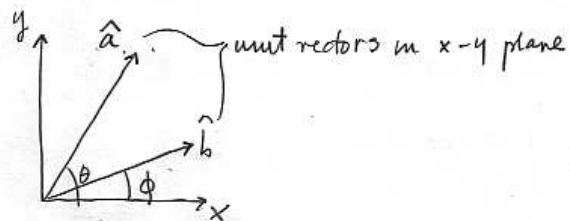
$$\rightarrow AB \sin \gamma = AC \sin \beta$$

$$\rightarrow \frac{B}{\sin \beta} = \frac{C}{\sin \gamma}, \text{ etc}$$

$\rightarrow$ Law of sines.

Problem 6

1-7



$$\begin{aligned}\hat{a} &= (\hat{a} \cdot \hat{i}) \hat{i} + (\hat{a} \cdot \hat{j}) \hat{j} = a_x \hat{i} + a_y \hat{j} \\ &= (1)(1) \cos \theta \hat{i} + (1)(1) \cos(\frac{\pi}{2} - \theta) \hat{j} \\ &= \cos \theta \hat{i} + \sin \theta \hat{j}\end{aligned}$$

Similarly,

$$\hat{b} = \cos \phi \hat{i} + \sin \phi \hat{j}$$

$$\hat{a} \cdot \hat{b} = (1)(1) \cos(\theta - \phi)$$

$$= a_x b_x + a_y b_y$$

$$= (\cos \theta)(\cos \phi) + (\sin \theta)(\sin \phi)$$

$$\rightarrow \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$$

Problem 7

$$1-8 \quad \vec{A} = \hat{i} + \hat{j} - \hat{k}, \quad \vec{B} = 2\hat{i} - \hat{j} + 3\hat{k}$$

$\hat{n}$ = unit vector such that

$$\left. \begin{array}{l} \vec{A} \cdot \hat{n} = 0 \\ \vec{B} \cdot \hat{n} = 0 \end{array} \right\} \hat{n} \text{ orthogonal to both } \vec{A} \text{ and } \vec{B}$$

$$\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$

$$\rightarrow \vec{A} \cdot \hat{n} = 0 = n_x(1) + n_y(1) + n_z(-1) = n_x + n_y - n_z$$

$$\vec{B} \cdot \hat{n} = 0 = n_x(2) + n_y(-1) + n_z(3) = 2n_x - n_y + 3n_z$$

$$\rightarrow \begin{array}{l} n_x + n_y - n_z = 0 \\ 2n_x - n_y + 3n_z = 0 \end{array}$$

$$\text{add equations} \rightarrow 3n_x + 2n_z = 0 \rightarrow n_x = -\frac{2}{3}n_z$$

$$\rightarrow -\frac{2}{3}n_z + n_y - n_z = 0 \rightarrow n_y = \frac{5}{3}n_z$$

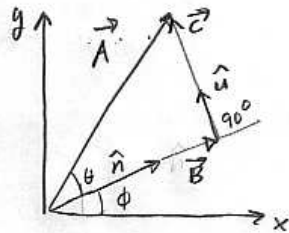
$$\hat{n} \cdot \hat{n} = 1 = n_x^2 + n_y^2 + n_z^2 = \left(\frac{4}{9} + \frac{25}{9} + 1 \right) n_z^2$$

$$\rightarrow n_z = \pm \sqrt{\frac{1}{38/9}} = \pm \frac{3}{\sqrt{38}}$$

$$\rightarrow \hat{n} = \pm \frac{1}{\sqrt{38}} (-2\hat{i} + 5\hat{j} + 3\hat{k})$$

Problem 8

1-11



From the diagram

$$\vec{A} = \vec{B} + \vec{C}$$

$$\vec{A} \cdot \hat{n} = A \cos \theta$$

$$B = A \cos \theta = \vec{A} \cdot \hat{n} \rightarrow \vec{B} = (\vec{A} \cdot \hat{n}) \hat{n}$$

$$C = A \sin \theta$$

$$\hat{n} \times \vec{A} = A \sin \theta \hat{k}$$

$$(\hat{n} \times \vec{A}) \times \hat{n} = A \sin \theta (\hat{k} \times \hat{n})$$

$$= A \sin \theta \hat{u} = \vec{C}$$

$$\therefore \vec{A} = \vec{B} + \vec{C}$$

$$= (\vec{A} \cdot \hat{n}) \hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$$

Problem 9

EPI

$$\vec{A} = 5\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{B} = -3\hat{i} + \hat{j} - 7\hat{k}$$

$$\vec{C} = 4\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\vec{D} = \vec{A} + \vec{B} + \vec{C} = 6\hat{i} + 3\hat{j} - 2\hat{k}$$

$$|\vec{D}| = D = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$$

$$\cos \theta_x = \frac{\vec{D} \cdot \hat{i}}{D} = \frac{6}{7} \rightarrow \theta_x = 31.0^\circ$$

$$\cos \theta_y = \frac{\vec{D} \cdot \hat{j}}{D} = \frac{3}{7} \rightarrow \theta_y = 64.6^\circ$$

$$\cos \theta_z = \frac{\vec{D} \cdot \hat{k}}{D} = -\frac{2}{7} \rightarrow \theta_z = 106.6^\circ$$

Problem 10

EP2

$$\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$$

$$\vec{B} = -\hat{i} + \hat{j} + 2\hat{k}$$

$\vec{A} \times \vec{B}$ is orthogonal to $\vec{A}$ and $\vec{B}$ and therefore it is orthogonal to the plane containing $\vec{A}$ and $\vec{B}$

a unit vector orthogonal to the plane is

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k} \\ &= (4)(2) - (-5)(1)\hat{i} + (-5)(-1) - (3)(2)\hat{j} + (3)(1) - (-4)(1)\hat{k} \\ &= 13\hat{i} - 11\hat{j} + 8\hat{k}\end{aligned}$$

$$|\vec{A} \times \vec{B}| = \sqrt{13^2 + 11^2 + 8^2} = \sqrt{354}$$

$$\therefore \hat{n} = \frac{1}{\sqrt{354}} (13\hat{i} - 11\hat{j} + 8\hat{k})$$

Problem 11

EP3

$$\vec{A} = 5\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{B} = -3\hat{i} + \hat{j} - 7\hat{k}$$

$$\vec{C} = 4\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\begin{aligned}\vec{B} \times \vec{C} &= (B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k} \\ &= ((1)(4) - (-7)(4))\hat{i} + ((-7)(4) - (-3)(4))\hat{j} + ((-3)(4) - (1)(4))\hat{k} \\ &= 32\hat{i} - 16\hat{j} - 16\hat{k}\end{aligned}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (5)(32) + (-2)(-16) + (1)(-16) = 112$$

$$\begin{aligned}\vec{A} \times (\vec{B} \times \vec{C}) &= ((-2)(-16) - (1)(-16))\hat{i} + ((1)(32) - (5)(-16))\hat{j} \\ &\quad + ((5)(-16) - (-2)(32))\hat{k} \\ &= 48\hat{i} + 112\hat{j} - 16\hat{k}\end{aligned}$$

$$\vec{A} \cdot \vec{C} = 20 - 8 + 4 = 16$$

$$\vec{A} \cdot \vec{B} = -15 - 2 - 7 = -24$$

$$(\vec{A} \cdot \vec{C})\vec{B} = -48\hat{i} + 16\hat{j} - 112\hat{k}$$

$$(\vec{A} \cdot \vec{B})\vec{C} = -96\hat{i} - 96\hat{j} - 96\hat{k}$$

$$(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = 48\hat{i} + 112\hat{j} - 16\hat{k} = \vec{A} \times (\vec{B} \times \vec{C})$$

Problem 12

EP4

$$\vec{r}(t) = 2t^2\hat{i} + 5t\hat{j} - t^3\hat{k}$$

$$(a) \quad \vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + 5\hat{j} - 3t^2\hat{k}$$

$$v(t) = \sqrt{16t^2 + 25 + 9t^4}$$

$$\begin{aligned} v(t=2) &= \sqrt{64 + 25 + 144} = \sqrt{333} \\ &= 18.2 \text{ m/s} \end{aligned}$$

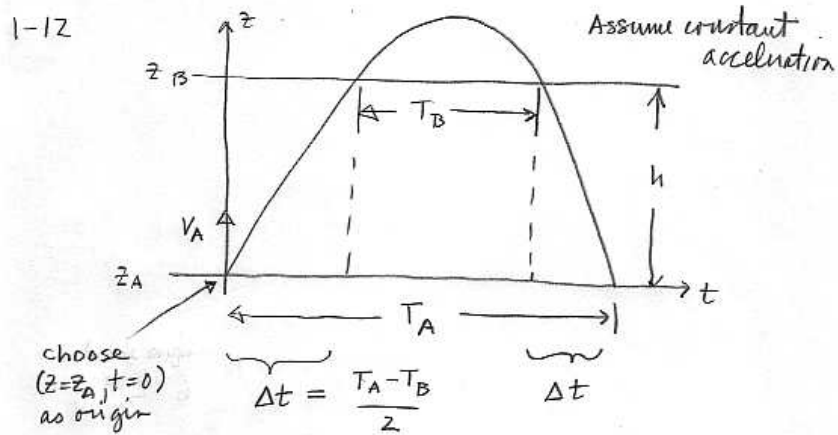
$$(b) \quad \vec{a} = \frac{d\vec{v}}{dt} = 4\hat{i} - 6t\hat{k}$$

$$\vec{a}(t=2) = 4\hat{i} - 12\hat{k}$$

$$\begin{aligned} a &= \sqrt{16 + 144} = \sqrt{160} \\ &= 12.6 \text{ m/s}^2 \end{aligned}$$

Assignment 2 - Solutions

Problem 1



We have $z = z_A + V_A t - \frac{1}{2} g t^2$

When $t = T_A$, $z = z_A$ (again)

$$\rightarrow z_A = z_A + V_A T_A - \frac{1}{2} g T_A^2 \rightarrow V_A = \frac{1}{2} g T_A^2$$

When $t = \frac{T_A - T_B}{2}$, $z = h + z_A$

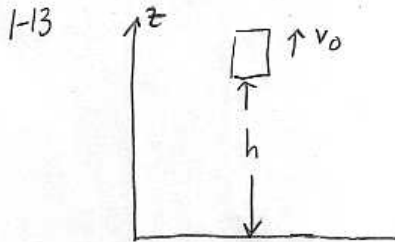
$$\rightarrow h + z_A = z_A + V_A \frac{T_A - T_B}{2} - \frac{1}{2} g \left(\frac{T_A - T_B}{2} \right)^2$$

$$\rightarrow h = \frac{1}{4} g \left[T_A^2 - T_A T_B - \frac{1}{2} T_A^2 + T_A T_B - \frac{1}{2} T_B^2 \right]$$

$$= \frac{1}{4} g \frac{(T_A^2 - T_B^2)}{2}$$

$$\rightarrow g = \frac{8h}{T_A^2 - T_B^2}$$

Problem 2



at $t = T_1$, $z = h$, $v = v_0$, $a = -g$

For $t > T_1$

$$z = h + v_0(t - T_1) - \frac{1}{2}g(t - T_1)^2$$

at $t = T_2 + T_1$, $z = 0$

$$\rightarrow 0 = h + v_0(T_2 + T_1 - T_1) - \frac{1}{2}g(T_2 + T_1 - T_1)^2$$

$$\text{or } 0 = h + v_0 T_2 - \frac{1}{2}g T_2^2$$

Now at $t = 0$, $z = 0$ and between 0 and T_1 elevator moves uniformly with v_0

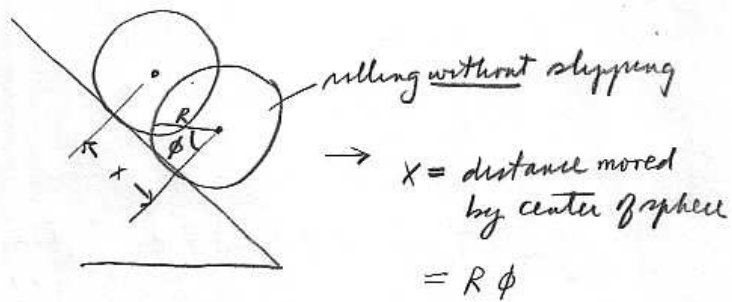
$$\rightarrow h = v_0 T_1$$

$$\therefore 0 = h + \frac{h}{T_1} T_2 - \frac{1}{2}g T_2^2$$

$$\text{or } h\left(1 + \frac{T_2}{T_1}\right) = \frac{1}{2}g T_2^2 \rightarrow h = \frac{1}{2}g \frac{T_1 T_2^2}{T_1 + T_2}$$

Problem 3

1-14



Where $\phi = \text{angle of rotation.}$

In this case, we then have

$$\dot{x} = \text{velocity of center of sphere}$$

$$= R \dot{\theta} = R \omega$$

Where $\dot{\theta} = \text{angular velocity} = \omega$

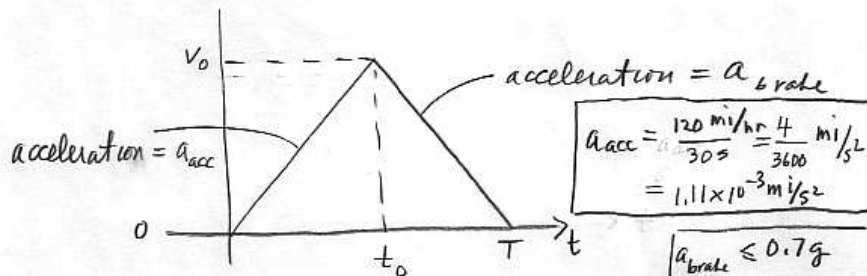
$$\text{and } \ddot{x} = \text{acceleration of center of sphere}$$

$$= R \ddot{\theta} = R \alpha = a$$

$$\rightarrow \alpha = \text{angular acceleration} = \frac{a}{R}$$

Problem 4

1-16 We assume that the car accelerates and then immediately decelerates (brakes) with no wasting in between.



$$\rightarrow v_0 = a_{acc} t_0 = a_{brake} (T - t_0)$$

$$\rightarrow t_0 = \frac{a_{brake} T}{a_{acc} + a_{brake}}$$

$$\begin{aligned} a_{brake} &\leq 0.7g \\ &\leq 0.7 \times 32 \text{ ft/sec}^2 \\ &\leq 22.4 \text{ ft/sec}^2 \\ &\leq 4.24 \times 10^{-3} \text{ mi/s}^2 \end{aligned}$$

For constant acceleration

$$\begin{aligned} s = \text{distance traveled} &= \frac{1}{2} a_{acc} t_0^2 + v_0 (T - t_0) - \frac{1}{2} a_{brake} (T - t_0)^2 \\ &= \frac{1}{2} v_0 t_0 + v_0 T - v_0 t_0 - \frac{1}{2} v_0 (T - t_0) = \frac{1}{2} v_0 T \end{aligned}$$

$$\rightarrow s = \frac{1}{2} a_{acc} t_0 T = \frac{1}{2} \frac{a_{acc} a_{brake}}{a_{acc} + a_{brake}} T^2 = \frac{1}{2} \text{ mi}$$

$$\rightarrow T = \left[\frac{2s (a_{acc} + a_{brake})}{a_{acc} a_{brake}} \right]^{1/2}$$

$$a_{acc} = 1.11 \times 10^{-3} \text{ mi/s}^2, \quad a_{brake} \leq 4.24 \times 10^{-3} \text{ mi/s}^2$$

T is minimized by making a_{brake} as large as possible $\rightarrow T_{min} = 33.7 \text{ s}$

Problem 5

$$\begin{array}{l} 1-17 \quad \dot{r} = 4 \text{ m/s} = \text{constant} \\ \dot{\theta} = 2 \text{ rad/s} = \text{constant} \end{array} \left. \vphantom{\begin{array}{l} \dot{r} = 4 \text{ m/s} = \text{constant} \\ \dot{\theta} = 2 \text{ rad/s} = \text{constant} \end{array}} \right\} \text{motion in a plane.}$$

$$(a) \quad \vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} = 4 \hat{r} + 2r \hat{\theta}$$

$$r = 3 \text{ m}$$

$$\vec{v} = 4 \hat{r} + 6 \hat{\theta}$$

$$\rightarrow v = \sqrt{4^2 + 6^2} = \sqrt{52} \text{ m/s}$$

$$(b) \quad \vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\theta}$$

$$= -4r \hat{r} + 2(4)(2) \hat{\theta}$$

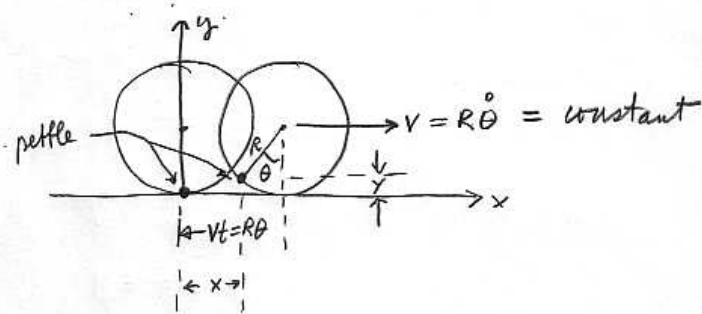
$$r = 3 \text{ m}$$

$$\vec{a} = -12 \hat{r} + 16 \hat{\theta}$$

$$\rightarrow a = \sqrt{12^2 + 16^2} = 20 \text{ m/s}^2$$

Problem 6

1-19



$$x = R\theta - R\sin\theta$$

$$y = R(1 - \cos\theta)$$

$$\dot{x} = R\dot{\theta} - R\cos\theta \dot{\theta} = v(1 - \cos\theta)$$

$$\dot{y} = R\sin\theta \dot{\theta} = v\sin\theta$$

$$\therefore \ddot{x} = v\dot{\theta} \sin\theta = v\omega \sin\theta = R\omega^2 \sin\theta$$

$$\ddot{y} = v\dot{\theta} \cos\theta = v\omega \cos\theta = R\omega^2 \cos\theta$$

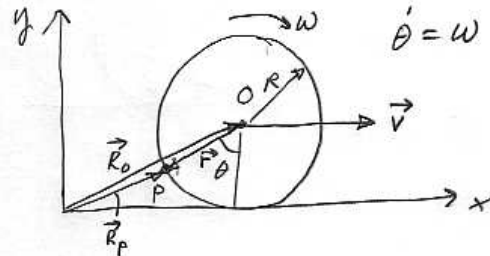
For pettle

$$\vec{r} = x\hat{i} + y\hat{j} = (R\theta - R\sin\theta)\hat{i} + R(1 - \cos\theta)\hat{j}$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = v(1 - \cos\theta)\hat{i} + v\sin\theta\hat{j}$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = R\omega^2 \sin\theta\hat{i} + R\omega^2 \cos\theta\hat{j}$$

alternatively using vectors



we have $\vec{V}_0 = \underline{v} = v \hat{i}$, $vt = R\theta$

$$\vec{R}_0 = \underline{v}t + R\hat{j} = vt\hat{i} + R\hat{j}$$

$$\vec{R}_p = \vec{R}_0 + \vec{r}$$

$$= (vt\hat{i} + R\hat{j}) + \underbrace{[-R\sin\theta\hat{i} - R\cos\theta\hat{j}]}_{\vec{r}}$$

$$= (vt - R\sin\theta)\hat{i} + R(1 - \cos\theta)\hat{j}$$

$$= (R\theta - R\sin\theta)\hat{i} + R(1 - \cos\theta)\hat{j}$$

as before

Problem 7

$$1-20 \quad r = A\theta, \quad A = \frac{1}{\pi} \text{ m/rad}$$

$$(a) \quad \theta = \frac{1}{2}\alpha t^2$$

$$\dot{r} = A\dot{\theta}$$

$$\dot{r} = \frac{\alpha t}{\pi}, \quad \dot{\theta} = \alpha t$$

$$\ddot{r} = \frac{\alpha}{\pi}, \quad \ddot{\theta} = \alpha$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$= \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{r} + \frac{5}{2}\frac{\alpha^2 t^2}{\pi}\hat{\theta}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$= \frac{\alpha t}{\pi}\hat{r} + \frac{\alpha^2 t^3}{2\pi}\hat{\theta}$$

$$(b) \quad a_r = \frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi} = 0 \rightarrow t^4 = \frac{2}{\alpha^2}$$

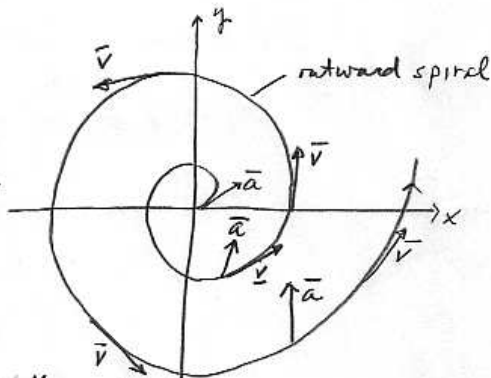
$$\rightarrow a_r = 0 \text{ when } \theta = \frac{\alpha t^2}{2} = \frac{\alpha\sqrt{2}}{2\alpha} = \frac{1}{\sqrt{2}} \text{ rad}$$

$$(c) \quad \vec{a} = \frac{\alpha}{\pi}((1-2\theta^2)\hat{r} + 5\theta\hat{\theta})$$

$$|a_r| = |a_\theta| \rightarrow |1-2\theta^2| = 5|\theta|$$

$$\text{For } \theta < \frac{1}{\sqrt{2}} \rightarrow 1-2\theta^2 = 5\theta \rightarrow \theta = \frac{-5+\sqrt{33}}{4} = 0.19 \text{ rad} = 11^\circ$$

$$\text{For } \theta > \frac{1}{\sqrt{2}} \rightarrow 2\theta^2 - 1 = 5\theta \rightarrow \theta = \frac{5+\sqrt{33}}{4} = 2.69 \text{ rad} = 154^\circ$$



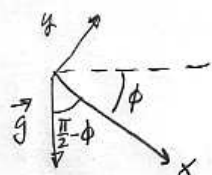
Problem 8

1-21

Approach #1

choose x-y axes
as shown.

acceleration components



$$\begin{aligned}\vec{g} &= \text{acceleration vector} \\ &= g \cos\left(\frac{\pi}{2} - \phi\right) \hat{i} - g \sin\left(\frac{\pi}{2} - \phi\right) \hat{j} \\ &= g \sin \phi \hat{i} - g \cos \phi \hat{j}\end{aligned}$$

$$\rightarrow a_x = g \sin \phi, \quad a_y = -g \cos \phi$$

Velocity components

$$\vec{v}_0 = v_0 \cos(\theta + \phi) \hat{i} + v_0 \sin(\theta + \phi) \hat{j}$$

$$\rightarrow v_{0x} = v_0 \cos(\theta + \phi), \quad v_{0y} = v_0 \sin(\theta + \phi)$$

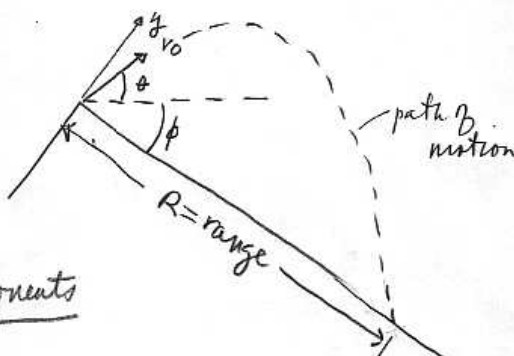
$y=0 \rightarrow$ starting and impact points

$$\rightarrow x = x_0^0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$y = y_0^0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$y=0 = (v_{0y} + \frac{1}{2} a_y t) t \rightarrow t=0 \text{ and } t = -\frac{2v_{0y}}{a_y}$$

$$\text{or } t_{\text{impact}} = \frac{2v_0 \sin(\theta + \phi)}{g \cos \phi}$$



$$x(t_{\text{impact}}) = R = v_{0x} t_{\text{impact}} + \frac{1}{2} a_x t_{\text{impact}}^2$$

$$\rightarrow R = \frac{2v_0^2 \cos(\theta+\phi) \sin(\theta+\phi)}{g \cos \phi} + \frac{2v_0^2 \sin^2(\theta+\phi) \sin \phi}{g \cos^2 \phi}$$

$$\rightarrow R = \frac{2v_0^2 \sin(\theta+\phi)}{g \cos^2 \phi} \left[\underbrace{\cos(\theta+\phi) \cos \phi + \sin(\theta+\phi) \sin \phi}_{\cos(\theta+\phi-\phi) = \cos \theta} \right]$$

$$\rightarrow R = \frac{2v_0^2 \sin(\theta+\phi) \cos \theta}{g \cos^2 \phi}$$

Maximum R is given by

$$\frac{dR}{d\theta} = 0 = \frac{2v_0^2}{g \cos^2 \phi} \left[\underbrace{\cos(\theta+\phi) \cos \theta - \sin(\theta+\phi) \sin \theta}_{\cos(\theta+\phi+\theta)} \right]$$

$$\rightarrow 0 = \cos(2\theta+\phi)$$

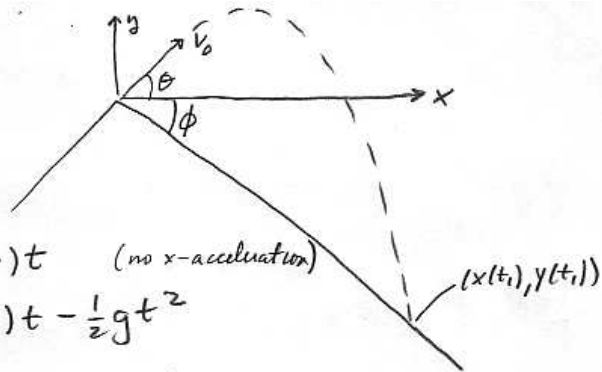
$$\rightarrow 2\theta+\phi = \frac{\pi}{2}$$

$$\rightarrow \theta = \frac{\pi}{4} - \frac{\phi}{2}$$

CHECK $\phi=0$ (hill horizontal)

$$\rightarrow \theta = \frac{\pi}{4} \text{ as expected}$$

Approach #2



$$x = (v_0 \cos \theta) t \quad (\text{no } x\text{-acceleration})$$

$$y = (v_0 \sin \theta) t - \frac{1}{2} g t^2$$

Rock impacts at $t = t_1 \rightarrow$

$$x(t_1) = (v_0 \cos \theta) t_1 \rightarrow t_1 = \frac{x(t_1)}{v_0 \cos \theta}$$

$$\text{and } y(t_1) = (v_0 \sin \theta) t_1 - \frac{1}{2} g t_1^2 = (\tan \theta) x(t_1) - \frac{1}{2} g \frac{x^2(t_1)}{v_0^2 \cos^2 \theta}$$

$$\text{Now } y(t_1) = -(\tan \phi) x(t_1)$$

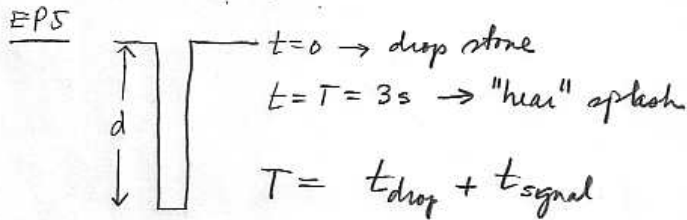
$$\rightarrow x(t_1) = \frac{2v_0^2}{g} (\cos^2 \theta \tan \phi + \cos \theta \sin \theta)$$

Maximize wrt $\theta \rightarrow \theta_{\max}$

$$\rightarrow \tan 2\theta_{\max} = \tan \left(\frac{\pi}{2} - \phi \right)$$

$$\text{or } \theta_{\max} = \frac{\pi}{4} - \frac{\phi}{2} \quad \text{as before}$$

Problem 9



$$d = \frac{1}{2} g t_{\text{drop}}^2$$

$$d = v_{\text{sound}} t_{\text{signal}}$$

$$\rightarrow \frac{1}{2} g t_{\text{drop}}^2 = v_{\text{sound}} t_{\text{signal}}$$

$$\frac{1}{2} g t_{\text{drop}}^2 = v_{\text{sound}} (T - t_{\text{drop}})$$

$$\rightarrow \frac{1}{2} g t_{\text{drop}}^2 + v_{\text{sound}} t_{\text{drop}} - v_{\text{sound}} T = 0$$

$$\rightarrow \text{solutions } t_{\text{drop}} = \frac{-v_s \pm \sqrt{v_s^2 + 2g v_s T}}{g}$$

Must choose + sign for physical solution ($t_{\text{drop}} > 0$)

$$\therefore d = \frac{1}{2} g t_{\text{drop}}^2 = \frac{1}{2g} \left(-v_s + v_s \left(1 + \frac{2gT}{v_s} \right)^{1/2} \right)^2$$

$$v_s \rightarrow \infty \rightarrow d \rightarrow \frac{1}{2g} \left(-v_s + v_s \left(1 + \frac{gT}{v_s} \right) \right)^2$$

$$\rightarrow \frac{1}{2g} (gT)^2 = \frac{1}{2} g T^2 \text{ as expected}$$

$$v_s = 340 \text{ m/s}, g = 10 \text{ m/s}^2, T = 3\text{s} \rightarrow d = 42.7 \text{ m}$$

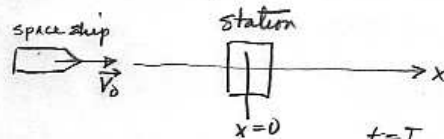
$$v_s = \infty \quad d_{\infty} = \frac{1}{2} g T^2 = 45 \text{ m}$$

Problem 10

EP6

$$\vec{V}_0 = 130 \text{ km/s} \hat{i} = 1.3 \times 10^5 \text{ m/s} \hat{i} = V_{0x} \hat{i}$$

$$a_x = b(t-T), \quad b = 0.26 \text{ m/s}^2$$



$$(a) \quad \frac{dv_x}{dt} = a_x \rightarrow v_x - V_{0x} = \int_{t=0}^{t=T} a_x dt$$

$$\overset{\nearrow v_x(t=T)}{v_x - V_{0x}} = b \int_0^T (t-T) dt = b \left[\frac{t^2}{2} - Tt \right]_0^T$$

$$v_x = 1.3 \times 10^5 - 0.13 T^2 = v_x(t=T)$$

$$= 0 \quad (\text{comes to rest at station})$$

$$\rightarrow T^2 = 10^6 \rightarrow T = 10^3 \text{ s}$$

$$(b) \quad v_x(t) = V_{0x} + \int_0^t a_x dt$$

$$= V_{0x} + b \left[\frac{t^2}{2} - Tt \right] \rightarrow a_x = \frac{dv_x}{dt} = b[t-T]$$

$$\text{Now } v_x(t) = \frac{dx}{dt} \rightarrow \underbrace{x(T)}_{=0 \text{ (at station)}} - \underbrace{x(0)}_{x_0} = \int_0^T v_x(t) dt$$

$$\rightarrow -x_0 = \int_0^T (V_{0x} + b \left[\frac{t^2}{2} - Tt \right]) dt = V_{0x} T - b \frac{T^3}{3}$$

$$\rightarrow x_0 = -4.4 \times 10^7 \text{ m}$$

Problem 11

EP7

$$V_{\text{car}} = 85 \text{ mi/hr} = 38 \text{ m/s}$$

$$a_{\text{police}} = \begin{cases} a_0 - bt^2 & 0 \leq t \leq t_0 = \sqrt{\frac{a_0}{b}} \rightarrow t_0 = 29.9 \text{ s} \\ 0 & t > t_0 \end{cases}$$

$$a_0 = 2.5 \text{ m/s}^2, b = 0.0028 \text{ m/s}^4$$

$$D_{\text{EXIT}} = 5.3 \text{ mi} = 8528 \text{ m}$$

At $t=0$, the two cars are at the same position

$$x_c(0) = x_p(0) = 0$$

The police car takes $t_{\text{start}} = 25 \text{ s}$ to get started.

Therefore

$$x_c(t_{\text{start}}) = V_{\text{car}} t_{\text{start}} = 38(25) = 950 \text{ m}$$

$$x_p(t_{\text{start}}) = 0$$

If we restart the clocks at t_{start} , the car moves according to

$$x_c(t) = x_c(t_{\text{start}}) + V_c t = 950 + 38 t$$

For the police car the situation is more complicated.

$$v_p(t) = \int_0^t a_p(t) dt = \int_0^t (a_0 - bt^2) dt = a_0 t - \frac{1}{3} bt^3 \quad t \leq t_0$$

$$v_p(t) = v_p(t_0) = a_0 t_0 - \frac{1}{3} bt_0^3 \quad t > t_0$$

$$\text{Finally } x_p(t) = \int_0^t v_p dt = \frac{1}{2} a_0 t^2 - \frac{1}{12} bt^4 \quad t \leq t_0$$

$$x_p(t) = x_p(t_0) + \int_{t_0}^t v_p dt = \frac{1}{2} a_0 t_0^2 - \frac{1}{12} bt_0^4 + \left(a_0 t_0 - \frac{1}{3} bt_0^3 \right) (t - t_0) \quad t > t_0$$

The police car catches the speeding car if

$$x_p(t_{\text{exit}}) \geq x_c(t_{\text{exit}}) = 8528 \text{ m}$$

$$\therefore 950 + 38 t_{\text{exit}} = 8528 \quad (\text{car reaches exit})$$

$$\rightarrow t_{\text{exit}} = \frac{7578}{38} = 199.2 \text{ s}$$

$$x_p(t_{\text{exit}}) = 9392 \text{ m} \quad (\text{see calculation below})$$

$\rightarrow$ police car catches speeding car

before the exit is reached

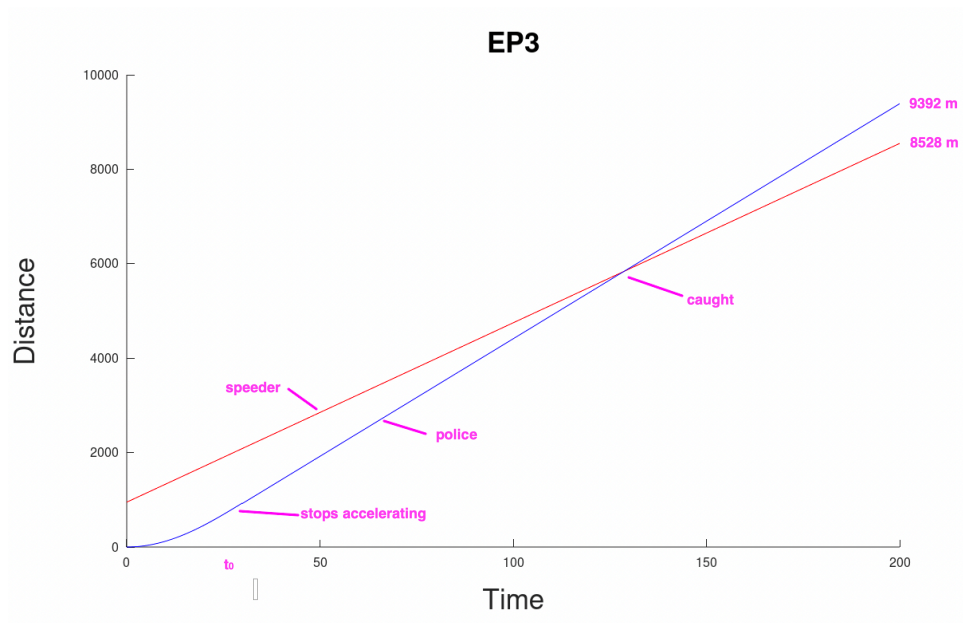
$$t_{\text{exit}} = 199.2 > t_0 = 29.9 \rightarrow$$

$$x_p(t_{\text{exit}}) = x_p(t_0) + (a_0 t_0 - \frac{1}{3} b t_0^3)(t_{\text{exit}} - t_0)$$

$$= \frac{1}{2} a_0 t_0^2 - \frac{1}{4} b t_0^4 + (a_0 t_0 - \frac{1}{3} b t_0^3)(t_{\text{exit}} - t_0)$$

$$= 9392 \text{ m}$$

Plot using Octave




```

clear;
%define parameters
a=2.5;
b=0.0028;
t0=29.9;
tmax=200;
xc0=950;
vc=38;
%define total time vector - steps 0.1 sec
t=0.1*(0:2000);
%calculate position of car
%stored in vector
xc=xc0+vc*t;
%define the vector up to t0 -steps os 0.1 sec
t1=0.1*(0:299);
%calculate the police car position up to t0
%stored in vector
xp=(a/2)*(t1.^2)-(b/12)*(t1.^4);
%define time vector after t0
t2=0.1*(0:1700);
%calculate the police care speed at t0
vt0=a*t0-(b/3)*(t0.^3);
%police care position is a vector made up of position vectors
%up to t0 + vector of positions after t0
xp=[xp , xp(length(xp)-1)+vt0*t2];
%open a plotting window
figure('position',[50,50,550,550])
%define axes
axis([0,200,0,10000]);
title('EP3','fontsize',20);
xlabel('Time','fontsize',20);
ylabel('Distance','fontsize',20);
%plot speeder position versus time
hold on;
%plot speeder position versus time
plot(t,xc,'-r','Markersize',2)
%plot police position versus time
plot(t,xp,'-b','Markersize',2)
set(gca, "fontsize", 16)
hold off;

```

Problem 12

EP8 $t=0, \theta=0, \dot{\theta}=\omega=0$

circular path $r=1.3 \text{ m}$

$\hookrightarrow \dot{r}=\ddot{r}=0$

$\ddot{\theta} = \alpha(t) = 120t^2 - 48t + 16$

(a) $\frac{d\dot{\theta}(t)}{dt} = \alpha(t) = 120t^2 - 48t + 16$

$\rightarrow \dot{\theta}(t) - \dot{\theta}(0) = \int_0^t \alpha(t) dt$

$\dot{\theta}(t) = 40t^3 - 24t^2 + 16t$

$\frac{d\theta(t)}{dt} = \dot{\theta}(t) = 40t^3 - 24t^2 + 16t$

$\rightarrow \theta(t) - \theta(0) = \int_0^t \dot{\theta}(t) dt$

$\theta(t) = 10t^4 - 8t^3 + 8t^2$

(b) $a_r = \ddot{r} - r\dot{\theta}^2 = -1.3(40t^3 - 24t^2 + 16t)^2$

$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 1.3(120t^2 - 48t + 16)$

Problem 13

EP9

circular motion : radius = R

$$s = t^3 + 2t^2 \text{ (measured along circle)}$$

$$a(t=2) = 16\sqrt{2} \text{ ft/s}^2$$

$$\text{circular motion} \rightarrow \dot{r} = \ddot{r} = 0$$

$$\rightarrow \vec{a} = -R\dot{\theta}^2 \hat{r} + R\ddot{\theta} \hat{\theta}$$

$$\frac{ds}{dt} = R\dot{\theta} = 3t^2 + 4t$$

$$\rightarrow R\ddot{\theta} = 6t + 4$$

$$\rightarrow \vec{a} = -R\left(\frac{3t^2 + 4t}{R}\right)^2 \hat{r} + R\left(\frac{6t + 4}{R}\right) \hat{\theta}$$

$$t=2 \quad \vec{a} = -\frac{1}{R}(20)^2 \hat{r} + (16) \hat{\theta}$$

$$a = \sqrt{16^2 + \frac{20^2}{R^2}} = 16\sqrt{2}$$

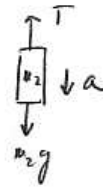
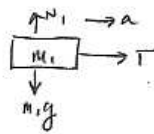
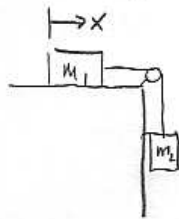
$$256 + \frac{400}{R^2} = 512 \rightarrow \frac{400}{R^2} = 256$$

$$\rightarrow R^2 = \frac{400}{256} \rightarrow R = \frac{20}{16} = 1.25 \text{ ft}$$

Assignment 3 - Solutions

Problem 1

2-02



$$T = m_1 a$$

$$m_2 g - T = m_2 a$$

$$\rightarrow m_2 g - m_1 a = m_2 a$$

$$\rightarrow a = \frac{m_2 g}{m_1 + m_2}$$

$$x(0) = 0, \quad v(0) = 0$$

$$x(t) = \frac{1}{2} a t^2 = \frac{1}{2} \frac{m_2 g}{m_1 + m_2} t^2$$

Problem 2

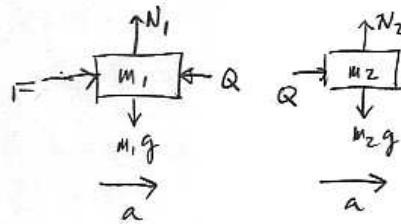
2-03



$$m_1 = 2 \text{ kg}$$

$$m_2 = 1 \text{ kg}$$

$$F = 3 \text{ N}$$



$$F - Q = m_1 a$$

$$Q = m_2 a$$

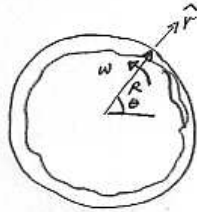
$$F - Q = m_1 \frac{Q}{m_2}$$

$$\rightarrow Q \left(1 + \frac{m_1}{m_2}\right) = F$$

$$\rightarrow Q = \frac{m_2}{m_1 + m_2} F = \frac{1}{3}(3) = 1 \text{ N}$$

Problem 3

2-06



The ingredients are most likely to fall from the top. Let us see why!

At arbitrary position (specified by θ) we have

$$\begin{aligned}
 & \text{Diagram showing forces: } N \text{ (normal force, radial outwards), } mg \text{ (weight, vertically down), and } \hat{r} \text{ (radial unit vector).} \\
 & \rightarrow N + mg \sin \theta = mR\omega^2 \\
 & \rightarrow \omega^2 = \frac{N + mg \sin \theta}{mR}
 \end{aligned}$$

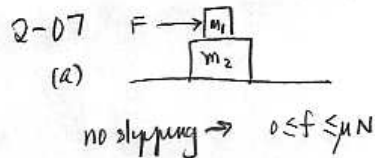
If cement is not sticking to the wall $\rightarrow$

$N = 0$ at some point (no reaction from wall)

$$\rightarrow \omega^2 = \frac{g \sin \theta}{R}$$

$$\rightarrow \omega_{\max}^2 = \omega^2 \text{ (at top, } \theta = \frac{\pi}{2}) = \frac{g}{R}$$

Problem 4



$$F - f = m_1 a$$

$$N = m_1 g$$

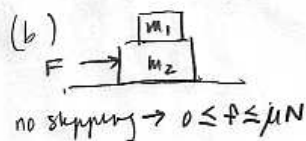
$$f = m_2 a$$

$$N + m_2 g = N_G$$

$$\rightarrow F - f = \frac{m_1}{m_2} f$$

$$\rightarrow F = f \left(1 + \frac{m_1}{m_2} \right)$$

$$\therefore F_{\max} = f_{\max} \left(1 + \frac{m_1}{m_2} \right) = \mu N \left(1 + \frac{m_1}{m_2} \right) = \mu m_1 g \left(1 + \frac{m_1}{m_2} \right)$$



$$f = m_1 a$$

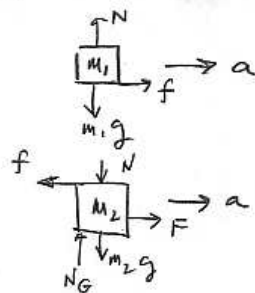
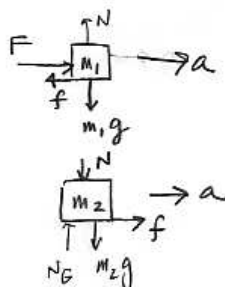
$$F - f = m_2 a$$

$$N = m_1 g$$

$$N + m_2 g = N_G$$

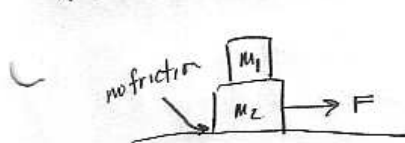
$$\rightarrow F - f = \frac{m_2}{m_1} f \rightarrow F = f \left(1 + \frac{m_2}{m_1} \right)$$

$$\rightarrow F_{\max} = f_{\max} \left(1 + \frac{m_2}{m_1} \right) = \mu m_1 g \left(1 + \frac{m_2}{m_1} \right)$$



Problem 5

2-08



$$m_2 = 5 \text{ kg}$$

$$m_1 = 4 \text{ kg}$$

from problem 2-07 we have

$$F_{\max} = \mu g (m_1 + m_2) = 27 \text{ N}$$

$$\rightarrow \mu = \frac{27}{9g} = \frac{3}{g} = 0.3$$

Now apply F to top block

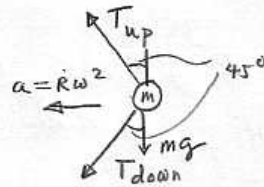
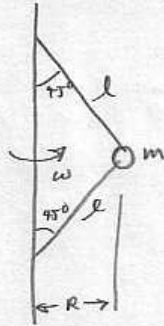
from problem 2-07

$$F_{\max} = \mu m_1 g \left(1 + \frac{m_1}{m_2}\right) \rightarrow \text{no slipping}$$

$$= \frac{3}{g} (4) g \left(1 + \frac{4}{5}\right) = \frac{108}{5} = 21.6 \text{ N}$$

Problem 6

2-11



$$T_{up} \sin 45^\circ + T_{down} \sin 45^\circ = m R \omega^2$$

$$T_{up} \cos 45^\circ - T_{down} \cos 45^\circ - mg = 0$$

$$\therefore T_{up} + T_{down} = \sqrt{2} m R \omega^2$$

$$T_{up} - T_{down} = \sqrt{2} m g$$

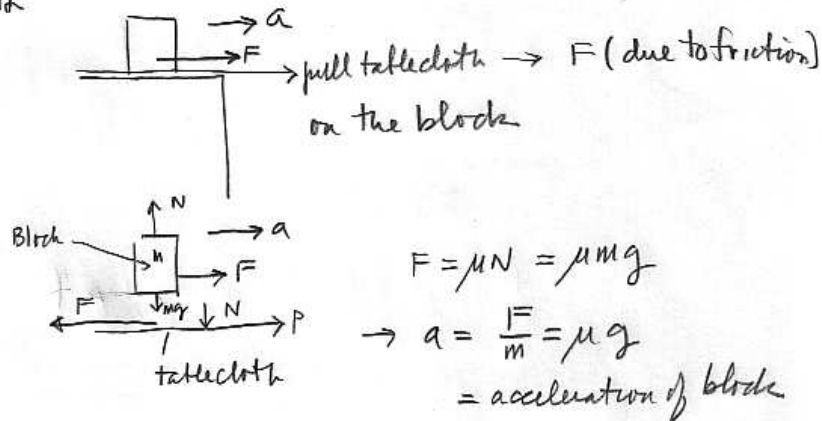
$$T_{up} = \frac{\sqrt{2}}{2} m (R \omega^2 + g)$$

$$T_{down} = \frac{\sqrt{2}}{2} m (R \omega^2 - g)$$

$$R = l \sin 45^\circ = \frac{l}{\sqrt{2}}$$

Problem 7

2-12



v_0 = velocity attained by block after time T $\rightarrow$ time for cloth to be pulled out

$= \mu g T$ (velocity of block after cloth pulled out)

We assume it has not moved!

At that point block is on table $\rightarrow$

$f = \mu N = \mu mg$ in direction shown (opposes motion)

$\rightarrow a' = -\mu g$ = deceleration due to table

$\rightarrow v = v_0 - \mu g t$ = velocity of block as it decelerates

Block (glass) comes to rest at $t_0 = \frac{v_0}{\mu g}$ ($v = 0$)

It travels a distance $s_0 = v_0 t_0 - \frac{1}{2} \mu g t_0^2 = \frac{1}{2} \frac{v_0^2}{\mu g} = \frac{1}{2} \mu g T^2$

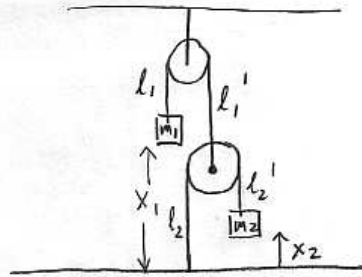
So for $s_0 < \frac{1}{2}$ foot we must have

$$T^2 < \frac{\frac{1}{2}}{\frac{1}{2} (0.5) (32)} = \frac{1}{4} \text{ second} \rightarrow \text{FAST!}$$

Assignment 4 - Solutions

Problem 1

2-13



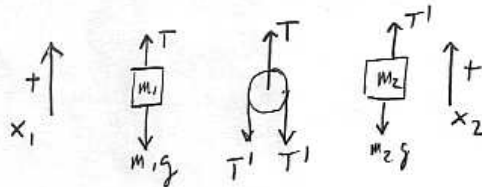
constraint

$$x_1 + l_1 + l_1' + \frac{l_2 + l_2' + x_2}{2} = \text{constant}$$

$$\rightarrow \dot{x}_1 + \frac{1}{2} \dot{x}_2 = 0$$

$$\rightarrow \ddot{x}_1 = -\frac{1}{2} \ddot{x}_2$$

$$\rightarrow \left. \begin{array}{l} a_1 = \text{acceleration of } m_1 \\ a_2 = \text{acceleration of } m_2 \end{array} \right\} \rightarrow a_1 = -\frac{1}{2} a_2$$



$$T - m_1 g = m_1 \ddot{x}_1$$

$$T' - m_2 g = m_2 \ddot{x}_2$$

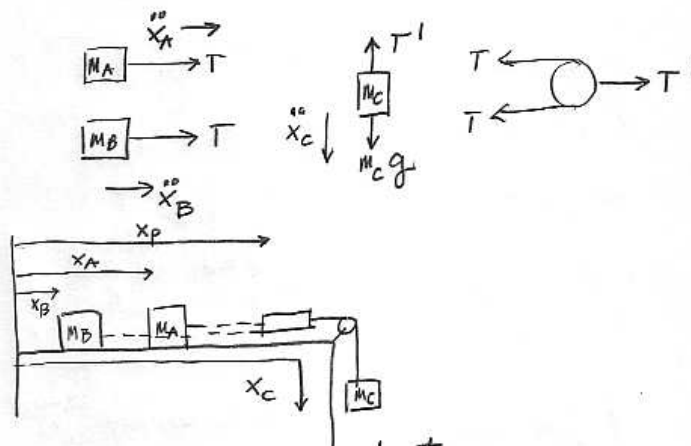
$$2T' - T = 0 \quad (\text{massless pulley})$$

$$\rightarrow \left. \begin{array}{l} T - m_1 g = m_1 \ddot{x}_1 \\ \frac{T}{2} - m_2 g = -2m_2 \ddot{x}_1 \end{array} \right\} \rightarrow (2m_2 - m_1)g = (4m_2 + m_1)\ddot{x}_1$$

$$\rightarrow \ddot{x}_1 = \frac{2m_2 - m_1}{4m_2 + m_1} g$$

Problem 2

2-14



Constraint $x_C - x_P = \text{constant}$
 $\rightarrow x_C'' = x_P''$ (accelerations the same)

$x_P - x_A + (x_P - x_B) = \text{constant} = \text{length of string}$
 $\rightarrow 2x_P'' = 2x_C'' = x_A'' + x_B''$

We also have

$$T = M_A x_A''$$

$$T = M_B x_B''$$

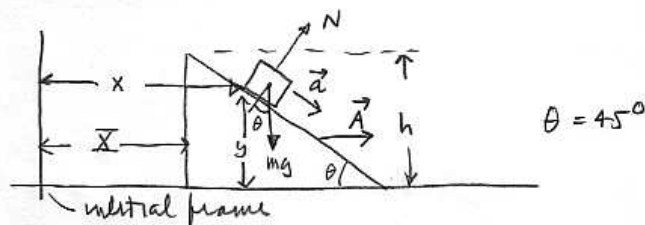
$$M_C g - T' = M_C x_C''$$

$$T' - 2T = 0 \text{ (massless pulley)}$$

$$\rightarrow \left. \begin{aligned} x_A'' &= \frac{M_B M_C g}{D} \\ x_B'' &= \frac{M_A M_C g}{D} \\ x_C'' &= \frac{1}{2} (M_A + M_B) \frac{M_C g}{D} \end{aligned} \right\} \text{where } D = 2M_A M_B + \frac{M_C^2}{2} + M_A M_C$$

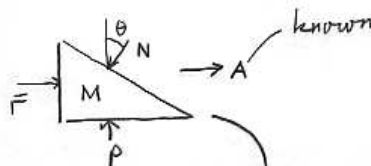
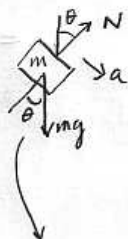
Problem 3

2-16



$$\vec{a}_{\text{plane}} = \vec{A} = A \hat{i} = \ddot{x} \hat{i}$$

$$\vec{a}_{\text{block}} = \vec{a} = a_x \hat{i} + a_y \hat{j} = \ddot{x} \hat{i} + \ddot{y} \hat{j}$$



$$\rightarrow N \cos \theta - mg = m a_y$$

$$N \sin \theta = m a_x$$

3 unknown N, a_x, a_y
2 equations

$$P - N \cos \theta = 0$$

$$F - N \sin \theta = M A$$

these eqs. not needed

Need a constraint equation:

$$x - \bar{x} = (h - y) \cot \theta = h - y \quad (\cot \theta = 1)$$

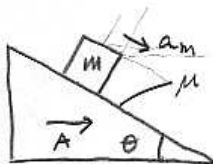
$$\rightarrow \ddot{x} - \ddot{\bar{x}} = -\ddot{y} \rightarrow a_x - A = -a_y \rightarrow \text{constraint eq.}$$

$$\frac{N \cos \theta}{N \sin \theta} = \cot \theta = 1 = \frac{m(g + a_y)}{m a_x} = \frac{g + a_y}{a_x} \rightarrow a_x = g + a_y$$

$$\rightarrow g + a_y - A = -a_y \rightarrow a_y = \frac{A - g}{2} \text{ and } a_x = g + a_y = \frac{A + g}{2}$$

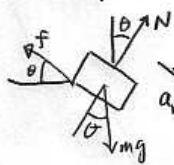
Problem 4

2-17



no motion w.r.t m

(a) wedge fixed. ($A=0$)



$$-f + mg \sin \theta = m a_m = 0$$

$$N - mg \cos \theta = 0$$

$$\rightarrow mg \sin \theta = f \leq \mu N = \mu mg \cos \theta$$

$$\rightarrow \tan \theta \leq \mu$$

$$\text{or } \tan \theta_{\max} = \mu$$

(b) wedge has horizontal acceleration ($A=a$)

$$\tan \theta > \mu$$

$a_m = 0$ (stays on wedge without sliding)

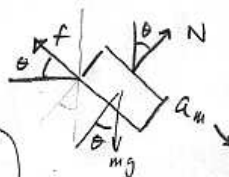
(same as 2-16) $\cot \theta \neq 1$ now

$$a_{mx} - a = -a_{my} \cot \theta$$

$$N \sin \theta - f \cos \theta = m a_{mx} = m a$$

$$N \cos \theta - mg + f \sin \theta = m a_y = 0$$

$\rightarrow$ eliminating N



not sliding $\rightarrow$
 $\vec{a}_m = \vec{a}$

Problem 5

$$N = m(g \cos \theta + a \sin \theta)$$

$$f = m(g \sin \theta - a \cos \theta)$$

$$\rightarrow a = \frac{mg \sin \theta - f}{m \cos \theta}$$

$$a = a_{\min} \text{ when } f = f_{\max} = \mu N$$

$$\rightarrow a_{\min} = \frac{mg \sin \theta - \mu m(g \cos \theta + a \sin \theta)}{m \cos \theta}$$

$$= g \frac{\sin \theta - \mu \cos \theta}{\cos \theta + \mu \sin \theta}$$

(c) part (b) assumed that 

$a_{\max}$ results when



or f is needed to keep block stationary

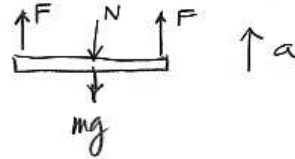
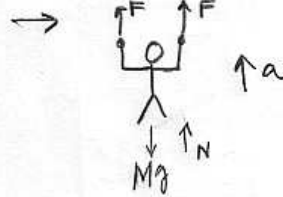
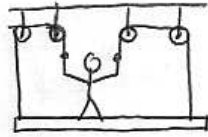
$\rightarrow$ redo (b) with $f \rightarrow -f$

$$\rightarrow a_{\max} = \frac{mg \sin \theta + \mu N}{m \cos \theta} = \frac{mg \sin \theta + \mu m(g \cos \theta + a_{\max} \sin \theta)}{m \cos \theta}$$

$$= g \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta}$$

Problem 6

2-18

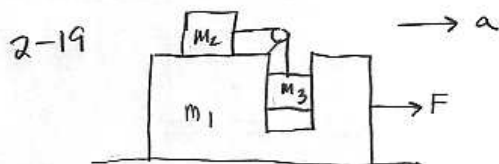


$$\begin{aligned} \rightarrow 2F + N - Mg &= Ma \\ 2F - N - mg &= ma \end{aligned}$$

$$\rightarrow 4F - (m+M)g = (m+M)a$$

$$\rightarrow a = \frac{4F}{m+M} g$$

Problem 7



m_3 is not moving vertically
 $\Rightarrow m_2$ is accelerating with m_1 } horizontally
 $\Rightarrow m_3$ is accelerating with m_1

$\therefore$ regarding $(m_1 + m_2 + m_3)$ as a single body
 we have

$$(m_1 + m_2 + m_3) a = F$$

But for m_2

$\rightarrow a$

$$T = m_2 a$$

and for m_3

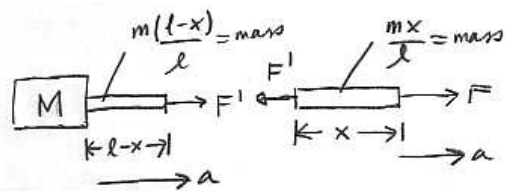
$$T = m_3 g$$

$$a = \frac{m_3}{m_2} g$$

$$\rightarrow F = (m_1 + m_2 + m_3) \frac{m_3}{m_2} g$$

Problem 8

2-21



$$F - F' = \frac{mx}{l} a$$

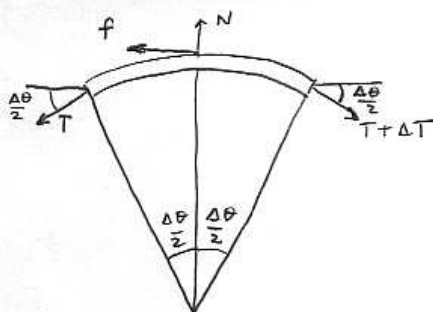
$$F' = \left(M + \frac{m(l-x)}{l} \right) a$$

$$\rightarrow F' = \frac{(M+m)l - mx}{(M+m)l} F$$

$$= \left(1 - \frac{m}{M+m} \frac{x}{l} \right) F$$

Problem 9

2-24



Force towards center $\Rightarrow$

$$T \sin \frac{\Delta\theta}{2} + (T + \Delta T) \sin \frac{\Delta\theta}{2} - N = 0 \quad (\text{balanced})$$

Horizontal forces $\Rightarrow$

$$(T + \Delta T) \cos \frac{\Delta\theta}{2} - T \cos \frac{\Delta\theta}{2} - f = 0 \quad (\text{balanced})$$

equilibrium

Assume rope just about to slip $\rightarrow$

$$f = \mu N$$

$$N = 2T \sin \frac{\Delta\theta}{2} + \Delta T \sin \frac{\Delta\theta}{2}$$

$$\approx 2T \frac{\Delta\theta}{2} + \Delta T \frac{\Delta\theta}{2} \rightarrow 0$$

$$f \approx \Delta T \cos \frac{\Delta\theta}{2} = \Delta T$$

$$\Delta T = \mu T \Delta\theta$$

Take limit $\Delta\theta \rightarrow 0, \Delta T \rightarrow 0 \rightarrow$

$$\frac{dT}{d\theta} = \mu T$$

Problem 10

$$\rightarrow \frac{dT}{T} = \mu d\theta$$

$$\int_{T_B}^{T_A} \frac{dT}{T} = \mu \int_0^\theta d\theta$$

$$\rightarrow \ln \frac{T_A}{T_B} = \mu \theta$$

$$\rightarrow T_A = T_B e^{\mu \theta}$$

$\therefore$ You pull with $T_B = 1 \text{ N}$

$\rightarrow$ can hold

$$T_A = e^{\mu \theta} \text{ N}$$

say $\mu = 0.2$

once around $\theta = 2\pi$: $T_A = e^{1.26} = 3.52 \text{ N}$

twice around $\theta = 4\pi$: $T_A = e^{2.52} = 12.43 \text{ N}$

3 times around $\theta = 6\pi$: $T_A = e^{3.78} = 43.82 \text{ N} !!$



Assignment 5 - Solutions

Problem 1

2-28

Minimum speed (car tries to slide down)

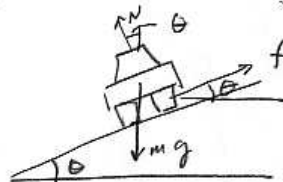
Forces as shown

$$N \cos \theta + f \sin \theta = mg$$

$$N \sin \theta - f \cos \theta = \frac{mv^2}{R}$$

$$\text{At limit } f_{\max} = \mu N$$

$$\rightarrow v_{\min}$$



moving in circle!

$$\rightarrow N (\cos \theta + \mu \sin \theta) = mg$$

$$N (\sin \theta - \mu \cos \theta) = \frac{mv_{\min}^2}{R}$$

$$\rightarrow v_{\min} = \left[gR \left(\frac{\sin \theta - \mu \cos \theta}{\cos \theta + \mu \sin \theta} \right) \right]^{1/2}$$

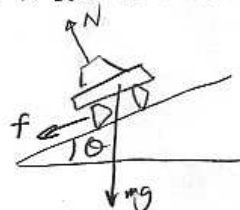
Maximum speed (car tries to slide up)

Forces as shown

$$N \cos \theta - f \sin \theta = mg$$

$$N \sin \theta + f \cos \theta = \frac{mv^2}{R}$$

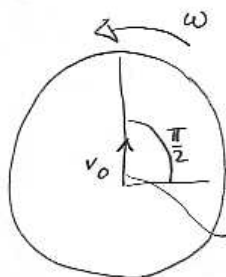
$$f_{\max} = \mu N$$



$$\rightarrow v_{\max} = \left[gR \left(\frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \right) \right]^{1/2}$$

Problem 2

2-29



car weight = w
friction $\rightarrow \mu$

car moves along
a painted line
which is rotating!

(a)

in rotating frame of reference:

$$\theta = \frac{\pi}{2} \text{ fixed}$$

$$r = v_0 t$$

in laboratory frame of reference: (inertial)

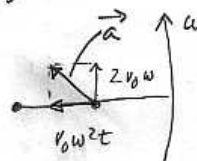
$$\theta = \frac{\pi}{2} + \omega t$$

$$r = v_0 t$$

$$\therefore \vec{r} = r \hat{r} = v_0 t \hat{r}$$

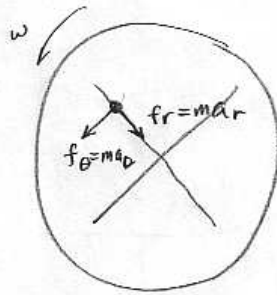
$$\begin{aligned} \vec{v} &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} = v_0 \hat{r} + r \omega \hat{\theta} \\ &= v_0 \hat{r} + v_0 \omega t \hat{\theta} \end{aligned}$$

$$\begin{aligned} \vec{a} &= (\ddot{r} - r \dot{\theta}^2) \hat{r} + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta} \\ &= (0 - (v_0 t) \omega^2) \hat{r} + (2v_0 \omega + 0) \hat{\theta} \\ &= -v_0 \omega^2 t \hat{r} + 2v_0 \omega \hat{\theta} \end{aligned}$$



(b) just starts to skid $\rightarrow$

$$f = f_{\max} = \mu N = \mu mg = \mu W$$



Car starts to skid when

$$f = \mu N = m a$$

$$\text{or } a = \mu g$$

$$f_r = m a_r = m v_0 \omega^2 t$$

$$f_\theta = m a_\theta = 2 m v_0 \omega$$

$$\rightarrow f = (f_r^2 + f_\theta^2)^{1/2} = m v_0 \omega \sqrt{4 + \omega^2 t^2} \leq \mu m g$$

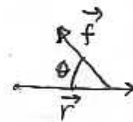
$$\text{or } 4 + \omega^2 t^2 \leq \frac{\mu^2 g^2}{v_0^2 \omega^2}$$

$$\text{or } t^2 \leq \frac{\mu^2 g^2}{v_0^2 \omega^4} - \frac{4}{\omega^2}$$

$$t_{\text{skid}} = \frac{1}{v_0 \omega^2} \sqrt{(\mu g)^2 - 4 v_0^2 \omega^2}$$

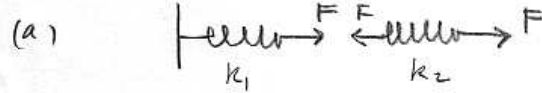
$$(c) \quad \tan \theta = \frac{f_\theta}{f_r} = \frac{2}{\omega t}$$

$$\tan \theta = \frac{2 v_0 \omega}{\sqrt{(\mu g)^2 - 4 v_0^2 \omega^2}}$$



Problem 3

2-31



$$\Delta x_1 = \frac{F}{k_1} \quad \Delta x_2 = \frac{F}{k_2}$$

$$\Delta x_{\text{total}} = \Delta x_1 + \Delta x_2 = F \left(\frac{1}{k_1} + \frac{1}{k_2} \right) = \frac{F}{k_{\text{eff}}}$$

$$\rightarrow k_{\text{eff}} = \frac{F}{\Delta x_{\text{total}}} \Rightarrow \frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\omega_a = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \frac{k_1 k_2}{k_1 + k_2}}$$



$$\left. \begin{array}{l} F_1 = k_1 \Delta x \\ F_2 = k_2 \Delta x \end{array} \right\} \rightarrow F_{\text{total}} = F_1 + F_2 = (k_1 + k_2) \Delta x = k_{\text{eff}} \Delta x$$

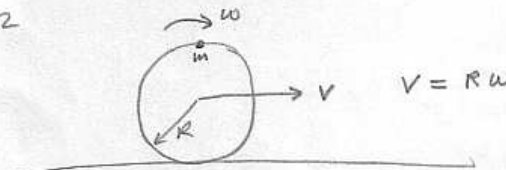
$$\rightarrow k_{\text{eff}} = k_1 + k_2$$

$$\rightarrow \omega_b = \sqrt{\frac{k_1 + k_2}{m}}$$

Problem 4

2-32

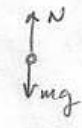
(a)



when released (instantaneously at rest) at top

$V_{\text{pettle}} = V \Rightarrow \text{just rotating} \Rightarrow V_{\text{top}} = R\omega = V$

$\therefore$ to continue moving on circle we must have



$$m \frac{V_{\text{pettle}}^2}{R} = mg - N$$

$$\text{or } m \frac{V^2}{R} = mg - N$$

if pettle is about to leave wheel $N = 0$ (losing contact)

$$\therefore V^2 = gR$$

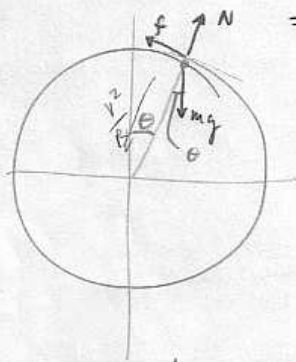
$$\text{or } V = \sqrt{gR}$$

$$\therefore V \geq \sqrt{gR} \rightarrow \text{leaves wheel immediately}$$

(b)

$$v < \sqrt{Rg}$$

$$\mu = 1$$



$\Rightarrow$ while in contact we have

$$mg \sin \theta - f = 0$$

$$mg \cos \theta - N = m \frac{v^2}{R}$$

$$f = \mu N \rightarrow \text{maximum}$$

$\rightarrow$ pebble about to slide!

$$mg \sin \theta = \mu \left(mg \cos \theta - m \frac{v^2}{R} \right)$$

$$\text{or } g (\sin \theta + \mu \cos \theta) = \frac{\mu v^2}{R}$$

$$\mu = 1$$

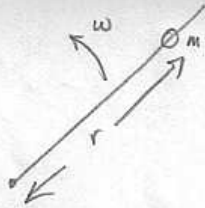
$$g (\cos \theta - \sin \theta) = \frac{v^2}{R}$$

$$g \left(\cos \left(\theta + \frac{\pi}{4} \right) \sqrt{2} \right) = \frac{v^2}{R}$$

$$\text{or } \cos \left(\theta + \frac{\pi}{4} \right) \frac{1}{\sqrt{2}} \frac{v^2}{gR} \Rightarrow \frac{\pi}{4} \theta = \cos^{-1} \left(\frac{1}{\sqrt{2}} \frac{v^2}{gR} - \frac{\pi}{4} \right)$$

Problem 5

2-33



$$m(\ddot{r} - r\dot{\theta}^2) = F_r = 0$$

$$\therefore \ddot{r} - \omega^2 r = 0$$

substitute $r = e^{\alpha t}$

$$\Rightarrow \alpha^2 e^{\alpha t} - \omega^2 e^{\alpha t} = 0$$

$$(\alpha^2 - \omega^2) e^{\alpha t} = 0$$

$$\text{or } \alpha = \pm \omega$$

$\therefore$ most general solution is

$$r(t) = A e^{\omega t} + B e^{-\omega t}$$

$$v(t) = \omega (A e^{\omega t} - B e^{-\omega t})$$

$$\begin{aligned} r(0) &= A + B \\ v(0) &= \omega(A - B) \end{aligned} \Rightarrow \begin{aligned} A &= \frac{r(0) + \frac{v(0)}{\omega}}{2} \\ B &= \frac{r(0) - \frac{v(0)}{\omega}}{2} \end{aligned}$$

$$r(t) = \frac{1}{2} \left(r(0) + \frac{v(0)}{\omega} \right) e^{\omega t} + \frac{1}{2} \left(r(0) - \frac{v(0)}{\omega} \right) e^{-\omega t}$$

(as problem assumes!)

suppose $v(0) = -\omega r(0)$

$$\Rightarrow r(t) = r(0) e^{-\omega t} \rightarrow \text{decreases continuously!}$$

$$\Rightarrow v(t) = -\omega r(0) e^{-\omega t} \Rightarrow \text{always inward!}$$

if $v(0) \neq -\omega r(0)$ say $v(0) = g \omega r(0)$

$\Rightarrow$ solution $\Rightarrow$

$$r(t) = \frac{r(0)}{2} (1+g) e^{\omega t} + \frac{r(0)}{2} (1-g) e^{-\omega t}$$

as $t \rightarrow \text{large}$ 1st term always dominates

$$\Rightarrow v(t) = \frac{r(0)}{2} \omega \left((1+g) e^{\omega t} - (1-g) e^{-\omega t} \right)$$

$$\Rightarrow \text{if } (1+g) < (1-g)$$

v is initially inward but will always reverse. Thus, particle moves inward on rotating rod!!

Problem 6

2-324

$$(a) \quad \ddot{\theta} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$= 2\dot{r}\omega + r\dot{\omega} = 0$$

$$\rightarrow 2\frac{dr}{dt}\omega = -r\frac{d\omega}{dt}$$

$$\rightarrow -2\frac{dr}{r} = \frac{d\omega}{\omega}$$

$$\rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = -2 \int_{r_0}^r \frac{dr}{r}$$

Starting value 0

$$\rightarrow \ln \frac{\omega}{\omega_0} = -2 \ln \frac{r}{r_0} = \ln \frac{r_0^2}{r^2}$$

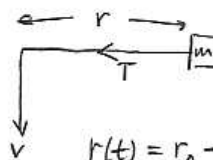
$$\rightarrow \frac{\omega}{\omega_0} = \frac{r_0^2}{r^2} \rightarrow \omega = \omega_0 \left(\frac{r_0}{r} \right)^2$$

$$\rightarrow \omega(t) = \left(\frac{r_0}{r_0 - vt} \right)^2$$

$$(b) \quad -T = m a_r = m (\ddot{r} - r\omega^2)$$

$$T = m r \omega^2 = m r \omega_0^2 \left(\frac{r_0}{r} \right)^4$$

$$= m \omega_0^2 \frac{r_0^4}{r^3}$$



$$r(t) = r_0 - vt \quad (\text{given})$$

Problem 7

2-36

$$m\vec{a} = \vec{F} \quad \text{opposite to motion}$$

$$m \frac{dv}{dt} = -b e^{\alpha v} \Rightarrow \underline{\text{not linear!}}$$

$$\text{or } \frac{dv}{dt} = -\frac{b}{m} e^{\alpha v}$$

$$\Rightarrow \frac{dv}{e^{\alpha v}} = -\frac{b}{m} dt$$

$$e^{-\alpha v} dv = -\frac{b}{m} dt$$

$$\int_{v(0)}^{v(t)} e^{-\alpha v} dv = -\frac{b}{m} \int_0^t dt = -\frac{b}{m} t$$

$$\frac{1}{(-\alpha)} \int_{-\alpha v(0)}^{-\alpha v(t)} e^x dx = -\frac{b}{m} t \quad (\text{change variables})$$

$$-\frac{1}{\alpha} (e^{-\alpha v(t)} - e^{-\alpha v(0)}) = -\frac{b}{m} t$$

$$e^{-\alpha V(t)} = + \frac{\alpha b}{m} t + e^{-\alpha V(0)}$$

$$\ln e^{-\alpha V(t)} = \ln \left(+ \frac{\alpha b}{m} t + e^{-\alpha V(0)} \right)$$

$$\text{or } -\alpha V(t) = \ln \left(\frac{\alpha b t}{m} + e^{-\alpha V(0)} \right)$$

$$\begin{aligned} \text{or } \alpha V(t) &= - \ln \left(\frac{\alpha b t}{m} + e^{-\alpha V(0)} \right) \\ &= \ln 1 - \ln \left(\frac{\alpha b t}{m} + e^{-\alpha V(0)} \right) \end{aligned}$$

$$= \ln \frac{1}{\frac{\alpha b t}{m} + e^{-\alpha V(0)}}$$

$$\text{or } V(t) = \frac{1}{\alpha} \ln \left[\frac{1}{\frac{\alpha b t}{m} + e^{-\alpha V(0)}} \right]$$

```

clear;
t=0.0;
v=10.0;
m=1.0;
b=0.5;
alpha=0.1;
h=0.01;
num=3000;
tth=0:30;
den=alpha *(b/m).*tth + exp(-alpha *v);
vth=(1/alpha)*log(1./den);
figure('Position',[50,50,800,600]);
hold on
xlabel('Time(seconds)','fontsize', 20);
ylabel('Velocity(meters/sec)','fontsize', 20);
title('Problem 2-36','fontsize', 20);
for j=1:num
    t=t+h;
    a=acc(b,alpha,m,v);
    v=v+a*h;
    plot(t,v,'.k')
end
plot(tth,vth,'or','Markersize', 10)
hold off

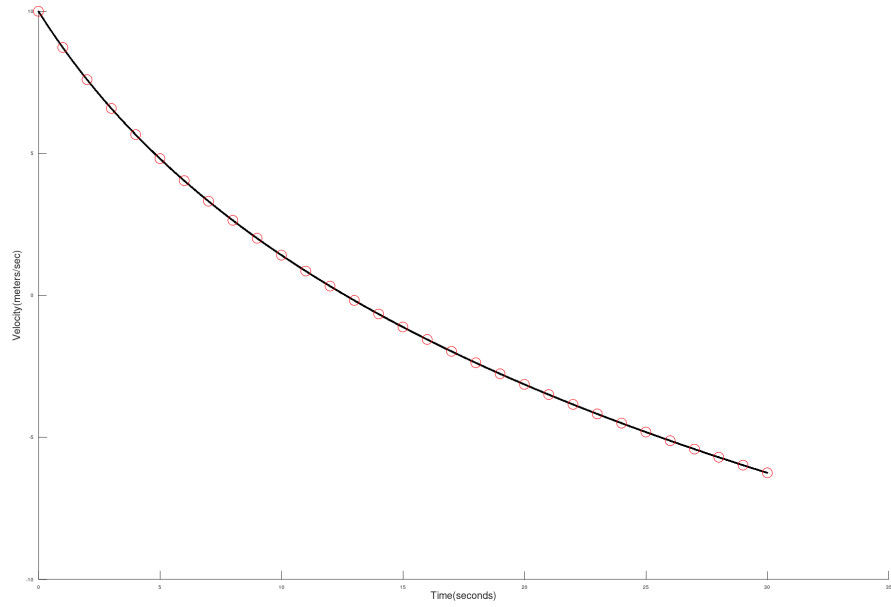
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```

function a=acc(b,alpha,m,v)
a=-(b/m)*exp(alpha*v);
endfunction

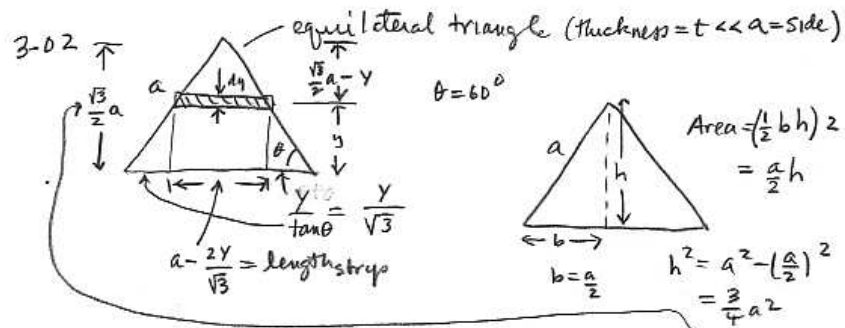
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Problem 2-36



Assignment 6 - Solutions

Problem 1



$$\rho = \text{density} = \frac{M}{\text{Area} \times \text{thickness}}$$

$$= \frac{4M}{\sqrt{3} a^2 t}$$

$$\Rightarrow h = \frac{\sqrt{3}}{2} a$$

$$\rightarrow \text{Area} = \frac{\sqrt{3}}{4} a^2$$

$$y_{cm} = \frac{1}{M} \int_0^h y \, dm_{\text{strip}}$$

distance to strip from bottom

$$dm_{\text{strip}} = \rho \, dA_{\text{strip}} \, t = \rho t \, dy \, \text{length strip}$$

$$= \rho t \left(a - \frac{2y}{\sqrt{3}} \right) dy$$

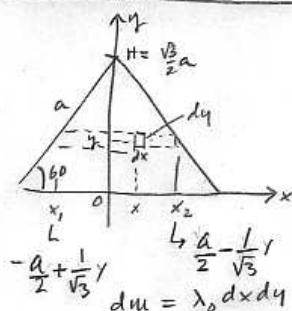
$$\therefore y_{cm} = \frac{\left(\frac{4M}{\sqrt{3} a^2 t} \right) t}{M} \int_0^{\frac{\sqrt{3}}{2} a} y \left(a - \frac{2y}{\sqrt{3}} \right) dy$$

$$= \frac{4}{\sqrt{3}} \frac{1}{a^2} \left[\frac{ay^2}{2} - \frac{2y^3}{3\sqrt{3}} \right]_0^{\frac{\sqrt{3}}{2} a}$$

$$= \frac{4}{\sqrt{3}} \frac{1}{a^2} \left[\frac{a}{2} \frac{3}{4} a^2 - \frac{2}{3\sqrt{3}} \frac{3\sqrt{3}}{2} a^3 \right]$$

$$= \frac{4}{\sqrt{3}} \left[\frac{3}{8} - \frac{2}{8} \right] a = \frac{1}{2\sqrt{3}} a$$

Double Integral Method



$$\vec{R} = \frac{1}{M} \int \vec{r} dm$$

$$\bar{x} = \frac{1}{M} \int x dm$$

$$\bar{y} = \frac{1}{M} \int y dm$$

$$dm = \lambda_0 dx dy$$

$$\begin{aligned} M &= \lambda_0 \int_0^H \int_{x_1}^{x_2} dx dy = \lambda_0 \int_0^{\frac{\sqrt{3}}{2}a} dy (x_2 - x_1) \\ &= \lambda_0 \int_0^{\frac{\sqrt{3}}{2}a} \left(a - \frac{2}{\sqrt{3}}y \right) dy \\ &= \lambda_0 \left(ay - \frac{y^2}{\sqrt{3}} \right) \Big|_0^{\frac{\sqrt{3}}{2}a} = \lambda_0 \left(\frac{\sqrt{3}}{2}a^2 - \frac{\sqrt{3}}{4}a^2 \right) \\ &= \lambda_0 \frac{\sqrt{3}}{4} a^2 \end{aligned}$$

$$\text{Area triangle} = \frac{1}{2} a H = \frac{\sqrt{3}}{4} a^2$$

$$\rightarrow M = \lambda_0 (\text{area}) \text{ as expected}$$

$$\int x dm = \lambda_0 \int_0^H \int_{x_1}^{x_2} x dx dy = \lambda_0 \int_0^H dy \frac{x_2^2 - x_1^2}{2} = 0$$

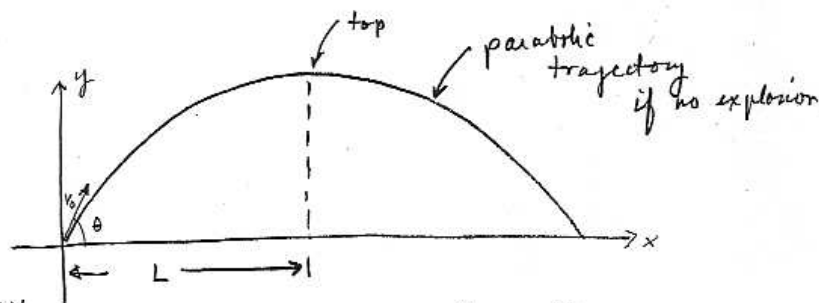
$$\rightarrow \bar{x} = 0$$

$$\begin{aligned} \int y dm &= \lambda_0 \int_0^H y dy \int_{x_1}^{x_2} dx = \lambda_0 \int_0^H y dy (x_2 - x_1) = \lambda_0 \int_0^{\frac{\sqrt{3}}{2}a} \left(ay - \frac{2}{\sqrt{3}}y^2 \right) dy \\ &= \lambda_0 \left(\frac{a}{2} \left(\frac{\sqrt{3}}{2}a \right)^2 - \frac{2}{3\sqrt{3}} \left(\frac{\sqrt{3}}{2}a \right)^3 \right) = \lambda_0 \frac{a^3}{8} \end{aligned}$$

$$\therefore \bar{y} = \frac{\lambda_0 \frac{a^3}{8}}{\lambda_0 \frac{\sqrt{3}}{4} a^2} = \frac{1}{2\sqrt{3}} a \quad (\text{as we found earlier})$$

Problem 2

3-4



HARD WAY

(1) original mass = M

(2) breaks into 2 pieces $\Rightarrow \frac{3}{4}M, \frac{1}{4}M$

(3) at top, velocity of original projectile is horizontal.

$$V_{\text{top}} = (V_0 \cos \theta) \hat{i}$$

$$\underbrace{M V_{\text{top}}}_{\text{before}} = \underbrace{\frac{3}{4}M V_1 + \frac{1}{4}M V_2}_{\text{after}}$$

$$\text{or } V_0 \cos \theta = \frac{3}{4} V_{1x} + \frac{1}{4} V_{2x}$$

$$0 = \frac{3}{4} V_{1y} + \frac{1}{4} V_{2y}$$

(4) pieces fly apart horizontally $\Rightarrow$

$$V_{1y} = V_{2y} = 0$$

(5) if smaller piece arrives back at starting point its velocity must be $-V_0 \cos \theta$ ($\Rightarrow$ follows same path backwards)

$$\therefore V_0 \cos \theta = \frac{3}{4} V_{1x} + \frac{1}{4} (-V_0 \cos \theta)$$

$$\text{or } V_{1x} = \frac{5}{3} V_0 \cos \theta$$

$$\text{time of travel} = \frac{L}{v_0 \cos \theta} \quad (\text{time to fall to ground for both pieces})$$

$\therefore$ larger piece ends up at

$$L + \underbrace{v_{ix} \frac{L}{v_0 \cos \theta}}_{\text{extra distance traveled}} = \frac{8}{3} L$$

EASY WAY

- (1) CM continues on same path
- (2) since no vertical velocities at breakup, both pieces down at ground at same time as CM.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

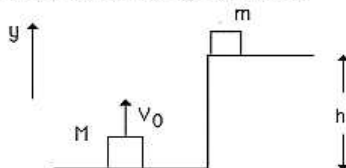
$$2L = \frac{m_1 x_1 + m_2 (0)}{m_1 + m_2} \quad \left(\begin{array}{l} \text{where CM lands} \\ = \text{original position} \\ \text{of projectile} \end{array} \right)$$

$$\text{or } x_1 = 2L \left(\frac{m_1 + m_2}{m_1} \right)$$

$$= 2L \frac{M}{\frac{3}{4}M} = \frac{8}{3} L$$

Problem 3

3.5 A circus acrobat of mass M leaps straight up with initial velocity v_0 from a trampoline. As she rises up, she takes a trained monkey of mass m off a perch at a height h above the trampoline. What is the maximum height attained by the pair?



Before grabbing monkey $\rightarrow$ getting to height h :

$$\frac{dv}{dt} = a = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy} \rightarrow v dv = a dy = -g dy$$

$$\int_{v_0}^{v_f} v dv = \frac{1}{2} (v_f^2 - v_0^2) = -g \int_{y_0}^{y_f} dy = -g(y_f - y_0)$$

$$v^2(h) - v_0^2 = -2g(h - 0) = -2gh \rightarrow v(h) = \sqrt{v_0^2 - 2gh}$$

Grab monkey $\rightarrow$ at h momentum is conserved:

$$P_{\text{initial}} = Mv = P_{\text{final}} = (M + m)v'$$

$$v' = \frac{Mv}{M + m}$$

is the new initial velocity of the pair. Let h' = extra height, then

$$v_{\text{max height}}^2 - v_{\text{after pickup}}^2 = 0 - v'^2 = -2g(y_{\text{max}} - h) = -2gh'$$

$$h' = \frac{v'^2}{2g} = \frac{1}{2g} \left(\frac{M}{M + m} \right)^2 (v_0^2 - 2gh)$$

Problem 4

3-6
(Part 1) conservation of momentum when sandbag is hoisted

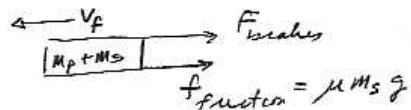
$$m_p v_p = (m_p + m_s) v_{final}$$

$$v_f = \frac{m_p}{m_p + m_s} v_p \quad v_p = 120$$

$$m_p = \frac{2500}{g}$$

$$m_s = \frac{250}{g}$$

(part 2)



$$F_r = F_{brake} + F_{friction} = (m_p + m_s) a$$

$$a = \frac{F_r}{m_p + m_s}$$

$$F_r = 300 + 0.4 \left(\frac{250}{g} \right) g$$

$$= 400$$

↑
total

$$v = v_f - at$$

stops at $t = \frac{v_f}{a} \rightarrow S = \text{distance traveled}$

$$= v_f t - \frac{1}{2} a t^2$$

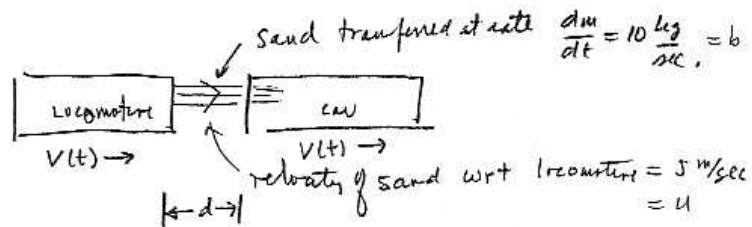
$$S = \frac{v_f^2}{a} - \frac{1}{2} a \frac{v_f^2}{a^2} = \frac{1}{2} \frac{v_f^2}{a}$$

$$S = \frac{1}{2} \frac{\left(\frac{m_p}{m_p + m_s} \right)^2 v_p^2}{\left(\frac{F_r}{m_p + m_s} \right)} = \frac{1}{2} \frac{m_p^2}{m_p + m_s} \frac{v_p^2}{F_r}$$

$$S = 1280 \text{ ft.}$$

Problem 5

3-12



Locomotive & Car have same velocity so that distance $d = \text{constant}$

$$V(0) = 0$$

$$M_{\text{car}}(0) = 2000 \text{ kg}$$

$$V(100 \text{ sec}) = ?$$

system = car + sand to enter in Δt

$$p(t) = (b \Delta t)(V(t) + u) + M_{\text{car}}(t) V(t)$$

↑ actual sand velocity wrt ground

$$p(t + \Delta t) = (M_{\text{car}}(t) + b \Delta t)(V(t) + \Delta V)$$

$$\Delta p = p(t + \Delta t) - p(t) = -bu \Delta t + M_{\text{car}}(t) \Delta V$$

$$\frac{\Delta p}{\Delta t} = \overset{M_{\text{car}}(0) + bt}{M_{\text{car}}(t)} \frac{\Delta V}{\Delta t} - bu = F_{\text{ext}} = 0$$

$$\therefore (M_{\text{car}}(0) + bt) \frac{dv}{dt} - bu = 0$$

$$\therefore \frac{b \, dt}{M_{\text{car}}(0) + bt} = \frac{1}{u} \, dv$$

$$\therefore \ln \frac{M_{\text{car}}(0) + bt}{M_{\text{car}}(0)} = \frac{1}{u} V(t)$$

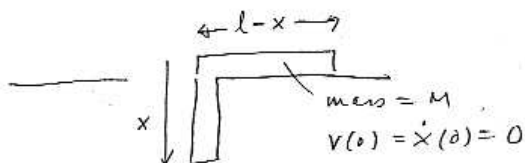
$$\Rightarrow V(t) = u \ln \frac{M_{\text{car}}(0) + bt}{M_{\text{car}}(0)}$$

$$V(100) = 5 \ln \frac{2000 + 1000}{2000}$$

$$= 5 \ln 1.5 = 2.03 \text{ m/sec}$$

Problem 6

3-15



$$M \ddot{x} = F = \frac{x}{l} M g$$

$$\text{or } \ddot{x} - \frac{g}{l} x = 0$$

$$x = e^{\alpha t}$$

$$\Rightarrow \alpha^2 - \frac{g}{l} = 0$$

$$\text{or } \alpha = \pm \sqrt{\frac{g}{l}}$$

$$\therefore x(t) = A e^{\sqrt{\frac{g}{l}} t} + B e^{-\sqrt{\frac{g}{l}} t} = \text{most general solution}$$

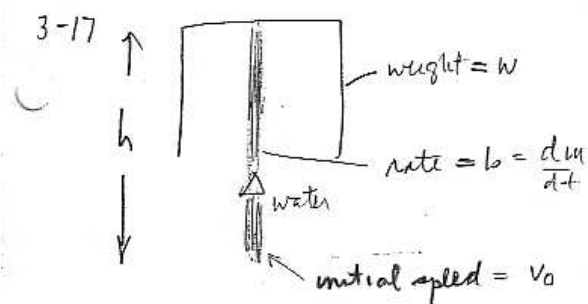
$$x(0) = l_0 = A + B$$

$$\dot{x}(0) = 0 = \sqrt{\frac{g}{l}} (A - B)$$

$$\therefore \begin{cases} A + B = l_0 \\ A - B = 0 \end{cases} \Rightarrow \begin{cases} A = l_0/2 \\ B = l_0/2 \end{cases}$$

$$\text{or } x(t) = \frac{l_0}{2} (e^{\sqrt{\frac{g}{l}} t} + e^{-\sqrt{\frac{g}{l}} t})$$

Problem 7



speed at height $h = v$

$$v^2 - v_0^2 = -2gh$$

$$v = \sqrt{v_0^2 - 2gh}$$

$$\int F dt = F_{ar} \Delta t = \Delta P \quad (\text{at collision with can})$$

$$F_{ar} = \frac{\Delta P}{\Delta t} = \frac{(b \Delta t) \Delta v}{\Delta t} = b \Delta v$$

mass hitting Δt

$$\therefore F_{ar}^{max} = b(\Delta v)^{max}$$

When water $\downarrow$ v afterward $\Rightarrow 2v$

$$\Rightarrow F_{ar}^{max} = 2bv = w$$

$$\text{or } v^2 = \frac{w^2}{4b} = v_0^2 - 2gh_{max} \text{ or } h_{max} = \frac{1}{2g} \left(v_0^2 - \frac{w^2}{4b} \right)$$

Problem 8

3-18

$$M(0) = M(\infty)$$

v positive $\downarrow$

$$F = Mg \downarrow$$

$$P(t) = M(t) V(t)$$

$$P(t + \Delta t) = (M(t) + \frac{dM}{dt} \Delta t) (V(t) + \Delta v)$$

$$\therefore \Delta P = \frac{dM}{dt} V(t) \Delta t + M(t) \Delta v$$

$$\text{or } \frac{\Delta P}{\Delta t} = M(t) g = V(t) \frac{dM}{dt} + M(t) \frac{dV}{dt} \quad \nearrow \frac{d(M(t)V(t))}{dt} \text{ works here!}$$

$$M(t) g = k M(t) v^2(t) + M(t) \frac{dv}{dt}$$

$$\text{or } g = k v^2 + \frac{dv}{dt}$$

$$\therefore g - k v^2 = \frac{dv}{dt}$$

$$\text{or } \frac{dv}{g - k v^2} = dt$$

$$\int_0^v \frac{dv}{g - k v^2} = \int_0^t dt = t$$

$$t = \int_0^v \frac{dv}{g - kv^2} = \frac{1}{2\sqrt{g}} \int_0^v \left[\frac{dv}{\sqrt{g - kv}v} + \frac{dv}{\sqrt{g + kv}v} \right]$$

$$t = \frac{1}{2\sqrt{g}} \left[\frac{1}{\sqrt{k}} \left[\ln(\sqrt{g - kv}) - \ln \sqrt{g} \right] + \frac{1}{\sqrt{k}} \left[\ln(\sqrt{g + kv}) - \ln \sqrt{g} \right] \right]$$

$$t = \frac{1}{2\sqrt{gk}} \ln \frac{\sqrt{g + kv}}{\sqrt{g - kv}} = \frac{1}{2\sqrt{gk}} \ln \frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v}$$

$$\Rightarrow \frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v} = e^{2\sqrt{gk}t}$$

$$v(1 + e^{2\sqrt{gk}t}) = \sqrt{\frac{g}{k}} (e^{2\sqrt{gk}t} - 1)$$

$$\Rightarrow v = \sqrt{\frac{g}{k}} \frac{e^{2\sqrt{gk}t} - 1}{e^{2\sqrt{gk}t} + 1}$$

$$\rightarrow \sqrt{\frac{g}{k}} \text{ as } t \rightarrow \infty$$

↳ terminal velocity

$$\frac{dy}{dt} = v = \sqrt{\frac{g}{k}} \frac{e^{\sqrt{gk}t} - e^{-\sqrt{gk}t}}{e^{\sqrt{gk}t} + e^{-\sqrt{gk}t}}$$

$$y(t) = \sqrt{\frac{g}{k}} \int_0^t \frac{e^{\sqrt{gk}t} - e^{-\sqrt{gk}t}}{e^{\sqrt{gk}t} + e^{-\sqrt{gk}t}} dt$$

$$= \sqrt{\frac{g}{k}} \frac{1}{\sqrt{gk}} \int_2^{e^{\sqrt{gk}t} + e^{-\sqrt{gk}t}} \frac{du}{u}$$

$$= \frac{1}{k} \ln(\cosh \sqrt{gk}t)$$

```

clear;
k=0.05;
g=9.80;
t=zeros(1,10000);
y=t;
y(1)=100;
v=t;
h=0.01;
count=1;
while(y(count) >= 0)
    t(count+1)=t(count)+h;
    y(count+1)=y(count)+h*v(count);
    v(count+1)=v(count)+h*acc1(g,k,v(count));
    count=count+1;
endwhile
tt=t(1:count);
yy=y(1:count);
vv=v(1:count);
figure('Position',[50,50,400,400])
hold on
axis([min(tt),max(tt),min(yy),max(yy)])
title('Raindrops','fontsize',40);
xlabel('Time(seconds)','fontsize',40)
ylabel('Position(meters)','fontsize',40)
plot(tt,yy,'-k')
tc=floor(length(tt)/20);
tcl=20*h*(0:tc);
ytheory=xth(g,k,tcl);
plot(tcl,100-ytheory,'or')
hold off
figure('Position',[500,50,950,400])
hold on
axis([min(tt),max(tt),min(vv),max(vv)])
title('Raindrops','fontsize',40);
xlabel('Time(seconds)','fontsize',40)
ylabel('Velocity(meters/seconds)','fontsize',40)
plot(tt,vv,'-k')
vtheory=vth(g,k,tcl);
hold on
plot(tcl,vtheory,'or')
hold off

```

```

function a=acc1(g,k,v)
a=-g+k*v*v;
endfunction

```

```

function a=xth(g,k,t)
a=(1/k)*log(0.5*(exp(sqrt(g*k) *t)+exp(-sqrt(g*k) *t)));
endfunction

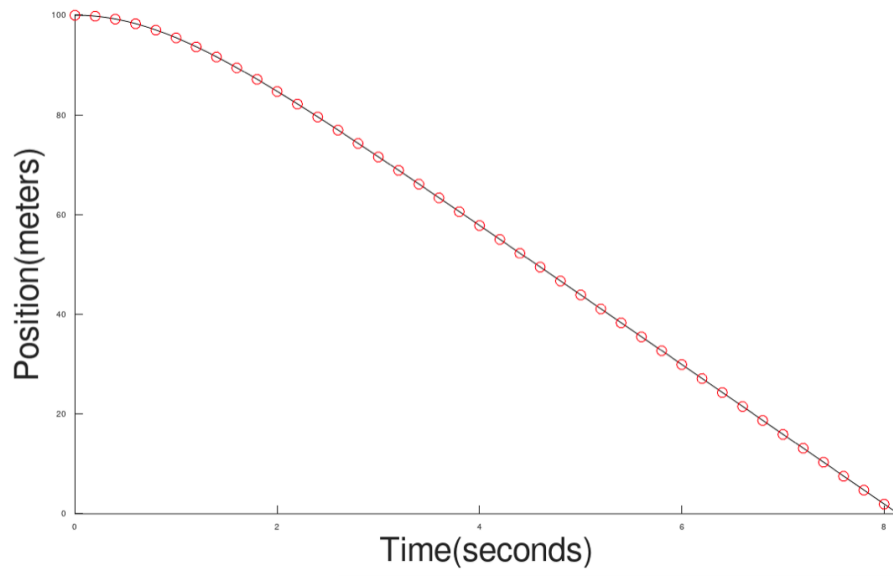
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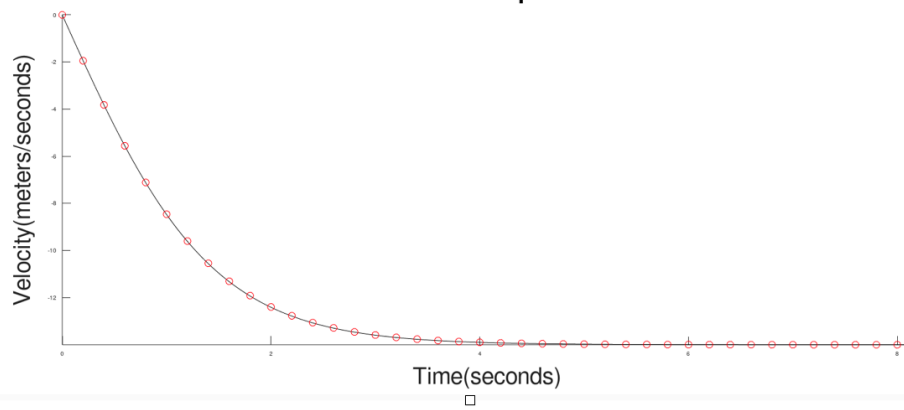
function a=vth(g,k,t)
a=-sqrt(g/k)*(exp(2*sqrt(g*k) *t)-1)./(exp(2*sqrt(g*k) *t)+1);
endfunction

```

Raindrops



Raindrops



Assignment 7 - Solutions

Problem 1

A-1



at a

$\downarrow m \downarrow a = \frac{v^2}{R} =$ acceleration needed to stay on circular path
 $N =$ reaction of track

Newton's 2nd Law

$$\therefore N + mg = ma = m \frac{v^2}{R}$$

we want to reach (a) such that $N = mg$

$$\rightarrow v^2 = 2gR$$

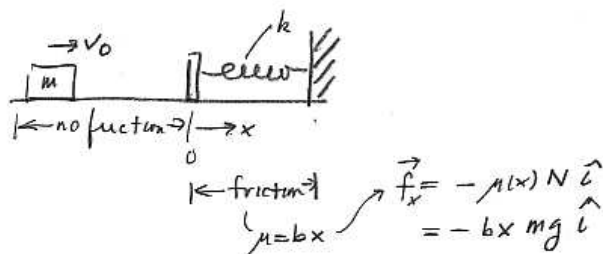
Conservation of Energy

$$\text{initial energy} = mgz = \text{final energy} = mg(2R) + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}}$$

$$\rightarrow z = 2R + \frac{v^2}{2g} = 3R$$

Problem 2

4-2



as block hits spring

$$\text{initial energy} = E_0 = \frac{1}{2} m v_0^2$$

as block compresses spring (when we are at x)

→ where block comes to rest

$$\text{energy} = E_x = \frac{1}{2} k x^2 - W_f$$

→ work done by friction (heat loss)

($E_{\text{block}} = 0$ at this point)

$$W_f = \int_0^x \vec{f} \cdot d\vec{r} = \int_0^x (-bx mg \hat{i}) \cdot (dx \hat{i})$$

$$= -bmg \int_0^x x dx \quad \text{using } (\hat{i} \cdot \hat{i}) = 1$$

$$= -\frac{bmg}{2} x^2$$

∴ $E_0 = E_x$ conservation of energy

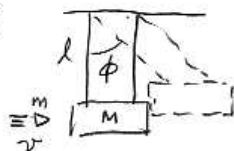
$$\rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} k x^2 + \frac{1}{2} bmg x^2$$

$$\rightarrow x = \sqrt{\frac{m v_0^2}{k + bmg}} \rightarrow \text{where it stops}$$

$$\Delta E = \text{loss of mechanical energy} = \frac{1}{2} m v_0^2 - \frac{1}{2} k x^2 = W_f = \frac{1}{2} bmg x^2$$

Problem 3

4.3



- (a) conservation of momentum during collision
(kinetic energy is NOT conserved)

$$\underbrace{mv}_{\text{bullet}} = \underbrace{(m+M)V_{\text{after}}}_{\text{bullet+block}} \rightarrow V_{\text{after}} = \frac{m}{m+M} v$$

- (b) after collision energy is conserved

initially $E_i = \frac{1}{2} (m+M) V_{\text{after}}^2$

(potential energy = 0 by choice)

finally $E_f = (m+M) g h$ final height system rises to as it comes to a momentary rest

(kinetic energy = 0 since at rest)

$$E_i = E_f$$

from diagram $h = l(1 - \cos \phi) \rightarrow$

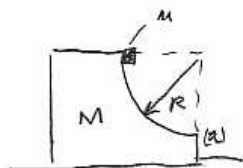
$$\frac{1}{2} (m+M) \left(\frac{m}{m+M} \right)^2 v^2 = (m+M) g l (1 - \cos \phi)$$

$$\rightarrow v = \frac{m+M}{m} \sqrt{2gl(1 - \cos \phi)}$$

$\therefore$ measure $m, M, l, \phi \rightarrow v = \text{speed of bullet}$

Problem 4

4-4



no friction
initially at rest

$$E_0 = m g R \quad (\text{potential energy} = 0 \text{ at (a)})$$

$$E_f = \frac{1}{2} m v_{\text{cube}}^2 + \frac{1}{2} M v_{\text{block}}^2 \quad (\text{both moving at end})$$

momentum conservation $\Rightarrow$

$$0 = P_x = m v_{\text{cube}} - M v_{\text{block}} = v_{\text{block}} = \frac{m}{M} v_{\text{cube}}$$

energy conservation $\Rightarrow$

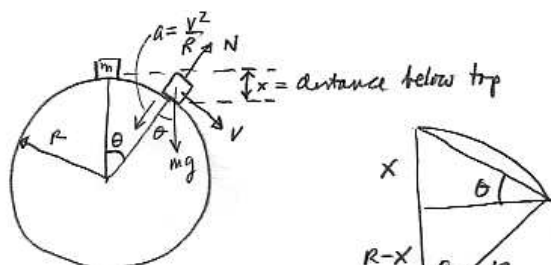
$$E_0 = m g R = E_f = \frac{1}{2} m v_{\text{cube}}^2 + \frac{1}{2} M \left(\frac{m}{M} \right)^2 v_{\text{cube}}^2$$

$$\rightarrow v_{\text{cube}}^2 = \frac{m g R}{\frac{1}{2} m + \frac{1}{2} \frac{m^2}{M}}$$

$$\text{or } v_{\text{cube}} = \sqrt{\frac{2 g R}{1 + \frac{m}{M}}}$$

Problem 5

4-6



$$E_0 = mgR \quad (\text{zero at center of circle})$$

$$E_f = \frac{1}{2}mv^2 + mg(R-x) \quad \text{see diagram}$$

Newton's 2nd law

$$mg \cos \theta - N = \frac{mv^2}{R}$$

$$N=0 \rightarrow \text{leaves surface} \rightarrow v^2 = gR \cos \theta$$

energy conservation

$$E_0 = E_f$$

$$mgR = \frac{1}{2}mgR \cos \theta + mg(R-x)$$

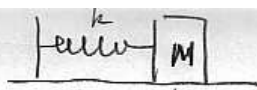
$$\rightarrow \frac{1}{2}R \cos \theta = x$$

$$\text{or } \frac{1}{2}R \left(\frac{R-x}{R} \right) = x \quad \cos \theta \quad (\text{see diagram})$$

$$\rightarrow 2x = R-x \rightarrow x = \frac{R}{3}$$

Problem 6

4-10



$x=0 \Rightarrow$ equilibrium

Block oscillation: amplitude = A_0 , $T_0 = 2\pi \sqrt{\frac{M}{k}}$

$$\rightarrow x = A_0 \cos \omega_0 t = A_0 \cos \frac{2\pi t}{T_0}$$

$$\dot{x} = v = -\frac{2\pi}{T_0} A_0 \sin \frac{2\pi t}{T_0}$$

$$v(t_1) = 0 \rightarrow \frac{2\pi}{T_0} t_1 = 0 \rightarrow t_1 = 0$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} E = \frac{1}{2} k A_0^2$$

$$\rightarrow x(t_1) = A_0$$

(a) at $t=t_1$, a mass m (putty) sticks to block (moving rectilinearly)

momentum conservation

$\Rightarrow$ like new system where system starts from rest at $x=A_0$

$$\nexists M \rightarrow m+M$$

$$\rightarrow (1) T_{\text{new}} = 2\pi \sqrt{\frac{m+M}{k}} = \sqrt{\frac{m+M}{M}} T_0$$

$$(2) A_{\text{new}} = A_0$$

$$(3) E = \frac{1}{2} k A_{\text{new}}^2 = \frac{1}{2} k A_0^2 \Rightarrow \Delta E = 0$$

(b) now mass sticks when $v = v_{\text{max}} = \frac{2\pi}{T_0} A_0$

$$(1) T_{\text{new}} = \sqrt{\frac{m+M}{M}} T_0 \quad (\text{period only function of } k \text{ \& mass})$$

$$(2) \text{ conservation of momentum } \rightarrow M v_{\text{max}} = (M+m) v_{\text{max, after}}$$

$$\rightarrow v_{\text{max, after}} = \frac{M}{M+m} v_{\text{max}}$$

$$\rightarrow \frac{1}{2} k A_{\text{new}}^2 = \frac{1}{2} (M+m) v_{\text{max, after}}^2 \Rightarrow \frac{1}{2} \frac{M^2}{m+M} v_{\text{max}}^2 = \frac{1}{2} \frac{M}{m+M} k A_0^2$$

$$\rightarrow A_{\text{new}} = \sqrt{\frac{M}{m+M}} A_0$$

$$(3) \Delta E = \frac{1}{2} k A_0^2 - \frac{1}{2} k A_{\text{new}}^2 = \frac{m}{m+M} \frac{1}{2} k A_0^2$$

Problem 7

4-12 drop from height h , acceleration $= g$
 speed at impact $= v_0 = \sqrt{2gh}$

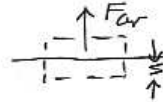
Let $\Delta t = \text{duration of impact}$

$$F_{ar} \Delta t = mv_0 - 0 = \Delta p$$

$$\text{depth marks} = s = v_0 \Delta t - \frac{1}{2} \frac{F_{ar}}{m} (\Delta t)^2$$

$$s = v_0 \Delta t - \frac{1}{2} \frac{v_0}{\Delta t} (\Delta t)^2 = \frac{1}{2} \frac{v_0}{\Delta t} \Delta t$$

$$\rightarrow \Delta t = \frac{2s}{v_0} \rightarrow F_{ar} = \frac{1}{2} \frac{mv_0^2}{s}$$



Alternatively

$$\frac{1}{2} mv_0^2 = \Delta K = \text{Work} = F_{ar} s$$

$$F_{ar} = \frac{1}{2} \frac{mv_0^2}{s} = \frac{mgh}{s}$$

$$p_{ar}^{\text{max}} = \frac{F_{ar}}{A} = \frac{mgh}{As} = \frac{(144)(150)}{5 \times 2} = 144 \times 15 \frac{\text{lb}}{\text{ft}^2}$$

$$= 15 \frac{\text{lb}}{\text{in}^2} < 30 \frac{\text{lb}}{\text{in}^2} \rightarrow \text{safe to drop}$$

Problem 8

4-17 uphill

speed = constant $\rightarrow a = 0$

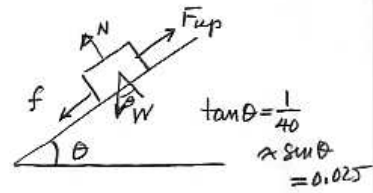
$$\rightarrow \sum \vec{F} = 0 \rightarrow$$

$$F_{up} = f + W \sin \theta$$

$$= \mu N + W \sin \theta \quad ; \quad N = W \cos \theta$$

$$F_{up} = W (\mu \cos \theta + \sin \theta)$$

$$= W (0.05 + \sin \theta)$$



$$f = \mu N = 0.05 W = \mu W \cos \theta / W$$

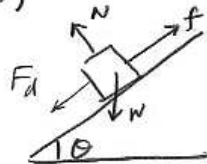
downhill

$$\text{Power} = F_{up} v_{up} = F_d v_d$$

speed = constant $\rightarrow a = 0$

$$\sum \vec{F} = 0 \rightarrow$$

$$F_{down} = f - W \sin \theta = W (0.05 - \sin \theta)$$

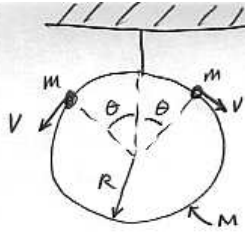


$$\therefore v_d = \frac{F_{up}}{F_{down}} v_{up} = \left(\frac{0.05 + \sin \theta}{0.05 - \sin \theta} \right) v_{up} = 3 v_{up} = 45 \text{ mph}$$

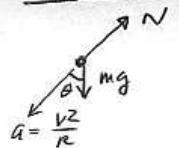
Assignment 8 - Solutions

Problem 1

4-7



for each bead

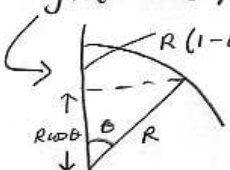


Newton's 2nd Law

$$\rightarrow mg \cos \theta - N = \frac{mv^2}{R}$$

change in $E_k = \text{change in } E_p$

conservation of energy $\rightarrow \frac{1}{2}mv^2 = mgR(1 - \cos \theta)$

$$\rightarrow N = mg \cos \theta - \frac{mv^2}{R} = mg(3 \cos \theta - 2)$$


Forces on Ring (ring at rest)

$$\rightarrow T = Mg + 2N \cos \theta$$

When $T=0$ ring starts to rise

$$\rightarrow 0 = Mg + 2 \cos \theta (mg(3 \cos \theta - 2))$$

$$\text{or } \cos \theta (3 \cos \theta - 2) = -\frac{M}{2m}$$

$$\text{or } 2 \cos \theta - 3 \cos^2 \theta = f(\cos \theta) = \frac{M}{2m}$$

Now $f(\cos \theta)$ has a maximum value when $\frac{df(\cos \theta)}{d(\cos \theta)} = 0$

$$\rightarrow 2 - 6 \cos \theta = 0 \rightarrow (\cos \theta)_{\max} = \frac{1}{3}$$

$$f((\cos \theta)_{\max}) = \frac{1}{3}$$

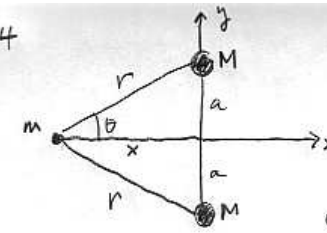
$$\frac{1}{3} \geq \frac{M}{2m} \rightarrow m \geq \frac{3}{2}M$$

$\rightarrow$ ring will not rise unless $\frac{1}{3} \geq \frac{M}{2m} \rightarrow m \geq \frac{3}{2}M$

$\rightarrow$ angle at which it rises given by $3 \cos^2 \theta - 2 \cos \theta + \frac{M}{2m} = 0 \rightarrow \cos \theta = \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{M}{2m}}$
(When $m \geq \frac{3}{2}M$)

Problem 2

4-14



(a) $U_{\text{head}} = -\frac{2GMm}{r}$
 $= -\frac{2GMm}{(a^2+x^2)^{1/2}}$
 where $r^2 = a^2+x^2$

Note: $\vec{F} = F_x \hat{i} = \frac{2GMm}{r^2} \cos\theta \hat{i} \Rightarrow F_x = \frac{2GMm}{a^2+x^2} \frac{x}{r} = \frac{2GMm x}{(a^2+x^2)^{3/2}}$

(b) head released at $x=3a$ with $v_x=v_0$
 $E_0 = \frac{1}{2}mv_0^2 + U(3a)$, $E_f = \frac{1}{2}mv^2 + U(0)$
 $E_0 = E_f \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - \frac{2GMm}{\sqrt{10}a} + \frac{2GMm}{a}$
 $\rightarrow v^2 = v_0^2 + \frac{4GM}{a} \left(1 - \frac{1}{\sqrt{10}}\right)$

(c) near any point x_0 , we can write
 $U(x) = U(x_0) + (x-x_0) \left. \frac{dU}{dx} \right|_{x=x_0} + \frac{(x-x_0)^2}{2} \left. \frac{d^2U}{dx^2} \right|_{x=x_0} + \dots$
 for $x_0 = 0 \Rightarrow U(x) = U(0) + x \left. \frac{dU}{dx} \right|_{x=0} + \frac{x^2}{2} \left. \frac{d^2U}{dx^2} \right|_{x=0} + \dots$
 Now $\frac{dU}{dx} = -2GMm \left(-\frac{1}{2}\right) (a^2+x^2)^{-3/2} (2x) = \frac{2GMm x}{(a^2+x^2)^{3/2}}$
 $\rightarrow \left. \frac{dU}{dx} \right|_{x=0} = 0 \rightarrow \text{equilibrium point } (F_x = 0)$
 $\frac{d^2U}{dx^2} = \frac{2GMm}{(a^2+x^2)^{3/2}} - \frac{6GMm x^2}{(a^2+x^2)^{5/2}} \Rightarrow \left. \frac{d^2U}{dx^2} \right|_{x=0} = \frac{2GMm}{a^3}$
 $\therefore U(x) - U(0) = \frac{1}{2}kx^2$ where $k = \frac{2GMm}{a^3}$
 $\rightarrow \text{SHM with frequency } f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{2GM}{a^3}}$

Problem 3

4-15 (a) $U = - \int \vec{F} \cdot d\vec{r}$

$\vec{F} \Rightarrow$ repulsive force with magnitude $\frac{A}{x^2} + \text{constant}$
 $\Rightarrow \vec{F} = \frac{A}{x^2} \hat{i} - B \hat{i}$ force towards origin
 $d\vec{r} = dx \hat{i}$

$$\therefore U(x) = -A \int \frac{dx}{x^2} + \int B dx$$

$$U(x) = Bx + \frac{A}{x} + \text{constant}$$

(b) choose constant = 0 (reference level is arbitrary)

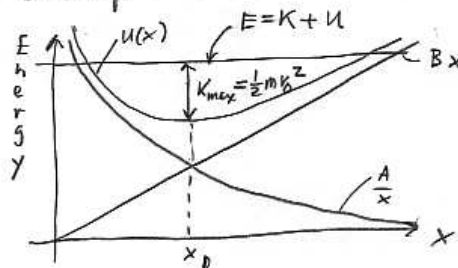
$\rightarrow$ energy diagram

$$x_0 \rightarrow Bx_0 = \frac{A}{x_0} \rightarrow x_0 = \sqrt{\frac{A}{B}}$$

(c) minimum of $U(x)$

$$\frac{dU}{dx} = 0 = B - \frac{A}{x^2}$$

$$\rightarrow x = \sqrt{\frac{A}{B}} = x_0$$



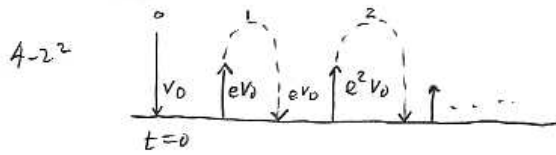
(d) $U(x) = U(x_0) + \frac{(x-x_0)^2}{2} \frac{d^2U}{dx^2} \Big|_{x=x_0} = \sqrt{\frac{A}{B}}$

$$\frac{d^2U}{dx^2} = \frac{2A}{x^3} \rightarrow \frac{d^2U}{dx^2} \Big|_{x=x_0} = \frac{2A}{(\sqrt{\frac{A}{B}})^3} = 2\sqrt{\frac{B^3}{A}}$$

$$U(x) - U(x_0) = \frac{1}{2} k (x-x_0)^2 \rightarrow k = \frac{d^2U}{dx^2} \Big|_{x=x_0} = 2\sqrt{\frac{B^3}{A}}$$

$$\rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{m}} \left(\frac{B^3}{A} \right)^{1/4}$$

Problem 4



at n^{th} stage $v \uparrow = e^n v_0$

if $v = v_{\text{up}}$, then Δt to return to ground is

$$\Delta t = \frac{2 v_{\text{up}}}{g}$$

at n^{th} stage g

$$\rightarrow \Delta t_n = \frac{2}{g} e^n v_0$$

$$\begin{aligned} \therefore (\Delta t)_{\text{total}} &= \sum_{n=1}^{\infty} \Delta t_n \\ &= \frac{2}{g} v_0 \sum_{n=1}^{\infty} e^n = \frac{2e}{g} v_0 (1 + e + e^2 + e^3 + \dots) \\ &= \frac{2}{g} v_0 \frac{e}{1-e} \end{aligned}$$

$= \frac{1}{1-e}$ geometric series

$$v_0 = 5 \text{ m/sec}, \quad e = \frac{1}{2}, \quad g \approx 10$$

$$\rightarrow (\Delta t)_{\text{total}} = \frac{2}{10} 5 \frac{\frac{1}{2}}{\frac{1}{2}} \approx 1 \text{ sec.}$$

Problem 5

4.23

collision with floor

USING CM

$\begin{matrix} \text{m} \\ \downarrow v \\ \text{m} \end{matrix}$ $\begin{matrix} \text{M} \\ \downarrow v \\ \text{M} \end{matrix}$ $\leftarrow$ M collides first

where $v = \sqrt{2gh}$ $\Rightarrow$ collision between balls

$$v_{cm} = \frac{Mv - mv}{M+m} = \frac{M-m}{M+m} v$$

$$v_{m,cm} = -v - v_{cm} = -\frac{2Mv}{M+m}, \quad v_{M,cm} = v - v_{cm} = \frac{2mv}{M+m}$$

elastic collision $\rightarrow$ velocities reverse

$$\Rightarrow v'_{m,cm} = \frac{2Mv}{M+m}, \quad v'_{M,cm} = -\frac{2mv}{M+m}$$

$$\rightarrow v'_{m,lat} = v'_{m,cm} + v_{cm} = \frac{2Mv}{M+m} + \frac{M-m}{M+m} v = v \left(\frac{3M-m}{M+m} \right)$$

$$\approx 3v \quad \text{for } M \gg m$$

$$\rightarrow h_{new} = \frac{v'^2_{m,lat}}{2g} = g \frac{v^2}{2g} = gh \quad \text{original height}$$

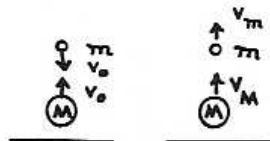
No CM

4.23 $v_0 = \sqrt{2gh}$

$$(M-m)v_0 = Mv_M + mv_m$$

$$\frac{1}{2}(M+m)v_0^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$$

Solving $v_M = \left(1 - \frac{m}{M}\right)v_0 - \frac{m}{M}v_m$



$$(M+m)v_0^2 = M \left[\left(1 - \frac{m}{M}\right)v_0 - \frac{m}{M}v_m \right]^2 + mv_m^2$$

$$= Mv_0^2 - 2mv_0^2 + \frac{m^2}{M}v_0^2 + \frac{m^2}{M}v_m^2 - 2mv_0v_m + 2\frac{m^2}{M}v_0v_m + mv_m^2$$

$$\text{OR } v_m^2 - 2v_0v_m - 3v_0^2 = 0$$

$$v_m = \frac{2v_0 + \sqrt{4v_0^2 + 12v_0^2}}{2} = 3v_0 = 3\sqrt{2gh}$$

Hence marble rises to a height H given by

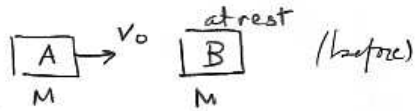
$$mgH = \frac{1}{2}mv_m^2$$

$$H = \frac{\frac{1}{2}v_m^2}{g} = 9h$$

Problem 6

4-24

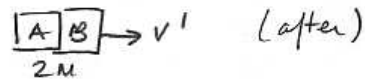
AB collision



$$Mv_0 = 2Mv'$$

inelastic $\Rightarrow$ stick together

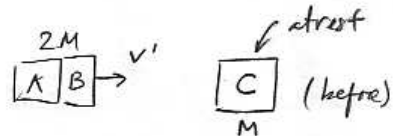
$$\rightarrow v' = \frac{1}{2}v_0$$



$$E_i = \frac{1}{2}Mv_0^2$$

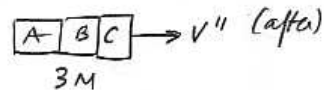
$$E' = \frac{1}{2}(2M)v'^2 = \frac{1}{2}\left(\frac{1}{2}Mv_0^2\right) = \frac{1}{2}E_i$$

(AB) - C collision



$$2Mv' = 3Mv''$$

$$v'' = \frac{2}{3}v' = \frac{1}{3}v_0$$

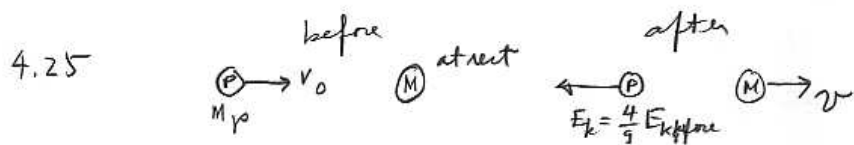


$$E'' = \frac{1}{2}(3M)v''^2 = \frac{1}{3}\left(\frac{1}{2}Mv_0^2\right)$$

$$= \frac{1}{3}E_i$$

Hence $\frac{1}{2}E_i - \frac{1}{3}E_i = \frac{1}{6}E_i$ is dissipated in the 2nd collision, divided between C and the other cars.

Problem 7



Using CM

$$v_{cm} = \frac{m_p v_0}{m_p + M}$$

$$v_{p,cm} = v_0 - v_{cm} = \frac{M v_0}{m_p + M}$$

after collision

$$v'_{p,cm} = -v_{p,cm} = -\frac{M v_0}{m_p + M}$$

after collision in LAB

$$\rightarrow v'_p = v'_{p,cm} + v_{cm} = \frac{v_0}{m_p + M} (m_p - M)$$

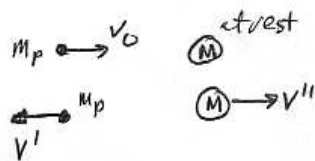
$$E_{k,after}^{proton} = \frac{1}{2} m_p v'^2_p = \frac{4}{9} E_{k,after}^{proton} = \frac{4}{9} \frac{1}{2} m_p v_0^2$$

$$\text{or } v'^2_p = \frac{v_0^2}{(m_p + M)^2} (M - m_p)^2 = \frac{4}{9} v_0^2$$

$$\rightarrow \frac{M - m_p}{M + m_p} = \frac{2}{3} \rightarrow M = 5m_p$$

4.25

Not using CM



$$m_p v_0 = M v'' - m_p v'$$

$$\frac{1}{2} m_p v_0^2 = \frac{1}{2} M v''^2 + \frac{1}{2} m_p v'^2$$

$$v'' = \frac{m_p}{M} (v_0 + v')$$

$$\rightarrow \frac{1}{2} m_p v_0^2 = \frac{1}{2} M \left(\frac{m_p}{M} \right)^2 (v_0 + v')^2 + \frac{1}{2} m_p v'^2$$

$$E_{k, \text{proton}} = \frac{1}{2} m_p v'^2 = \frac{4}{9} E_{k, \text{before}} = \frac{4}{9} \left(\frac{1}{2} m_p v_0^2 \right)$$

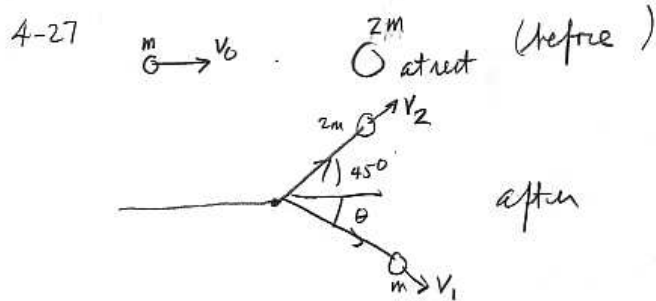
$$\rightarrow v' = \frac{2}{3} v_0$$

$$\rightarrow \frac{1}{2} m_p v_0^2 = \frac{1}{2} \frac{m_p^2}{M} \left(\frac{5}{3} v_0 \right)^2 + \frac{1}{2} m_p \left(\frac{2}{3} v_0 \right)^2$$

$$\text{or } 1 = \frac{25 m_p}{9 M} + \frac{4}{9}$$

$$\frac{25 m_p}{9 M} = \frac{5}{9} \rightarrow M = 5 m_p$$

Problem 8



momentum conservation

$$(X) \quad m v_0 = 2m v_2 \cos 45^\circ + m v_1 \cos \theta$$

$\swarrow \sqrt{2}/2$

$$(Y) \quad 0 = 2m v_2 \sin 45^\circ - m v_1 \sin \theta$$

energy conservation

$$\frac{1}{2} m v_0^2 = \frac{1}{2} (2m) v_2^2 + \frac{1}{2} m v_1^2$$

→ 3 eqs in 3 unknowns v_1, v_2, θ

$$\rightarrow v_2 = \frac{\sqrt{2}}{2} v_1 \sin \theta$$

$$v_0^2 = 2 \frac{1}{2} v_1^2 \sin^2 \theta + v_1^2$$

$$\rightarrow v_0^2 = v_1^2 (1 + \sin^2 \theta)$$

$$\rightarrow v_1^2 = \frac{v_0^2}{1 + \sin^2 \theta}$$

finally using x-momentum eq.

$$V_0 = 2 \left(\frac{\sqrt{2}}{2} \right)^2 V_1 \sin \theta + V_1 \cos \theta$$
$$= V_1 (\cos \theta + \sin \theta)$$

$$\text{or } V_0^2 = V_1^2 (\cos \theta + \sin \theta)^2$$

$$\rightarrow 1 = \frac{(\cos \theta + \sin \theta)^2}{1 + \sin^2 \theta}$$

$$\text{or } 1 + \sin^2 \theta = \cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta$$

$$\text{or } \sin \theta (\sin \theta - 2 \cos \theta) = 0$$

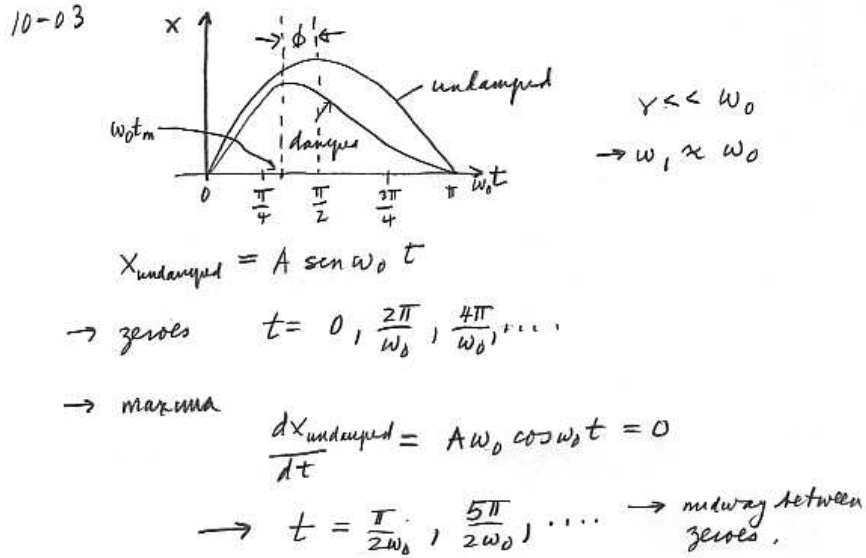
$$\rightarrow \sin \theta = 2 \cos \theta$$

$$\text{or } \tan \theta = 2$$

$$\Rightarrow \theta = 63^\circ$$

Assignment 9 - Solutions

Problem 1



For damped oscillator:

$$x_d = x_0 e^{-\frac{\gamma}{2}t} \sin \omega_0 t$$

$\rightarrow$ zeroes $t = 0, \frac{2\pi}{\omega_0}, \frac{4\pi}{\omega_0}, \dots \rightarrow$ same as undamped case

Maxima occur when $\frac{dx_d}{dt} = 0$

$$\frac{dx_d}{dt} = x_0 \left[\omega_0 \cos \omega_0 t_m - \frac{\gamma}{2} \sin \omega_0 t_m \right] e^{-\frac{\gamma}{2}t_m} = 0$$

$$\rightarrow \frac{\cos \omega_0 t_m}{\sin \omega_0 t_m} = \frac{\gamma}{2\omega_0} \ll 1$$

Let $q = \omega_0 t_m$ and $\phi = \frac{\pi}{2} - \epsilon \rightarrow \begin{cases} \cos \omega_0 t_m = \sin \phi \approx \phi \\ \sin \omega_0 t_m = \cos \phi \approx 1 \end{cases}$

$\underbrace{q \approx \frac{\pi}{2}}$

$$\rightarrow \frac{\cos \omega_0 t}{\sin \omega_0 t} \approx \phi = \frac{\gamma}{2\omega_0} = \frac{1}{2Q}$$

Problem 2

10-04 For damped oscillator, motion is given by
 $x = x_0 e^{-\frac{\gamma}{2}t} \sin \omega_0 t$ $\gamma < \omega_0 \rightarrow \omega_1 \approx \omega_0$

Ratio of two maxima is

$$R = \frac{x_1}{x_2} = \frac{x_0 e^{-\frac{\gamma}{2}t_1} \sin \omega_0 t_1}{x_0 e^{-\frac{\gamma}{2}t_2} \sin \omega_0 t_2}$$

Now $\omega_0 t_2 \approx \omega_0 t_1 + 2\pi \rightarrow \sin \omega_0 t_1 = \sin \omega_0 t_2$

$$\rightarrow R = e^{-\frac{\gamma}{2}(t_1 - t_2)} \approx e^{\frac{\gamma}{2}(\frac{2\pi}{\omega_0})} = e^{\frac{\gamma\pi}{\omega_0}} = e^{\frac{\pi}{Q}}$$

$\delta = \text{logarithmic decrement}$

$$= \ln R = \frac{\pi}{Q} = \frac{\pi\gamma}{\omega_0}$$

$$\rightarrow \gamma = \frac{\delta \omega_0}{\pi} = \frac{0.2\pi}{\pi} = 0.2 \text{ sec}^{-1}$$

$$b = \gamma m = 0.2 \text{ sec}^{-1} \times 5 \text{ kg} = 0.1 \text{ kg/s} = 0.1 \text{ N sec/m}$$

$$k = m\omega_0^2 = 5 \text{ kg} (\pi \text{ sec}^{-1})^2 = 49 \frac{\text{N}}{\text{m}}$$

Problem 3

10-05

equation of motion is $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$

For $\gamma = 2\omega_0^2$ we have

$$\ddot{x} + 2\omega_0 \dot{x} + \omega_0^2 x = 0$$

Assume $x = (A+Bt)e^{-\frac{\gamma}{2}t} = (A+Bt)e^{-\omega_0 t}$

$$\rightarrow \dot{x} = -\omega_0(A+Bt)e^{-\omega_0 t} + Be^{-\omega_0 t}$$

$$\ddot{x} = \omega_0^2(A+Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t}$$

$$\rightarrow \ddot{x} + 2\omega_0 \dot{x} + \omega_0^2 x = 0 \rightarrow \text{it is a solution!}$$

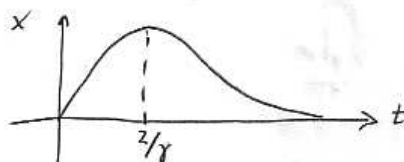
Initial conditions are:

$$x(0) = 0, \quad \dot{x}(0) = \frac{I}{m} \quad \text{impulse} = \Delta p$$

$$\begin{aligned} \rightarrow \quad 0 &= A \\ \frac{I}{m} &= -\omega_0 A + B \end{aligned} \quad \left. \vphantom{\begin{aligned} 0 &= A \\ \frac{I}{m} &= -\omega_0 A + B \end{aligned}} \right\} \rightarrow \begin{aligned} A &= 0 \\ B &= \frac{I}{m} \end{aligned}$$

$$\rightarrow x(t) = \frac{I}{m} t e^{-\frac{\gamma}{2}t}$$

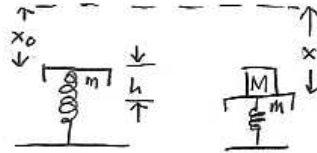
$$\frac{dx}{dt} = 0 = \frac{I}{m} \left(e^{-\frac{\gamma}{2}t} - \frac{\gamma}{2} t e^{-\frac{\gamma}{2}t} \right) = 0 \rightarrow t = \frac{2}{\gamma} \rightarrow \text{maximum}$$



Problem 4

10-06

$$\begin{aligned}x_0 &= 50 \text{ cm} \\ M &= 10 \text{ kg} \\ m &= 2 \text{ kg} \\ h &= 10 \text{ cm}\end{aligned}$$



(a) x = displacement of pan from equilibrium position of spring

m = mass of pan, M = mass of block

$$kx_0 = mg$$

$$kx_1 = (m+M)g$$

$$k(x_1 - x_0) = kh = Mg$$

$$k = \frac{Mg}{h} = \frac{10 \times 9.8}{0.1} = 980 \text{ N/m}$$

(b) The equation of motion for a critically damped oscillator is

$$x = (A + Bt)e^{-\frac{\gamma}{2}t} \rightarrow \text{comes to rest in minimum time}$$

Now measuring x from the final resting place

$$x(0) = h = A$$

$$\dot{x}(0) = v_0 = \left(-\frac{\gamma}{2}(A + Bt)e^{-\frac{\gamma}{2}t} + B e^{-\frac{\gamma}{2}t} \right)_{t=0}$$

$$\rightarrow v_0 = -\frac{\gamma}{2}A + B = -\frac{\gamma h}{2} + B$$

Numtus

$$\gamma = 2\omega_0 = 2\sqrt{\frac{k}{M+m}} = 2\sqrt{\frac{750}{12}} = 18 \text{ sec}^{-1}$$

$$A = h = 0.1 \text{ m}$$

For v_0 , we use conservation of momentum when block hits (energy is not conserved here)

$$MV = (m+M)v_0$$

$$\frac{1}{2}MV^2 = Mgh \text{ where } H = \text{distance block falls}$$

$$\rightarrow v_0 = \frac{M}{m+M} V = \frac{M}{m+M} \sqrt{2gH} = \frac{10}{2} \sqrt{2 \times 9.8 \times \frac{1}{2}} = 2.6 \text{ m/s}$$

Since motion is downwards, $v_0 = -2.6 \text{ m/s}$.

$$B = v_0 + \frac{1}{2}\gamma h = -2.6 + \frac{1}{2} \times 18 \times 0.1 = -1.7 \text{ m/s}$$

$$\rightarrow x = [0.1 - 1.7t] e^{-9t} \text{ m}$$

Note: for $x < 0$ for $t > \frac{1}{17} \text{ s}$.

$\rightarrow$ the system does overshoot
contrary to the statement of the
problem

Problem 5

10-09

The energy dissipated by a damped oscillator is the energy converted to heat by the retarding force bv .

Assume the motion is given by $x = A \cos(\omega t + \phi)$

→ velocity is $v = -A\omega \sin(\omega t + \phi)$

→ power dissipated is $P = v \times bv$
 $= bv^2$
 $= bA^2\omega^2 \sin^2(\omega t + \phi)$

→ $\langle P \rangle = \text{average power dissipated} = \frac{1}{2} bA^2\omega^2$

→ average energy dissipated during 1 radian is

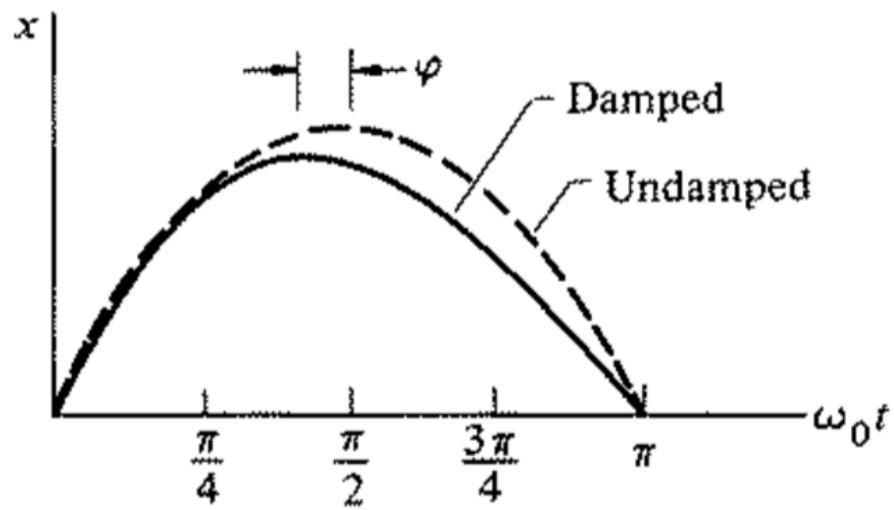
$$\Delta E = \langle P \rangle \times \text{time} = \frac{\langle P \rangle}{\omega} = \frac{1}{2} \omega b A^2$$

For a lightly damped oscillator

$$\omega^2 \approx \omega_0^2 = \frac{k}{m}$$

→ $\langle E \rangle = \text{average energy of oscillator}$
 $= \frac{1}{2} k \langle x^2 \rangle + \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k \left(\frac{1}{2} A^2 \right) + \frac{1}{2} m \left(\frac{1}{2} \omega^2 A^2 \right)$
 $= \frac{1}{2} k A^2$

$$\rightarrow \frac{\langle E \rangle}{\Delta E} = \frac{\frac{1}{2} k A^2}{\frac{1}{2} \omega b A^2} = \frac{k}{\omega b} = \frac{\sqrt{m}}{\sqrt{k}} \frac{k}{m \gamma} = \frac{\omega_0}{\gamma} = Q$$



Problem 6

$$\begin{aligned} \text{EP-9 SHM} \rightarrow x &= A \cos \omega t \\ v &= \dot{x} = -A\omega \sin \omega t \\ a &= \ddot{x} = -A\omega^2 \cos \omega t \\ F &= ma = -Am\omega^2 \cos \omega t \end{aligned}$$

(a) force is maximum when

$$\omega t = 0 \rightarrow |F| = Am\omega^2$$

$$\rightarrow a = -A\omega^2 \Rightarrow \text{maximum magnitude}$$

$$\rightarrow v = 0 \Rightarrow \text{minimum magnitude}$$

$$\rightarrow x = A \Rightarrow \text{maximum magnitude}$$

$\hookrightarrow$ endpoint
 $\therefore$ force is maximum at end points

(b) velocity is maximum when

$$\omega t = \frac{\pi}{2}$$

$$\rightarrow a = 0 \Rightarrow \text{minimum magnitude}$$

$$x = 0 \Rightarrow \text{minimum magnitude}$$

$\hookrightarrow$ equilibrium position

$\therefore$ velocity is maximum at equilibrium position

(c) acceleration is maximum when force is maximum

$\rightarrow$ at endpoints

(d) to double maximum speed ($= A\omega$)

$$V_{\max} = A\omega = A\sqrt{\frac{k}{m}}$$

$$\rightarrow 2V_{\max} \text{ if } \begin{array}{l} A \rightarrow 2A \\ k \rightarrow 4k \\ m \rightarrow m/4 \end{array}$$

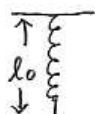
(e) to double maximum acceleration ($= A\omega^2$)

$$a_{\max} = A\omega^2 = A\frac{k}{m}$$

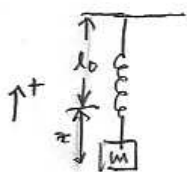
$$\rightarrow 2a_{\max} \text{ if } \begin{array}{l} A \rightarrow 2A \\ k \rightarrow 2k \\ m \rightarrow m/2 \end{array}$$

Problem 7

EP-10



$l_0 = \text{unstretched length}$



measured from equilibrium (l_0)

$$F_z = -kz + mg = m \ddot{z}$$

let $mg = kz_0$

$$\rightarrow m \ddot{z} + k(z - z_0) = 0$$

let $q = z - z_0$

$$\rightarrow \ddot{q} = \ddot{z}$$

$$\rightarrow \ddot{q} + \frac{k}{m} q = 0$$

$$\rightarrow q = A \cos \omega t \quad \omega = \sqrt{\frac{k}{m}}$$

$$\rightarrow z = z_0 + q = z_0 + A \cos \omega t$$

$\rightarrow$ SHM around new equilibrium position

$$z_0 = \frac{mg}{k}$$

with frequency $\omega = \sqrt{\frac{k}{m}}$ which is independent of g .

Problem 8

EP-11

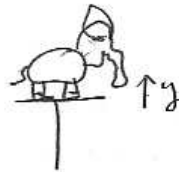
$$y = A \cos \omega t$$

$$A = 0.5 \text{ m}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \quad T = \text{period of motion}$$

$$a = \ddot{y} = -A\omega^2 \cos \omega t$$

$$= -\frac{4\pi^2 A}{T^2} \cos \omega t$$



forces on elephant

When platform $a \uparrow$

$$M_E \frac{4\pi^2 A}{T^2} \geq N - M_E g \Rightarrow N > 0 \text{ always}$$

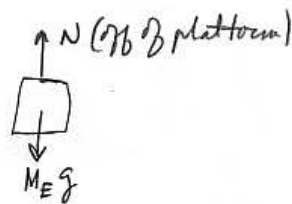
→ elephant cannot lose contact

When platform $a \downarrow$

$$M_E \frac{4\pi^2 A}{T^2} \geq M_E g - N$$

$$\rightarrow \text{if } \frac{4\pi^2 A}{T^2} = g \Rightarrow N = 0 \text{ at top} \rightarrow \text{loses contact}$$

$$\text{if } \frac{4\pi^2 A}{T^2} < g \Rightarrow N > 0 \text{ always} \rightarrow \text{never loses contact}$$



$$\therefore T^2 = \frac{4\pi^2 A}{g} \rightarrow \text{smallest } T \text{ such that just does not lose contact}$$

$$\rightarrow T_{\min} = 2\pi \sqrt{\frac{A}{g}} = 6.28 \sqrt{\frac{1}{19.6}} = 27.8 \text{ sec}$$

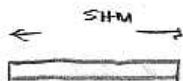
if $T < T_{\min}$ elephant will lose contact for some position near top (platform accelerating downward faster than elephant (g)).

if $T > T_{\min}$ elephant never loses contact

$\Rightarrow T_{\min}$ is smallest period to not lose contact!

Problem 9

EP-12



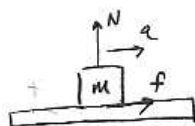
$$x = A \cos \omega t$$

$$A = 1.5 \text{ m}$$

$$f = 15 \text{ min}^{-1}$$

$$= \frac{1}{4} \text{ sec}^{-1}$$

$$\rightarrow \omega = 2\pi f = \frac{\pi}{2} \text{ sec}^{-1}$$



$$N = mg$$

$$f \leq \mu N = \mu mg$$

$$f = ma = m \ddot{x} = -mA\omega^2 \cos \omega t$$

in order that block does not slide

$$f_{\max} = mA\omega^2 \rightarrow \mu_{\min} mg = mA\omega^2$$

$$\rightarrow \mu_{\min} = \frac{A\omega^2}{g} = \frac{1.5 \left(\frac{\pi}{2}\right)^2}{9.8}$$

$$\mu_{\min} = 0.38$$

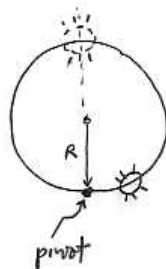
Assignment 10 - Solutions

Problem 1

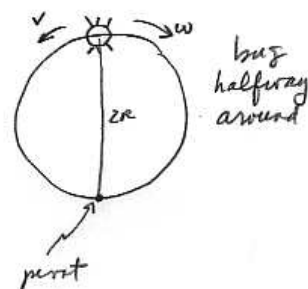
6-03

The vertical component of the angular momentum ($\perp$ to plane of ring)
 $\vec{L}$ is constant and equal to its initial value

$$\vec{L} = 0$$



(a) angular momentum of bug L_{bug} about pivot is



$$L_{\text{bug}} = \underset{\substack{\uparrow \\ \text{mass}}}{m} (\underset{\substack{\downarrow \\ \text{velocity wrt} \\ \text{ground}}}{v - 2Rw}) \underset{\substack{\uparrow \\ \text{radius}}}{2R}$$

angular momentum of ring L_{ring} about pivot is

$$L_{\text{ring}} = -I_{\text{pivot}} \omega = -(I_0 + MR^2) \omega = -2MR^2 \omega$$

$$L = 0 = L_{\text{bug}} + L_{\text{ring}} \Rightarrow m(v - 2Rw)2R = 2MR^2 \omega$$

$$\rightarrow \omega = \frac{mv}{(M + 2m)R}$$

(b) about pivot $L_{\text{bug}} = 0$ (radius = 0)

$$\therefore L_{\text{ring}} = L - L_{\text{bug}} = 0 \rightarrow \omega = 0$$

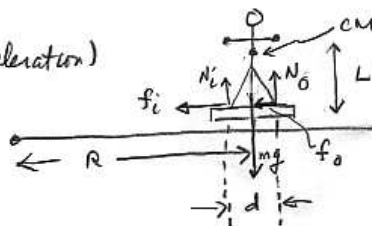
Problem 2

6-06

$$N_i + N_o - mg = 0 \text{ (no acceleration)}$$

$$f_i + f_o = m \frac{v^2}{R}$$

(moving in a circle)



take torques about CM

$$N_o \frac{d}{2} - N_i \frac{d}{2} - (f_i + f_o)L = 0 \quad \leftarrow \text{not rotating about CM}$$

$$\rightarrow N_i + N_o = mg$$

$$N_i - N_o = \frac{2}{d} \left(m \frac{v^2}{R} \right) L$$

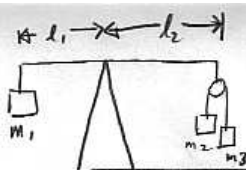
$$\rightarrow 2N_i = \frac{2}{d} \frac{mv^2}{R} L + mg$$

$$2N_o = mg - \frac{2}{d} \left(m \frac{v^2}{R} \right) L$$

$$\text{or } \left. \begin{aligned} N_i &= \frac{1}{2} \left[mg + \frac{2MV^2L}{dR} \right] \\ N_o &= \frac{1}{2} \left[mg - \frac{2MV^2L}{dR} \right] \end{aligned} \right\} \Rightarrow \text{amount of weight on each foot}$$

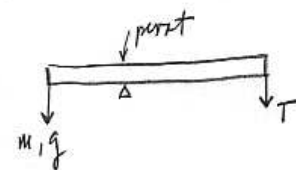
Problem 3

6-12 $m_2 > m_3$



$\Rightarrow T = 2T'$
 $m_2 g - T' = m_2 a$
 $T' - m_3 g = m_3 a$

if $\sum \tau = 0$, beam will not rotate

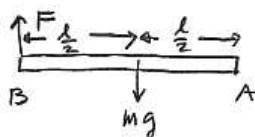


$\rightarrow$ about pivot $m_1 g l_1 - T l_2 = 0 \Rightarrow$ condition for no rotation!

$2T' = (m_3 - m_2) a + (m_2 + m_3) g$
 $(m_2 - m_3) g = (m_2 + m_3) a \rightarrow a = \frac{m_2 - m_3}{m_2 + m_3} g$
 $\rightarrow 2T' = T = (m_2 + m_3) g - \frac{(m_2 - m_3)^2}{m_2 + m_3} g$
 $T = \frac{4 m_2 m_3 g}{m_2 + m_3}$
 $\rightarrow$ condition: $m_1 l_1 = \frac{4 m_2 m_3 l_2}{m_2 + m_3}$

Problem 4

6-14



(a) at release

$$\tau = mg \frac{l}{2}$$

$$(b) \tau = I \alpha \rightarrow \alpha = \frac{\tau}{I} = \frac{mg \frac{l}{2}}{\frac{1}{3} m l^2} = \frac{3}{2} \frac{g}{l}$$

$$(c) a = R \alpha = \frac{l}{2} \alpha = \frac{3}{4} g = a_{cm}$$

(d) in vertical direction

$$mg - F_v = ma$$

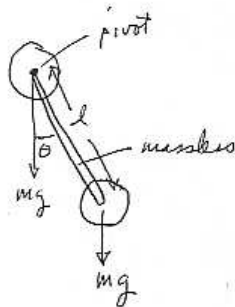
$$F_v = mg - ma = \frac{mg}{4}$$

Problem 5

6.15

$$\tau = I \frac{d\omega}{dt}$$

calculate everything wrt pivot



$$-mgl \sin \theta \approx -mgl \theta = I \ddot{\theta}$$

$$\ddot{\theta} + \frac{mgl}{I} \theta = 0$$

$$\Rightarrow \omega = \sqrt{\frac{mgl}{I}}$$

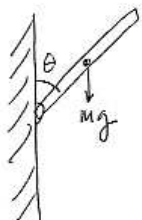
$$I = \frac{1}{2} MR^2 + \underbrace{\left(\frac{1}{2} MR^2 + Ml^2 \right)}_{\text{parallel axis theorem}}$$

$$= M(R^2 + l^2)$$

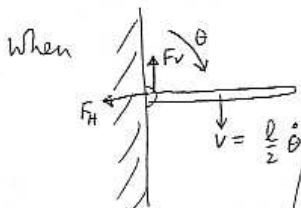
$$\therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R^2 + l^2}{gl}}$$

Problem 6

6.20



$$I\ddot{\theta} = \tau = mg \frac{l}{2} \sin \theta$$



$$Mg - F_V = Ma = m \frac{l}{2} \ddot{\theta}$$

$$F_H = \frac{Mv^2}{(\frac{l}{2})^2} = M \frac{l}{2} \dot{\theta}^2$$

use this relation

$$\therefore \ddot{\theta} \Big|_{\theta=90^\circ} = \frac{Mg \frac{l}{2}}{\frac{1}{3} ml^2} = \frac{3}{2} \frac{g}{l}$$

energy conservation

$$E(\theta=60^\circ) = E(\theta=90^\circ)$$

$$\underbrace{mg \frac{l}{2} \cos 60^\circ}_U = \underbrace{\frac{1}{2} I \dot{\theta}_{at 90^\circ}^2}_K \quad (\text{reference level at } \theta=90^\circ)$$

$$\text{or } \frac{Mgl}{4} = \frac{1}{6} Ml^2 \dot{\theta}_{at 90^\circ}^2$$

$$\Rightarrow \dot{\theta} \Big|_{\theta=90} = \sqrt{\frac{3g}{2l}} \Rightarrow F_H = \frac{3}{4} Mg, \quad F_V = \frac{1}{4} Mg$$

Assignment 11 - Solutions

Problem 1

6-21

torque: about CM

$$fR = \left(\frac{1}{2}MR^2\right)\alpha = I_{CM}\alpha$$

$$\rightarrow f = \frac{1}{2}MR\alpha$$

$$\text{but } R\alpha = a \rightarrow f = \frac{1}{2}Ma$$

rolling without slipping!

$$\text{forces: } N = Mg \cos \theta$$

$$Mg \sin \theta - f = Ma$$

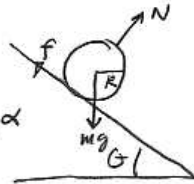
$$\rightarrow Mg \sin \theta = \frac{3}{2}Ma$$

$$\text{but } f \leq \mu N = \mu Mg \cos \theta$$

$$\therefore \frac{1}{2}Ma \leq \mu Mg \cos \theta$$

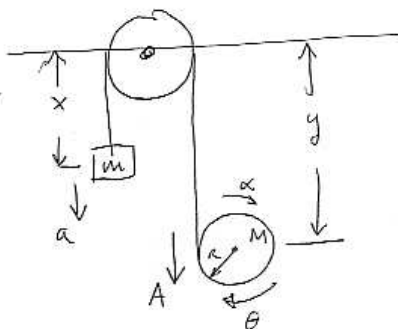
$$\text{or } \frac{1}{3}Mg \sin \theta \leq \mu Mg \cos \theta$$

$$\Rightarrow \tan \theta \leq 3\mu$$



Problem 2

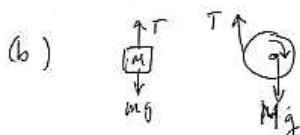
6.23



(a) $x + y = l_0 + R\theta$
 $\uparrow$ string length when $\theta = 0$ $\nwarrow$ amount of string unwound.

$$\ddot{x} + \ddot{y} = R\ddot{\theta}$$

or $a + A = R\alpha$



$$mg - T = ma$$

$$Mg - T = MA$$

$$TR = \underbrace{\frac{1}{2}MR^2}_{I}\alpha$$

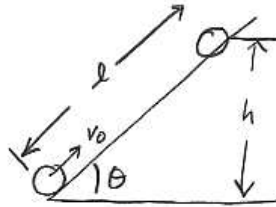
$$\Rightarrow a = \left(\frac{3m - M}{3m + M} \right) g$$

$$A = \left(\frac{m + M}{3m + M} \right) g$$

$$\alpha = \left(\frac{4m}{3m + M} \right) \frac{g}{R}$$

Problem 3

6-25



$$E_{\text{initial}} = E_{\text{final}}$$

$$\frac{1}{2} m v_0^2 + \frac{1}{2} I \dot{\theta}_0^2 = mgh$$

rolling without slipping

$$\frac{1}{2} m v_0^2 + \frac{1}{2} \left(\frac{2}{5} m R^2 \right) \left(\frac{v_0}{R} \right)^2 = mgl \sin \theta$$

$$\Rightarrow \frac{7}{10} m v_0^2 = mgl \sin \theta$$

$$\text{or } l = \frac{7 v_0^2}{10 g \sin \theta}$$

Problem 4

6.30

Method #1

sliding

$$f = -MA$$

$$fR = I_0 \alpha$$

Ball rolls without slipping when $v = R\dot{\theta}$ ($\Rightarrow$ pure rolling)

$$v(t) = v_0 - \int_0^t \frac{f}{M} dt \quad \dot{\theta} = \int_0^t \frac{fR}{I_0} dt$$

v decreases and $\dot{\theta}$ increases until $v = R\dot{\theta}$

$$R\dot{\theta} = \int_0^t \frac{fR^2}{I_0} dt = \frac{R^2}{\frac{2}{5}MR^2} \int_0^t f dt = v = v_0 - \frac{1}{M} \int_0^t f dt$$

hence ball rolls without slipping when

$$\frac{5}{2} \frac{1}{M} \int_0^t f dt = v_0 - \frac{1}{M} \int_0^t f dt$$

$$\text{or } \frac{1}{M} \int_0^t f dt = \frac{2}{7} v_0$$

$\therefore$ when rolling begins $v_f = v_0 - \frac{2}{7} v_0 = \frac{5}{7} v_0$. After this time, $f = 0$ and rolling without slipping continues!

Method #2 - use conservation of angular momentum about contact point.

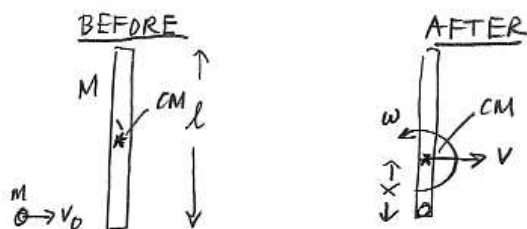
$$L_i = Mv_0 R \quad L_f = Mv_f R + I_0 \omega$$

$$\omega = \frac{v}{R}, \quad I_0 = \frac{2}{5} MR^2 \Rightarrow L_i = L_f \text{ or } Mv_0 R = Mv_f R + \frac{2}{5} Mv_f R$$

$$\text{or } v = \frac{5}{7} v_0$$

Problem 5

6-39



- (a) The boy-plank system translates and rotates about its NEW CM.

$$m v_0 = (M+m) V \quad \text{conservation of momentum}$$

$$\rightarrow V = \frac{m v_0}{M+m}$$

Position of new CM is at

$$M \left(\frac{l}{2} - x \right) = m x \quad \left(x = \frac{M \frac{l}{2} + m(0)}{m+M} \right)$$

$$\rightarrow x = \frac{M \frac{l}{2}}{M+m}$$

Take angular momentum about new CM

$$m v_0 x = \underbrace{\left[m x^2 + I_0 + M \left(\frac{l}{2} - x \right)^2 \right]}_{\substack{\text{by parallel axis theorem} \\ I_{CM}}} \omega$$

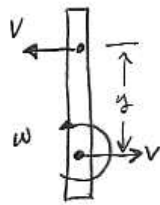
$$\begin{aligned} \text{Now } \left[m x^2 + I_0 + M \left(\frac{l}{2} - x \right)^2 \right] &= \frac{m M^2}{(m+M)^2} \frac{l^2}{4} + \frac{1}{12} M l^2 + \frac{1}{4} M l^2 \left(\frac{m}{m+M} \right)^2 \\ &= \frac{1}{12} M l^2 + \frac{m M l^2}{4(m+M)} = \frac{M l^2}{12} \left(\frac{M+4m}{M+m} \right) \end{aligned}$$

Hence

$$\omega = \frac{m v_0 x}{\frac{M l^2}{12} (M+4m)} = \frac{m M \frac{l}{2} v_0}{\frac{M l^2}{12} (M+4m)}$$

$$\rightarrow \omega = \left(\frac{m}{M+4m} \right) \frac{6 v_0}{l}$$

(b)



The point at rest immediately after the impact has speed $-v$ due to rotation

$$\Rightarrow v = \gamma \omega$$

$$\Rightarrow \gamma = \frac{v}{\omega} = \frac{\frac{m v_0}{m+M}}{\frac{6 v_0}{l} \left(\frac{m}{M+4m} \right)} = \frac{l}{6} \frac{M+4m}{m+M}$$

$$\gamma + x = \frac{\left(\frac{M}{6} + \frac{2}{3} m + \frac{M}{2} \right) l}{M+m} = \frac{2}{3} l \text{ from the boy!}$$

Problem 6

6.41

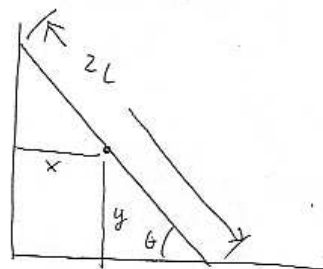
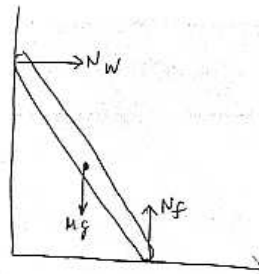
Since there is no friction, only normal forces act at surfaces.

The position of the CM and the angular position are given by a single coordinate θ

Location of CM is given by

$$x = L \cos \theta$$

$$y = L \sin \theta$$



Horizontal force is given by

$$N_W = M \ddot{x} = ML (-\cos \theta \dot{\theta}^2 - \sin \theta \ddot{\theta})$$

contact is lost when $N_W = 0$ or

$$\dot{\theta}^2 = -(\tan \theta) \ddot{\theta} \quad (1)$$

Energy is conserved, so we can find $\dot{\theta}^2$ from the energy equation. We set $\ddot{\theta}$ by $\frac{d}{dt}(\text{energy equation})$

$$\begin{aligned}
 E = Mgy_0 &= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_0\dot{\theta}^2 + Mgy \\
 &= \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\dot{\theta}^2 + Mgy \\
 &= \frac{2}{3}ML^2\dot{\theta}^2 + Mgy \quad (2)
 \end{aligned}$$

$$\frac{dE}{dt} = 0 = \frac{4}{3}ML^2\dot{\theta}\ddot{\theta} + Mgy\cos\theta\dot{\theta}$$

$$\text{or } \ddot{\theta} = -\frac{3}{4}\frac{g}{L}\cos\theta \quad (3)$$

Substitute (3) into (1) $\Rightarrow$

$$\dot{\theta}^2 = \frac{3}{4}\frac{g}{L}\sin\theta$$

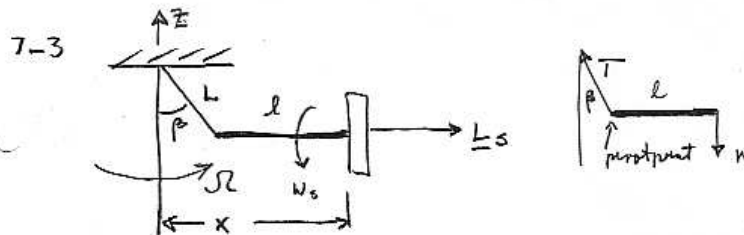
Substitute into (2) $\Rightarrow$

$$Mgy_0 = \frac{1}{2}Mgy\sin\theta + Mgy\sin\theta = \frac{3}{2}Mgy\sin\theta$$

$$\Rightarrow y = L\sin\theta = \frac{2}{3}y_0$$

$\therefore$ board leaves wall when top has slipped $\frac{1}{3}$ of the way down.

Problem 7



linear eqs. of motion

$$T \cos \beta - W = 0 \quad (\text{no acceleration vertically})$$

$$T \sin \beta = M \Omega^2 x \quad (\text{circular motion})$$

angular eq. (wrt pivot point)

$$\tau_z = \frac{dL_z}{dt} = \Omega L_s \quad , \quad L_s = I_0 \omega_s$$

$$= Wl = (T \cos \beta) l$$

for small x β

$$T = W \quad , \quad \beta = \frac{M \Omega^2 x}{W} = \frac{\Omega^2 x}{g}$$

$$\Omega = \frac{T l}{L_s} = \frac{W l}{L_s} = \frac{W l}{I_0 \omega_s}$$

$$\therefore \beta = \frac{W^2 l^2 x}{g I_0^2 \omega_s^2} \quad , \quad \text{but } x = l + L \beta \Rightarrow$$

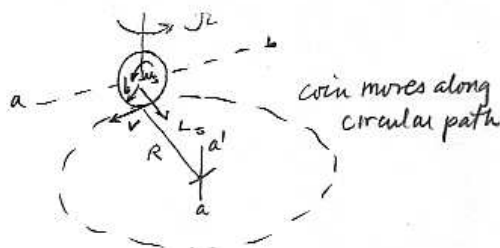
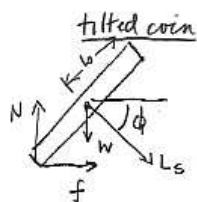
$$\beta = \frac{W^2 l^2 (l + L \beta)}{g (I_0 \omega_s)^2} = \frac{W^2 l^3 (1 + \frac{L}{l} \beta)}{g (I_0^2 \omega_s^2)} = \beta_0 (1 + \frac{L}{l} \beta)$$

$$\Rightarrow \beta (1 - \frac{L}{l} \beta_0) = \beta_0 \quad \text{or} \quad \beta = \frac{\beta_0}{1 - \frac{L}{l} \beta_0}$$

$$\beta_0 = \frac{M^2 g L^3}{(I_0 \omega_s)^2}$$

Problem 8

7.6



the coin is accelerating $\Rightarrow$ use CM as origin for torque required!
eqs $\Rightarrow$

$$N = \text{normal force} = Mg$$

$$f = \text{frictional force} = \frac{Mv^2}{R} \quad \text{direction as shown above!}$$

$$L_s = \text{spin angular mom} = I\omega = \frac{1}{2}Mb^2 \frac{v}{b} = \frac{1}{2}Mbv \quad \text{pure rolling}$$

$$\omega = \text{angular speed of coin rotation about vertical axis } a'a' = \frac{v}{R} \quad (\text{pure rotation})$$

$$\omega L_s \cos \phi = \text{rate of change of } L \text{ mom in horizontal plane} \quad (\text{see gyroscope discussion in text})$$

torque eq along a-b axis is

$$Nb \sin \phi - fb \cos \phi = \omega L_s \cos \phi$$

$$\Rightarrow \tan \phi = \frac{3v^2}{2gR}$$

Assignment 12 - Solutions

Problem 1

9-02 $M = 50 \text{ g}$
 $\vec{F} = -4r^3 \hat{r}$ (- sign $\Rightarrow$ attractive)

$$\rightarrow U(r) = r^4 \quad (U(0) = 0)$$

i.e., $\frac{dU}{dr} = -f(r) = +4r^3$

$$\int_0^r du = +4 \int_0^r r^3 dr$$

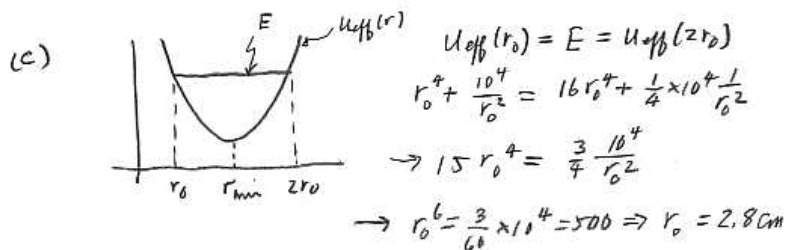
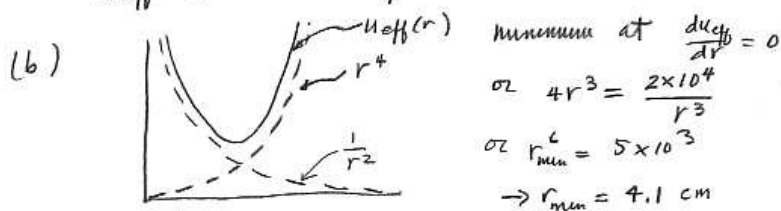
$$U(r) - U(0) = +r^4$$

$$\rightarrow U(r) = U(0) + r^4 = r^4 \quad (U(0) = 0)$$

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2mr^2}$$

$$l = 1000 \text{ g cm}^2/\text{s}$$

(a) $U_{\text{eff}}(r) = r^4 + \frac{10^4}{r^2}$



Problem 2

9-03 $\vec{F}(r) = -\frac{2A}{r^3} \hat{r}$ (attractive inverse cube force)

$$\Rightarrow u(r) = -\frac{A}{r^2}$$

$$\Rightarrow u_{\text{eff}}(r) = -\frac{A}{r^2} + \frac{l^2}{2\mu r^2}$$

If $l^2 = 2\mu A$, then $u_{\text{eff}} = 0$ and radial motion is like that of a free particle ($\frac{1}{2}\mu \dot{r}^2 = E - u_{\text{eff}}$)

$$\Rightarrow \dot{r} = \text{constant} \Rightarrow r = r_0 + vt$$

Now $l^2 = \mu^2 r^2 \dot{\theta}^2 = 2\mu A$

$$\Rightarrow \dot{\theta} = \left(\frac{2A}{\mu}\right)^{1/2} \frac{1}{r^2} = \frac{d\theta}{dt}$$

$$\Rightarrow \int_{\theta_0}^{\theta} d\theta = \left(\frac{2A}{\mu}\right)^{1/2} \int \frac{dt}{r^2} = \left(\frac{2A}{\mu}\right)^{1/2} \int_{r_0}^r \frac{dt}{dr} \frac{dr}{r^2}$$

for free radial motion $\frac{dt}{dr} = \frac{1}{v} = \text{constant}$

$$\Rightarrow \theta - \theta_0 = \left(\frac{2A}{\mu}\right)^{1/2} \frac{1}{v} \left(\frac{1}{r_0} - \frac{1}{r}\right)$$

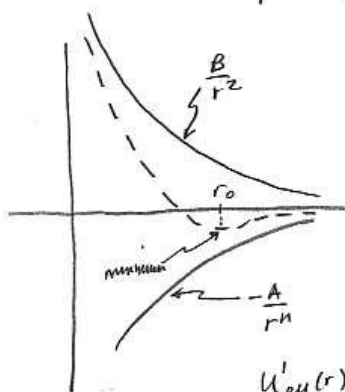
Problem 3

9-04

$$U(r) = -\frac{A}{r^n}, \quad A > 0$$

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

$$= -\frac{A}{r^n} + \frac{B}{r^2}, \quad B = \frac{l^2}{2\mu}$$



For a stable circular orbit to exist, the potential must have a minimum at $r=r_0$

$$\rightarrow U_{\text{eff}}''(r_0) > 0$$

$$U_{\text{eff}}'(r) = \frac{nA}{r^{n+1}} - \frac{2B}{r^3}$$

$$U_{\text{eff}}'(r) = 0 \Rightarrow \frac{nA}{r_0^{n+1}} = \frac{2B}{r_0^3} \Rightarrow r_0$$

$$U_{\text{eff}}''(r_0) = \frac{-n(n+1)A}{r_0^{n+2}} + \frac{6B}{r_0^4} = -\frac{n+1}{r_0} \left(\frac{2B}{r_0^3} \right) + \frac{6B}{r_0^4} > 0$$

$$\text{or } \frac{2B}{r_0^4} \left(-(n+1) + 3 \right) > 0$$

$$\text{or } n+1 < 3 \Rightarrow n < 2$$

Problem 4

$$9-05 \quad (a) \quad u(r) = \frac{1}{2} k r^2 \quad F = -kr \quad |F| = 3r \Rightarrow k = 3 \frac{N}{m}$$

$$E = \frac{1}{2} k r^2 + \frac{1}{2} m v^2 \quad (\text{circular orbit})$$

for equilibrium

$$\frac{m v^2}{r} = k r$$

$$\Rightarrow E = \frac{1}{2} k r^2 + \frac{1}{2} k r^2 = k r^2$$

$$\text{For } E = 12 \text{ J} = 3 r_0^2 \rightarrow r_0 = 2 \text{ m}$$

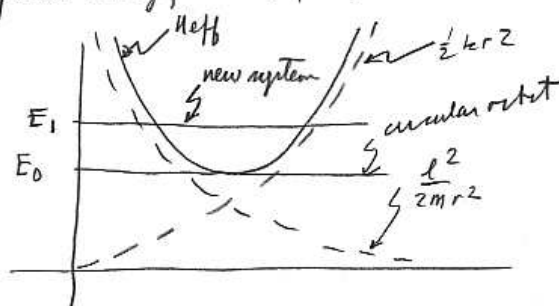
$$\Rightarrow v = \sqrt{\frac{k}{m}} r = \sqrt{\frac{3}{2}} 2 = 2.45 \text{ m/s}$$

(b) The blow changes the energy but not the angular momentum (force in radial direction which is $\perp$ to $\vec{L}$).

Since the effective potential only depends on L , it also does not change

$$\text{The initial energy is } E_0 = \frac{1}{2} k r^2 + \frac{1}{2} m v^2 = k r^2 = 12 \text{ J}$$

$$\text{The final energy is } E_1 = E_0 + \frac{1}{2} m \dot{r}^2 = 12 + \frac{1}{2} (2) 1^2 = 13 \text{ J}$$



$$(c) \quad u_{\text{eff}}(r) = \frac{1}{2} k r^2 + \frac{l^2}{2mr^2}, \quad l = mvr \text{ (in initial circular orbit)}$$

$$l = m v_0 r_0 \text{ in initial circular orbit} = \text{constant}$$

$$\rightarrow u_{\text{eff}}(r) = \frac{1}{2} k r^2 + \frac{1}{2} m v_0^2 \frac{r_0^2}{r^2}$$

$$\text{At turning points } u_{\text{eff}}(r) = E$$

$$\rightarrow \frac{1}{2} k r^2 + \frac{1}{2} m v_0^2 \frac{r_0^2}{r^2} = E$$

$$\frac{3}{2} r^2 + \frac{24}{r^2} = 13 \quad (\text{new energy})$$

$$\rightarrow r^4 - \frac{26}{3} r^2 + 16 = 0$$

$$\rightarrow r^2 = \frac{1}{2} \left[\frac{26}{3} \pm \sqrt{\left(\frac{26}{3}\right)^2 - 4(16)} \right]$$

$$= \frac{1}{2} \left[8.67 \pm \sqrt{11.17} \right]$$

$$r^2 = \frac{1}{2} \left[8.67 \pm 3.34 \right]$$

$$\rightarrow r_{\text{min}} = \sqrt{2.67} = 1.63 \text{ m}$$

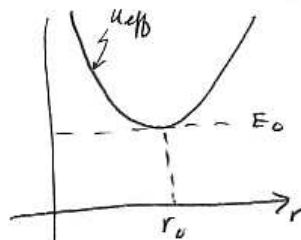
$$r_{\text{max}} = \sqrt{6} = 2.45 \text{ m}$$

Problem 5

9-06 $\vec{F} = -Kr^4 \hat{r}$

$$U(r) = \frac{1}{5} Kr^5$$

$$U_{\text{eff}}(r) = \frac{1}{5} Kr^5 + \frac{l^2}{2\mu r^2}$$



For circular motion $E_0 = U_{\text{eff}}(r_0)$

$$U'_{\text{eff}}(r) = Kr^4 - \frac{l^2}{\mu r^3}$$

$$U'_{\text{eff}}(r_0) = 0 \rightarrow Kr_0^4 = \frac{l^2}{\mu r_0^3} \Rightarrow r_0 = \left(\frac{l^2}{\mu K} \right)^{1/7}$$

$$\Rightarrow E_0 = U_{\text{eff}}(r_0) = \frac{1}{5} Kr_0^5 + \frac{l^2}{2\mu r_0^2} = \frac{7}{10} \left(\frac{l^2 K^6}{\mu} \right)^{5/7}$$

frequency of oscillation is $\omega = \sqrt{\frac{U''_{\text{eff}}(r_0)}{\mu}}$

Now $U''_{\text{eff}} = 4Kr^3 + \frac{3l^2}{\mu r^4}$

$$\rightarrow U''_{\text{eff}}(r_0) = 4Kr_0^3 + \frac{3l^2}{\mu r_0^4}$$

but $Kr_0^4 = \frac{l^2}{\mu r_0^3} \rightarrow U''_{\text{eff}}(r_0) = \frac{7l^2}{\mu r_0^4} = \frac{7l^2}{\mu} \left(\frac{K\mu}{l^2} \right)^{4/7}$

$$\text{or } U''_{\text{eff}}(r_0) = \frac{7K^{4/7} l^{6/7}}{\mu^{3/7}}$$

$$\rightarrow \omega = \left(\frac{7K^{4/7} l^{6/7}}{\mu^{10/7}} \right)^{1/2}$$

Problem 6

9-07

The increase in total energy is

$$\Delta E = (K+U)_{\text{final}} - (K+U)_{\text{initial}}$$

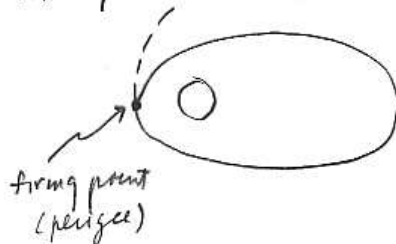
The potential energy is constant during the firing (short duration). This is true for $U(r)$ but not for $U_{\text{effective}}(r)$.

Using $\vec{V}_f = \vec{V}_i + \Delta \vec{V}$

$$\begin{aligned} \rightarrow \Delta E &= K_f - K_i = \frac{1}{2} m (\vec{V}_i + \Delta \vec{V})^2 - \frac{1}{2} m V_i^2 \\ &= m \vec{V}_i \cdot \Delta \vec{V} + \frac{1}{2} m (\Delta V)^2 \end{aligned}$$

$\rightarrow \Delta E$ is a maximum when $\Delta \vec{V}$ is parallel to $\vec{V}_i$ and when V_i is a maximum.

The speed V_i is greatest at perigee



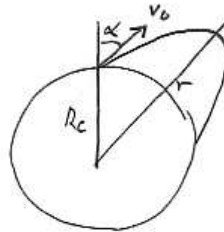
Assignment 13- Solutions

Problem 1

9-08 The initial energy is

$$E = U(R_e) + \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mR_e^2}$$

$$= -\frac{GMm}{R_e} + \frac{1}{2} m v_0^2$$



At the top of the trajectory, $\dot{r} = 0$ and

$$E = -\frac{GMm}{r} + \frac{L^2}{2mr^2}$$

$$\vec{L} = \text{constant} = \vec{R}_e \times \vec{P}_0 \Rightarrow L = m v_0 R_e \sin \alpha$$

$$\rightarrow E = -\frac{GMm}{r} + \frac{1}{2} m v_0^2 \sin^2 \alpha \left(\frac{R_e}{r} \right)^2$$

equating initial and final energies $\Rightarrow$

$$-\frac{GMm}{R_e} + \frac{1}{2} m v_0^2 = -\frac{GMm}{r} + \frac{1}{2} m v_0^2 \sin^2 \alpha \left(\frac{R_e}{r} \right)^2$$

$$\text{Now } v_0 = \sqrt{\frac{GM}{R_e}} \Rightarrow \frac{v_0^2 R_e}{GM} = 1$$

$$\rightarrow -1 + \frac{1}{2} \frac{v_0^2 R_e}{GM} = -\frac{R_e}{r} + \frac{1}{2} \frac{R_e v_0^2}{GM} \sin^2 \alpha \left(\frac{R_e}{r} \right)^2$$

$$\rightarrow -\frac{1}{2} = -\frac{R_e}{r} + \frac{1}{2} \sin^2 \alpha \left(\frac{R_e}{r} \right)^2$$

$$\text{Let } x = \frac{r}{R_e} \rightarrow -\frac{1}{2} x^2 = -x + \frac{1}{2} \sin^2 \alpha$$

$$\text{or } x^2 - 2x + \sin^2 \alpha = 0$$

$$\Rightarrow x = \frac{1}{2} [2 \pm \sqrt{4 - 4 \sin^2 \alpha}] \Rightarrow + \text{ for positive root}$$

$$\Rightarrow x = 1 + \cos \alpha \Rightarrow r = R_e (1 + \cos \alpha) = \text{maximum height}$$

Problem 2

9-09 The period of a satellite depends only on its major axis A . [Eq. 9.30] $\rightarrow$

$$A = \left[2 \left(\frac{T}{\pi} \right)^2 (M+m) G \right]^{1/3}$$

$$T = 76 \text{ years} = 76 \times 3.16 \times 10^7 = 2.4 \times 10^9 \text{ s}$$

$$M+m = M_{\text{sun}} = 2 \times 10^{30} \text{ kg}$$

$$\Rightarrow A = \left[2 \times \left(\frac{2.4 \times 10^9}{\pi} \right)^2 \times 2 \times 10^{30} \times 6.67 \times 10^{-11} \right]^{1/3} = 5.4 \times 10^{12} \text{ m}$$

[Eq. 9.25] $\rightarrow r_0 = \frac{1}{2} A (1 - \epsilon^2) \quad \epsilon = 0.967$

$$\Rightarrow r_0 = \frac{1}{2} (5.4 \times 10^{12}) \times [1 - (0.967)^2] = 1.76 \times 10^{11} \text{ m}$$

$$(a) \text{ perihelion} = \frac{r_0}{1 + \epsilon} = \frac{1.76 \times 10^{11}}{1.967} = 8.88 \times 10^{10} \text{ m}$$

$$\text{aphelion} = \frac{r_0}{1 - \epsilon} = \frac{1.76 \times 10^{11}}{0.033} = 5.29 \times 10^{12} \text{ m}$$

$$(\text{check: perihelion} + \text{aphelion} = 5.4 \times 10^{12} = A)$$

(b) [Eq. 9.19] $\Rightarrow l^2 = r_0 \mu C$
 $\mu = \text{reduced mass of comet} \approx M_{\text{comet}}$ and $C = GM_{\text{sun}}$

$$\text{At perihelion } l = m v r_{\text{min}} \Rightarrow m^2 v^2 r_{\text{min}}^2 = r_0^2 m^2 G M$$

$$\Rightarrow v = \left(\frac{r_0 G M}{r_{\text{min}}^2} \right)^{1/2}$$

$$\text{Now } r_{\text{min}} = \text{perihelion} = \frac{r_0}{1 + \epsilon} \approx \frac{r_0}{2}$$

$$\Rightarrow v = \left(\frac{4 G M}{r_0} \right)^{1/2} = 4.2 \times 10^4 \text{ m/s}$$

Problem 3

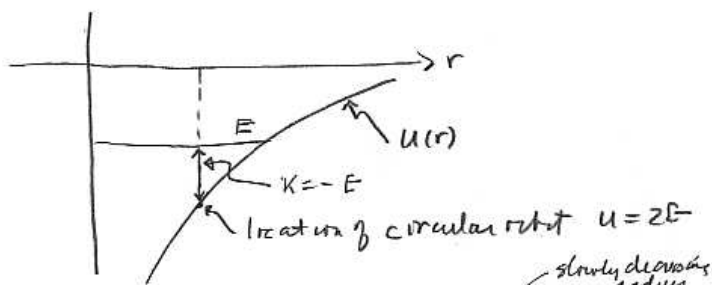
9-10 In circular orbit under an attractive force

$$F = \frac{C}{r^2}, \text{ we have}$$

$$\frac{mv^2}{r} = \frac{C}{r^2} \text{ or } \frac{1}{2}mv^2 = \frac{1}{2}\frac{C}{r} = -\frac{1}{2}U(r) = K$$

(a) The total energy is

$$E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U = -K = -\frac{1}{2}\frac{C}{r}$$



(b) Work done by friction is $\Delta W = \underbrace{-2\pi r f}_{\text{for each orbit}} = \Delta E$

$$\therefore \Delta E = \left(\frac{dE}{dr}\right) \Delta r$$

$$\Rightarrow \Delta r = \frac{\Delta E}{(dE/dr)} = \frac{-2\pi r f}{\frac{1}{2}\frac{C}{r^2}} = -\frac{4\pi r^3 f}{C}$$

$$(c) \Delta K = -\Delta E = +2\pi r f$$

$\rightarrow$ friction results in satellite speeding up!

Problem 4

9-11

$$\text{Kepler's 3rd law} \Rightarrow T^2 = \frac{\pi^2}{2(M+m)} \frac{A^3}{G}$$

$$\text{or } M = \frac{\pi^2}{2} \frac{A^3}{T^2} \frac{1}{G}$$

$$A = \text{major axis} = r_{\max} + r_{\min} = (1861 + 1838) \text{ km} \\ = 3,699 \times 10^6 \text{ m}$$

$$T = 119 \text{ min} = 7,14 \times 10^3 \text{ s}$$

$$G = 6.67 \times 10^{-11}$$

$$\Rightarrow M = \frac{(3,114)^2}{2} \frac{(3,699)^3 \times 10^{18}}{(7,14)^2 \times 10^6} \frac{10^{11}}{6.67} = 1.98 \times 10^{22} \text{ kg.}$$

Problem 5

9-12

$$m = 3000 \text{ kg}$$

$$R_e = 6400 \text{ km}$$

In general,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

for circular orbit $\frac{mv^2}{r} = \frac{GMm}{r^2}$

$$\text{and } E = -\frac{1}{2} \frac{GMm}{r} = -\frac{1}{2} \frac{GMm}{R_e^2} \frac{R_e^2}{r} = -\frac{1}{2} R_e W \frac{R_e}{r}$$

where $W = mg = \text{weight at earth surface}$,

$$(a) \quad E_A = -\frac{1}{2} R_e W \left(\frac{R_e}{2R_e} \right) = -\frac{1}{4} R_e W$$

$$E_B = -\frac{1}{2} R_e W \left(\frac{R_e}{4R_e} \right) = -\frac{1}{8} R_e W$$

$$\Delta E = E_B - E_A = \frac{1}{8} R_e W = 2.35 \times 10^{10} \text{ J}$$

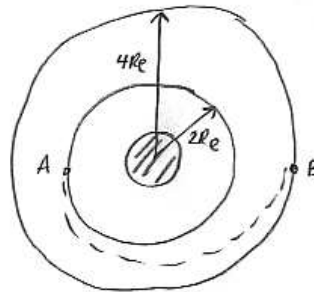
(b) The transfer orbit is an ellipse with perigee at A and apogee at B. The semimajor axis is $A = r_A + r_B = 6R_e$

$$\text{The energy is } E' = -\frac{GMm}{A} = -\frac{WR_e^2}{A} = -\frac{1}{6} WR_e$$

The initial speed at A is given by

$$\frac{1}{2} m v_0^2 = K_A = -E_A = \frac{1}{4} R_e mg$$

$$\rightarrow v_0 = \sqrt{\frac{1}{2}} \sqrt{g R_e}$$



The final speed at A is given by

$$K_A' + U_A = E'$$

$$\frac{1}{2} m v'^2 = -\frac{1}{6} mg R_e + \frac{1}{2} mg R_e = \frac{1}{3} mg R_e$$

$$\rightarrow v' = \sqrt{\frac{2}{3}} \sqrt{g R_e}$$

$$\text{hence } \Delta v_A = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}} \right) \sqrt{g R_e} = 850 \text{ m/s}$$

at B, the final speed is

↳ final circular orbit

$$\frac{1}{2} m v_f^2 = K_B = -E_B = \frac{1}{8} R_e m g$$

$$\rightarrow v_f = \frac{1}{2} \sqrt{g R_e}$$

The initial speed at B (when arrive in elliptical orbit) is

$$\frac{1}{2} m v_B'^2 = E' - U_B = -\frac{1}{6} mg R_e + \frac{1}{4} mg R_e = \frac{1}{12} mg R_e$$

$$v_B' = \sqrt{\frac{1}{6}} \sqrt{g R_e}$$

$$\rightarrow \Delta v_B = \left(\frac{1}{2} - \sqrt{\frac{1}{6}} \right) \sqrt{g R_e} = 730 \text{ m/s}$$

Problem 6

EP-13 Choose $m=1$ for convenience.

$$\rightarrow V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{1}{2}\left(\frac{L}{c}\right)^2 \frac{GM}{r^3}$$

conservation of energy $\Rightarrow$

$$E = \frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r)$$

(1)

$$\frac{\partial V_{\text{eff}}(r)}{\partial r} = 0 \rightarrow \text{maxima, minima}$$

$$0 = \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3}{2}\left(\frac{L}{c}\right)^2 \frac{GM}{r^4}$$

$$\text{or } \frac{1}{r^4} (GM r^2 - L^2 r + \frac{3}{2} \left(\frac{L}{c}\right)^2 GM) = 0$$

$$\rightarrow r = \infty \text{ (trivial case - reference level)}$$

$$\rightarrow r = \frac{L^2 \pm \sqrt{L^4 - 6G^2 M^2 \left(\frac{L}{c}\right)^2}}{2GM}$$

$$\text{or } r = \frac{L^2 \pm L^2 \sqrt{1 - 6\left(\frac{GM}{cL}\right)^2}}{2GM}$$

$$\text{Let } R = \frac{2GM}{c^2}, L = \beta \frac{GM}{c} = \beta \frac{RC}{2}$$

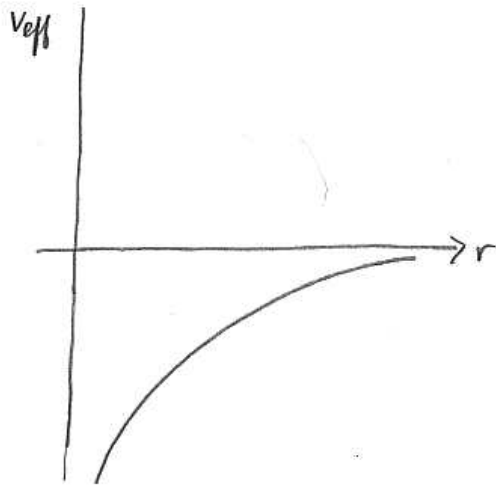
$$\rightarrow r = \frac{\beta^2 \left(\frac{GM}{c}\right)^2 \pm \beta^2 \left(\frac{GM}{c}\right)^2 \sqrt{1 - \frac{6}{\beta^2}}}{2GM}$$

$$= \beta^2 \frac{GM}{2c^2} (1 \pm \sqrt{1 - 6/\beta^2})$$

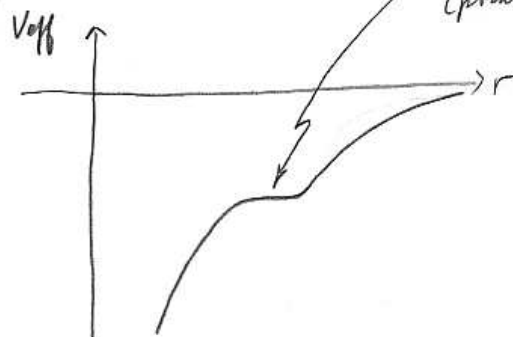
$$\text{or } r = \frac{1}{4} R \beta^2 (1 \pm \sqrt{1 - 6/\beta^2})$$

Cases

(a) $\beta < \sqrt{6} \rightarrow$ no maxima, minima

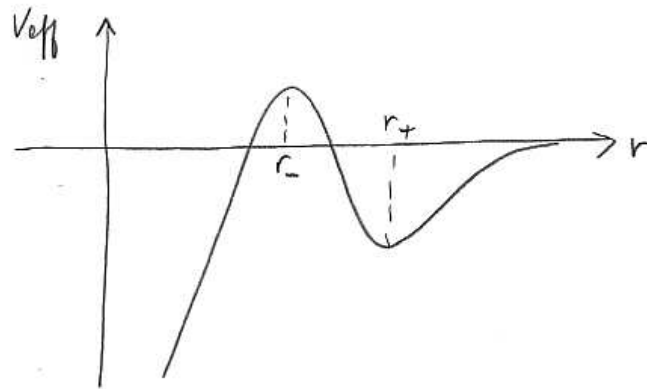


(b) $\beta = \sqrt{6} \rightarrow r = \frac{3}{2} R$ (point of inflection)



$$(c) \beta > \sqrt{6}$$

$$\Rightarrow r_{\pm} = \frac{1}{4} R \beta^2 \left(1 \pm \sqrt{1 - \frac{6}{\beta^2}} \right)$$



(2)

Plots

$$V_{\text{eff}}(r) = -\frac{RC^2}{2r} + \beta^2 \frac{R^2 C^2}{8r^2} - \frac{\beta^2 \frac{R^2}{4} \frac{RC^2}{2}}{2r^3}$$

$$= -\frac{C^2}{2} \left[\frac{R}{r} - \frac{\beta^2 \left(\frac{R}{r}\right)^2}{4} + \frac{\beta^2 \left(\frac{R}{r}\right)^3}{8} \right]$$

$$\text{Let } x = \frac{r}{R}$$

$$V_{\text{eff}}(x) = -\frac{C^2}{2} \left[\frac{1}{x} - \frac{1}{4} \beta^2 \left(\frac{1}{x}\right)^2 + \frac{1}{8} \beta^2 \left(\frac{1}{x}\right)^3 \right]$$

$$\frac{2GM}{c} \leq L \leq 30 \frac{GM}{c}$$

$$\Rightarrow \frac{2GM}{c} \leq \beta \frac{GM}{c} \leq 30 \frac{GM}{c}$$

$$\Rightarrow 2 \leq \beta \leq 30$$

$$\frac{1}{X_{\pm}} = \frac{1}{4} \beta^2 \left(1 \pm \sqrt{1 - \frac{6}{\beta^2}} \right) \quad \beta > \sqrt{6} = 2.45$$

$$\frac{1}{X_{\text{inflection}}} = \frac{3}{2} \quad \beta = \sqrt{6} = 2.45$$

$$\text{no special point} \quad \beta < \sqrt{6} = 2.45$$

plots for $\beta = 2.0, 2.25, 2.45, 2.60$
 $2.85, 3.00, 3.50, 5.00$

we note that as $\beta \rightarrow \infty$

$$\begin{aligned} \frac{1}{X_{\pm}} &= \frac{1}{4} \beta^2 \left(1 \pm \left(1 - \frac{3}{\beta^2} \right) \right) \\ &= \frac{1}{4} \beta^2 \begin{pmatrix} 2 - \frac{3}{\beta^2} \\ \frac{3}{\beta^2} \end{pmatrix} = \begin{cases} \frac{1}{2} \beta^2 \\ \frac{3}{4} \end{cases} \end{aligned}$$

```

1 clear;
2 %bvalues=[2,2.25,2.45,2.6,2.85,3,3.5,5]
3 x=0.6+0.1*(0:100);
4 b=5.0;
5 v=veff(x,b);
6 figure('Position',[50,50,400,400])
7 axis([0,12,-0.5,2.5]);
8 xlabel('r/R','fontsize',40)
9 ylabel('Effective Potential','fontsize',40)
10 title('Black-Hole Physics: beta= ', 'fontsize',40)
11 hold on
12 plot(x, v, '-r')
13 hold off

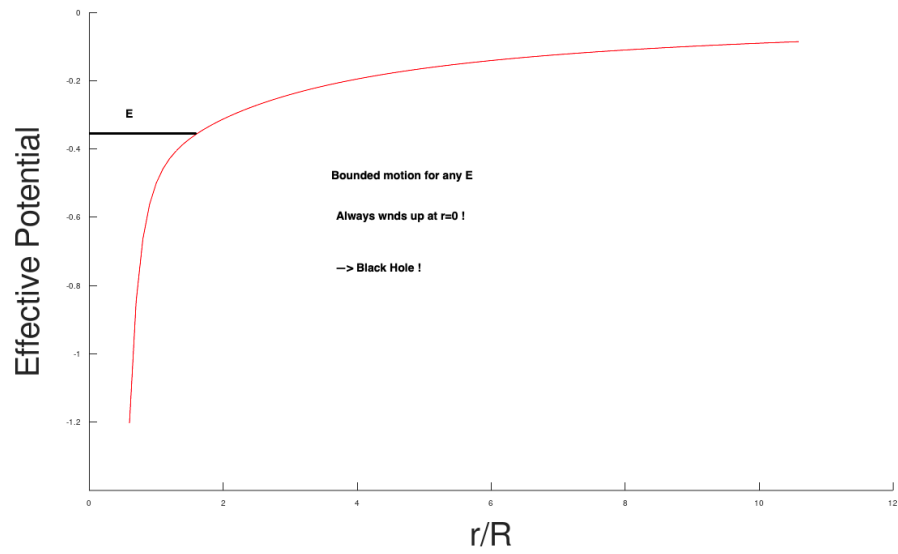
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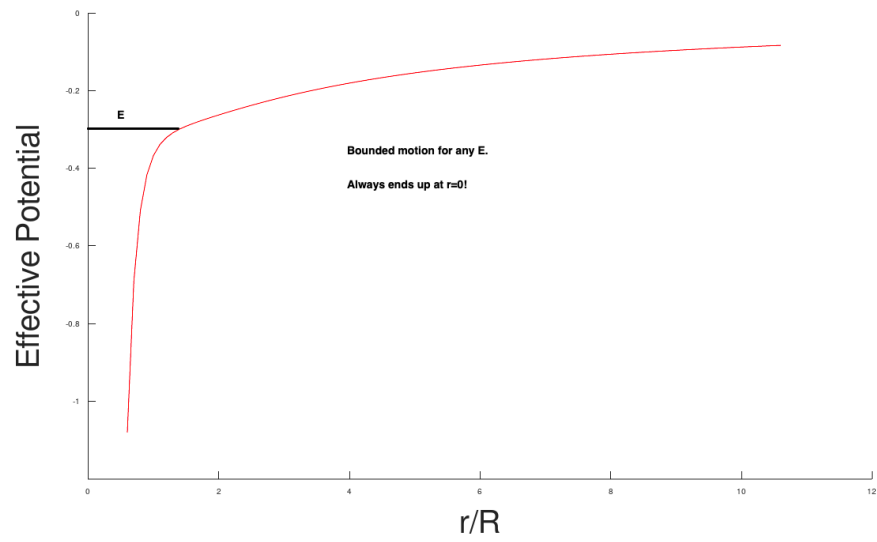
function a=veff(x,b)
a=0.25*(b^2)./(x.^2)-1./x-0.125*(b^2)./(x.^3);
endfunction

```

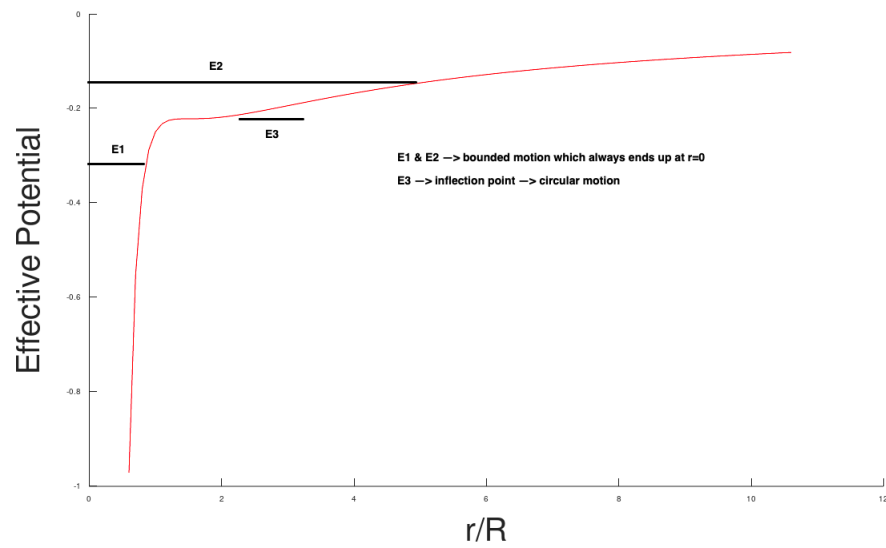
Black-Hole Physics: beta= 2.00



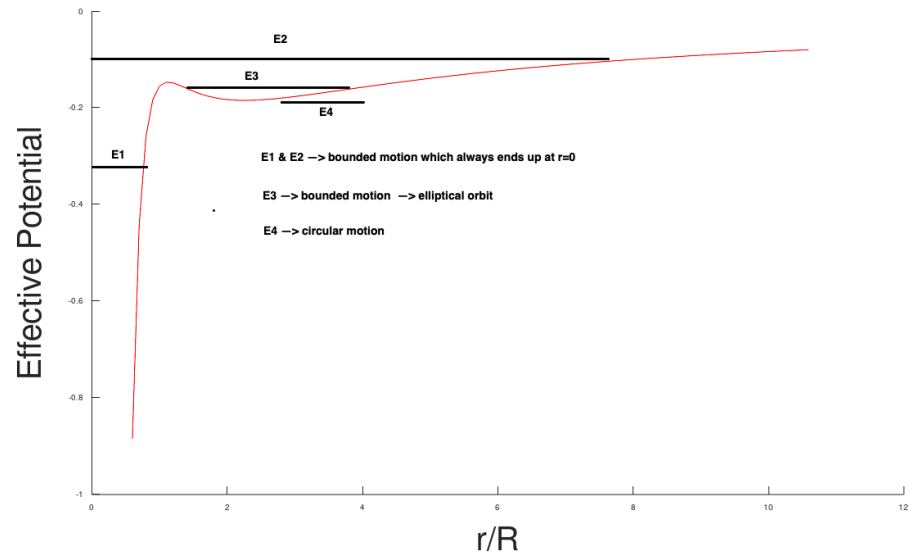
Black-Hole Physics: $\beta = 2.25$



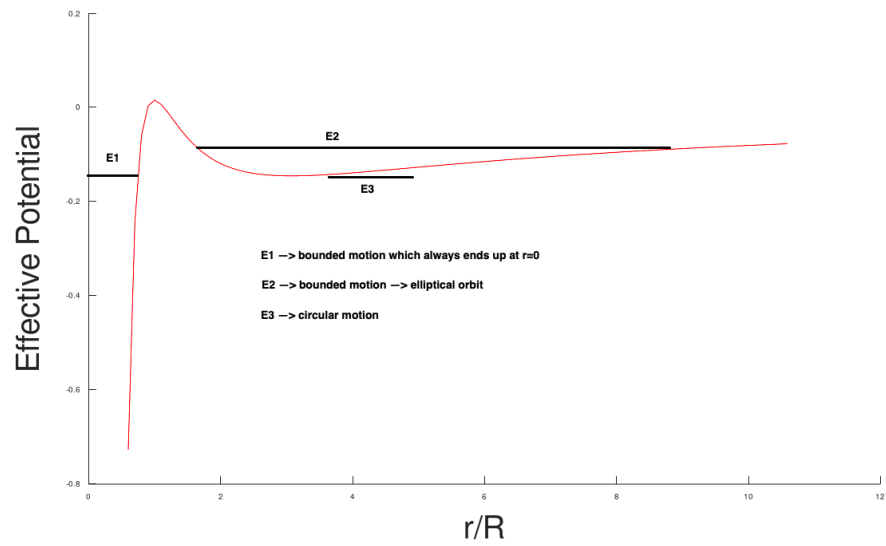
Black-Hole Physics: $\beta = 2.45$



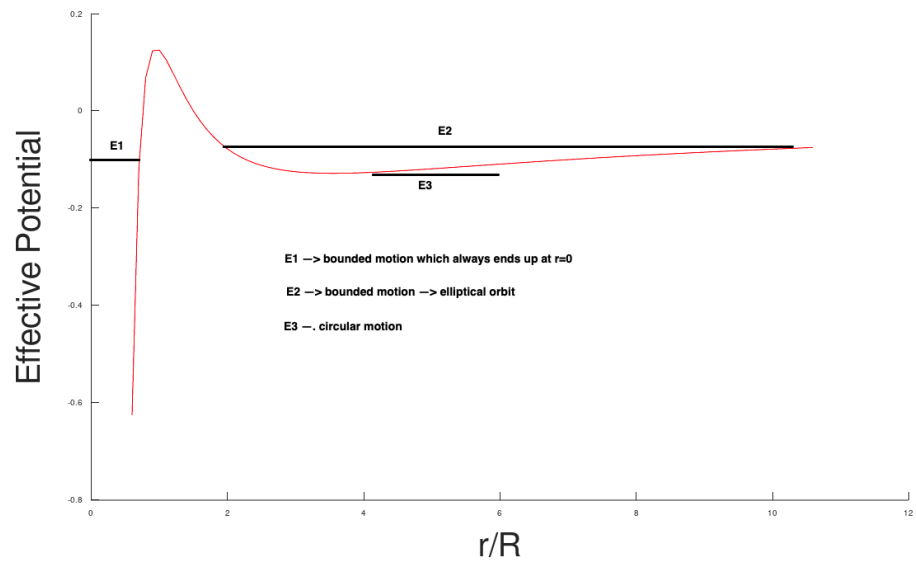
Black-Hole Physics: beta= 2.60



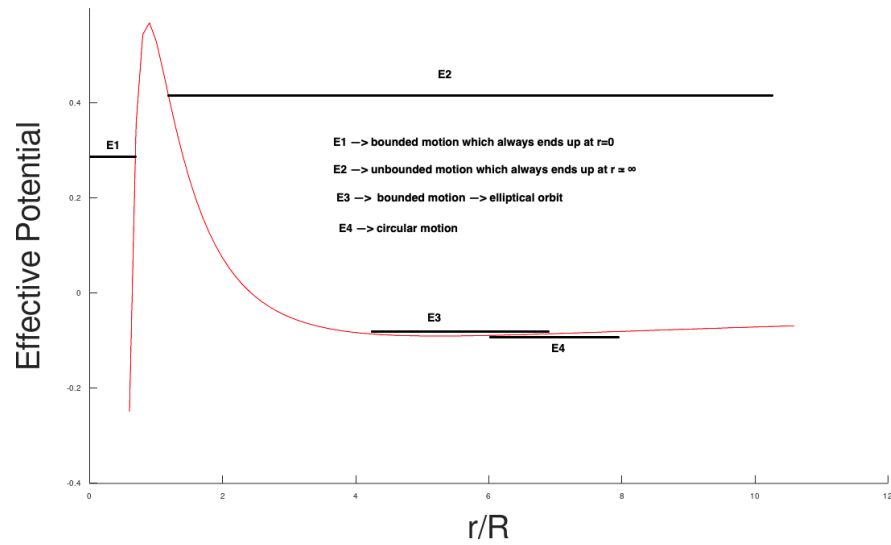
Black-Hole Physics: beta= 2.80



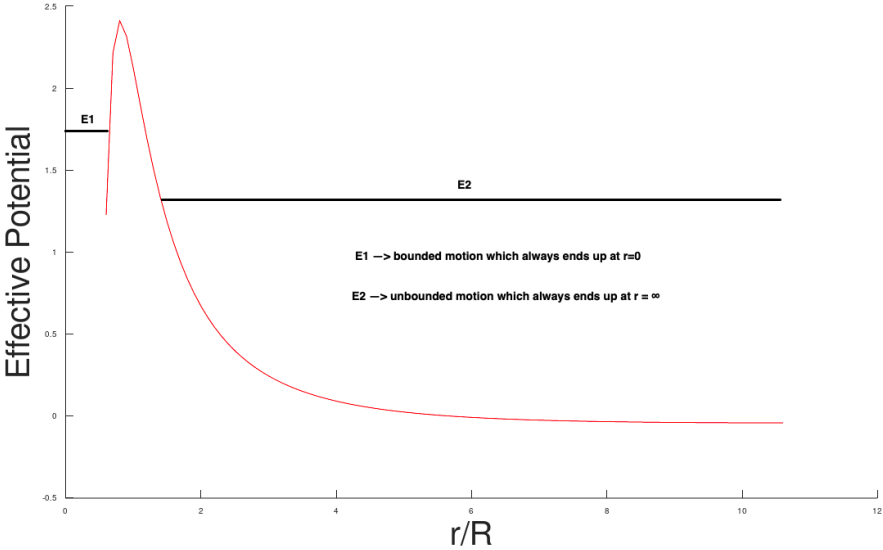
Black-Hole Physics: beta= 3.00



Black-Hole Physics: beta= 3.50



Black-Hole Physics: beta= 5.00



Assignment 14 - Solutions

Problem 1

12.2

$$x = \gamma(x' + vt')$$

$$6 \times 10^8 = \gamma(6 \times 10^8 + 4 \frac{v}{c} \times 3 \times 10^8)$$

$$1 = \gamma(1 + 2 \frac{v}{c})$$

$$1 + 2 \frac{v}{c} = (1 - \frac{v^2}{c^2})^{\frac{1}{2}}$$

$$1 + 4 \frac{v}{c} + 4 \frac{v^2}{c^2} = 1 - \frac{v^2}{c^2}$$

$$5 \frac{v^2}{c^2} + 4 \frac{v}{c} = 0$$

$$\frac{v}{c} = -\frac{4}{5}$$

Problem 2

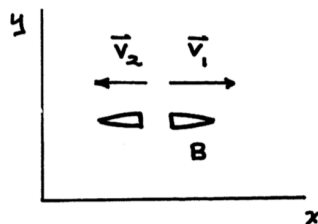
12.5

In S

$$v_1 = v_2 = 0.99c$$

In S' , moving with B ,

$$v_2' = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}}$$



Let $v = c(1-x)$ Then

$$\begin{aligned} v_2' &= \frac{2c(1-x)}{1 + (1-x)^2} = \frac{2c(1-x)}{2 - 2x + x^2} = \frac{c(1-x)}{1 - x + x^2/2} \\ &= c(1-x) \left[1 + (x - x^2/2) + (x - x^2/2)^2 \right] \\ &= c(1-x) \left[1 + x - x^2/2 + x^2 + \mathcal{O}(x^3) \right] \\ &= c(1 - x^2 + x^2/2) \\ &= c(1 - x^2/2) \end{aligned}$$

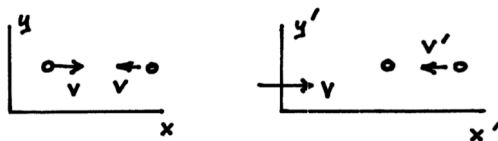
For $x = 0.01$

$$\frac{x^2}{2} = 5 \times 10^{-5}$$

$$v_2' = c(1 - 5 \times 10^{-5}) = 0.99995c$$

Problem 3

13,4



$$v' = \frac{2v}{1+v^2/c^2}$$

$$E' = m_0 c^2 \gamma$$

$$\gamma = 1 / \sqrt{1 - v'^2/c^2} = \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{1 + v^2/c^2}{\sqrt{1 + \frac{2v^2}{c^2} + \left(\frac{v^2}{c^2}\right)^2 - \frac{4v^2}{c^2}}}$$

$$\gamma = \frac{1 + v^2/c^2}{1 - v^2/c^2} \quad \text{hence} \quad E' = m_0 c^2 \left(\frac{1 + v^2/c^2}{1 - v^2/c^2} \right)$$

Problem 4

13.6

Initial

Final

$$\begin{array}{c} \circ \rightarrow v \\ m_o \end{array} \quad \begin{array}{c} \circ \\ m_o \end{array}$$

$$\begin{array}{c} \circ \rightarrow v \\ m' \end{array}$$

$$K = x m_o c^2$$

$$E_i = m c^2 + m_o c^2 = m_o c^2 (\gamma + 1)$$

$$P_i = m_o v \gamma$$

$$E_f = m' c^2 = m_o' c^2 \Gamma$$

$$P_f = m_o' v \Gamma$$

$$K = x m_o c^2 = (\gamma - 1) m_o c^2 \Rightarrow x = \gamma - 1$$

$$\gamma = x + 1$$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = \frac{1}{(x+1)^2}$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(x+1)^2}} = \frac{\sqrt{x^2 + 2x + 1 - 1}}{x+1} = \frac{\sqrt{x^2 + 2x}}{x+1}$$

$$P_i = P_f \Rightarrow \frac{m_o c \sqrt{x^2 + 2x}}{(x+1)} (x+1) = m_o' v \Gamma$$

$$E_i = E_f \Rightarrow m_o c^2 (x+2) = m_o' c^2 \Gamma$$

$$\text{Dividing, } \frac{\sqrt{x^2 + 2x}}{x+2} = \frac{v}{c} \equiv r$$

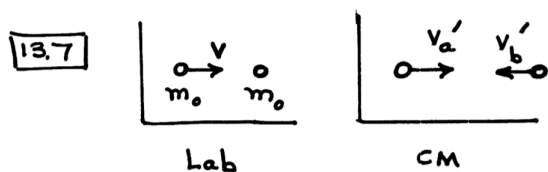
$$v \Gamma = r c \frac{1}{\sqrt{1-r^2}} = \frac{\sqrt{x^2 + 2x}}{x+2} \frac{c}{\sqrt{1 - \frac{x^2 + 2x}{(x+2)^2}}} = \frac{c \sqrt{x^2 + 2x}}{\sqrt{x^2 + 4x + 4 - x^2 - 2x}}$$

$$v \Gamma = c \frac{\sqrt{x^2 + 2x}}{\sqrt{2x + 4}} = c \sqrt{\frac{x}{2}}$$

$$\text{From } P_i = P_f$$

$$m_o' = m_o c \sqrt{x^2 + 2x} / v \Gamma = m_o \sqrt{\frac{2}{x}} \sqrt{x^2 + 2x} = m_o \sqrt{2} \sqrt{x+2}$$

Problem 5



The CM frame moves to the right with speed V . The speeds of the particles are

$$v'_a = \frac{v - V}{1 - \frac{vV}{c^2}}$$

$$v'_b = -V$$

For zero momentum in CM frame, $v'_a = -v'_b$

$$\frac{v - V}{1 - \frac{vV}{c^2}} = V$$

$$v - V = V - \frac{vV^2}{c^2}$$

$$V^2 - 2\frac{c^2}{v}V + c^2 = 0$$

$$\Rightarrow V = \frac{c^2}{v} \pm \sqrt{\left(\frac{c^2}{v}\right)^2 - c^2} = \frac{c^2}{v} (1 \pm \sqrt{1 - \frac{v^2}{c^2}})$$

To determine the correct root, take the limit $v \rightarrow 0$

$$V \cong \frac{c^2}{v} (1 \pm (1 - \frac{1}{2}\frac{v^2}{c^2}))$$

The negative root gives $V \cong \frac{c^2}{v} (\frac{1}{2}\frac{v^2}{c^2}) = \frac{v}{2}$ as required

$$\text{Hence } V = \frac{c^2}{v} (1 - \sqrt{1 - v^2/c^2})$$

Problem 6

19,10 Assume that the light is reflected from the sphere so that

$$F = 2 \frac{\text{Power}}{c} = \frac{2 \times 10^{-3}}{3 \times 10^8} = 6.7 \times 10^{-6} \text{ N}$$

At equilibrium, $F = mg$

$$m = \frac{6.7 \times 10^{-6}}{9.8} = 6.8 \times 10^{-7} \text{ kg} = 6.8 \times 10^{-4} \text{ g}$$

$$m = \frac{4}{3} \pi r^3 \rho, \quad \rho = \text{density} = 2.7 \text{ g/cm}^3$$

$$r = \left(\frac{3m}{4\pi\rho} \right)^{1/3} = \left(\frac{3 \times 6.8 \times 10^{-4}}{4\pi \times 2.7} \right)^{1/3} = (6.0 \times 10^{-5})^{1/3}$$

$$\Rightarrow r = 3.9 \times 10^{-3} \text{ cm}$$

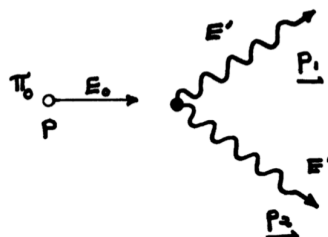
Problem 7

14.1

$$\underline{P} = \Gamma M_0 c \left(\frac{v}{c}, 0, 0, i \right)$$

$$\underline{P}_1 = \frac{E'}{c} (\cos \theta, \sin \theta, 0, i)$$

$$\underline{P}_2 = \frac{E'}{c} (\cos \theta, -\sin \theta, 0, i)$$



$$(a) \quad \underline{P} = \underline{P}_1 + \underline{P}_2$$

$$\Gamma M_0 v = \frac{2E'}{c} \cos \theta$$

$$\Gamma \frac{v}{c} = \frac{2E'}{M_0 c^2} \cos \theta$$

$$(b) \quad \Gamma M_0 c = 2 \frac{E'}{c}$$

$$\Gamma = \frac{2E'}{M_0 c^2}$$

Let

$$\chi = \frac{E'}{M_0 c^2} = \frac{1}{1.25}$$

From (b)

$$\Gamma = 2\chi$$

$$1 - \frac{v^2}{c^2} = \left(\frac{1}{2\chi} \right)^2$$

$$\frac{v^2}{c^2} = 1 - \left(\frac{1.25}{2} \right)^2 = .546$$

$$\frac{v}{c} = 0.74$$

From (a)

$$\cos \theta = \Gamma \frac{v}{c} \frac{1}{2\chi} = \frac{v}{c} = 0.74$$

$$\theta = 42^\circ$$

Problem 8

14.2 $\gamma + p \rightarrow p + \pi_0$

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4$$

$$p_1^2 + p_2^2 + 2 \underline{p}_1 \cdot \underline{p}_2 = (\underline{p}_3 + \underline{p}_4)^2$$

In lab frame $\underline{p}_1 = \frac{E_\gamma}{c} (1, 0, 0, i)$ and $\underline{p}_2 = M_p c (0, 0, 0, i)$

$$l.h.s. = 0 - (M_p c)^2 - 2 M_p E_\gamma$$

Evaluate r.h.s. in c of m frame :

$$(\underline{p}_3 + \underline{p}_4)^2 = - (M_p + M_{\pi_0})^2 c^2$$

$$\text{so } (M_p c^2)^2 + 2 M_p c^2 E_\gamma = (M_p c^2)^2 + (M_{\pi_0} c^2)^2 + 2 M_p c^2 M_{\pi_0} c^2$$

$$E_\gamma = M_{\pi_0} c^2 \left(1 + \frac{M_{\pi_0} c^2}{2 M_p c^2} \right) = 135 \text{ MeV} \left(1 + \frac{135}{2 \times 938} \right)$$

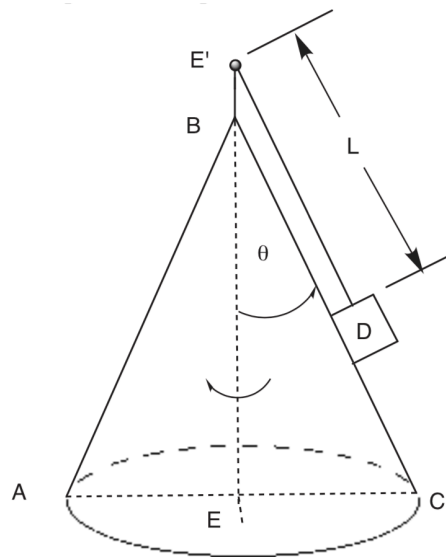
$$= 144 \text{ MeV}$$

Examination Problems

Examination 1

Problem 1-1 - Kinematics - An RA in Wharton dormitory sees an illegal water-filled balloon fall vertically past her window. Having lightning reflexes, she observes that the balloon took 0.15 seconds to pass from the top to the bottom of her window - a distance of 2 meters. Assuming that the balloon was released from rest, how high above the bottom of her window was the guilty party?

Problem 1-2 - Circular Motion - A body D which has a mass of 25 kg is on a smooth conical surface ABC as shown below.



The body D is rotating about the axis EE? with an angular velocity of 10 rev/min. Its path of motion is a circle.

$$\theta = 60^\circ, \quad L = 4.5 \text{ m}$$

Calculate

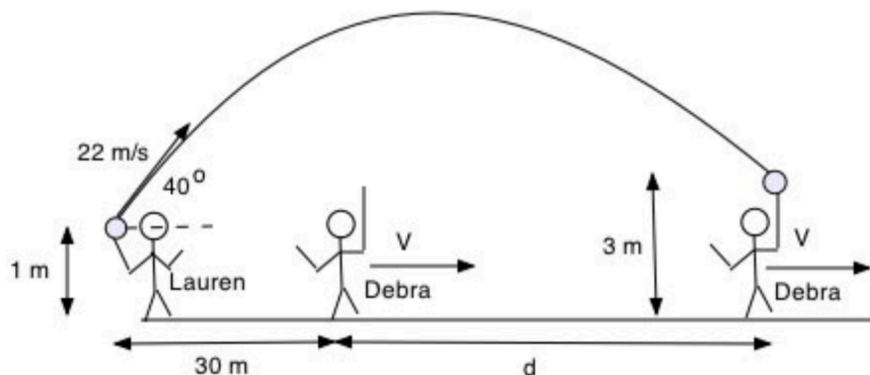
- (a) linear velocity of D

- (b) reaction force of the surface on D
- (c) tension in the string
- (d) angular velocity necessary to reduce the reaction force to zero

Problem 1-3 - Kinematics + Newton's Laws - A block of mass 0.2 kg starts up a plane inclined at an angle of 30° with respect to the horizontal with a velocity of 12 m/sec. If the coefficient of friction is 0.16, determine how far up the plane the block travels before stopping.

Problem 1-4 - Plane-Polar Coordinates - The angular position of a particle moving on a circle of radius $r = 5$ m is given by the expression $\theta = 3t^2$, where θ is given in radians and t in seconds. Calculate the velocity and acceleration vectors when $t = 0.5$ s.

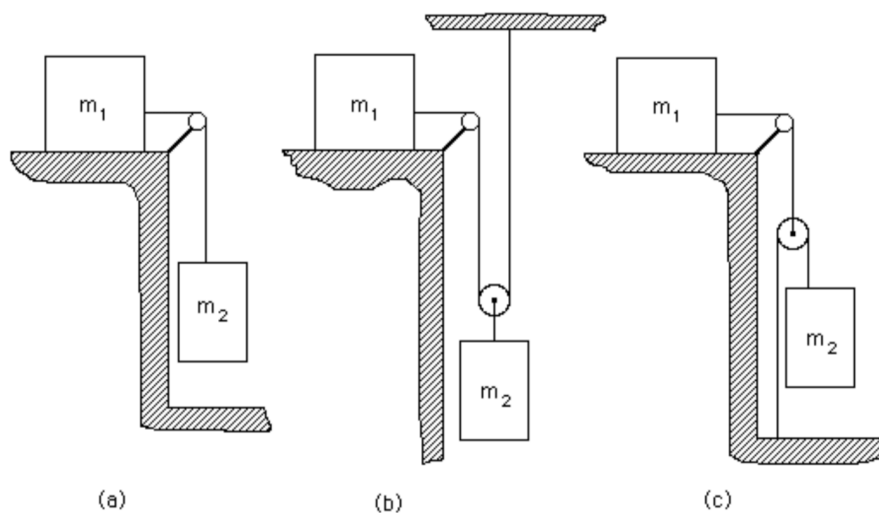
Problem 1-5 - Kinematics - Lauren throws a basketball so that it has a velocity of 22 m/sec at an angle of 40° above the horizontal. Debra, standing 30 m from Lauren, and in the same plane as the ball's trajectory begins to run away from Lauren the instant the ball is thrown. Calculate the velocity that Debra needs to have in order to catch the ball as high above the ground as she can reach. What is the distance d that Debra had to run?



Debra can reach up to 3 m above the ground and the ball was 1 m above the ground when it was thrown.

Problem 1-6 - Newton's Laws - Calculate the accelerations of the bodies

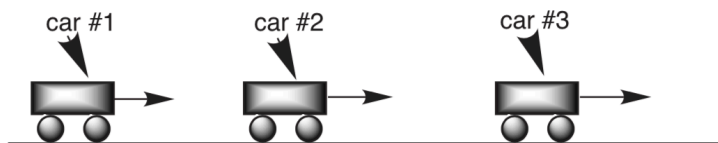
of mass m_1 and m_2 and the tensions in the ropes in the three cases shown below. All pulleys are massless and frictionless and the bodies slide on the tables without friction.



Which device can accelerate m_1 faster than in free fall? Show your reasoning.

Examination 2

Problem 2-1 - Three cars are moving along a straight road as shown below:



$$m_1 = 1000 \text{ kg}, v_1 = 18 \text{ m/s} ; \quad m_2 = 2000 \text{ kg}, v_2 = 9 \text{ m/s} ; \quad m_3 = 3000 \text{ kg}, v_3 = 6 \text{ m/s}$$

Assume only elastic collisions. Assume that the first collision is between car #1 and car #2. After **ALL** collisions are over, how are the cars moving?

Problem 2-2 - A child sleds down a snowy hillside, starting from rest. The hill has a 15° slope, with a long stretch of level field at the bottom. The child

starts from rest 50 ft up the slope and continues for 100 ft on the level field before coming to a complete stop. Find the coefficient of friction between the sled and the snow, assuming it is constant throughout the ride.

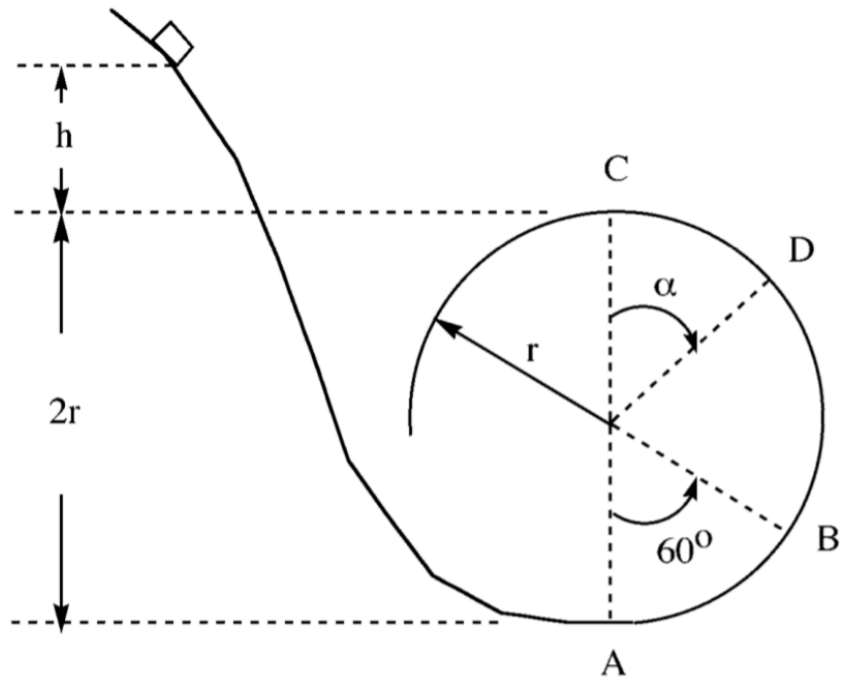
Problem 2-3 - A particle moves in a region where the potential energy is given by

$$U(x) = 3x^2 - x^3 \quad (x \text{ in m, } U \text{ in J})$$

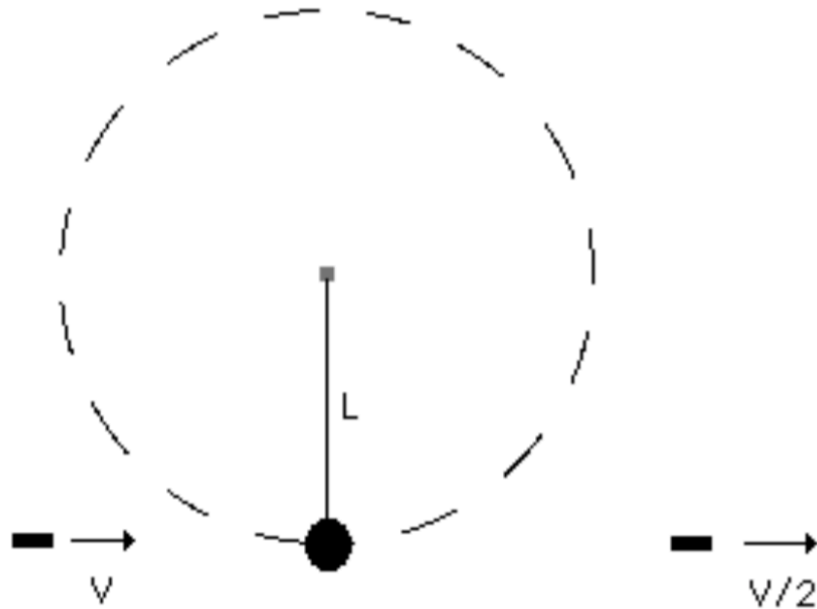
- (a) Sketch a graph of the potential energy for both positive and negative values of x . Indicate maxima and minima clearly on the graph and whether they represent stable or unstable equilibrium points.
- (b) In what ranges of x is the force on the particle in the positive x -direction? In the negative x -direction. Indicate clearly on the graph.
- (c) Choose a set of energy values that illustrate the different possible motions for this system. Indicate clearly whether the motion is bounded, periodic, unbounded, rest, etc. If the motion is bounded, how do you determine the turning points?
- (d) What is the maximum value of the total energy such that oscillatory motion is possible?

Problem 2-4 - An object of mass m slide on a frictionless loop-the-loop apparatus. The object is released from rest at a height h above the top of the loop (see the figure below).

- (a) What is the magnitude and direction of the force exerted on the object by the track as the object passes through point A ?
- (b) Draw a force diagram showing the forces acting on the object at point B and find the radial acceleration at that point.
- (c) Show that the object must start from $h \geq r/2$ to successfully complete the loop (reach the top at C).
- (d) For $h < r/2$ the object will begin to fall away from the track before reaching the top of the circle (at some point D). Show that this happens at a position such that $3 \cos \alpha = 2 + 2h/r$, where α is an angle measured from the upward vertical.

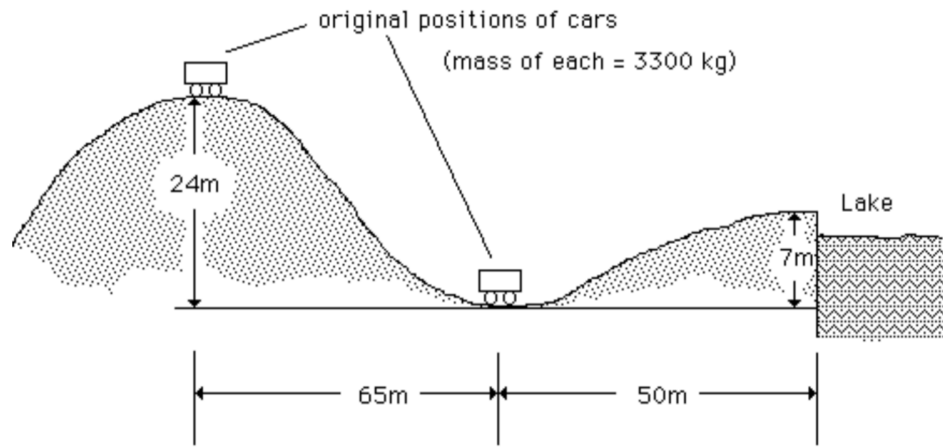


Problem 2-5 - A bullet of mass m and velocity V passes through a mass M hanging on a string and emerges with velocity $V/2$ as shown below.



The length of the string is L . What is the **minimum** value of V such that the pendulum bob will just swing through a complete circle?

Problem 2-6 - You are investigating a mishap on a mountain railroad. Last night, one railroad car was left parked at the top of a hill above another car of the same mass. This morning both cars were found coupled together in a lake, as shown in the figure below.



The tracks go right up to the lake (in order to load ore boats). The supervisor claims that the engineer must have not set the brake on the upper car, and during the night, it rolled down, hit and coupled with the lower car, and its momentum carried both into the lake. The engineer disputes this, saying that someone must have released the brake and then **pushed** the upper car. The collision is totally inelastic. **Use physics to resolve this dispute?**

Examination 3

Problem 3-1

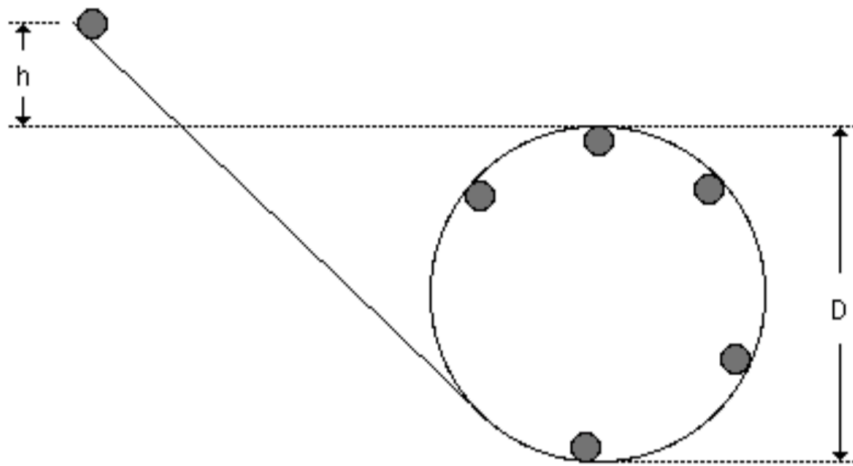
- A simple harmonic oscillator of mass 0.40 kg has a frequency of 15 Hz and a total energy of 30 J. Find the amplitude of the motion.
- A particular simple harmonic motion has an amplitude of 8×10^{-2} m and a period of 4s. Calculate the velocity and the acceleration 0.5s after the particle reaches maximum amplitude.

Problem 3-2 - The springs of a car of mass 1200 kg give it a period of 0.5s for small vertical oscillations.

- How far does the car sink down when the driver and three passengers, each of mass 75kg, get into the car?
- The car with its passengers is traveling along a horizontal road when it suddenly runs onto a piece of new road surface, raised 5 cm above

the old surface. Assume that this suddenly raises the wheels of and the bottom end of the springs through 5 cm before the body of the car (including the passengers) begins to move upward. In the ensuing rebound, are the passengers thrown clear of their seats? Assume simple harmonic motion.

Problem 3-3 - A spherical ball of radius R and mass M rolls without slipping along the loop-the-loop as shown in the figure below.

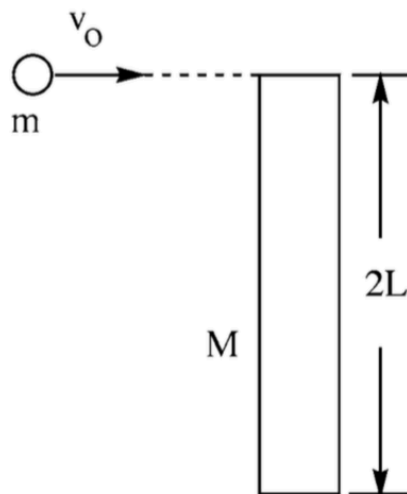


Find the minimum height h (as shown) that will permit the ball to maintain contact with the track at all times. The moment of inertia of the spherical ball is $\frac{2}{5}MR^2$.

Problem 3-4 - A skier is enjoying the mountain air while standing on a 30° snow slope when suddenly she notices a huge snowball rolling without slipping down the hill at her. By the time she notices the ball, it is only 100 meters away and is traveling at 25 m/s. The skier gives herself a speed of 10 m/s almost instantaneously and proceeds to accelerate down the slope at $g/2$. Assume that the snowball has a constant acceleration corresponding to that of a sphere rolling, without slipping, down the slope and that the moment of inertia of the snowball about an axis through its center is $\frac{2}{5}MR^2$. Does she get away?

Problem 3-5

- (a) A plank of length $2L$ and mass M lies on a frictionless plane. A ball of mass m and speed v_0 strikes its end as shown.

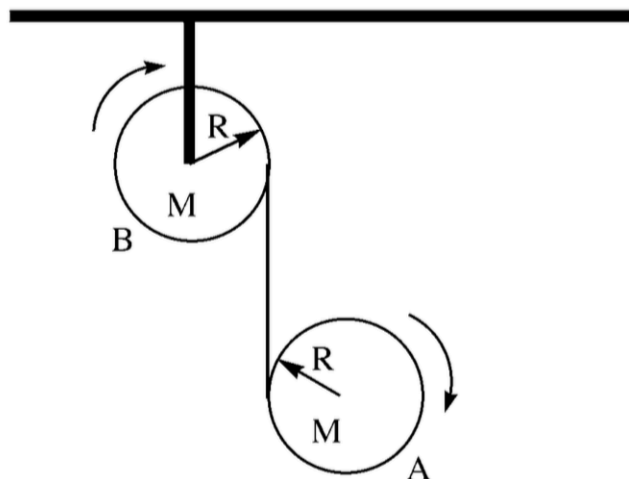


$$I_0 = \frac{1}{12} M(2L)^2$$

Find the final velocity of the ball, v_f , assuming that mechanical energy is conserved and that v_f is along the original line of motion.

- (b) Find v_f assuming that the stick is pivoted at the lower end.

Problem 3-6 - Drum A of mass M and radius R is suspended from a drum B also of mass M and radius R , which is free to rotate about its axis (see diagram below).



$$I_0 = \frac{1}{2}MR^2$$

The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind, as shown in the diagram. Gravity is directed downward. Both drums are initially at rest. Find the initial acceleration of drum A, assuming that it moves straight down.

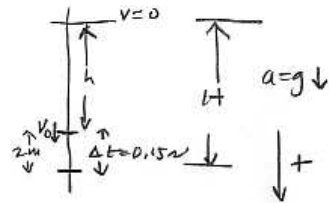
Examination Solutions

Examination 1

$$1. \quad V_f^2 = V_o^2 + 2g(x - x_o)$$

$$x = x_o + V_o t + \frac{1}{2} g t^2$$

top to bottom of window



$$x - x_o = 2 = V_o \Delta t + \frac{1}{2} g (\Delta t)^2$$

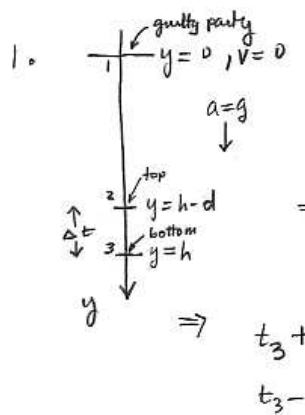
$$\rightarrow V_o = \frac{2 - \frac{1}{2} g (\Delta t)^2}{\Delta t} = 12.6 \text{ m/s}$$

= velocity at top of window

$$V_{top}^2 = V_{start}^2 + 2g h \quad h = \text{start} \rightarrow \text{top of window}$$

$$V_o^2 = 2g h \rightarrow h = \frac{V_o^2}{2g} = 8.1 \text{ m}$$

$$\rightarrow H = \text{distance to bottom} = h + 2 = 10.1 \text{ m}$$

1. 

$$v_3 - v_2 = g(t_3 - t_2) = g \Delta t$$

$$h = \frac{1}{2} g t_3^2$$

$$h - d = \frac{1}{2} g t_2^2$$

$$\Rightarrow d = \frac{1}{2} g (t_3^2 - t_2^2) = \frac{1}{2} g (t_3 - t_2)(t_3 + t_2)$$

$$d = \frac{1}{2} g \Delta t (t_3 + t_2)$$

$$\Rightarrow t_3 + t_2 = \frac{2d}{g \Delta t}$$

$$t_3 - t_2 = \Delta t$$

$$\rightarrow 2t_3 = \frac{2d}{g \Delta t} + \Delta t$$

$$t_3 = \frac{d}{g \Delta t} + \frac{\Delta t}{2}$$

$$\left. \begin{array}{l} d = 2 \text{ m} \\ g = 9.8 \text{ m/s}^2 \\ \Delta t = 0.15 \text{ s} \end{array} \right\} \Rightarrow t_3 = \frac{2}{9.8(0.15)} + \frac{0.15}{2} = 1.44 \text{ s}$$

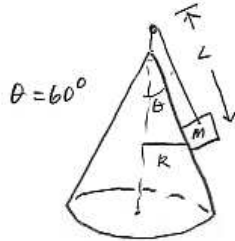
$$h = \frac{1}{2} g t_3^2 = 4.9(1.44)^2 = 10.16 \text{ m}$$

= height above bottom of window

check: $h - d = 8.16 = \frac{1}{2} g t_2^2 \Rightarrow t_2 = 1.29$

$$\rightarrow t_3 - t_2 = 0.15 = \Delta t \checkmark$$

2.



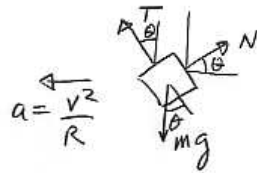
$$m = 25 \text{ kg}, \quad L = 4.5 \text{ m}$$

$$R = L \sin \theta = 4.5 (0.866) = 3.9 \text{ m}$$

$$\omega = 10 \text{ rev/min} = 20\pi \frac{\text{rad}}{\text{min}} = \frac{20\pi}{60} \text{ rad/sec} = \frac{\pi}{3} \text{ sec}^{-1}$$

$$(a) \quad V = R\omega = (3.9) \left(\frac{\pi}{3} \right) = 1.3\pi \text{ m/s} = 4.08 \text{ m/s}$$

(b)



$$T \sin \theta - N \cos \theta = m \frac{v^2}{R}$$

$$T \cos \theta + N \sin \theta - mg = 0$$

$$T \sin \theta \cos \theta - N \cos^2 \theta = m \frac{v^2}{R} \cos \theta$$

$$-T \sin \theta \cos \theta - N \sin^2 \theta = -mg \sin \theta$$

$$\therefore -N = \frac{mv^2}{R} \cos \theta - mg \sin \theta$$

$$N = mg \sin \theta - \frac{mv^2}{R} \cos \theta = 159.3 \text{ Newtons}$$

$$(c) \quad T = \frac{mg - N \sin \theta}{\cos \theta} = 215 \text{ Newtons}$$

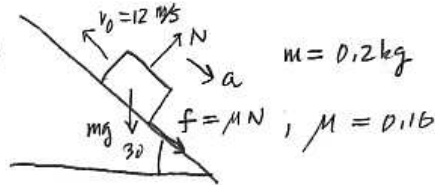
$$(d) \quad N = 0 \rightarrow g \sin \theta = R \omega^2 \cos \theta$$

$$\rightarrow \omega^2 = \frac{g}{R} \tan \theta = 4.35$$

$$\rightarrow \omega = 2.09 \text{ sec}^{-1}$$

3.

Going up:



$$(a) \quad N - mg \cos 30^\circ = 0 \Rightarrow N = mg \cos 30^\circ$$

$$mg \sin 30^\circ + f = ma, \quad f = \mu N$$

$$\rightarrow a = mg \sin 30^\circ + \mu g \cos 30^\circ$$

$$= (9.8) \frac{1}{2} + 0.16(9.8)(0.866) = 6.26 \text{ m/s}^2$$

stops as it goes up plane

$$v_f^2 = 0 = v_0^2 - 2ad \quad \rightarrow \text{distance traveled}$$

$$\therefore d = \frac{v_0^2}{2a} = \frac{144}{12.52} = 11.59 \text{ m}$$

$$4. \quad r = 5 \quad \dot{r} = \ddot{r} = 0$$

$$\theta = 3t^2$$

$$\dot{\theta} = 6t$$

$$\ddot{\theta} = 6$$

$$\rightarrow \vec{V} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} = r \dot{\theta} \hat{\theta}$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{\theta}$$

$$= -r \dot{\theta}^2 \hat{r} + r \ddot{\theta} \hat{\theta}$$

$$\text{at } t = 0.5 \text{ sec}$$

$$r = 5$$

$$\theta = \frac{3}{4}$$

$$\dot{\theta} = 3$$

$$\ddot{\theta} = 6$$

$$\rightarrow \vec{V} = 15 \hat{\theta} \text{ m/s}$$

$$\vec{a} = -45 \hat{r} + 30 \hat{\theta} \text{ m/s}^2$$

$$\begin{aligned}
 5. \quad & \text{Ball} \begin{cases} Y_B = Y_0 + (V_0 \sin \theta) t - \frac{1}{2} g t^2 \\ X_B = (V_0 \cos \theta) t \end{cases} & \begin{aligned} Y_0 &= 1 \text{ m} \\ V_0 &= 22 \text{ m/s} \\ \theta &= 40^\circ \\ g &= 9.8 \text{ m/s}^2 \\ X_0 &= 30 \text{ m} \end{aligned} \\
 & \text{Debra} \begin{cases} Y_D = 3 \text{ m} \\ X_D = X_0 + V t \end{cases}
 \end{aligned}$$

Ball is caught at $t = t_c$ when $Y_B = Y_D$

$$\Rightarrow 3 = 1 + 14.14 t_c - 4.9 t_c^2$$

$$\text{or } 4.9 t_c^2 - 14.14 t_c + 2 = 0$$

$$\begin{aligned}
 \Rightarrow t_c &= \frac{14.14 \pm \sqrt{(14.14)^2 - 39.2}}{9.8} \\
 &= \frac{14.14 \pm 12.68}{9.8}
 \end{aligned}$$

For direction Debra is running we choose larger time

$$\Rightarrow + \text{ sign} \Rightarrow t_c = \frac{26.82}{9.8} = 2.74 \text{ s}$$

$$\Rightarrow X_B = 16.85 t_c = 46.18 \text{ m}$$

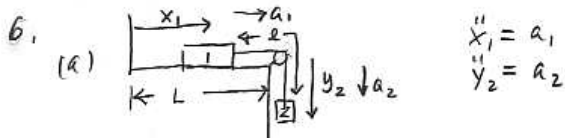
Debra catches ball when

$$X_D = 30 + V t_c = X_B = 46.18$$

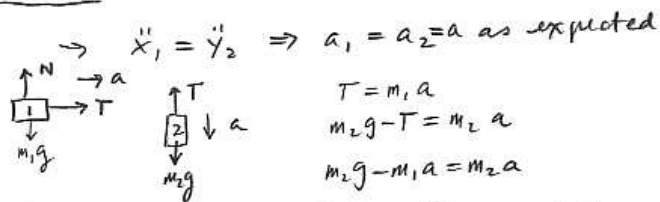
$$\text{or } V = \frac{16.18}{t_c} = 5.91 \text{ m/s}$$

The distance Debra had to run is

$$X_B - 30 = V t_c = 16.18 \text{ m}$$

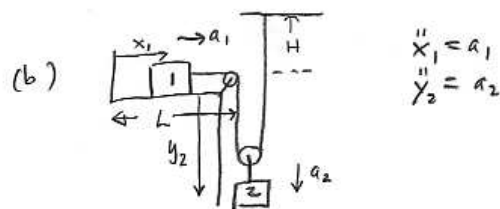


constraint: $L = x_1 + l - y_2$



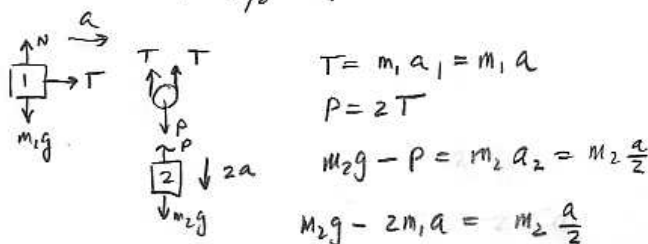
$\rightarrow a = \frac{m_2}{m_1 + m_2} g \leq g$

$T = m_1 a = \frac{m_1 m_2}{m_1 + m_2} g$



constraint: $2y_2 + H + (L - x_1) = l_{\text{string}}$

$\rightarrow 2\ddot{y}_2 - \ddot{x}_1 = 0 \Rightarrow a_1 = 2a_2 = a$



$\rightarrow a = \frac{m_2 g}{2m_1 + \frac{m_2}{2}} \quad T = m_1 a = \frac{m_1 m_2 g}{2m_1 + \frac{m_2}{2}}$

$$a > g$$

$$\Rightarrow \frac{m_2}{2m_1 + \frac{m_2}{2}} > 1$$

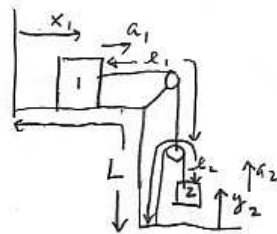
$$\Rightarrow m_2 > 2m_1 + \frac{m_2}{2}$$

$$\frac{m_2}{2} > 2m_1$$

$$m_2 > 4m_1$$

$$\Rightarrow a_1 = a \text{ can be greater than } g!$$

(c)

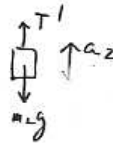
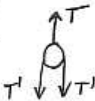
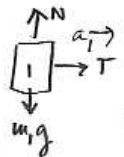


$$\ddot{x}_1 = a_1, \quad \ddot{y} = a_2$$

$$x_1 + l_1 + \frac{l_2 + y}{2} = L$$

$$\Rightarrow \ddot{x}_1 + \frac{\ddot{y}}{2} = 0$$

$$\Rightarrow a_1 = a = -\frac{a_2}{2}$$



$$T = m_1 a_1 = m_1 a, \quad T = 2T', \quad T' - m_2 g = m_2 a_2 = -2m_2 a$$

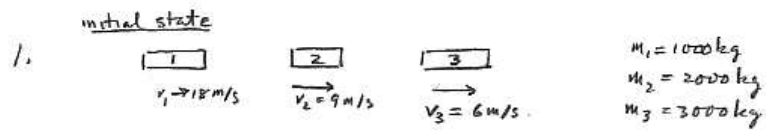
$$\Rightarrow \frac{m_1 a}{2} - m_2 g = -2m_2 a$$

$$\Rightarrow a = \frac{m_2 g}{\frac{m_1}{2} + 2m_2} < g \Rightarrow$$

$$T = m_1 a = \frac{m_1 m_2 g}{\frac{m_1}{2} + 2m_2}$$

$$T' = \frac{1}{2} T = \frac{m_1 m_2 g}{\frac{m_1}{2} + m_2}$$

Examination 2



1st collision (1,2)

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{1}{3} v_1 + \frac{2}{3} v_2 = 12 \text{ m/s}$$

$$v_{1,cm} = v_1 - V_{cm} = 6 \text{ m/s}$$

$$v_{2,cm} = v_2 - V_{cm} = -3 \text{ m/s}$$

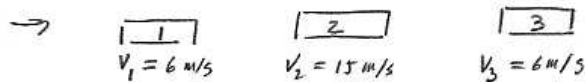
$$v'_{1,cm} = -v_{1,cm} = -6 \text{ m/s}$$

$$v'_{2,cm} = -v_{2,cm} = 3 \text{ m/s}$$

$$v'_1 = v'_{1,cm} + V_{cm} = 6 \text{ m/s}$$

$$v'_2 = v'_{2,cm} + V_{cm} = 15 \text{ m/s}$$

} elastic collision



2nd collision (2,3)

$$V_{cm} = \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} = \frac{2}{5} v_2 + \frac{3}{5} v_3 = \frac{48}{5} = 9.6 \text{ m/s}$$

$$v_{2,cm} = v_2 - V_{cm} = 5.4 \text{ m/s}$$

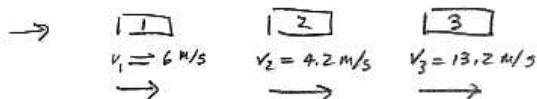
$$v_{3,cm} = v_3 - V_{cm} = -3.6 \text{ m/s}$$

$$v'_{2,cm} = -v_{2,cm} = -5.4 \text{ m/s}$$

$$v'_{3,cm} = -v_{3,cm} = 3.6 \text{ m/s}$$

$$v'_2 = v'_{2,cm} + V_{cm} = 4.2 \text{ m/s}$$

$$v'_3 = v'_{3,cm} + V_{cm} = 13.2 \text{ m/s}$$



3rd collision (1,2)

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{1}{3} v_1 + \frac{2}{3} v_2 = 4.8 \text{ m/s}$$

$$V_{1cm} = v_1 - V_{cm} = 1.2 \text{ m/s}$$

$$V_{2cm} = v_2 - V_{cm} = -0.6 \text{ m/s}$$

$$V_{1cm}' = -V_{1cm} = -1.2 \text{ m/s}$$

$$V_{2cm}' = -V_{2cm} = 0.6 \text{ m/s}$$

$$V_1' = V_{1cm}' + V_{cm} = 3.6 \text{ m/s}$$

$$V_2' = V_{2cm}' + V_{cm} = 5.4 \text{ m/s}$$

→



$$V_1 = 3.6 \text{ m/s}$$



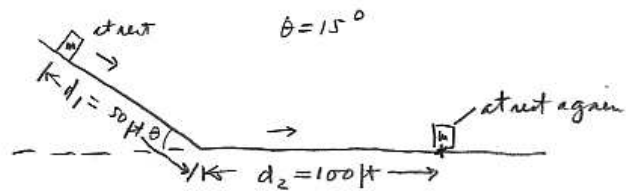
$$V_2 = 5.4 \text{ m/s}$$



$$V_3 = 13.2 \text{ m/s}$$

⇒ no more collisions

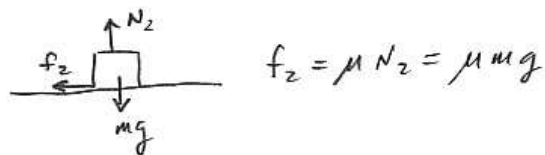
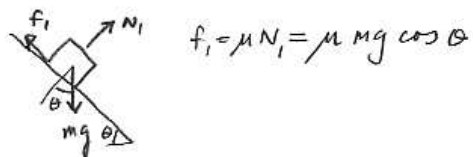
2.



$$E_o = mg d_1 \sin \theta$$

$$E_f = 0$$

$$W_f = f_1 d_1 + f_2 d_2$$



$$\therefore E_o = W_f$$

$$mg d_1 \sin \theta = \mu mg \cos \theta d_1 + \mu mg d_2$$

$$\Rightarrow \mu = \frac{d_1 \sin \theta}{d_1 \cos \theta + d_2} = \frac{50 (.259)}{50 (.966) + 100} = \frac{12.95}{148.3}$$

$$\rightarrow \mu = 0.087$$

3.

$$u(x) = 3x^2 - x^3$$

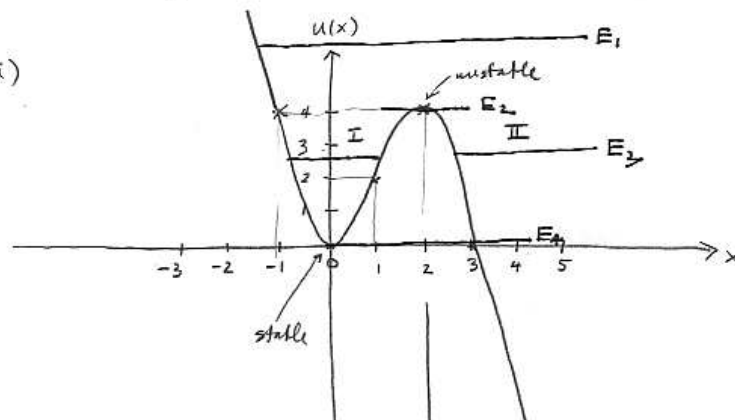
$$u(x) = 0 = 3x^2 - x^3 = x^2(3-x) \rightarrow x = 0, 3$$

$$\frac{du}{dx} = 0 = 6x - 3x^2 = 3x(2-x) \rightarrow x = 0, 2 \rightarrow \text{extrema}$$

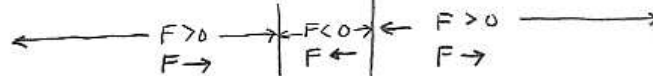
$$\frac{d^2u}{dx^2} = 0 = 6 - 6x = \begin{cases} 6 & x = 0 \rightarrow \text{minimum} \\ -6 & x = 2 \rightarrow \text{maximum} \end{cases}$$

$$u(0) = 0, u(3) = 0, u(2) = 4$$

(a)



(b)



(c)

E_1 : motion unbounded

E_2 : rest (unstable equilibrium)

E_3 : region II motion unbounded; region I motion bounded, periodic

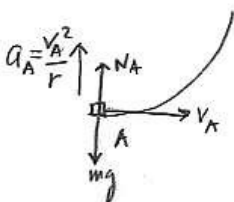
turning points given by $E_3 = u(x)$

E_4 : rest (stable equilibrium)

(d) $E_2 = u(2) = 4 \text{ J}$

4. at A

(a)



conservation of energy

$$mg(2r+h) = \frac{1}{2}mv_A^2$$

Newton's 2nd law

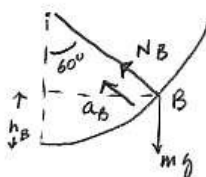
$$N_A - mg = m \frac{v_A^2}{r}$$

$$N_A = mg + \frac{m}{r}(2g(2r+h))$$

$$N_A = 5mg + 2mg \frac{h}{r}$$

vertically upwards

(b) at B



$$a_B = \frac{v_B^2}{r}$$

$$mg(2r+h) = \frac{1}{2}mv_B^2 + mgh_B$$

$$h_B = r(1 - \cos 60^\circ) = r/2$$

$$a_B = \frac{v_B^2}{r} = \frac{2g(2r+h) - gr}{r} = 3g + \frac{2h}{r}g$$

$$= \left(3 + \frac{2h}{r}\right)g$$

(c) at top



$$a_c = \frac{v_c^2}{r}$$

$$N + mg = ma_c = m \frac{v_c^2}{r}$$

just makes it $\Rightarrow N = 0$

$$\text{or } v_c^2 = gr$$

but $mg(2r + h_{\min}) = \frac{1}{2}mv_c^2 + 2mgr$

$$\rightarrow 2gr + gh_{\min} = \frac{1}{2}gr + 2gr = \frac{5}{2}gr$$

$$\rightarrow h_{\min} = \frac{r}{2}$$

$\rightarrow h \geq \frac{r}{2}$ to complete loop

(d) $h < \frac{r}{2}$

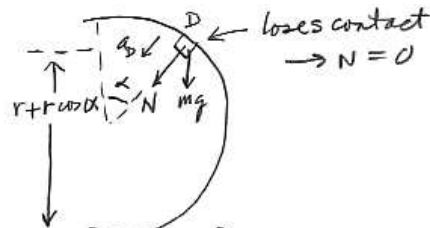
$$\Rightarrow m \frac{v_D^2}{r} = mg \cos \alpha$$

$$\text{or } v_D^2 = gr \cos \alpha$$

but $mg(2r + h) = \frac{1}{2}mv_D^2 + mgr(1 + \cos \alpha)$

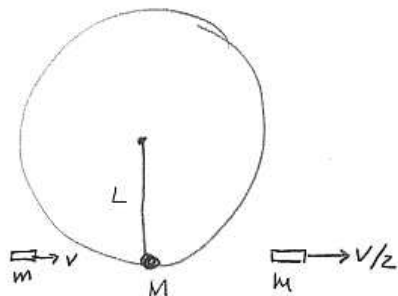
$$\Rightarrow 2gr + gh = \frac{1}{2}gr \cos \alpha + gr + gr \cos \alpha$$

$$\text{or } 3 \cos \alpha = 2 + \frac{zh}{r}$$



(e) final path is a parabola

5.



Momentum conservation in collision

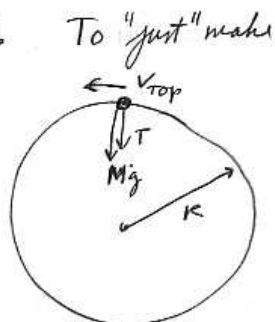
$$mv = m\frac{v}{2} + Mu$$

$$\rightarrow u = \frac{mv}{2M}$$

After collision energy conserved.
a complete circle needs to get
past top as tension = 0

$$T + Mg = \frac{Mv_{top}^2}{R}$$

$$T = 0 \Rightarrow v_{top} = \sqrt{gR}$$

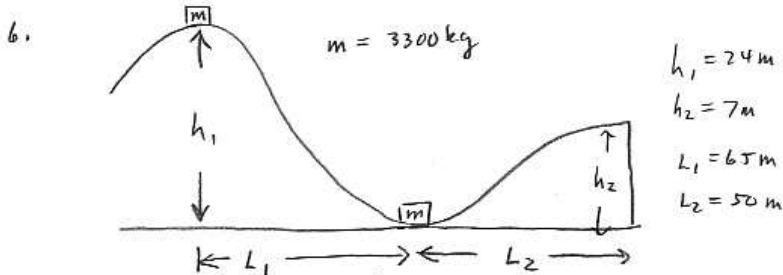


$$E_{bottom} = \frac{1}{2}Mu^2, \quad E_{top} = \frac{1}{2}Mv_{top}^2 + 2MgR$$

$$\therefore \frac{1}{2}Mu^2 = \frac{1}{2}Mv_{top}^2 + 2MgR$$

$$\rightarrow v_{top}^2 = u^2 - 4gR = gR \rightarrow u^2 = 5gR \rightarrow u = \sqrt{5gR}$$

$$\rightarrow v = \frac{2M}{m}u = \frac{2M}{m}\sqrt{5gR}$$



Assume no push at top!

conservation of energy as m goes down hill

$$mgh_1 = \frac{1}{2}mv_1^2 \rightarrow v_1 = \sqrt{2gh_1}$$

collision (inelastic)

$$\begin{array}{ccc} \boxed{m} & \boxed{m} & \Rightarrow \quad \boxed{2m} \\ \xrightarrow{v_1} & \text{at rest} & \xrightarrow{v_2} \end{array}$$

$$mv_1 = 2mV_2 \rightarrow V_2 = \frac{v_1}{2} = \sqrt{\frac{gh_1}{2}}$$

conservation of energy going up hill

$$2mgH \overset{\text{height reached}}{=} \frac{1}{2}(2m)V_2^2 = \frac{mgh_1}{2}$$

$$\text{or } H = \frac{h_1}{4}$$

if $H > h_2 \Rightarrow$ no push necessary

$$\text{or } h_1 > 4h_2$$

actually $24 > 28$ not true $\Rightarrow$ needed a push!

Examination 3

$$1a \quad x(t) = A \sin \omega t$$

$$m = 0.4 \text{ kg}$$

$$f = 15 \text{ sec}^{-1}$$

$$E = 30 \text{ J}$$

$$\omega = 2\pi f = 30\pi \text{ sec}^{-1}$$

$$\omega = 30\pi = \sqrt{\frac{k}{m}}$$

$$k = m\omega^2$$

$$E = \frac{1}{2} k A^2$$

$$= \frac{1}{2} m \omega^2 A^2$$

$$\Rightarrow A = \frac{1}{\omega} \sqrt{\frac{2E}{m}} = \frac{1}{30\pi} \sqrt{150}$$

$$= 0.13 \text{ m}$$

1b. $x(t) = A \sin \omega t$

$$A = 0.08 \text{ m}$$

$$T = 4 \text{ sec}$$

$$\omega = \frac{2\pi}{T} = \frac{\pi}{2} \text{ sec}^{-1}$$

$$x(t) = 0.08 \sin \frac{\pi t}{2}$$

particle reaches maximum amplitude at
 $t = 1 \text{ sec}$.

(a) $v(t) = \dot{x}(t) = 0.04\pi \cos \frac{\pi t}{2}$

$$\begin{aligned} \rightarrow v(1.5) &= 0.04\pi \cos 0.75\pi \\ &= 0.04(3.14)(-0.707) = -0.089 \text{ m/sec} \end{aligned}$$

(b) $a(t) = \ddot{x}(t) = -0.02\pi^2 \sin \frac{\pi t}{2}$

$$\begin{aligned} \rightarrow a(1.5) &= -0.02\pi^2 \sin 0.75\pi \\ &= -0.02(3.14)^2(0.707) = -0.14 \text{ m/sec}^2 \end{aligned}$$

2.

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$

$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 (1200)}{(0.5)^2} = 189304 \text{ N/m}$$

$$(a) \quad \Delta m = 300 \text{ kg}$$

$$k \Delta y = (\Delta m) g$$

$$\Delta y = \frac{(\Delta m) g}{k} = \frac{300 (9.8)}{189304} = 0.0155 \text{ m}$$

$$(b) \quad A = 0.05 \text{ m}$$

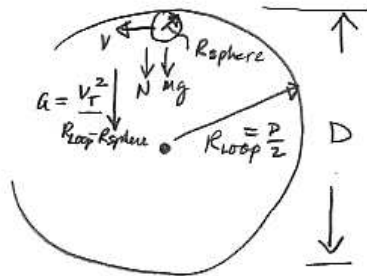
$$a_{\max} = A \omega^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{189304}{1500}} = 11.23 \text{ rad/s}$$

$$\rightarrow a_{\max} = 0.05 (11.23)^2 = 6.31 \text{ m/s}^2$$

Since $a_{\max} < g$, the passengers will remain in contact with seats

3, (include radius of ball) $\rightarrow$ correct



$$\rightarrow a = \frac{v_T^2}{\frac{D}{2} - R_{\text{sph.}}}$$

$$N=0 \rightarrow mg = ma \rightarrow v_T^2 = \left(\frac{D}{2} - R_{\text{sph.}}\right)g$$

conservation of energy $\rightarrow$

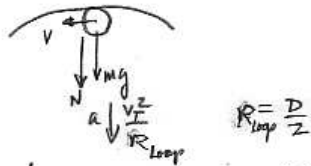
$$mgh = \frac{1}{2}mv_T^2 + \frac{1}{2}I_0\omega^2 - mgR_{\text{sph}}$$

$$\omega = \frac{v_T}{R_{\text{sph}}}, I_0 = \frac{2}{5}mR_{\text{sph}}^2$$

$$\rightarrow gh = \frac{1}{2}v_T^2 + \frac{1}{5}v_T^2 - gR_{\text{sph}} = \frac{7}{10}v_T^2 - gR_{\text{sph}}$$

$$\begin{aligned} \rightarrow h_{\text{min}} &= \frac{7}{10g}v_T^2 - R_{\text{sph}} = \frac{7}{10}\left(\frac{D}{2} - R_{\text{sph}}\right) - R_{\text{sph}} \\ &= \frac{7D}{20} - \frac{17}{10}R_{\text{sph}} \end{aligned}$$

3. (ignored radius of ball) $\rightarrow$ incorrect at top



just makes it $\rightarrow N=0 \rightarrow mg = \frac{mv_T^2}{D/2} \rightarrow v_T^2 = \frac{gD}{2}$

conservation of energy $\rightarrow$

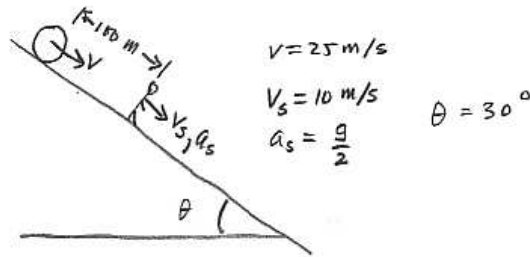
$$mgh = \frac{1}{2} m v_T^2 + \frac{1}{2} I_0 \omega^2$$

$$\omega = \frac{v_T}{R_{\text{sphere}}}, \quad I_0 = \frac{2}{5} m R_{\text{sphere}}^2$$

$$\rightarrow gh = \frac{1}{2} v_T^2 + \frac{1}{5} v_T^2 = \frac{7}{10} v_T^2$$

$$\rightarrow h_{\text{min}} = \frac{7}{10g} v_T^2 = \frac{7}{10g} \frac{gD}{2} = \frac{7}{20} D$$

4,



sphere rolling without slipping



$$N = W \cos \theta$$

with respect to CM

$$fR = I \alpha = I_0 \alpha$$

$$W \sin \theta - f = ma$$

$$f = \frac{I_0 \alpha}{R} = \frac{2}{5} \frac{MR^2}{R} \alpha$$

$$\alpha = \frac{a}{R} \quad (\text{rolling no slipping})$$

$$\rightarrow f = \frac{2}{5} ma$$

$$\rightarrow mg \sin \theta - \frac{2}{5} ma = ma \rightarrow a = \frac{5}{7} g \sin \theta$$

$$\theta = 30^\circ \rightarrow a = \frac{5}{14} g$$

$$x_{\text{snowball}} = vt + \frac{1}{2} at^2 = 25t + \frac{5g}{28} t^2$$

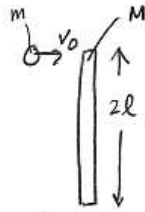
$$x_{\text{shier}} = 100 + v_s t + \frac{1}{2} a_s t^2 = 100 + 10t + \frac{g}{4} t^2$$

$$\text{Does } x_{\text{shier}} = x_{\text{snowball}} \Rightarrow 100 - 15t + \frac{1}{14} g t^2 = 0$$

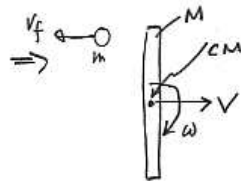
$$\text{or } \frac{1}{14} g t^2 - 15t + 100 = 0 \rightarrow t = \frac{15 \pm \sqrt{225 - \frac{400g}{14}}}{\frac{g}{14}} \Rightarrow t =$$

$\rightarrow$ no solution with t real $\rightarrow$ shier escapes.

5.



assume energy conservation
 & after collision (m) ←



$$(a) \quad m v_0 = M V - m v_f \quad (\text{momentum})$$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_f^2 + \frac{1}{2} M V^2 + \frac{1}{2} I_0 \omega^2 \quad (\text{energy})$$

about CM

$$m v_0 l = -m v_f l + I_0 \omega \quad (\text{angular momentum})$$

$$\rightarrow V = \frac{m}{M} (v_0 + v_f)$$

$$\omega = m (v_0 + v_f) \frac{l}{I_0}$$

$$I_0 = \frac{1}{12} M (2l)^2 = \frac{1}{3} M l^2$$

$$\rightarrow \frac{1}{2} m v_0^2 + \frac{1}{2} m v_f^2 + \frac{1}{2} \frac{m^2}{M^2} M (v_0 + v_f)^2 + \frac{1}{2} \left(\frac{1}{3} M l^2 \right) \frac{m^2 l^2}{I_0^2} (v_0 + v_f)^2$$

$$\rightarrow \left(1 + \frac{4m}{M} \right) v_f^2 + \left(\frac{8m}{M} v_0 \right) v_f - \left(1 - \frac{4m}{M} \right) v_0^2 = 0$$

$$\rightarrow v_f = \left(\frac{1 - \frac{4m}{M}}{1 + \frac{4m}{M}} \right) v_0$$

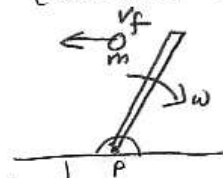
(b) now stick is pivoted at bottom end

→ momentum need not be conserved because forces act at the pivot

$$\rightarrow m v_0 (2l) = -m v_f (2l) + I_p \omega \quad (\text{angular mom})$$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_f^2 + \frac{1}{2} I_p \omega^2 \quad (\text{no translation of CM})$$

$$\rightarrow \omega = \frac{2ml (v_0 + v_f)}{I_p}$$

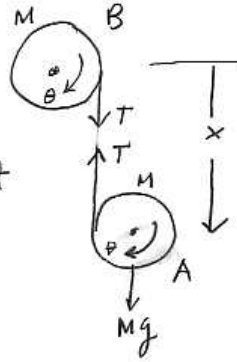


$$\rightarrow \left(1 + \frac{3m}{M}\right) v_f^2 + \left(\frac{6m}{M} v_0\right) v_f - \left(1 - \frac{3m}{M}\right) v_0^2 = 0$$

$$\Rightarrow v_f = \left(\frac{1 - \frac{3m}{M}}{1 + \frac{3m}{M}} \right) v_0$$

$$\begin{aligned} I_p &= \frac{1}{12} M (2l)^2 + M l^2 \\ &= \frac{1}{3} M l^2 + M l^2 \\ &= \frac{4}{3} M l^2 \end{aligned}$$

6. Both drums turn through the same angle and therefore have the same angular acceleration (the torques about the centers are identical)



$$\text{Now } x = l_0 + 2R\theta$$

$$\rightarrow \ddot{x} = A = 2R\alpha$$

Drum A:

$$Mg - T = MA$$

$$TR = \frac{1}{2}MR^2\alpha$$

Drum B:

$$TR = \frac{1}{2}MR^2\alpha$$

$$\Rightarrow Mg - \frac{1}{2}MR\alpha = MA$$

$$Mg - \frac{1}{4}MA = MA$$

$$\rightarrow A = \frac{4}{5}g$$