

Mathematical Methods Solutions

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Assignment 1

Problem 1-1

Since no factor x may appear in a numerator, all repeated factors appearing in the denominator give rise to as many terms in the partial fraction expansion as the power to which that factor is raised in the denominator.

(a) The denominator is already factorised but contains the repeated factor $(x-1)^2$. Thus the expansion will contain a term of the form $(x-1)^{-1}$, as well as one of the form $(x-1)^{-2}$. So,

$$\frac{2x^2 + x + 1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

We can evaluate A and B using the third method given in section 1.4:

$$A = \frac{2(-3)^2 - 3 + 1}{(-3-1)^2} = 1 \quad \text{and} \quad B = \frac{2(1)^2 + 1 + 1}{1+3} = 1.$$

We now evaluate C by setting $x = 0$ (say):

$$\frac{1}{(-1)^2 \cdot 3} = \frac{1}{3} + \frac{1}{(-1)^2} + \frac{C}{-1},$$

giving $C = 1$ and the full expansion as

$$\frac{2x^2 + x + 1}{(x-1)^2(x+3)} = \frac{1}{x+3} + \frac{1}{(x-1)^2} + \frac{1}{x-1}.$$

(b) Here the denominator needs factorising, but this is elementary,

$$\frac{x^2 - 2}{x^3 + 8x^2 + 16x} = \frac{x^2 - 2}{x(x+4)^2} = \frac{A}{x} + \frac{B}{(x+4)^2} + \frac{C}{x+4}.$$

Now, using the same method as in part (a):

$$A = \frac{0-2}{(0+4)^2} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-4)^2 - 2}{-4} = -\frac{7}{2}.$$

Setting $x = 1$ (say) determines C through

$$\frac{-1}{25} = -\frac{1}{8(1)} - \frac{7}{2(5)^2} + \frac{C}{5}.$$

Thus $C = 9/8$, and the full expression is

$$\frac{x^2 - 2}{x^3 + 8x^2 + 16x} = -\frac{1}{8x} - \frac{7}{2(x+4)^2} + \frac{9}{8(x+4)}.$$

(c)

$$\frac{x^3 - x - 1}{(x+3)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x+3)^3} + \frac{C}{(x+3)^2} + \frac{D}{x+3}.$$

As in parts (a) and (b), the third method in section 1.4 gives A and B as

$$A = \frac{(-1)^3 - (-1) - 1}{(-1+3)^3} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-3)^3 - (-3) - 1}{-3+1} = \frac{25}{2}.$$

Setting $x = 0$ requires that

$$\frac{-1}{27} = -\frac{1}{8} + \frac{25}{54} + \frac{C}{9} + \frac{D}{3} \quad \text{i.e. } C + 3D = -\frac{27}{8}.$$

Setting $x = 1$ gives the additional requirement that

$$\frac{-1}{128} = -\frac{1}{16} + \frac{25}{128} + \frac{C}{16} + \frac{D}{4} \quad \text{i.e. } C + 4D = -\frac{18}{8}.$$

Solving these two equations for C and D now yields $D = 9/8$ and $C = -54/8$.

Thus,

$$\frac{x^3 - x - 1}{(x+3)^3(x+1)} = -\frac{1}{8(x+1)} + \frac{1}{8} \left[\frac{100}{(x+3)^3} - \frac{54}{(x+3)^2} + \frac{9}{x+3} \right].$$

If necessary, that the expansion is valid for all x (and not just for 0 and 1) can be checked by writing all of its terms so as to have the common denominator $(x+3)^3(x+1)$.

Problem 1-2

To use the binomial expansion, we need to express the inverse square root in the form $(1 + a)^{-1/2}$ with $|a| < 1$. We do this as follows.

$$\begin{aligned}\frac{1}{\sqrt{4.2}} &= \frac{1}{(4 + 0.2)^{1/2}} = \frac{1}{2(1 + 0.05)^{1/2}} \\ &= \frac{1}{2} \left[1 - \frac{1}{2}(0.05) + \frac{3}{8}(0.05)^2 - \frac{15}{48}(0.05)^3 + \cdots \right] \\ &= 0.487949218.\end{aligned}$$

This four-term sum and the accurate value differ by about 8×10^{-7} .

Problem 1-3

We need to write the numerator in such a way that every term in it that involves x contains a factor $ax + b$. Therefore, write x^2 as

$$x^2 = \frac{x}{a}(ax + b) - \frac{b}{a^2}(ax + b) + \frac{b^2}{a^2}.$$

Then,

$$\begin{aligned} \int_0^{b/a} \frac{x^2}{ax + b} dx &= \int_0^{b/a} \left(\frac{x}{a} - \frac{b}{a^2} + \frac{b^2}{a^2(ax + b)} \right) dx \\ &= \left[\frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3} \ln(ax + b) \right]_0^{b/a} \\ &= \frac{b^2}{a^3} \left(\ln 2 - \frac{1}{2} \right). \end{aligned}$$

An alternative approach, consistent with the wording of the question, is to use the binomial theorem to write the integrand as

$$\frac{x^2}{ax + b} = \frac{x^2}{b} \left(1 + \frac{ax}{b} \right)^{-1} = \frac{x^2}{b} \sum_{n=0}^{\infty} \left(-\frac{ax}{b} \right)^n.$$

Then the integral is

$$\begin{aligned} \int_0^{b/a} \frac{x^2}{ax + b} dx &= \frac{1}{b} \int_0^{b/a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{b} \right)^n x^{n+2} dx \\ &= \frac{1}{b} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{b} \right)^n \frac{1}{n+3} \left(\frac{b}{a} \right)^{n+3} \\ &= \frac{b^2}{a^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}. \end{aligned}$$

That these two solutions are the same can be seen by writing $\ln 2 - \frac{1}{2}$ as

$$\begin{aligned} \ln 2 - \frac{1}{2} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \right) - \frac{1}{2} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}. \end{aligned}$$

Problem 1-4

Taking $\sin n\theta$ as u and $\cos \theta$ as v and noting that with this choice $u'' = -n^2u$

and $v'' = -v$, we expect that after two integrations by parts we will recover (a multiple of) I_n .

$$\begin{aligned}
 I_n &= \int_0^{\pi/2} \sin n\theta \cos \theta \, d\theta \\
 &= [\sin n\theta \sin \theta]_0^{\pi/2} - \int_0^{\pi/2} n \cos n\theta \sin \theta \, d\theta \\
 &= \sin \frac{n\pi}{2} - n \left\{ [-\cos n\theta \cos \theta]_0^{\pi/2} - \int_0^{\pi/2} (-n \sin n\theta)(-\cos \theta) \, d\theta \right\} \\
 &= \sin \frac{n\pi}{2} - n[(-1) - nI_n].
 \end{aligned}$$

Rearranging this gives

$$I_n(1 - n^2) = \sin \frac{n\pi}{2} - n,$$

and hence the stated result.

Problem 1-5

All of these evaluations rely directly on the definitions of the various functions involved as applied to complex numbers; these should be known to the reader. There are too many to give every one individually at each step and, if the justification for any particular step is unclear, reference should be made to a textbook.

$$(a) \operatorname{Re}(\exp 2iz) = \operatorname{Re}[\exp(2ix - 2y)] = \exp(-2y) \cos 2x.$$

$$\begin{aligned} (b) \operatorname{Im}(\cosh^2 z) &= \operatorname{Im} \left[\frac{1}{2}(\cosh 2z + 1) \right] \\ &= \frac{1}{2} \operatorname{Im} [\cosh(2x + 2iy)] \\ &= \frac{1}{2} \operatorname{Im} (\cosh 2x \cosh 2iy + \sinh 2x \sinh 2iy) \\ &= \frac{1}{2} \operatorname{Im} (\cosh 2x \cos 2y + i \sinh 2x \sin 2y) \\ &= \frac{1}{2} \sinh 2x \sin 2y. \end{aligned}$$

$$\begin{aligned} (c) (-1 + \sqrt{3}i)^{1/2} &= [(-1)^2 + (\sqrt{3})^2]^{1/4} \exp \left[i \frac{1}{2} (\tan^{-1}(\frac{\sqrt{3}}{-1}) + 2n\pi) \right] \\ &= \sqrt{2} \exp \left[i \frac{1}{2} (\frac{2}{3}\pi + 2n\pi) \right] \\ &= \sqrt{2} \exp \left(\frac{\pi i}{3} \right) \quad \text{or} \quad \sqrt{2} \exp \left(\frac{4\pi i}{3} \right). \end{aligned}$$

$$\begin{aligned} (d) \left| \exp(i^{1/2}) \right| &= \left| \exp \left((e^{i\pi/4})^{1/2} \right) \right| \\ &= \left| \exp \left(e^{i\pi/8} \right) \right| \\ &= \left| \exp \left[\cos(n + \frac{1}{4})\pi + i \sin(n + \frac{1}{4})\pi \right] \right| \\ &= \exp[\cos(n + \frac{1}{4})\pi] \\ &= \exp \left(\frac{1}{\sqrt{2}} \right) \quad \text{or} \quad \exp \left(-\frac{1}{\sqrt{2}} \right). \end{aligned}$$

$$\begin{aligned} (e) \exp(i^3) &= \exp \left(e^{3i(\pi/4)} \right) = \exp \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \\ &= \exp(0 - i) = \cos(-1) + i \sin(-1) = 0.540 - 0.841i. \end{aligned}$$

$$(f) \operatorname{Im}(2^{i+3}) = \operatorname{Im}(8 \times 2^i) = 8 \operatorname{Im}(2^i) = 8 \operatorname{Im}(e^{i \ln 2}) = 8 \sin(\ln 2) = 5.11.$$

$$(g) i^i = [\exp i(\frac{1}{2}\pi + 2n\pi)]^i = [\exp i^2(\frac{1}{2}\pi + 2n\pi)] = \exp[-(2n + \frac{1}{2})\pi].$$

$$\begin{aligned} (h) \ln[(\sqrt{3} + i)^3] &= 3 \ln(\sqrt{3} + i) \\ &= 3 \left(\ln 2 + i \tan^{-1} \frac{1}{\sqrt{3}} \right) \\ &= \ln 8 + 3i \left(\frac{\pi}{6} + 2n\pi \right) \\ &= \ln 8 + i(6n + \frac{1}{2})\pi. \end{aligned}$$

Problem 1-6

(a) Expressing each term of

$$\cosh x = \sinh x + 2 \operatorname{sech} x$$

in exponential form:

$$\begin{aligned}\frac{1}{2}(e^x + e^{-x}) &= \frac{1}{2}(e^x - e^{-x}) + \frac{4}{e^x + e^{-x}}, \\ e^{-x} &= \frac{4}{e^x + e^{-x}}, \\ 1 + e^{-2x} &= 4.\end{aligned}$$

Taking logarithms of this, after simplification, yields

$$x = \frac{1}{2}(-\ln 3) = -\ln \sqrt{3} = -0.5493.$$

(b) Expressing the defining equality in terms of sinh and cosh functions,

$$\tanh x = \operatorname{cosech} x \quad \Rightarrow \quad \sinh^2 x = \cosh x.$$

Writing $\cosh x$ as u , this equation can be put in the form $u^2 - 1 = u$. It follows from substituting this into the standard logarithmic expression for an inverse hyperbolic cosine that

$$x = \cosh^{-1} u = \ln(\sqrt{u^2 - 1} + u) = \ln(\sqrt{u} + u).$$

Since $u^2 - u - 1 = 0$, we also have that

$$u = \frac{1 \pm \sqrt{5}}{2}, \quad (\text{i.e. the golden mean and minus its reciprocal!})$$

with the positive sign being taken to make $\cosh x > 0$, as is required if x is to be real.

(c) Using a double angle formula for hyperbolic functions to replace $\cosh 2x$ in

$$\cosh 2x = 2 \cosh x,$$

we have

$$\begin{aligned}2 \cosh^2 x - 1 &= 2 \cosh x, \\ \cosh x &= \frac{1 \pm \sqrt{3}}{2},\end{aligned}$$

with the positive sign needed for x to be real. It then follows that

$$\begin{aligned}\sinh x &= \pm \left(\frac{1 + 2\sqrt{3} + 3}{4} - 1 \right)^{1/2} = \pm \left(\frac{\sqrt{3}}{2} \right)^{1/2}, \\ \Rightarrow \tanh x &= \frac{\pm 2 \left(\frac{\sqrt{3}}{2} \right)^{1/2}}{1 + \sqrt{3}} = \pm \frac{(12)^{1/4}}{1 + \sqrt{3}}.\end{aligned}$$

Problem 1-7

(a) Using the Maclaurin series for $\ln(1+x)$,

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right) \\ &= 2x + 2\frac{x^3}{3} + \dots \\ &= 2 \sum_{n \text{ odd}} \frac{x^n}{n}.\end{aligned}$$

This series is convergent only for $-1 < x < 1$.

(b) Here the binomial expansion can be employed directly. This saves the trouble of having to find the expansion coefficients by differentiation.

$$\begin{aligned}(x^2 + 4)^{-1} &= \frac{1}{4} \left(1 + \frac{x^2}{4}\right)^{-1} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}.\end{aligned}$$

(c) We calculate the derivatives of $\sin^2 x$ at $x = 0$.

$$\begin{aligned}y &= \sin^2 x, \\ y' &= 2 \sin x \cos x = \sin 2x, \\ y'' &= 2 \cos 2x, \\ y''' &= -4 \sin 2x, \\ \vdots &= \vdots \\ y^{(2m)} &= (-1)^{m+1} 2^{2m-1} \cos 2x, \\ y^{(2m+1)} &= (-1)^m 2^{2m} \sin 2x.\end{aligned}$$

At $x = 0$ all odd derivatives vanish and $\cos 2x = 1$ in the even ones. So the Maclaurin expansion is

$$\sin^2 x = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} 2^{2m-1} x^{2m}}{(2m)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2x)^{2n}}{2(2n)!}.$$

This result could also have been obtained by writing $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and using the known Maclaurin series for a cosine function.

Problem 1-8

Using L'Hôpital's rule whenever both the numerator and denominator of a fraction become zero in the limit, we have for cases (a)-(c):

$$(a) \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{\sinh x} = \lim_{x \rightarrow 0} \frac{3 \cos 3x}{\cosh x} = 3.$$

$$\begin{aligned} (b) \quad L &= \lim_{x \rightarrow 0} \frac{\tan x - \tanh x}{\sinh x - x} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - \operatorname{sech}^2 x}{\cosh x - 1} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x + 2 \operatorname{sech}^2 x \tanh x}{\sinh x} \\ &= \lim_{x \rightarrow 0} \frac{2(\tan x + \tan^3 x + \tanh x - \tanh^3 x)}{\sinh x} \\ &= \lim_{x \rightarrow 0} \frac{2(\sec^2 x + 3 \tan^2 x \sec^2 x + \operatorname{sech}^2 x - 3 \tanh^2 x \operatorname{sech}^2 x)}{\cosh x} \\ &= 4. \end{aligned}$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{\tan x - x}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{-\sin x} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{-\cos x} = 0.$$

(d) In order to avoid undetermined combinations such as $x \operatorname{cosec} x$, we arrange the function so that all terms in it tend to 0 or a finite quantity as $x \rightarrow 0$. For this particular case this also allows us to use known Maclaurin expansions of the factors involved.

[Inspection of the function shows that both fractions behave as x^{-4} as $x \rightarrow 0$, but that their difference will be less divergent. Finding the limit (if it is finite) using L'Hôpital's rule will take up to six differentiations; hence our preference for Maclaurin expansions.]

$$\begin{aligned} L &= \frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5} \\ &= \frac{x^2 - \sin x \sinh x}{x^5 \sin x} \\ &= \frac{x^2 - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}{x^5 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} \\ &= \frac{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) \left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)}{x^4 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)} \\ &= \frac{(1-1) + x^2 \left(\frac{1}{3!} - \frac{1}{3!}\right) + x^4 \left(-\frac{1}{5!} - \frac{1}{5!} + \frac{1}{(3!)^2}\right) + \dots}{x^4 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)}, \end{aligned}$$

which has as its limit, when $x \rightarrow 0$, the value

$$\frac{1}{(3!)^2} - \frac{2}{5!} = \frac{20-12}{720} = \frac{1}{90}.$$

Problem 1-9

The required differentiations are most easily carried out after taking the logarithms of both sides of the original equation (for typographical convenience use P instead of p):

$$\ln P + \ln V - \ln R - \ln T = -\frac{\alpha}{VRT}. \quad (*)$$

First, differentiating (*) with respect to V with T held fixed:

$$\begin{aligned} \frac{1}{P} \left(\frac{\partial P}{\partial V} \right)_T + \frac{1}{V} &= \frac{\alpha}{V^2 RT}, \\ \left(\frac{\partial P}{\partial V} \right)_T &= \left(\frac{\alpha}{V^2 RT} - \frac{1}{V} \right) P \\ &= \frac{P(\alpha - VRT)}{V^2 RT}. \end{aligned}$$

Next, differentiating (*) with respect to T with P held fixed:

$$\begin{aligned} \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P - \frac{1}{T} &= \frac{\alpha}{VRT^2} + \frac{\alpha}{V^2 RT} \left(\frac{\partial V}{\partial T} \right)_P, \\ \left(\frac{\partial V}{\partial T} \right)_P &= \frac{\frac{\alpha}{VRT^2} + \frac{1}{T}}{\frac{1}{V} - \frac{\alpha}{V^2 RT}} \\ &= \frac{V(\alpha + VRT)}{T(VRT - \alpha)}. \end{aligned}$$

Finally, differentiating (*) with respect to P with V held fixed.

$$\begin{aligned} \frac{1}{P} - \frac{1}{T} \left(\frac{\partial T}{\partial P} \right)_V &= \frac{\alpha}{VRT^2} \left(\frac{\partial T}{\partial P} \right)_V, \\ \left(\frac{\partial T}{\partial P} \right)_V &= \frac{1}{P \left(\frac{\alpha}{VRT^2} + \frac{1}{T} \right)} \\ &= \frac{VRT^2}{P(\alpha + VRT)}. \end{aligned}$$

These are the three partial derivatives and their product,

$$\left(\frac{\partial P}{\partial V} \right)_T \left(\frac{\partial V}{\partial T} \right)_P \left(\frac{\partial T}{\partial P} \right)_V,$$

is

$$\frac{P(\alpha - VRT)}{V^2 RT} \frac{V(\alpha + VRT)}{T(VRT - \alpha)} \frac{VRT^2}{P(\alpha + VRT)} = -1,$$

as expected.

Problem 1-10

To find the stationary points, we set each of the two first partial derivatives,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left[2x - \frac{2x}{a^2}(x^2 - 2y^2)\right] \exp\left(-\frac{x^2 + y^2}{a^2}\right), \\ \frac{\partial f}{\partial y} &= \left[-4y - \frac{2y}{a^2}(x^2 - 2y^2)\right] \exp\left(-\frac{x^2 + y^2}{a^2}\right),\end{aligned}$$

equal to zero:

$$\begin{aligned}\frac{\partial f}{\partial x} = 0 &\Rightarrow x = 0 \text{ or } x^2 - 2y^2 = a^2; \\ \frac{\partial f}{\partial y} = 0 &\Rightarrow y = 0 \text{ or } x^2 - 2y^2 = -2a^2.\end{aligned}$$

Since $a \neq 0$, possible solutions for (x, y) are $(0, 0)$, $(0, \pm a)$ and $(\pm a, 0)$. The corresponding values are $f(0, 0) = 0$, $f(0, \pm a) = -2a^2 e^{-1}$ and $f(\pm a, 0) = a^2 e^{-1}$. These results, taken together with the observation that $|f(x, y)| \rightarrow 0$ as either or both of $|x|$ and $|y| \rightarrow \infty$, show that $f(x, y)$ has maxima at $(\pm a, 0)$, minima at $(0, \pm a)$ and a saddle point at the origin.

Sketches of $f(x, 0)$ and $f(0, y)$, whilst hardly necessary, illustrate rather than confirm these conclusions.

Problem 1-11

Starting from

$$dU = TdS - PdV,$$

□

□

□

the partial derivatives of U with respect to X and Y are

$$\frac{\partial U}{\partial X} = T \frac{\partial S}{\partial X} - P \frac{\partial V}{\partial X} \quad \text{and} \quad \frac{\partial U}{\partial Y} = T \frac{\partial S}{\partial Y} - P \frac{\partial V}{\partial Y}.$$

We next differentiate these two expressions to obtain two (equal) second derivatives. Note that, since X and Y can be any pair drawn from P , V , T and S , we must differentiate all four terms on the RHS as products, giving rise to two terms each. The resulting equations are

$$\begin{aligned} \frac{\partial^2 U}{\partial Y \partial X} &= T \frac{\partial^2 S}{\partial Y \partial X} + \frac{\partial T}{\partial Y} \frac{\partial S}{\partial X} - P \frac{\partial^2 V}{\partial Y \partial X} - \frac{\partial P}{\partial Y} \frac{\partial V}{\partial X}, \\ \frac{\partial^2 U}{\partial X \partial Y} &= T \frac{\partial^2 S}{\partial X \partial Y} + \frac{\partial T}{\partial X} \frac{\partial S}{\partial Y} - P \frac{\partial^2 V}{\partial X \partial Y} - \frac{\partial P}{\partial X} \frac{\partial V}{\partial Y}. \end{aligned}$$

Equating the two expressions, and then cancelling the terms that appear on both side of the equality, yields

□

$$\begin{aligned} \frac{\partial T}{\partial Y} \frac{\partial S}{\partial X} - \frac{\partial P}{\partial Y} \frac{\partial V}{\partial X} &= \frac{\partial T}{\partial X} \frac{\partial S}{\partial Y} - \frac{\partial P}{\partial X} \frac{\partial V}{\partial Y}, \\ \Rightarrow \frac{\partial T}{\partial Y} \frac{\partial S}{\partial X} - \frac{\partial T}{\partial X} \frac{\partial S}{\partial Y} &= \frac{\partial P}{\partial Y} \frac{\partial V}{\partial X} - \frac{\partial P}{\partial X} \frac{\partial V}{\partial Y}, \\ \Rightarrow \frac{\partial(S, T)}{\partial(X, Y)} &= \frac{\partial(V, P)}{\partial(X, Y)}. \end{aligned}$$

□

Now, using this result and the properties of Jacobians ($J_{pr} = J_{pq}J_{qr}$ and $J_{pq} = [J_{qp}]^{-1}$), we can write

$$\begin{aligned} \frac{\partial(S, T)}{\partial(V, P)} &= \frac{\partial(S, T)}{\partial(X, Y)} \frac{\partial(X, Y)}{\partial(V, P)} \\ &= \frac{\partial(S, T)}{\partial(X, Y)} \left[\frac{\partial(V, P)}{\partial(X, Y)} \right]^{-1} \\ &= \frac{\partial(S, T)}{\partial(X, Y)} \left[\frac{\partial(S, T)}{\partial(X, Y)} \right]^{-1} \\ &= 1. \end{aligned}$$

Problem 1-12

We compute only the x-component of each side of the equation. The corresponding results for other components can be obtained by cyclic permutation of x , y and z .

$$\begin{aligned}
 \mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \\
 [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]_x &= (a_z b_x - a_x b_z)c_z - (a_x b_y - a_y b_x)c_y \\
 &= b_x(a_z c_z + a_y c_y) - a_x(b_z c_z + b_y c_y) \\
 &= b_x(a_z c_z + a_y c_y + a_x c_x) - a_x(b_x c_x + b_z c_z + b_y c_y) \\
 &= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}]_x.
 \end{aligned}$$

To obtain the penultimate line we both added and subtracted $a_x b_x c_x$ on the

RHS. This establishes the result for the x-component and hence for all three components.

We have shown that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Now consider

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -(\mathbf{b} \cdot \mathbf{a})\mathbf{c} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}.$$

The last terms on the RHSs of the two equations are not equal, showing that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Assignment 2

Problem 2-1

As we have only the definitions to work with, these basic products must be found by explicit matrix multiplication:

$$\begin{aligned} S_x^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \\ S_x S_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iS_z, \\ S_y S_z &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = iS_x. \end{aligned}$$

Similarly $S_y^2 = S_z^2 = I$ and $S_z S_x = iS_y$. We also note that

$$S_y S_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -iS_z,$$

i.e. that S_x and S_y anticommute. This applies to any pair of the matrices (excluding I of course).

We first note that if $\mathbf{0}$ is the zero vector then $\mathbf{S}(\mathbf{0}) = \mathbf{0}$, the zero matrix; conversely, if $\mathbf{S}(\mathbf{v}) = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$. Now consider the product of the two matrices $\mathbf{S}(\mathbf{a})$ and $\mathbf{S}(\mathbf{b})$.

$$\begin{aligned} \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b}) &= (a_x S_x + a_y S_y + a_z S_z)(b_x S_x + b_y S_y + b_z S_z) \\ &= (a_x b_x + a_y b_y + a_z b_z)I + a_x b_y (iS_z) + a_y b_x (-iS_z) + \dots \\ &= (a_x b_x + a_y b_y + a_z b_z)I + i(\mathbf{a} \times \mathbf{b})_z S_z + \dots \\ &= (\mathbf{a} \cdot \mathbf{b})I + i\mathbf{S}(\mathbf{a} \times \mathbf{b}), \quad \text{as stated in the question.} \end{aligned}$$

Interchanging $\mathbf{a}$ and $\mathbf{b}$ gives $\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a}) = (\mathbf{b} \cdot \mathbf{a})I + i\mathbf{S}(\mathbf{b} \times \mathbf{a})$. It then follows that

$$\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b}) - \mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a}) = 2i\mathbf{S}(\mathbf{a} \times \mathbf{b}).$$

The matrix on the RHS is the zero matrix if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

Problem 2-2

The general procedure is to construct the orthonormal base set $\{\hat{z}_i\}$ using iteration procedure

$$\mathbf{z}_n = \mathbf{x}_n - \sum_{r=1}^{n-1} [\hat{z}_r^\dagger \mathbf{x}_n] \hat{z}_r, \text{ with } \mathbf{z}_1 = \mathbf{x}_1.$$

The vector $\hat{z}$ is the vector z after normalisation and the expression in square brackets is the (complex) inner product of $\hat{z}_r$ and $\mathbf{x}_n$.

(a) We start with $\hat{z}_1 = 2^{-1/2} \mathbf{x}_1 = 2^{-1/2} [0 \ 0 \ 1 \ 1]^\top$.

Next we calculate $(\hat{z}_1)^\dagger \mathbf{x}_2$ as $-2^{-1/2}$ and then form $\mathbf{z}_2$ as

$$\mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

□ The normalised vector $\hat{z}_2$ is $6^{-1/2} (2 \ 0 \ -1)^T 1$. □

Proceeding in this way, but without detailed description, we obtain

$$\mathbf{z}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 2 \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

The normalised vector $\hat{z}_3$ is $(39)^{-1/2} (-1 \ 6 \ -1)^T 1$.

Finally,

$$\mathbf{z}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{39}} \frac{1}{\sqrt{39}} \begin{bmatrix} -1 \\ 6 \\ -1 \\ 1 \end{bmatrix}.$$

□ □ □

The normalised vector $\hat{z}_4$ is $(13)^{-1/2} (2 \ 1 \ 2)^T -1$.

[Note that if the only requirement had been to find an orthonormal set of base vectors then the obvious $(1 \ 0 \ 0 \ 0)^T$, $(0 \ 1 \ 0 \ 0)^T$, etc. could have been chosen.]

(b) The procedure is as in part (a) except that we require only three orthonormal vectors. However, we *must* begin with the given vectors so as to ensure that the correct subspace is spanned.

We start with $\hat{z}_1 = 5^{-1/2} \mathbf{x}_1 = 5^{-1/2} [1 \ 2 \ 0 \ 0]^T$.

Next we calculate $(\hat{z}_1)^\dagger \mathbf{x}_2$ as $-5^{-1/2}$ and then form $\mathbf{z}_2$ as

$$\mathbf{z}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ -\frac{7}{5} \\ 2 \\ 0 \end{bmatrix}.$$

The normalised vector $\hat{z}_2$ is $(345)^{-1/2} (14 \ -7 \ 10)^T 0$.

As the final base vector for the subspace we obtain

$$\mathbf{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} - 0 \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{20}{\sqrt{345}} \frac{1}{\sqrt{345}} \begin{bmatrix} 14 \\ -7 \\ 10 \\ 0 \end{bmatrix} = \frac{1}{345} \begin{bmatrix} -280 \\ 140 \\ 490 \\ 345 \end{bmatrix}.$$

Thus, the normalised vector $\hat{z}_3$ is $(18285)^{-1/2} (-56 \ 28 \ 98)^T 69$. The fact that three orthonormal vectors can be found shows that the subspace is 3-dimensional and that the three original vectors are not linearly dependent.

Problem 2-3

We start by finding the eigenvalues of H using

$$\begin{vmatrix} 10 - \lambda & 3i \\ -3i & 2 - \lambda \end{vmatrix} = 0, \\ 20 - 12\lambda + \lambda^2 - 3 = 0, \\ \lambda = 1 \quad \text{or} \quad 11.$$

As expected for an hermitian matrix, the eigenvalues are real.

For $\lambda = 1$ and normalised eigenvector $(x, y)^T$,

$$9x + 3iy = 0 \quad \Rightarrow \quad \mathbf{x}^1 = (10)^{-1/2} (1, 3i)^T.$$

For $\lambda = 11$ and normalised eigenvector $(x, y)^T$,

$$-x + 3iy = 0 \quad \Rightarrow \quad \mathbf{x}^2 = (10)^{-1/2} (3i, 1)^T.$$

Again as expected, $(\mathbf{x}^1)^\dagger \mathbf{x}^2 = 0$, thus verifying the mutual orthogonality of the eigenvectors. It should be noted that the normalisation factor is determined by $(\mathbf{x}^i)^\dagger \mathbf{x}^i = 1$ (and *not* by $(\mathbf{x}^i)^T \mathbf{x}^i = 1$).

We now use these normalised eigenvectors of H as the columns of the matrix U and check that it is unitary:

$$U = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}, \quad U^\dagger = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix},$$

$$UU^\dagger = \frac{1}{10} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = I.$$

U has the further property that

$$\begin{aligned} U^\dagger H U &= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} \begin{pmatrix} 1 & 33i \\ 3i & 11 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 110 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix} = \Lambda. \end{aligned}$$

That the diagonal entries of Λ are the eigenvalues of H is in accord with the general theory of normal matrices.

Problem 2-4

We need to invert the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix},$$

whose determinant is $1(1) + 2(-7) + 3(5) = 2$. This is non-zero and so A has an inverse. The matrix of cofactors is

$$C = \begin{pmatrix} 1 & -7 & 5 \\ 1 & 1 & -1 \\ -2 & 4 & -2 \end{pmatrix},$$

from which it follows that $A^{-1} = (1/|A|)C^T$ is given by

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix}.$$

Finally, rewriting the given equation $Ax = y$ as $x = A^{-1}y$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}.$$

Thus $x_1 = -\frac{3}{2}$, $x_2 = \frac{7}{2}$ and $x_3 = -\frac{3}{2}$.

Problem 2-5

The proof of this standard result for the divergence of a vector product is most easily carried out in Cartesian coordinates though, of course, the result is valid in any three-dimensional coordinate system.

$$\begin{aligned}
 \text{LHS} &= \nabla \cdot (\mathbf{a} \times \mathbf{b}) \\
 &= \frac{\partial}{\partial x}(a_y b_z - a_z b_y) + \frac{\partial}{\partial y}(a_z b_x - a_x b_z) + \frac{\partial}{\partial z}(a_x b_y - a_y b_x) \\
 &= a_x \left(-\frac{\partial b_z}{\partial y} + \frac{\partial b_y}{\partial z} \right) + a_y \left(\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z} \right) + a_z \left(-\frac{\partial b_y}{\partial x} + \frac{\partial b_x}{\partial y} \right) \\
 &\quad + b_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + b_y \left(-\frac{\partial a_z}{\partial x} + \frac{\partial a_x}{\partial z} \right) + b_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \\
 &= -\mathbf{a} \cdot (\nabla \times \mathbf{b}) + \mathbf{b} \cdot (\nabla \times \mathbf{a}) = \text{RHS}.
 \end{aligned}$$

Problem 2-6

With $\mathbf{F}$ as given, we calculate the curl of $\mathbf{F}$ to see whether or not it is the zero vector:

$$\nabla \times \mathbf{F} = (4yz - 4yz, 2x - 2x, 0 - 0) = \mathbf{0}.$$

The fact that it is implies that $\mathbf{F}$ can be written as $\nabla\phi$ for some scalar ϕ .

The form of $\phi(x, y, z)$ is found by integrating, in turn, the components of $\mathbf{F}$ until consistency is achieved, i.e. until a ϕ is found that has partial derivatives equal to the corresponding components of $\mathbf{F}$:

$$\begin{aligned} 2xz = F_x = \frac{\partial\phi}{\partial x} &\Rightarrow \phi(x, y, z) = x^2z + g(y, z), \\ 2yz^2 = F_y = \frac{\partial}{\partial y}[x^2z + g(y, z)] &\Rightarrow g(y, z) = y^2z^2 + h(z), \\ x^2 + 2y^2z - 1 = F_z = \frac{\partial}{\partial z}[x^2z + y^2z^2 + h(z)] &= \\ &\Rightarrow h(z) = -z + k. \end{aligned}$$

Hence, to within an unimportant constant, the form of ϕ is

$$\phi(x, y, z) = x^2z + y^2z^2 - z.$$

Problem 2-7

This is an odd function in x and so a sine series with period 2π is appropriate. The coefficient of $\sin nx$ will be given by

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \\ &= \frac{1}{\pi} \left\{ \left[-\frac{x \cos nx}{n} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} \, dx \right\} \\ &= \frac{1}{\pi} \left[-\frac{\pi(-1)^n - (-\pi)(-1)^n}{n} + 0 \right] = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Thus,
$$x = f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

We note in passing that although this series is convergent, as it must be, it has poor (i.e. n^{-1}) convergence; this can be put down to the periodic version of the function having a discontinuity (of 2π) at the end of each basic period.

To obtain the sum of a series from such a Fourier representation, we must make a judicious choice for the value of x – making such a choice is rather more of an art than a science! Here, setting $x = \pi/2$ gives

$$\begin{aligned} \frac{\pi}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n\pi/2)}{n} \\ &= 2 \sum_{n \text{ odd}} \frac{(-1)^{n+1} (-1)^{(n-1)/2}}{n}, \\ \Rightarrow \frac{\pi}{4} &= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \end{aligned}$$

Problem 2-8

Since the Fourier series will have period 2, we can say immediately that at $x = 2$ the series will converge to the value it has at $x = 0$, namely 1.

As the function $f(x) = \exp x$ is neither even nor odd, its Fourier series will contain

both sine and cosine terms. The cosine coefficients are given by

$$\begin{aligned}
 a_n &= \frac{2}{2} \int_{-1}^1 e^x \cos(n\pi x) dx \\
 &= [\cos(n\pi x) e^x]_{-1}^1 + \int_{-1}^1 n\pi \sin(n\pi x) e^x dx \\
 &= (-1)^n (e^1 - e^{-1}) + [n\pi \sin(n\pi x) e^x]_{-1}^1 \\
 &\quad - \int_{-1}^1 n^2 \pi^2 \cos(n\pi x) e^x dx \\
 &= 2(-1)^n \sinh 1 - n^2 \pi^2 a_n, \\
 \Rightarrow a_n &= \frac{2(-1)^n \sinh 1}{1 + n^2 \pi^2}.
 \end{aligned}$$

Similarly, the sine coefficients are given by

$$\begin{aligned}
 b_n &= \frac{2}{2} \int_{-1}^1 e^x \sin(n\pi x) dx \\
 &= [\sin(n\pi x) e^x]_{-1}^1 - \int_{-1}^1 n\pi \cos(n\pi x) e^x dx \\
 &= 0 + [-n\pi \cos(n\pi x) e^x]_{-1}^1 - \int_{-1}^1 n^2 \pi^2 \sin(n\pi x) e^x dx \\
 &= 2(-1)^{n+1} n\pi \sinh 1 - n^2 \pi^2 b_n, \\
 \Rightarrow b_n &= \frac{2(-1)^{n+1} n\pi \sinh 1}{1 + n^2 \pi^2}.
 \end{aligned}$$

Problem 2-9

The function $y(x) = |x|$ is an even function and its Fourier series will therefore contain only cosine terms. They are given, for $n \geq 1$, by

$$\begin{aligned} a_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \\ &= \left\{ \left[\frac{x \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \, dx \right\} \\ &= \frac{2}{\pi} \left[\frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= -\frac{4}{\pi n^2} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.} \end{aligned}$$

The constant term is $a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi$. Thus

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}.$$

We now consider the integral of $y(x)$ from 0 to x .

$$(i) \text{ For } x < 0, \int_0^x |x'| \, dx' = \int_0^x (-x') \, dx' = -\frac{1}{2}x^2.$$

$$(ii) \text{ For } x > 0, \int_0^x |x'| \, dx' = \int_0^x x' \, dx' = \frac{1}{2}x^2.$$

Integrating the series gives

$$\frac{\pi x}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}.$$

Equating these two results and isolating the series gives

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3} = \begin{cases} \frac{1}{2}x(\pi+x) & \text{for } x \leq 0, \\ \frac{1}{2}x(\pi-x) & \text{for } x \geq 0. \end{cases}$$

If we set $x = \pi/2$ in this result, the sine terms have values $(-1)^m$ and we obtain

$$\frac{1}{2} \frac{\pi}{2} \left(\pi - \frac{\pi}{2} \right) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{4}{\pi} S.$$

It follows that $S = \pi^3/32$.

Problem 2-10

In order to obtain a cosine series we must make an even continuation $f(t) = \cosh(t+1)$ for $-1 \leq t \leq 0$.

The constant term in the series is $a_0/2$ with

$$a_0 = 2 \int_0^1 \cosh(t-1) dt = [2 \sinh(t-1)]_0^1 = 2 \sinh(1).$$

The general coefficient is

$$\begin{aligned} a_n &= 2 \int_0^1 \cosh(t-1) \cos(n\pi t) dt \\ &= 2 \left[\frac{\cosh(t-1) \sin(n\pi t)}{n\pi} \right]_0^1 - 2 \int_0^1 \frac{\sinh(t-1) \sin(n\pi t)}{n\pi} dt \\ &= 0 - 2 \left[-\frac{\sinh(t-1) \cos(n\pi t)}{n^2 \pi^2} \right]_0^1 - 2 \int_0^1 \frac{\cosh(t-1) \cos(n\pi t)}{n^2 \pi^2} dt \end{aligned}$$

Hence,

$$a_n \left(1 + \frac{1}{n^2 \pi^2} \right) = -\frac{2 \sinh(-1)}{n^2 \pi^2} \Rightarrow a_n = \frac{2 \sinh(1)}{n^2 \pi^2 + 1}.$$

The Fourier expansion for $f(t)$ is thus

$$\cosh(t-1) = \sinh(1) \left(1 + 2 \sum_{n=1}^{\infty} \frac{\cos n\pi t}{1 + n^2 \pi^2} \right).$$

Setting $t = 0$ gives

$$\begin{aligned} \cosh(1) &= \sinh(1) + 2 \sinh(1) \sum_{n=1}^{\infty} \frac{1}{1 + n^2 \pi^2} \quad (*) \\ \sum_{n=1}^{\infty} \frac{1}{1 + n^2 \pi^2} &= \frac{\cosh(1) - \sinh(1)}{2 \sinh(1)} \\ &= \frac{e^{-1}}{e - e^{-1}} = \frac{1}{e^2 - 1}. \end{aligned}$$

Now, to separate the contributions to the series from the odd and the even integers, we need an extra factor of $(-1)^n$ in each term. We get this, in the form $\cos n\pi$, by setting $t = 1$ and so obtain

$$\cosh(0) = \sinh(1) + 2 \sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2 \pi^2} \quad (**).$$

Adding (*) and (**) gives, with an obvious notation,

$$\frac{\cosh(1) + \cosh(0)}{\sinh(1)} = 2 + 4 \Sigma_{\text{even}}$$

Re-arrangement and substitution of explicit expressions for the hyperbolic sinuoids give

$$\begin{aligned} \Sigma_{\text{even}} &= \frac{1}{4} \frac{\frac{1}{2}(e + e^{-1}) + 1 - (e - e^{-1})}{\frac{1}{2}(e - e^{-1})} \\ &= \frac{3 - e^2 + 2e}{4(e^2 - 1)} = \frac{(3 - e)(1 + e)}{4(e - 1)(e + 1)} = \frac{3 - e}{4(e - 1)}. \end{aligned}$$

It then follows that

$$\begin{aligned} \Sigma_{\text{odd}} &= \Sigma_{\text{all}} - \Sigma_{\text{even}} \\ &= \frac{1}{e^2 - 1} - \frac{3 - e}{4(e - 1)} = \frac{4 - 2e + e^2 - 3}{4(e^2 - 1)} \\ &= \frac{(1 - e)^2}{4(e^2 - 1)} = \frac{e - 1}{4(e + 1)}. \end{aligned}$$

Problem 2-11

To ensure a sine series we take $f(x) = -x^2$ for $-2 < x \leq 0$. This means that $f(-2) = -4$ and so, since $f(2) = +4$, we expect the series to converge to the average value of $\frac{1}{2}(-4 + 4) = 0$ at $x = \pm 2$.

The coefficients in the sine series $\sum b_n \sin(n\pi x/2)$ are

$$b_n = 2 \int_0^2 x^2 \sin \frac{n\pi x}{2} dx.$$

Setting $n\pi x/2 = y$ gives $b_n = 8I_n/(n\pi)^3$ with

$$\begin{aligned} I_n &= \int_0^{n\pi} y^2 \sin y \, dy \\ &= [-y^2 \cos y]_0^{n\pi} + \int_0^{n\pi} 2y \cos y \, dy \\ &= (-1)^{n+1} n^2 \pi^2 + [2y \sin y]_0^{n\pi} - \int_0^{n\pi} 2 \sin y \, dy \\ &= (-1)^{n+1} n^2 \pi^2 + 0 + [2 \cos y]_0^{n\pi} \\ &= (-1)^{n+1} n^2 \pi^2 + 2(-1)^n - 2. \end{aligned}$$

Thus,

$$\begin{aligned} b_n &= \frac{(-1)^{n+1} 8}{n\pi} - \frac{32}{n^3 \pi^3} \text{ for } n \text{ odd,} \\ &= \frac{(-1)^{n+1} 8}{n\pi} \text{ for } n \text{ even.} \end{aligned}$$

For $x = \pm 2$ all terms in the series are zero and so this is the value of the expansion at these points. This is not simply as expected, but inevitable, because, for a pure Fourier sine series, the arguments of all the sine functions are bound to be of the form $n\pi$ at the end points of the period.

Problem 2-12

odd square-wave function has only the odd harmonics present in its Fourier sine series representation. The coefficient of the $\sin(2m+1)\pi x$ term is

$$b_{2m+1} = \frac{4}{(2m+1)\pi}.$$

For a periodic function of period L whose complex Fourier coefficients are c_r , or whose cosine and sine coefficients are a_r and b_r , respectively, Parseval's theorem for one function states that

$$\begin{aligned} \frac{1}{L} \int_{x_0}^{x_0+L} |f(x)|^2 dx &= \sum_{r=-\infty}^{\infty} |c_r|^2 \\ &= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2} \sum_{r=1}^{\infty} (a_r^2 + b_r^2), \end{aligned}$$

and therefore requires in this particular case, in which all the a_r are zero and $L = 2$, that

$$\frac{1}{2} \sum_{m=0}^{\infty} \frac{16}{(2m+1)^2 \pi^2} = \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 = \frac{1}{2} \int_{-1}^1 |\pm 1|^2 dx = 1.$$

Since

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8},$$

this reduces to the identity

$$\frac{1}{2} \frac{16}{\pi^2} \frac{\pi^2}{8} = 1.$$

The power at any particular frequency in an electrical signal is proportional to the square of the amplitude at that frequency, i.e. to $|b_n|^2$ in the present case. If the filter passes only frequencies up to $8\pi/T = 4\omega$, then only the $n = 1$ and the $n = 3$ components will be passed. They contribute a fraction

$$\left(\frac{1}{1} + \frac{1}{9}\right) \div \frac{\pi^2}{8} = 0.901$$

of the total, i.e. more than 90%.

Problem 2-13

We first note that, although $\cosh x$ is an even function of x , e^{-inx} is neither even nor odd. Consequently it will not be possible to convert the integral into one over the range $0 \leq x \leq \pi$. The complex Fourier coefficients c_n ($-\infty < n < \infty$) are therefore calculated as

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh x e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (e^{-inx+x} + e^{-inx-x}) dx \\ &= \frac{1}{4\pi} \left[\frac{e^{(1-in)x}}{1-in} \right]_{-\pi}^{\pi} + \frac{1}{4\pi} \left[\frac{e^{(-1-in)x}}{-1-in} \right]_{-\pi}^{\pi} \\ &= \frac{1}{4\pi} \frac{(1+in)(-1)^n(2 \sinh \pi) - (1-in)(-1)^n(-2 \sinh \pi)}{1+n^2} \\ &= \frac{(-1)^n 4 \sinh(\pi)}{4\pi(1+n^2)}. \end{aligned}$$

Thus,

$$\cosh x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1+n^2)} e^{inx}.$$

We now set $x = 0$ on both sides of the equation:

$$\begin{aligned} 1 &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1+n^2)}, \\ \Rightarrow \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2} &= \frac{\pi}{\sinh \pi}. \end{aligned}$$

Separating out the $n = 0$ term, and noting that $(-1)^n = (-1)^{-n}$, now gives

$$1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh \pi}$$

and hence the stated result.

Assignment 3

Problem 3-1

As the function $|t|$ is not representable by the same integrable function throughout the integration range, we must divide the range into two sections and use different explicit expressions for the integrand in each:

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(1+i\omega)t} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{(1-i\omega)t} dt \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{1+\omega^2}.\end{aligned}$$

(a) Substituting this result into the inversion theorem gives

$$\exp^{-|t|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}(1+\omega^2)} e^{i\omega t} d\omega.$$

Equating the real parts on the two sides of this equation and noting that the resulting integrand is symmetric in ω , shows that

$$\exp^{-|t|} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega t}{(1+\omega^2)} d\omega,$$

as given in the question.

(b) For Parseval's theorem, which states that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega,$$

we first evaluate

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2 dt &= \int_{-\infty}^0 e^{2t} dt + \int_0^{\infty} e^{-2t} dt \\ &= 2 \int_0^{\infty} e^{-2t} dt \\ &= 2 \left[\frac{e^{-2t}}{-2} \right]_0^{\infty} = 1.\end{aligned}$$

The second integral, over ω , is

$$\begin{aligned}\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega &= 2 \int_0^{\infty} \frac{2}{\pi(1+\omega^2)^2} d\omega, \quad \text{set } \omega \text{ equal to } \tan \theta, \\ &= \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{\sec^4 \theta} \sec^2 \theta d\theta \\ &= \frac{4}{\pi} \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{4}{\pi} \frac{1}{2} \frac{\pi}{2} = 1,\end{aligned}$$

i.e. the same as the first one, thus verifying the theorem for this function.

Problem 3-2

(a) As f is periodic with period a ,

$$f(x) = f(x - ma),$$

for any integer m . However, from the general translation property of Fourier transforms,

$$\tilde{f}(k) = \mathcal{F}[f(x)] = \mathcal{F}[f(x - ma)] = e^{-imka} \tilde{f}(k).$$

Thus

$$0 = \tilde{f}(k)(1 - e^{-imka}),$$

implying, in the particular case $m = 1$, that either $\tilde{f}(k) = 0$ or $ka = 2\pi n$ where n is an integer.

(b) This result is immediate, since differentiating under the integral sign gives

$$i \frac{d\tilde{f}(\omega)}{d\omega} = \frac{i}{\sqrt{2\pi}} \frac{\partial}{\partial \omega} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t f(t) e^{-i\omega t} dt.$$

(c) From the definition of a Fourier transform,

$$\mathcal{F}[f(mt + c)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(mt + c) e^{-i\omega t} dt.$$

We make a change of integration variable by setting $mt + c = u$, with $dt = du/m$ and $-\infty < u < \infty$. This yields

$$\begin{aligned} \mathcal{F}[f(mt + c)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega(u-c)/m} \frac{du}{m} \\ &= \frac{e^{i\omega c/m}}{m} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i(\omega/m)u} du \\ &= \frac{e^{i\omega c/m}}{m} \tilde{f}\left(\frac{\omega}{m}\right), \end{aligned}$$

as stated in the question.

Problem 3-3

The Heaviside function $H(x)$ has value 0 for $x < 0$ and value 1 for $x \geq 0$. Write $H(x - a)e^{-bx} = h(x)$ with b assumed > 0 . Then,

$$\begin{aligned}\tilde{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x - a)e^{-bx} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-bx - ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-bx - ikx}}{-b - ik} \right]_a^{\infty} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-ba} e^{-ika}}{b + ik} = e^{-ika} \frac{e^{-ba}}{\sqrt{2\pi}} \frac{b - ik}{b^2 + k^2}.\end{aligned}$$

This same result could be obtained by setting $y = x - a$, finding the transform of $e^{-ba} e^{-by}$, and then using the translation property of Fourier transforms.

Problem 3-4

The function to be transformed is unity in the range $-1 \leq t \leq 1$ and so

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 1 e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right] = \frac{2 \sin \omega}{\sqrt{2\pi}\omega}.$$

Denote by $p(t)$ the convolution of f with itself and, in the second line of the calculation below, change the integration variable from s to $u = t - s$:

$$\begin{aligned} p(t) &\equiv \int_{-\infty}^{\infty} f(t-s)f(s) ds = \int_{-1}^1 f(t-s) 1 ds \\ &= \int_{t+1}^{t-1} f(u)(-du) = \int_{t-1}^{t+1} f(u)du. \end{aligned}$$

It follows that

$$p(t) = \begin{cases} (t+1) - (-1) & 0 > t > -2 \\ 1 - (t-1) & 2 > t > 0 \end{cases} = \begin{cases} 2 - |t| & 0 < |t| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

The transform of p is given directly by the convolution theorem [which states that if $h(t)$, given by $h = f * g$, is the convolution of f and g , then $\tilde{h} = \sqrt{2\pi} \tilde{f} \tilde{g}$] as

$$\tilde{p}(\omega) = \sqrt{2\pi} \frac{2 \sin \omega}{\sqrt{2\pi}\omega} \frac{2 \sin \omega}{\sqrt{2\pi}\omega} = \frac{4}{\sqrt{2\pi}} \frac{\sin^2 \omega}{\omega^2}.$$

Noting that the two integrals to be evaluated have as integrands the squares of functions that are essentially the known transforms of simple functions, we are led to apply Parseval's theorem to each. Applying the theorem to $f(t)$ and $p(t)$ yields

$$\int_{-\infty}^{\infty} \frac{4 \sin^2 \omega}{2\pi\omega^2} d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt = 2 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi,$$

$$\begin{aligned} \text{and} \quad \int_{-\infty}^{\infty} \frac{16}{2\pi} \frac{\sin^4 \omega}{\omega^4} d\omega &= \int_{-2}^0 (2+t)^2 dt + \int_0^2 (2-t)^2 dt \\ &= \left[\frac{(2+t)^3}{3} \right]_{-2}^0 - \left[\frac{(2-t)^3}{3} \right]_0^2 \\ &= \frac{8}{3} + \frac{8}{3}, \\ \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4} d\omega &= \frac{2\pi}{3}. \end{aligned}$$

Problem 3-5

(a) Write the given sine function in terms of exponential functions. Its Fourier transform is then easily calculated as

$$\begin{aligned}\tilde{f}(\omega, \gamma, p) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{(-\gamma-i\omega+ip)t} - e^{(-\gamma-i\omega-ip)t}}{2i} dt \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left(\frac{-1}{-\gamma-i\omega+ip} + \frac{1}{-\gamma-i\omega-ip} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{p}{(\gamma+i\omega)^2 + p^2}.\end{aligned}$$

(b) Since the current is given by the convolution

$$I(t) = \int_{-\infty}^\infty K(t-u)V(u) du,$$

the convolution theorem implies that the Fourier transforms of I , K and V are related by $\tilde{I}(\omega) = \sqrt{2\pi} \tilde{K}(\omega) \tilde{V}(\omega)$ with, from part (a),

$$\tilde{K}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{a_1 p_1}{(\gamma_1 + i\omega)^2 + p_1^2} + \frac{a_2 p_2}{(\gamma_2 + i\omega)^2 + p_2^2} \right].$$

Now, by expressing $I(t')$ in its Fourier integral form, we can write

$$Q(\infty) = \int_{-\infty}^\infty I(t') dt' = \int_{-\infty}^\infty dt' \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \tilde{I}(\omega) e^{i\omega t'} d\omega.$$

But $\int_{-\infty}^\infty e^{i\omega t'} dt' = 2\pi \delta(\omega)$ and so

$$\begin{aligned}Q(\infty) &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \tilde{I}(\omega) 2\pi \delta(\omega) d\omega \\ &= \frac{2\pi}{\sqrt{2\pi}} \tilde{I}(0) = \sqrt{2\pi} \sqrt{2\pi} \tilde{K}(0) \tilde{V}(0) \\ &= 2\pi \frac{1}{\sqrt{2\pi}} \left[\frac{a_1 p_1}{\gamma_1^2 + p_1^2} + \frac{a_2 p_2}{\gamma_2^2 + p_2^2} \right] \tilde{V}(0).\end{aligned}$$

For $Q(\infty)$ to be zero for an arbitrary $V(t)$, we must have

$$\frac{a_1 p_1}{\gamma_1^2 + p_1^2} + \frac{a_2 p_2}{\gamma_2^2 + p_2^2} = 0,$$

and so this is the required relationship.

Problem 3-6

We utilise the first result of the previous exercise (13.13) in the special case where $\gamma = p = a$, $f(t) = e^{-at} \sin at$ and consequently

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{a}{(a + i\omega)^2 + a^2}.$$

Applying Parseval's theorem,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega,$$

to this function and its transform:

$$\begin{aligned} \int_0^{\infty} e^{-2at} \sin^2 at dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{a}{(a + i\omega)^2 + a^2} \right] \left[\frac{a}{(a - i\omega)^2 + a^2} \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a^2}{(a^2 + \omega^2)^2 + 2a^2(a^2 - \omega^2) + a^4} d\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{a^2}{4a^4 + \omega^4} d\omega. \end{aligned}$$

Problem 3-7

By definition,

$$\begin{aligned} a(z) &= \int_{-\infty}^{\infty} \frac{1}{\lambda} e^{-\lambda t} H(t) \frac{1}{\lambda} e^{-\lambda(t+z)} H(t+z) dt \\ &= \frac{e^{-\lambda z}}{\lambda^2} \int_{z_0}^{\infty} e^{-2\lambda t} dt, \end{aligned}$$

where $z_0 = 0$ for $z > 0$ and $z_0 = |z|$ for $z < 0$; so

$$\begin{aligned} a(z) &= \frac{e^{-\lambda z}}{\lambda^2} \left[\frac{e^{-2\lambda t}}{-2\lambda} \right]_{z_0}^{\infty} \\ &= \frac{e^{-\lambda(z+2z_0)}}{2\lambda^3} = \frac{e^{-\lambda|z|}}{2\lambda^3}. \end{aligned}$$

The Fourier transform of $f(t)$ is given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\lambda} e^{-\lambda t} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}\lambda(\lambda + i\omega)}.$$

The special case of the Wiener-Kinchin theorem in which both functions are the same shows that the inverse Fourier transform of the energy spectrum, $\sqrt{2\pi}|\tilde{f}(\omega)|^2$, is equal to the auto-correlation function, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi} \frac{e^{i\omega z}}{2\pi\lambda^2(\lambda^2 + \omega^2)} d\omega = \frac{e^{-\lambda|z|}}{2\lambda^3},$$

from which the stated result follows immediately.

Problem 3-8

Setting $x^2 = st$, and hence $2x \, dx = s \, dt$ and $dx = s \, dt / (2\sqrt{st})$, we obtain

$$\int_0^\infty e^{-st} \frac{\sqrt{s}}{2} t^{-1/2} \, dt = \frac{\sqrt{\pi}}{2},$$

$$\Rightarrow \mathcal{L} \left[t^{-1/2} \right] \equiv \int_0^\infty t^{-1/2} e^{-st} \, dt = \sqrt{\frac{\pi}{s}}.$$

Integrating the LHS of this result by parts yields

$$\left[e^{-st} 2t^{1/2} \right]_0^\infty - \int_0^\infty (-s) e^{-st} 2t^{1/2} \, dt = \sqrt{\frac{\pi}{s}}.$$

The first term vanishes at both limits, whilst the second is a multiple of the required Laplace transform of $t^{1/2}$. Hence,

$$\mathcal{L} \left[t^{1/2} \right] \equiv \int_0^\infty e^{-st} t^{1/2} \, dt = \frac{1}{2s} \sqrt{\frac{\pi}{s}}.$$

Problem 3-9

To find the original functions we must express the transforms in terms of those given in table 13.1. Partial fraction expansions (chapter 1) are needed for (a) and (b).

(a) Factorising the denominator and expressing the transform as partial fractions:

$$\tilde{f}(s) = \frac{1}{s^2 - s - 2} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)},$$

and from the table of Laplace transforms and the linearity of the process of taking Laplace transforms, it follows that

$$f(t) = \frac{1}{3}(e^{2t} - e^{-t}).$$

(b) The quadratic term in the denominator cannot be factorised further without involving complex roots (and, in any case, transforms containing $(s^2 + a^2)^{-2}$ appear in the table) and so we express the transform in partial fractions as

$$\begin{aligned} \frac{2s}{(s+1)(s^2+4)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+4}, \\ 2s &= s^2(A+B) + s(B+C) + (4A+C), \\ \Rightarrow A &= -\frac{2}{5}, \quad B = \frac{2}{5}, \quad C = \frac{8}{5}. \end{aligned}$$

Thus, we may write

$$\tilde{f}(s) = -\frac{2}{5(s+1)} + \frac{2s+8}{5(s^2+4)},$$

and from the table of Laplace transforms can read off that

$$f(t) = \frac{2}{5}(-e^{-t} + \cos 2t + 4 \sin 2t).$$

(c) Apart from the factor $e^{-\gamma t_0}$ (which indicates a change of origin to $t = t_0$), $\tilde{f}(s)$ is the product of the Laplace transforms of $\delta(t - t_0)$ and $b^{-1}e^{-\gamma t} \sin bt$. Thus, by

the convolution theorem,

$$\begin{aligned} f(t) &= \frac{e^{-\gamma t_0}}{b} \int_0^t e^{-\gamma u} \sin(bu) H(u) \delta(t - u - t_0) du \\ &= \frac{e^{-\gamma t_0}}{b} e^{-\gamma(t-t_0)} \sin[b(t-t_0)] H(t-t_0) \\ &= \frac{1}{b} e^{-\gamma t} \sin[b(t-t_0)] H(t-t_0). \end{aligned}$$

Note that $f(t) = 0$ for $t < t_0$.

Problem 3-10

(a) We use the general result for Laplace transforms that

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, \quad \text{for } n = 1, 2, 3, \dots$$

If we take $n = 2$, then $f(t)$ becomes $t^{1/2}$, for which we found the Laplace transform in exercise 13.21:

$$\begin{aligned} \mathcal{L}[t^{5/2}] &= \mathcal{L}[t^2 t^{1/2}] = (-1)^2 \frac{d^2}{ds^2} \left(\frac{\sqrt{\pi} s^{-3/2}}{2} \right) \\ &= \frac{\sqrt{\pi}}{2} \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) s^{-7/2} = \frac{15\sqrt{\pi}}{8} s^{-7/2}. \end{aligned}$$

(b) Here we apply a second general result for Laplace transforms which states that

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u) du,$$

provided $\lim_{t \rightarrow 0} [f(t)/t]$ exists, which it does in this case.

$$\begin{aligned} \mathcal{L}\left[\frac{\sinh(at)}{t}\right] &= \int_s^\infty \frac{a}{u^2 - a^2} du, \quad u > |a|, \\ &= \frac{1}{2} \int_s^\infty \left(\frac{1}{u-a} - \frac{1}{u+a} \right) du \\ &= \frac{1}{2} \ln \left(\frac{s+a}{s-a} \right), \quad s > |a|. \end{aligned}$$

(c) The translation property of Laplace transforms can be used here to deal with

□

the $\sinh(at)$ factor, as it can be expressed in terms of exponential functions:

$$\begin{aligned} \mathcal{L}[\sinh(at) \cos(bt)] &= \mathcal{L}\left[\frac{1}{2} e^{at} \cos(bt)\right] - \mathcal{L}\left[\frac{1}{2} e^{-at} \cos(bt)\right] \\ &= \frac{1}{2} \frac{s-a}{(s-a)^2 + b^2} - \frac{1}{2} \frac{s+a}{(s+a)^2 + b^2} \\ &= \frac{1}{2} \frac{(s^2 - a^2)2a + 2ab^2}{[(s-a)^2 + b^2][(s+a)^2 + b^2]} \\ &= \frac{a(s^2 - a^2 + b^2)}{[(s-a)^2 + b^2][(s+a)^2 + b^2]}. \end{aligned}$$

The result is valid for $s > |a|$.

Problem 3-11 p1

From the definition of a Laplace transform,

$$\mathcal{L}[f(t-a)H(t-a)] = \int_0^\infty f(t-a)H(t-a)e^{-st} dt.$$

We start by changing the integration variable to $u = t - a$, with a corresponding change in the integration limits:

$$\begin{aligned}\mathcal{L}[f(t-a)H(t-a)] &= \int_{-a}^\infty f(u)H(u)e^{-su}e^{-sa} du \\ &= \int_0^\infty f(u)e^{-su}e^{-sa} du\end{aligned}$$

□ With $g(t)$ periodic, $g(t) = g(t-T)$ and the Laplace transform can be written as □

$$\int_0^\infty H(t)g(t)e^{-st} dt = \int_0^T H(t)g(t)e^{-st} dt + \int_T^\infty H(t)g(t)e^{-st} dt.$$

However, it follows from the properties of the Heaviside function that

$$\int_T^\infty H(t) \cdots dt = \int_0^\infty H(t-T) \cdots dt$$

and so, using the previous result, we can rewrite the above equation as

$$\begin{aligned}\bar{g}(s) &= \int_0^\infty g(t)e^{-st} dt = \int_0^T g(t)e^{-st} dt + \int_0^\infty H(t-T)g(t-T)e^{-st} dt \\ &= \int_0^T g(t)e^{-st} dt + e^{-sT}\bar{g}(s) \\ \Rightarrow \bar{g}(s) &= \frac{1}{1-e^{-sT}} \int_0^T g(t)e^{-st} dt.\end{aligned}$$

□ (a) The graph of $g(t)$ consists of a continuously repeating pattern of isosceles triangles, each of unit height and base width T . Any one of these triangles has the same shape as the function $g_a(x)$ found in Exercise 13.27, except that a has been replaced by $T/2$ and the height of the triangle is unity rather than a . Its Laplace transform is therefore □

$$\frac{1}{(T/2)} \frac{1}{s^2} \left(1 - e^{-sT/2}\right)^2.$$

From our earlier result it now follows that the Laplace transform of $g(t)$ is

$$\begin{aligned}\bar{g}(s) &= \frac{2}{Ts^2} \frac{(1 - e^{-sT/2})^2}{1 - e^{-sT}} \\ &= \frac{2}{Ts^2} \frac{1 - e^{-sT/2}}{1 + e^{-sT/2}} \\ &= \frac{2}{Ts^2} \tanh\left(\frac{sT}{4}\right).\end{aligned}$$

Problem 3-11 p2

(b) The contributions to

$$f(t) = \sum_{n=0}^{\infty} f_n(t) = \frac{2}{T} [tH(t) + 2 \sum_{n=1}^{\infty} (-1)^n (t - \frac{1}{2}nT)H(t - \frac{1}{2}nT)]$$

are shown in the sketch (figure 13.1).

For $0 \leq t \leq T/2$, only f_0 contributes; it is identical to $g(t)$ with $f(T/2) = 1$.

For $T/2 \leq t \leq T$, f_0 and f_1 contribute with *net* slope $-2/T$ and $f_1 = -f_0$ at $t = T$, making $f(T) = 0$.

For $T \leq t \leq 3T/2$, $f_2(t) + f_1(t) = f_1(T)$, since the two terms contribute equal and opposite changes as t varies; the change in $f(t)$ is entirely due to that in f_0 .

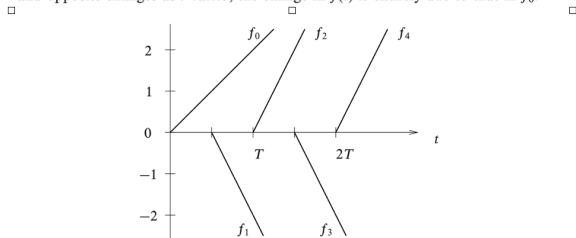


Figure 13.1 The contributions to the function considered in exercise 13.28. Their analytic forms in terms of the Heaviside function are $f_0 = 2tH(t)/T$, with slope $2/T$; $f_n = 4(-1)^n(t - \frac{1}{2}nT)H(t - \frac{1}{2}nT)/T$, with slope $4/T$.

For $3T/2 \leq t \leq 2T$, the change in $f(t)$ is due to those in f_0 and f_3 , i.e. has a net slope $-2/T$, making $f(2T) = 0$.

This sequence is then repeated in successive blocks of length $2T$.

Hence $f(t)$ is an alternative representation of $g(t)$ with

$$\tilde{f}(s) = \frac{2}{T} \left[\mathcal{L}[tH(t)] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-nTs/2} \mathcal{L}[tH(t)] \right],$$

where we have used the result from part (a). But,

$$\mathcal{L}[tH(t)] = \int_0^{\infty} t e^{-st} dt = \left[\frac{t e^{-st}}{-s} \right]_0^{\infty} - \left[\frac{e^{-st}}{s^2} \right]_0^{\infty} = \frac{1}{s^2}.$$

Thus,

$$\frac{2}{Ts^2} \tanh\left(\frac{sT}{4}\right) = \frac{2}{Ts^2} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-nTs/2} \right].$$

Finally, setting $sT = 4x$ gives

$$\tanh x = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-2nx}.$$

Assignment 4

Problem 4-1

In each case we re-arrange the equation so that all terms involving y appear on one side of an equality sign and all those involving x appear on the other. To save space we write two equations on each line.

$$\begin{aligned} \text{(a)} \quad y' &= xy^3, \quad \Rightarrow \quad \frac{dy}{y^3} = x \, dx, \\ \Rightarrow \quad -\frac{1}{2y^2} &= \frac{1}{2}x^2 + A, \quad \Rightarrow \quad y = \frac{\pm 1}{\sqrt{c - x^2}}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y' \tan^{-1} x &= \frac{y}{1+x^2}, \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{(1+x^2) \tan^{-1} x}, \\ \Rightarrow \quad \ln y &= \ln(\tan^{-1} x) + A, \quad \Rightarrow \quad y = c \tan^{-1} x. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad x^2 y' &= y^2(4-x), \quad \Rightarrow \quad \frac{dy}{y^2} = \left(\frac{4}{x^2} - \frac{1}{x} \right) dx, \\ \Rightarrow \quad -\frac{1}{y} &= -\frac{4}{x} - \ln x + c, \quad \Rightarrow \quad y = \frac{x}{4 + x \ln x - cx}. \end{aligned}$$

Problem 4-2 p1

In general, given an equation expressed in the form $A dx + B dy = 0$, we consider

the function

$$h(x, y) = \frac{1}{B} \left[\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} \right].$$

If this expression is zero, then the equation is exact and can be integrated as it stands to give a solution of the form $f(x, y) = c$. Even if $g(x, y)$ is non-zero, if it is a function of x alone then

$$\mu(x) = \exp \left\{ \int g(x) dx \right\}$$

provides an integrating factor (IF) that will make the equation exact. Similar considerations apply if $g(x, y)$ is a function of y alone. If g does actually depend on both x and y , then, in general, no further progress can be made using this method.

(a) Following the above procedure, we consider

$$h(x, y) = \frac{1}{2x^2y^3 + y} \left[\frac{\partial}{\partial y} (xy^4 + x) - \frac{\partial}{\partial x} (2x^2y^3 + y) \right] = \frac{4xy^3 - 4xy^3}{2x^2y^3 + y} = 0.$$

It follows that the equation is exact and can be integrated as it stands:

$$\begin{aligned} c = f(x, y) &= \int (2x^2y^3 + y) dy + g(x) \\ &= \frac{1}{2}x^2y^4 + \frac{1}{2}y^2 + g(x), \text{ where} \\ xy^4 + x &= \frac{\partial f}{\partial x} = xy^4 + 0 + g'(x), \Rightarrow g(x) = \frac{1}{2}x^2 + k, \\ \Rightarrow c = f(x, y) &= \frac{1}{2}(x^2y^4 + y^2 + x^2). \end{aligned}$$

The common factor of $\frac{1}{2}$ on the RHS can, of course, be absorbed into the constant on the LHS and has no particular significance.

(b) Again following the procedure, we consider

$$h(x, y) = \frac{1}{2x} \left[\frac{\partial}{\partial y} (3x + y) - \frac{\partial}{\partial x} (2x) \right] = -\frac{1}{2x}.$$

This is non-zero and implies that the equation is not exact. However, it is a function of x alone and so there is an IF given by

$$\mu(x) = \exp \left\{ \int -\frac{1}{2x} dx \right\} = \exp \left(-\frac{1}{2} \ln x \right) = \frac{1}{x^{1/2}}.$$

The exact equation is thus

$$2x^{1/2} dy + (3x^{1/2} + yx^{-1/2}) dx = 0,$$

Problem 4-2 p2

and this can now be integrated:

$$\begin{aligned}
 c = f(x, y) &= \int 2x^{1/2} dy + g(x) \\
 &= 2x^{1/2}y + g(x), \text{ where} \\
 3x^{1/2} + yx^{-1/2} &= \frac{\partial f}{\partial x} = x^{-1/2}y + g'(x), \Rightarrow g(x) = 2x^{3/2} + k, \\
 \Rightarrow c &= f(x, y) = 2(x^{1/2}y + x^{3/2}).
 \end{aligned}$$

Again, the overall numerical multiplicative factor on the RHS has no particular significance.

(c) Following the same general procedure,

$$\begin{aligned}
 h(x, y) &= \frac{1}{\cos^2 x + y \sin 2x} \left[\frac{\partial}{\partial y} (y^2) - \frac{\partial}{\partial x} (\cos^2 x + y \sin 2x) \right] \\
 &= \frac{1}{\cos^2 x + y \sin 2x} (2y + \sin 2x - 2y \cos 2x) \\
 &= \frac{4y \sin^2 x + 2 \sin x \cos x}{\cos^2 x + y \sin 2x} \\
 &= \frac{2 \sin x (2y \sin x + \cos x)}{\cos x (\cos x + 2y \sin x)} = 2 \tan x.
 \end{aligned}$$

This is non-zero and implies that the equation is not exact. However, it is a function of x alone and so there is an IF given by

$$\mu(x) = \exp \left\{ \int 2 \tan x dx \right\} = \exp(-2 \ln \cos x) = \frac{1}{\cos^2 x}.$$

The exact equation is thus

$$(1 + 2y \tan x) dy + y^2 \sec^2 x dx = 0,$$

and this can now be integrated:

$$\begin{aligned}
 c = f(x, y) &= \int (1 + 2y \tan x) dy + g(x) \\
 &= y + y^2 \tan x + g(x), \text{ where} \\
 y^2 \sec^2 x &= \frac{\partial f}{\partial x} = 0 + y^2 \sec^2 x + g'(x), \Rightarrow g(x) = k, \\
 \Rightarrow c &= f(x, y) = y + y^2 \tan x.
 \end{aligned}$$

Problem 4-3

Arrange the equation in the form

$$xy \, dy + (2x^2 + y^2 + x) \, dx = 0.$$

Now apply the standard prescription for determining whether a suitable IF exists:

$$\frac{1}{xy} \left[\frac{\partial}{\partial y} (2x^2 + y^2 + x) - \frac{\partial}{\partial x} (xy) \right] = \frac{1}{xy} (2y - y) = \frac{1}{x}.$$

This is a function of x only, thus showing that one does and that the IF needed is

$$\mu(x) = \exp \left\{ \int \frac{1}{x} \, dx \right\} = \exp(\ln x) = x.$$

The exact equation is thus

$$x^2 y \, dy + (2x^3 + xy^2 + x^2) \, dx = 0.$$

If this is to integrate to $f(x, y) = c$ then

$$\frac{\partial f}{\partial x} = 2x^3 + xy^2 + x^2 \quad \Rightarrow \quad f(x, y) = \frac{1}{2}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 + g(y).$$

The further requirement that $\partial f / \partial y = x^2 y$ shows that $g(y) = 0$ and so, on multiplying through by 6, we obtain the solution

$$3x^4 + 2x^3 + 3x^2y^2 = c.$$

Problem 4-4

We first test whether the equation is exact, or can be made so with the help of an integrating factor. To do this, we write the equation as

$$(y - x) dy + (2x + 3y) dx = 0$$

and consider

$$h_x(x, y) = \frac{1}{y - x} \left[\frac{\partial}{\partial y} (2x + 3y) - \frac{\partial}{\partial x} (y - x) \right] = \frac{4}{y - x}.$$

This is not a function of x alone. Equally

$$h_y(x, y) = \frac{1}{2x + 3y} \left[-\frac{\partial}{\partial y} (2x + 3y) + \frac{\partial}{\partial x} (y - x) \right] = \frac{-4}{2x + 3y}$$

is not a function of y alone. We conclude that there is no straightforward IF and that another method has to be tried.

We note that the equation is homogeneous in x and y and so we set $y = vx$, with $\frac{\partial y}{\partial x} = v + x \frac{\partial v}{\partial x}$, and obtain

$$\begin{aligned} v + x \frac{\partial v}{\partial x} &= -\frac{2 + 3v}{v - 1}, \\ x \frac{\partial v}{\partial x} &= \frac{-2 - 3v - v^2 + v}{v - 1} = -\frac{v^2 + 2v + 2}{v - 1}, \\ \frac{dx}{x} &= \frac{(1 - v) dv}{v^2 + 2v + 2} \\ &= \frac{2}{(v + 1)^2 + 1} - \frac{v + 1}{(v + 1)^2 + 1}, \\ \Rightarrow \ln Ax &= 2 \tan^{-1}(v + 1) - \frac{1}{2} \ln[1 + (v + 1)^2], \\ \ln \{Bx^2[1 + (v + 1)^2]\} &= 4 \tan^{-1}(v + 1). \end{aligned}$$

On setting $v = y/x$ this becomes

$$B[x^2 + (y + x)^2] = \exp \left[4 \tan^{-1} \left(\frac{y + x}{x} \right) \right],$$

the final form of the solution.

Problem 4-5

Using the stated property of derivatives with $n = 1$ and the result from the exercise, we Laplace transform the equation and obtain

$$\begin{aligned} -\frac{d}{ds}[s\bar{y}(s) - y(0)] - \frac{d\bar{y}(s)}{ds} - \bar{y}(s) &= 0, \\ -s\frac{d\bar{y}}{ds} - \bar{y} + 0 - \frac{d\bar{y}}{ds} - \bar{y} &= 0, \\ (1+s)\frac{d\bar{y}}{ds} + 2\bar{y} &= 0, \\ \frac{d\bar{y}}{\bar{y}} + \frac{2ds}{1+s} &= 0, \\ \Rightarrow \ln \bar{y} + 2\ln(1+s) &= k, \\ \Rightarrow \bar{y} &= \frac{C}{(1+s)^2}. \end{aligned}$$

As a power series, $\bar{y}(s)$ takes the form

$$\begin{aligned} \bar{y}(s) &= \frac{C}{s^2} \left(1 + \frac{1}{s}\right)^{-2} \\ &= \frac{C}{s^2} \left(1 - \frac{2}{s} + \frac{(-2)(-3)}{2!} \frac{1}{s^2} + \dots\right) \\ &= \frac{C}{s^2} + \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{s^{n+2}}. \end{aligned}$$

But, transforming the given solution,

$$y(t) = t + \sum_{m=2}^{\infty} a_m t^m,$$

yields

$$\bar{y} = \frac{1}{s^2} + \sum_{m=2}^{\infty} a_m \frac{m!}{s^{m+1}}.$$

Comparing coefficients in the two expressions for $\bar{y}$ shows that $C = 1$ and that $a_{m+1} = (-1)^m/m!$, i.e. $a_m = (-1)^{m-1}/(m-1)!$.

Direct integration of (*) by separating the variables gives

$$\begin{aligned} 0 &= \frac{dy}{y} + \left(1 - \frac{1}{t}\right) dt, \\ \Rightarrow A &= \ln y + t - \ln t, \\ \Rightarrow y &= Bte^{-t} \\ &= Bt + B \sum_{n=1}^{\infty} \frac{(-1)^n t^{n+1}}{n!} \\ &= Bt + B \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{(n-1)!}. \end{aligned}$$

With B determined by the linear term as unity, the two solutions agree.

Problem 4-6

Though this is clearly not a homogeneous equation, we test whether it might be an isobaric one by giving x a weight 1 and y a weight m and then seeing whether a suitable value for m can be found. From the presence of the term $1 - 2x^2y$ it is clear that the only possible value of m is -2 , since $2x^2y$ must have the same weight as unity, namely weight 0. For this value of m the three terms in the equation have weights

$$1 + 0 + (-2) = -1, \quad -2, \quad 2 + 2(-2).$$

These are all the same (at -2) and so the equation is isobaric.

To find its solution we set $y = vx^m = vx^{-2}$ with

$$\frac{dy}{dx} = -\frac{2v}{x^3} + \frac{1}{x^2} \frac{dv}{dx}.$$

Substituting in the original equation produces

$$\begin{aligned} x(1-2v) \left(-\frac{2v}{x^3} + \frac{1}{x^2} \frac{dv}{dx} \right) + \frac{v}{x^2} &= \frac{3x^2v^2}{x^4}, \\ (1-2v) \left(-2v + x \frac{dv}{dx} \right) + v &= 3v^2, \\ (1-2v)x \frac{dv}{dx} &= v(1-v), \\ \frac{1-2v}{v(1-v)} dv &= \frac{dx}{x}, \\ \left(\frac{1}{v} - \frac{1}{1-v} \right) dv &= \frac{dx}{x}, \\ \Rightarrow \ln v + \ln(1-v) &= \ln x + A \quad \Rightarrow \quad v(1-v) = Cx. \end{aligned}$$

Expressing this in terms of the original variables by substituting $v = yx^2$ gives $yx^2(1 - yx^2) = Cx$, with $\frac{1}{2}(1 - \frac{1}{2}) = C$. Thus, after cancelling x from both sides, the solution is

$$4yx(1 - yx^2) = 1.$$

Problem 4-7

(a) As in part (a) of the previous exercise (14.23), this equation needs an integrating factor, given in this case by

$$\mu(x) = \exp \left\{ \int -\frac{1}{x} dx \right\} = \exp(-\ln x) = \frac{1}{x}.$$

We then have

$$\frac{d}{dx} \left(\frac{y}{x} \right) = \frac{1}{x} \Rightarrow \frac{y}{x} = \ln x + A.$$

Since $y(1) = -1$, A must have the value -1 and so $y = x \ln x - x$.

(b) Again, an IF is needed; this time given by

$$\mu(x) = \exp \left\{ \int -\tan x dx \right\} = \exp(\ln \cos x) = \cos x.$$

The equation now reads

$$\frac{d}{dx} (y \cos x) = \cos x \Rightarrow y \cos x = \sin x + A.$$

The given boundary condition is that $3/\sqrt{2} = 1/\sqrt{2} + A$, establishing A as $\sqrt{2}$. The final answer is therefore $y = \tan x + \sqrt{2} \sec x$.

(c) This is not a linear equation, though it is homogeneous, and we therefore set $y = ux$. The equation then becomes

$$u + x \frac{du}{dx} = u^2 + \frac{1}{4},$$

$$x \frac{du}{dx} = \left(\frac{1}{2} - u \right)^2.$$

The equation is now separable and gives

$$\frac{du}{(\frac{1}{2} - u)^2} = \frac{dx}{x} \Rightarrow \frac{1}{\frac{1}{2} - u} = \ln x + A.$$

The boundary condition is that $u = 1/1 = 1$ when $x = 1$, implying that $A = -2$.

Now, substituting $u = y/x$ gives

$$\frac{1}{2} - \frac{y}{x} = \frac{1}{\ln x - 2} \Rightarrow y = \frac{x}{2 - \ln x} + \frac{x}{2}.$$

(d) For the boundary condition $y(1) = 1/2$ the method of solution used in part (c) fails, as it requires A to satisfy the impossible equation $1/0 = 0 + A$. We must therefore try to find a singular solution.

The first equation of the solution in part (c) would be automatically satisfied with u independent of x if

$$\frac{du}{dx} = 0 \Rightarrow u = u^2 + \frac{1}{4} \Rightarrow u = \frac{1}{2}.$$

The conclusion is that u is a constant and, since this satisfies the initial assumption that its derivative is zero, the 'circle is complete'. Hence there is a singular solution $u = \frac{1}{2}$, i.e. $y = \frac{1}{2}x$, that satisfies both the differential equation and the boundary condition.

Problem 4-8

Since we have coupled equations, working with their Laplace transforms suggests itself. This will convert the equations into simultaneous algebraic equations – though there may be some difficulty in converting the solution back into t -space.

The transform of the Heaviside function is s^{-1} , and so the two transformed equations (incorporating the initial conditions and using the translation property of Laplace transforms) are

$$\begin{aligned}s\bar{x} - 1 + 2\bar{y} &= \frac{1}{s}, \\ s\bar{y} - 0 - 2\bar{x} &= \frac{1}{s}e^{-3s}.\end{aligned}$$

Since it is $x(t)$ that we require, we eliminate $\bar{y}$ to obtain

$$s^2\bar{x} - s + \frac{2}{s}e^{-3s} + 4\bar{x} = 1,$$

from which

$$\begin{aligned}\bar{x} &= \frac{s^2 + s - 2e^{-3s}}{s(s^2 + 4)}, \\ &= \frac{s+1}{s^2+4} + \left[-\frac{1}{2s} + \frac{s}{2(s^2+4)} \right] e^{-3s}.\end{aligned}$$

For the first term in square brackets, the coefficient in the partial fractions expansion was determined by considering the limit $s \rightarrow 0$; that for the second term was found by inspection.

Now, using a look-up table if necessary, we find that, in t -space, the function corresponding to the $\bar{x}$ found above is

$$x(t) = \frac{1}{2} \sin 2t + \cos 2t - \frac{1}{2} H(t-3) + \frac{1}{2} H(t-3) \cos 2(t-3).$$

Problem 4-9

Either

By inspection, the IF for this equation is $\sin x$.

or

After dividing through by $\sin x$ this becomes a standard first-order linear equation in need of the integrating factor

$$\exp \left\{ \int \frac{2 \cos x}{\sin x} dx \right\} = \exp(2 \ln \sin x) = \sin^2 x.$$

By either method, multiplying the original equation through by $\sin x$ or the standardised one by $\sin^2 x$, the exact equation is

$$\sin^2 x \frac{dy}{dx} + 2y \cos x \sin x = \sin x,$$

$$\frac{d}{dx}(y \sin^2 x) = \sin x,$$

$$y \sin^2 x = -\cos x + A.$$

The condition $y(\pi/2) = 1$ implies that $A = 1$ and hence

$$y = \frac{1 - \cos x}{\sin^2 x} = \frac{1 - \cos x}{1 - \cos^2 x} = \frac{1}{1 + \cos x}.$$

Problem 4-10

Since x appears only in the combination $x dy/dx$ it will probably make the solution simpler to take y as the independent variable and x as the dependent one. With this in mind, we re-arrange the equation as

$$\tan y \frac{dx}{dy} + x = 2 \sin y,$$

or, in standard form, as

$$\frac{dx}{dy} + x \cot y = 2 \cos y.$$

The IF is clearly $\exp(\ln \sin y) = \sin y$, and the equation can be written

$$\begin{aligned} \frac{d}{dy}(x \sin y) &= \sin 2y, \\ x \sin y &= -\frac{1}{2} \cos 2y + k. \end{aligned}$$

(a) For $y(0) = 0$ we must have $k = \frac{1}{2}$, and the solution becomes

$$x = \frac{1 - \cos 2y}{2 \sin y} = \frac{2 \sin^2 y}{2 \sin y} = \sin y.$$

(b) If $y(0) = \pi/2$ then $k = -\frac{1}{2}$ and the solution is

$$x = \frac{-1 - \cos 2y}{2 \sin y} = \frac{-2 \cos^2 y}{2 \sin y} = -\cos y \cot y.$$

Problem 4-11

For

$$\frac{dy}{dx} - y = \cos x$$

So, $P(x) = -1$ and therefore

$$\alpha(x) = e^{\int -dx} = e^{-x}$$

Then

$$y(x) = \frac{\int e^{-x} \cos x dx + C_0}{e^{-x}}$$

Now, integral table says

$$\int e^{-x} \cos x dx = \frac{1}{2} e^{-x} (\sin x - \cos x)$$

Therefore

$$y(x) = \frac{1}{2} (\sin x - \cos x) + C_0 e^x$$

Assignment 5

Problem 5-1

The auxiliary equation and resulting CF are

$$\begin{aligned}m^2 + 6m + 9 = 0 &\Rightarrow m = -3 \text{ (repeated root),} \\&\Rightarrow f(t) = (A + Bt)e^{-3t}.\end{aligned}$$

For a particular integral, since e^{-t} does not appear in the CF, we try $f = Ce^{-t}$ and obtain

$$Ce^{-t} - 6Ce^{-t} + 9Ce^{-t} = e^{-t} \Rightarrow C = \frac{1}{4}.$$

The general solution is therefore

$$f(t) = (A + Bt)e^{-3t} + \frac{1}{4}e^{-t}.$$

We now incorporate the boundary conditions:

$$\begin{aligned}f(0) = 0 &\Rightarrow 0 = (A + 0)1 + \frac{1}{4} \Rightarrow A = -\frac{1}{4}, \\f'(0) = \lambda &\Rightarrow \lambda = -3A + B - 0 - \frac{1}{4} \Rightarrow B = \lambda - \frac{1}{2}.\end{aligned}$$

Therefore the solution matching these boundary conditions is

$$f(t) = [-\frac{1}{4} + (\lambda - \frac{1}{2})t]e^{-3t} + \frac{1}{4}e^{-t}.$$

The turning points of the solution are given by $f'(t) = 0$, i.e.

$$\begin{aligned}\frac{3}{4}e^{-3t} + (\lambda - \frac{1}{2})e^{-3t} - 3(\lambda - \frac{1}{2})te^{-3t} - \frac{1}{4}e^{-t} &= 0, \\e^{-2t} [(\lambda + \frac{1}{4}) + (\frac{3}{2} - 3\lambda)t] &= \frac{1}{4}, \\(4\lambda + 1) + (6 - 12\lambda)t &= e^{2t}.\end{aligned}$$

(a) For $\lambda = \frac{1}{4}$ the equation becomes $e^{2t} = 2 + 3t$. Consider the behaviours of the functions on either side of this equation:

$$\begin{aligned}\text{At } t = 0, \quad 2 + 3t &= 2 > 1 = e^{2t}. \\ \text{For large } t, \quad e^{2t} &> 2 + 3t.\end{aligned}$$

Both functions are monotonic and thus there is one, and only one, solution to $e^{2t} = 2 + 3t$ in $t > 0$. It follows that the solution to the original differential equation has only one turning point in this range.

(b) For $\lambda = -\frac{1}{4}$ the equation becomes $e^{2t} = 9t$. Again consider the behaviours of the two sides of the equation.

$$\begin{aligned}\text{At } t = 0, \quad e^{2t} &= 1 > 0 = 9t. \\ \text{At } t = 1, \quad e^{2t} &= e^2 < 9 = 9t. \\ \text{For large } t, \quad e^{2t} &> 9t.\end{aligned}$$

Both functions are monotonic and thus there are two solutions to $e^{2t} = 9t$ in $t > 0$. It follows that the solution to the original differential equation has two turning points in this range.

Problem 5-2

The auxiliary equation, $m^2 + 2m + 1 = 0$, has repeated roots $m = -1$, and so the general CF has the special form $y(x) = (A + Bx)e^{-x}$.

Turning to the PI, we note that the form of the RHS of the original equation is contained in the CF, and (to make matters worse) so is x times the RHS. We therefore need to take x^2 times the RHS as a trial PI:

$$y(x) = Cx^2e^{-x}, \quad y' = C(2x - x^2)e^{-x}, \quad y'' = C(2 - 4x + x^2)e^{-x}.$$

Substituting these into the original equation shows that

$$2Ce^{-x} = 4e^{-x} \Rightarrow C = 2$$

and that the full general solution is given by

$$y(x) = (A + Bx)e^{-x} + 2x^2e^{-x}.$$

We now determine the unknown constants using the information given about the solution. Since $y(0) = 1$, $A = 1$. Further, $y(1) = e^{-1}$ requires

$$e^{-1} = (1 + B)e^{-1} + 2e^{-1} \Rightarrow B = -2.$$

Finally, we conclude that $y(x) = (1 - 2x + 2x^2)e^{-x}$ and, therefore, that $y(2) = 5e^{-2}$.

Problem 5-3

(a) As this is a third-order equation, we expect three terms in the CF.

Since it is linear with constant coefficients, we can make use of the auxiliary equation, which is

$$m^3 - 12m + 16 = 0.$$

By inspection, $m = 2$ is one root; the other two can be found by factorisation:

$$m^3 - 12m + 16 = (m - 2)(m^2 + 2m - 8) = (m - 2)(m + 4)(m - 2) = 0.$$

Thus we have one repeated root ($m = 2$) and one other ($m = -4$) leading to a CF of the form

$$y(x) = (A + Bx)e^{2x} + Ce^{-4x}.$$

As the RHS contains no exponentials, we try $y(x) = Dx + E$ for the PI. We then need $16D = 32$ and $-12D + 16E = -8$, giving $D = 2$ and $E = 1$.

The general solution is therefore

$$y(x) = (A + Bx)e^{2x} + Ce^{-4x} + 2x + 1.$$

(b) The equation is already arranged in the form

$$\frac{dg(y)}{dx} + h(x)g(y) = j(x)$$

and so needs only an integrating factor to allow the first integration step to be made. For this equation the IF is

$$\exp \left\{ \int 2a \coth 2ax \, dx \right\} = \exp(\ln \sinh 2ax) = \sinh 2ax.$$

After multiplication through by this factor, the equation can be written

$$\begin{aligned} \sinh 2ax \frac{d}{dx} \left(\frac{1}{y} \frac{dy}{dx} \right) + (2a \cosh 2ax) \left(\frac{1}{y} \frac{dy}{dx} \right) &= 2a^2 \sinh 2ax, \\ \frac{d}{dx} \left(\sinh 2ax \frac{1}{y} \frac{dy}{dx} \right) &= 2a^2 \sinh 2ax. \end{aligned}$$

Integrating this gives

$$\begin{aligned} \sinh 2ax \frac{1}{y} \frac{dy}{dx} &= \frac{2a^2}{2a} \cosh 2ax + A, \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= a \coth 2ax + \frac{A}{\sinh 2ax}. \end{aligned}$$

$$\begin{aligned} \text{Integrating again, } \ln y &= \frac{1}{2} \ln(\sinh 2ax) + \int \frac{A}{\sinh 2ax} \, dx + B \\ &= \frac{1}{2} \ln(\sinh 2ax) + \frac{A}{2a} \ln(|\tanh ax|) + B, \\ \Rightarrow y &= C(\sinh 2ax)^{1/2} (|\tanh ax|)^B. \end{aligned}$$

The indefinite integral of $(\sinh 2ax)^{-1}$ appearing in the fourth line can be verified by differentiating $y = \ln |\tanh ax|$ in the form $y = \frac{1}{2} \ln(\tanh^2 ax)$ and recalling that

$$\cosh ax \sinh ax = \frac{1}{2} \sinh 2ax.$$

Problem 5-4

(a) Recalling that $\mathcal{L}[f'] = s\bar{f} - f(0)$ and $\mathcal{L}[f''] = s^2\bar{f} - sf(0) - f'(0)$, we have

$$\begin{aligned} 0 &= s^2\bar{f} - sf(0) - f'(0) + 5[s\bar{f} - f(0)] + 6\bar{f} \\ &= s^2\bar{f} - s + 4 + 5(s\bar{f} - 1) + 6\bar{f}, \\ \Rightarrow \quad \bar{f} &= \frac{s+1}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}, \\ &= \frac{-1}{s+2} + \frac{2}{s+3}, \text{ using one of the standard methods,} \\ \Rightarrow \quad f(t) &= 2e^{-3t} - e^{-2t}, \text{ from the look-up table.} \end{aligned}$$

(b) Using the same method as in (a), we have

$$\begin{aligned} 0 &= s^2\bar{f} - sf(0) - f'(0) + 2[s\bar{f} - f(0)] + 5\bar{f} \\ &= s^2\bar{f} - s + 2(s\bar{f} - 1) + 5\bar{f}, \\ \Rightarrow \quad \bar{f} &= \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+4} \\ &= \frac{(s+1)+1}{(s+1)^2+4}, \\ \Rightarrow \quad f(t) &= e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t, \text{ from the look-up table.} \end{aligned}$$

Problem 5-5

(a) The CF is clearly

$$y(x) = Ae^x + Be^{-x},$$

and so we take as the PI

$$y(x) = k_1(x)e^x + k_2(x)e^{-x}.$$

The two simultaneous equations generated using the method of variation of parameters are

$$\begin{aligned} k_1'e^x + k_2'e^{-x} &= 0, \\ k_1'e^x - k_2'e^{-x} &= x^n. \end{aligned}$$

Solving for k_1' and integrating gives

$$\begin{aligned} k_1' &= -\frac{x^n e^{-x}}{2}, \\ k_1 &= \left[-\frac{x^n e^{-x}}{2} \right]^x + \int^x \frac{nx^{n-1}}{2} e^{-x} dx \\ &= -\frac{e^{-x}}{2} (x^n + nx^{n-1} + n(n-1)x^{n-2} + \dots + n!) \end{aligned}$$

Similarly, k_2 is given by

$$\begin{aligned} k_2' &= \frac{x^n e^x}{2}, \\ k_2 &= \left[\frac{x^n e^x}{2} \right]^x - \int^x \frac{nx^{n-1}}{2} e^x dx \\ &= \frac{e^x}{2} (x^n - nx^{n-1} + n(n-1)x^{n-2} - \dots + (-1)^n n!) \\ &= \frac{e^x}{2} n! (-1)^n \sum_{m=0}^n \frac{(-x)^m}{m!}. \end{aligned}$$

The full PI, $k_1(x)e^x + k_2(x)e^{-x}$, has no explicit exponential factors, since each term in it contains the product $e^x e^{-x}$. It takes the form

$$\begin{aligned} y(x) &= -\frac{n!}{2} \sum_{m=0}^n \frac{x^m}{m!} - \frac{n!}{2} (-1)^n \sum_{m=0}^n \frac{(-x)^m}{m!} \\ &= -\frac{n!}{2} \sum_{m=0}^n \frac{x^m}{m!} [1 + (-1)^{n+m}]. \end{aligned}$$

This n -th order polynomial is added to the CF, $y(x) = Ae^x + Be^{-x}$, to give the general solution.

(b) The auxiliary equation for the CF is $m^2 - 2m + 1 = 0$, which has repeated roots $m = 1$. Thus the CF is $y(x) = (A + Bx)e^x$ and, since the RHS of the original equation is contained in this, the PI is to be taken as $y(x) = k_1(x)e^x + k_2(x)xe^x$. The simultaneous equations generated by the variation of parameters method are

$$\begin{aligned} k_1'e^x + k_2'xe^x &= 0, \\ k_1'e^x + k_2'(e^x + xe^x) &= 2xe^x, \\ k_2'e^x &= 2xe^x \Rightarrow k_2(x) = x^2, \\ k_1' &= -k_2'x = -2x^2 \Rightarrow k_1(x) = -\frac{2}{3}x^3. \end{aligned}$$

A PI is therefore

$$y(x) = -\frac{2}{3}x^3 e^x + x^2 x e^x = \frac{1}{3}x^3 e^x,$$

giving the general solution as

$$y(x) = (A + Bx + \frac{1}{3}x^3)e^x.$$

Problem 5-6

The only realistic hope for this non-linear equation is to try to arrange it as an exact equation! We note that the second and fourth terms can be written as the derivative of a product, and that adding and subtracting $2y'y''$ will enable the first term to be written in a similar way. We therefore rewrite the equation as

$$\begin{aligned}\frac{d}{dx} \left(2y \frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left(2y \frac{dy}{dx} \right) + (6-2) \frac{dy}{dx} \frac{d^2 y}{dx^2} &= \sin x, \\ \frac{d}{dx} \left(2y \frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left(2y \frac{dy}{dx} \right) + \frac{d}{dx} \left[2 \left(\frac{dy}{dx} \right)^2 \right] &= \sin x.\end{aligned}$$

This second form is obtained by noting that the final term on the LHS of the first equation happens to be an exact differential. Thus the whole of the LHS is an exact differential and one stage of integration can be carried out:

$$2y \frac{d^2 y}{dx^2} + 2y \frac{dy}{dx} + 2 \left(\frac{dy}{dx} \right)^2 = -\cos x + A.$$

We now note that the first and third terms of this integrated equation can be combined as the derivative of a product, whilst the second term is the derivative

of y^2 . This allows a further step of integration:

$$\begin{aligned}\frac{d}{dx} \left(2y \frac{dy}{dx} \right) + 2y \frac{dy}{dx} &= -\cos x + A, \\ \frac{d}{dx} \left(2y \frac{dy}{dx} \right) + \frac{d(y^2)}{dx} &= -\cos x + A, \\ \Rightarrow 2y \frac{dy}{dx} + y^2 &= -\sin x + Ax + B, \\ \frac{d(y^2)}{dx} + y^2 &= -\sin x + Ax + B.\end{aligned}$$

At this stage an integrating factor is needed. However, as the LHS consists of the sum of the differentiated and undifferentiated forms of the same function, the required IF is simply e^x . After multiplying through by this, we obtain

$$\begin{aligned}\frac{d}{dx} (e^x y^2) &= -e^x \sin x + A x e^x + B e^x, \\ \Rightarrow y^2 &= e^{-x} \left[C + \int^x (B + Au - \sin u) e^u du \right] \\ &= C e^{-x} + B + A(x-1) - \frac{1}{2}(\sin x - \cos x).\end{aligned}$$

The last term in this final solution is obtained by considering

$$\begin{aligned}\int^x e^u \sin u du &= \text{Im} \int^x e^{(1+i)u} du \\ &= \text{Im} \left[\frac{e^{(1+i)u}}{1+i} \right]^x \\ &= \text{Im} \left[\frac{1}{2}(1-i)e^{(1+i)x} \right] \\ &= \frac{1}{2}e^x(\sin x - \cos x).\end{aligned}$$

Problem 5-7

Consider the equation

$$x^2 y'' - x(1-x)y' + (1-x)y = 0$$

or

$$y'' - \frac{1-x}{x} y' + \frac{1-x}{x^2} y = 0 = y'' + p_0(x)y' + p_1(x)y \quad (\text{A})$$

One solution is $y_1(x) = x$. To find the second solution we use the wronskian method as shown below. We know there exists a second solution $y_2(x)$.

The Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is defined as

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Now, in general, we have

$$\begin{aligned} W'(x) &= y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1'' \\ &= y_1(-p_0 y_2' - p_1 y_2) - y_2(-p_0 y_1' - p_1 y_1) \\ &= -p_0 W(x) \end{aligned}$$

Hence

$$W(x) = K e^{-\int [p_0(x)] dx}$$

Setting the two expression for $W(x)$ equal to each other we get

$$\frac{y_2(x)}{y_1(x)} = K \int \frac{1}{y_1^2(x)} e^{-\int [p_0(x)/x] dx} dx$$

so that we can obtain a second solution if we know the first solution. We get

$$y_2 = y_1 \int \frac{1}{y_1^2(x)} e^{\int \frac{1-x}{x} dx} dx$$

or

$$y_2 = x \int \frac{1}{x^2} e^{\int \frac{1-x}{x} dx} dx$$

$$y_2 = x \int \frac{1}{x^2} e^{\ln x - x} dx$$

$$y_2 = x \int \frac{1}{x} e^{-x} dx = x(\ln x + \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!})$$

Problem 5-8 p1

We have the equation

$$y'' + 6y' + 5y = x^2 + 2\sin x$$

We first solve the homogeneous equation

$$y'' + 6y' + 5y = 0$$

We assume a solution of the form

$$y = \sum_{n=0}^{\infty} b_n x^n$$

Substitution gives

$$\sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} + 6 \sum_{n=1}^{\infty} n b_n x^{n-1} + 5 \sum_{n=0}^{\infty} b_n x^n = 0$$

Therefore

$$\sum_{n=0}^{\infty} [(n+2)(n+1)b_{n+2} + 6(n+1)b_{n+1} + 5b_n]x^n = 0$$

Therefore

$$(2b_2 + 6b_1 + 5b_0) + (6b_3 + 12b_2 + 5b_1)x + (12b_4 + 18b_3 + 4b_2)x^2 + (20b_5 + 24b_4 + 5b_3)x^3 + \dots = 0$$

Hence

$$b_2 = -3b_1 - \frac{5}{2}b_0, \quad b_3 = \frac{31}{6}b_1 + 5b_0, \quad b_4 = -\frac{13}{2}b_1 - \frac{155}{24}b_0, \quad \dots$$

Therefore

$$u_1(x) = b_0 \left[1 - \frac{5}{2}x^2 + 5x^3 - \frac{155}{24}x^4 + \dots \right]$$

and

$$u_2(x) = b_1 \left[x - 3x^2 + \frac{31}{6}x^3 - \frac{13}{2}x^4 + \dots \right]$$

Then we use the full equation to find a particular solution. Clever method(always works)....

$$y'' + 6y' + 5y = x^2 + 2\sin x \quad (\text{A})$$

We assume that

$$u_p(0) = u_p^{(0)}(0) = 0, \quad u_p'(0) = u_p^{(1)}(0) = 0$$

Problem 5-8 p2

Then we have from (A)

$$u^{(2)}(x) + 6u^{(1)}(x) + 5u^{(0)}(x) = x^2 + 2\sin x \rightarrow u^{(2)}(0) + 6u^{(1)}(0) + 5u^{(0)}(0) = 0 \rightarrow u^{(2)}(0) = 0$$

Continuing we have

$$u^{(3)}(x) + 6u^{(2)}(x) + 5u_p^{(1)}(x) = 2x + 2\cos x \rightarrow u^{(3)}(0) + 6u^{(2)}(0) + 5u_p^{(1)}(0) = 2 \rightarrow u^{(3)}(0) = 2$$

$$u^{(4)}(x) + 6u^{(3)}(x) + 5u_p^{(2)}(x) = 2 - 2\sin x \rightarrow u^{(4)}(0) + 6u^{(3)}(0) + 5u_p^{(2)}(0) = 2 \rightarrow u^{(4)}(0) = -10$$

$$u^{(5)}(x) + 6u^{(4)}(x) + 5u_p^{(3)}(x) = -2\cos x \rightarrow u^{(5)}(0) + 6u^{(4)}(0) + 5u_p^{(3)}(0) = 2 \rightarrow u^{(5)}(0) = -48$$

Therefore using the standard form of a power series we have

$$u_p(x) = u_p^{(0)}(0) + u_p^{(1)}(0)\frac{x}{1!} + u_p^{(2)}(0)\frac{x^2}{2!} + u_p^{(3)}(0)\frac{x^3}{3!} + u_p^{(4)}(0)\frac{x^4}{4!} + u_p^{(5)}(0)\frac{x^5}{5!} + \dots$$

or

$$u_p(x) = 2\frac{x^3}{3!} - 10\frac{x^4}{4!} + 48\frac{x^5}{5!} + \dots = \frac{x^3}{3} - \frac{5}{12}x^4 + \frac{2}{5}x^5 + \dots$$

General solution is

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x)$$

Problem 5-9

For the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t \quad y(0) = 0, y'(0) = 0$$

using the Laplace transform method gives

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{2}{s^2}$$

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{2}{s^2}$$

Therefore

$$Y(s) = \frac{2}{s^2(s^2 + 2s + 1)} = \frac{2}{s^2(s+1)^2}$$

or (using partial fractions)

$$Y(s) = \frac{-4}{s} + \frac{2}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}$$

so that

$$y(t) = -4 + 2t + 2(1+t)e^{-t}$$

Problem 5-10

We have the equation

$$u'' + 9u = x^2 + \sin 2x$$

The solution is $u(x) = u_h(x) + u_p(x)$ where $u_h(x)$ solves

$$u_h'' + 9u_h = 0$$

and $u_p(x)$ is any solution of full equation. Now exponential substitution $u_h(x) = e^{\alpha x}$ gives

$$\alpha^2 + 9 = 0 \rightarrow \alpha = \pm 3i$$

so that the homogeneous solution is

$$u_h(x) = (C_1 \cos 3x + C_2 \sin 3x)$$

and we let $u_p(x) = Ax^2 + Bx + C + D \sin 2x$ so that we get by substitution

$$2A - 4D \sin 2x + 9Ax^2 + 9Bx + 9C + 9D \sin 2x = x^2 + \sin 2x$$

or

$$2A + 9C = 0; 9A = 1; -4D + 9D = 1; 9B = 0 \rightarrow A = 1/9, B = 0, C = -2/81, D = 1/5$$

Therefore

$$u(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{81}(9x^2 - 2) + \frac{1}{5} \sin 2x$$

Problem 5-11

We have the equation

$$u'' + 6u' + 10u = \cos 2x \quad u(0) = u'(0) = 0$$

The solution is $u(x) = u_h(x) + u_p(x)$ where $u_h(x)$ solves

$$u_h'' + 6u_h' + 10u_h = 0$$

and $u_p(x)$ is any solution of full equation. Now exponential substitution $u_h(x) = e^{\alpha x}$ gives

$$\alpha^2 + 6\alpha + 10 = 0 \rightarrow \alpha = -3 \pm i$$

so that the homogeneous solution is

$$u_h(x) = e^{-3x}(C_1 \cos x + C_2 \sin x)$$

and we let $u_p(x) = A \cos 2x + B \sin 2x$ so that we get by substitution

$$-4A \cos 2x - 4B \sin 2x - 12A \sin 2x + 12B \cos 2x + 10A \cos 2x + 10B \sin 2x = \cos 2x$$

or

$$-4A + 12B + 10A = 1 \quad \text{and} \quad -4B - 12A + 10B = 0 \rightarrow A = 1/30, B = 1/15$$

Therefore

$$u(x) = e^{-3x}(C_1 \cos x + C_2 \sin x) - \frac{1}{30}(\cos 2x + 2 \sin 2x)$$

This gives

$$u(0) = 0 = C_1 + 1/30 \quad \text{and} \quad u'(0) = 0 = -3C_1 + C_2 + 4/30 \rightarrow C_1 = -1/30, C_2 = -7/30$$

Thus,

$$u(x) = -\frac{1}{30} [e^{-3x}(\cos x + 7 \sin x) - \cos 2x - 2 \sin 2x]$$

Problem 5-12

We have the equation

$$\ddot{y} + 2\dot{y} + 10y = 26 \sin 2t$$

The solution is $y(xt) = y_h(t) + uy_p t$ where $uy_h(t)$ solves

$$\ddot{y}_p + 2\dot{y}_p + 10y_p = 0$$

and $y_p(t)$ is any solution of full equation. Now exponential substitution $y_h(t) = e^{\alpha t}$ gives

$$\alpha^2 + 2\alpha + 10 = 0 \rightarrow \alpha = -1 \pm 3i$$

so that the homogeneous solution is

$$y_h(t) = e^{-t}(C_1 \cos 3t + C_2 \sin 3t)$$

and we let $y_p(t) = A \cos 2t + B \sin 2t$ so that we get by substitution

$$-4A + -4B + 10A = 26 \quad \text{and} \quad -4B + 4A + 10B = 0 \rightarrow A = 3, B = -2$$

Therefore

$$y(t) = e^{-t}(C_1 \cos 3t + C_2 \sin 3t) + 3 \sin 2t - 2 \cos 2t$$

The initial conditions then give

$$y(0) = 1 = C_1 + 2 \rightarrow C_1 = 3; \dot{y}(0) = 0 = 3C_2 - 3C_1 + 6 \rightarrow C_2 = 1$$

Thus,

$$y(t) = e^{-t}(C_3 \cos 3t + \sin 3t) + 3 \sin 2t - 2 \cos 2t$$

Problem 5-13 p1

We have the equation

$$x^2 u'' + xu' - u = 9$$

We use the "variation of parameters" method involving the Wronskian. Let me review the method. Consider the equation

$$u'' + P_1(x)u' + P_1(x)u = g(x) \quad (\text{A})$$

A general solution is found by adding a particular solution $u_p(x)$ to a general solution of the homogeneous equation, to obtain

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x)$$

where $u_1(x)$ and $u_2(x)$ are solutions to the homogeneous equation.

$$u'' + P_1(x)u' + P_1(x)u = 0$$

To find a particular solution, assume that the solution has the form

$$u_p(x) = v_1(x)u_1(x) + v_2(x)u_2(x)$$

If we differentiate we get

$$u'_p = v_1 u'_1 + v_2 u'_2 + u_1 v'_1 + u_2 v'_2$$

We seek a solution such that

$$u_1 v'_1 + u_2 v'_2 = 0 \quad (\text{B})$$

We are free to impose this one restriction on $v_1(x)$ and $v_2(x)$ without loss of generality as the following analysis shows. We have

$$u'_p = v_1 u'_1 + v_2 u'_2$$

Differentiating we get

$$u''_p = v_1 u''_1 + v_2 u''_2 + v'_1 u'_1 + v'_2 u'_2$$

Substituting into (A) we get

$$v_1 (u''_1 + P_0 u'_1 + P_1 u_1) + v_2 (u''_2 + P_0 u'_2 + P_1 u_2) + v'_1 u'_1 + v'_2 u'_2 = g(x) \quad (1)$$

Problem 5-13 p2

The quantities in parentheses are both zero since u_1 and u_2 are solutions of the homogeneous equation. Hence,

$$v'_1 u'_1 + v'_2 u'_2 = g(x) \quad (C)$$

(B) and (C) are now solved simultaneously to find

$$v'_1 = -\frac{u_2 g(x)}{u_1 u'_2 - u_2 u'_1}, \quad v'_2 = \frac{u_1 g(x)}{u_1 u'_2 - u_2 u'_1} \quad (D)$$

The quantity in the denominator is the Wronskian W of $u_1(x)$ and $u_2(x)$,

$$W(x) = u_1 u'_2 - u_2 u'_1$$

We now integrate (D) and obtain

$$v_1(x) = -\int \frac{u_2 g}{W} dx, \quad v_2(x) = \int \frac{u_1 g}{W} dx$$

A particular solution is then

$$u_p(x) = -u_1 \int \frac{u_2 g}{W} dx + u_2 \int \frac{u_1 g}{W} dx \quad (E)$$

and adding the homogeneous solution gives a general solution.

Now returning to the problem, this equation has basic solutions

$$u_1 = x, u_2 = 1/x$$

Therefore the Wronskian is given by

$$W = \begin{vmatrix} u_1 & u_2 \\ \frac{du_1}{dx} & \frac{du_2}{dx} \end{vmatrix} = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x}$$

We get $g(x)$ from (A)

$$x^2 u'' + x u' - u = 9 \rightarrow u'' + \frac{1}{x} u' - \frac{1}{x^2} u = \frac{9}{x^2} = g(x)$$

Hence, using (E) we have

$$u_p(x) = -x \int \frac{\frac{1}{x} \frac{9}{x^2}}{-\frac{2}{x}} dx + \frac{1}{x} \int \frac{x \frac{9}{x^2}}{-\frac{2}{x}} dx = -\frac{9}{2} - \frac{9}{2} = -9$$

Finally,

$$u(x) = c_1 x + \frac{c_2}{x} - 9$$

Assignment 6

Problem 6-1

If the equation is imagined divided through by $(1 - z^2)$ it is straightforward to see that, although $z = \pm 1$ are singular points of the equation, the point $z = 0$ is an *ordinary* point. We therefore expect two (uncomplicated!) series solutions with indicial values $\sigma = 0$ and $\sigma = 1$.

(a) $\sigma = 0$ and $y(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 \neq 0$.

Substituting and equating the coefficients of z^m ,

$$(1 - z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 3 \sum_{n=0}^{\infty} n a_n z^n + \lambda \sum_{n=0}^{\infty} a_n z^n = 0,$$

$$(m+2)(m+1) a_{m+2} - m(m-1) a_m - 3m a_m + \lambda a_m = 0,$$

gives as the recurrence relation

$$a_{m+2} = \frac{m(m-1) + 3m - \lambda}{(m+2)(m+1)} a_m = \frac{m(m+2) - \lambda}{(m+1)(m+2)} a_m.$$

Since this recurrence relation connects alternate coefficients a_m , and $a_0 \neq 0$, only the coefficients with even indices are generated. All such coefficients with index higher than m will become zero, and the series will become an N th-degree polynomial $U_N(z)$, if $\lambda = m(m+2) = N(N+2)$ for some (even) m appearing in the series; here, this means any positive even integer N .

□

To construct $U_2(z)$ we need to take $\lambda = 2(2+2) = 8$. The recurrence relation gives a_2 as

$$a_2 = \frac{0-8}{(0+1)(0+2)} a_0 = -4a_0 \Rightarrow U_2(z) = a_0(1 - 4z^2).$$

(b) $\sigma = 1$ and $y(z) = z \sum_{n=0}^{\infty} a_n z^n$ with $a_0 \neq 0$.

Substituting and equating the coefficients of z^{m+1} ,

$$(1 - z^2) \sum_{n=0}^{\infty} (n+1) n a_n z^{n-1} - 3 \sum_{n=0}^{\infty} (n+1) a_n z^{n+1} + \lambda \sum_{n=0}^{\infty} a_n z^{n+1} = 0,$$

$$(m+3)(m+2) a_{m+2} - (m+1) m a_m - 3(m+1) a_m + \lambda a_m = 0,$$

gives as the recurrence relation

□

$$a_{m+2} = \frac{m(m+1) + 3(m+1) - \lambda}{(m+2)(m+3)} a_m = \frac{(m+1)(m+3) - \lambda}{(m+2)(m+3)} a_m.$$

Again, all coefficients with index higher than m will become zero, and the series will become an N th-degree polynomial $U_N(z)$, if $\lambda = (m+1)(m+3) = N(N+2)$ for some (even) m appearing in the series; here, this means any positive odd integer N .

To construct $U_3(z)$ we need to take $\lambda = 3(3+2) = 15$. The recurrence relation gives a_2 as

$$a_2 = \frac{3-15}{(0+2)(0+3)} a_0 = -2a_0.$$

Thus,

$$U_3(z) = a_0(z - 2z^3).$$

Problem 6-2

Putting the equation in its standard form shows that $z = 0$ is a singular point of the equation but, as $2z(1-z)/4z$ and $-z^2/4z$ are finite as $z \rightarrow 0$, it is a regular singular point. We therefore substitute a Frobenius type solution,

$$y(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n \text{ with } a_0 \neq 0,$$

and obtain

$$4 \sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma-1} + 2(1-z) \sum_{n=0}^{\infty} (n+\sigma) a_n z^{n+\sigma-1} - \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero gives the indicial equation as

$$4\sigma(\sigma-1)a_0 + 2\sigma a_0 = 0 \Rightarrow \sigma = 0, \frac{1}{2}.$$

These two indicial values do not differ by an integer and so we expect both to yield (independent) series solutions.

(a) $\sigma = 0$.

Equating the general coefficient of z^m to zero (with $\sigma = 0$),

$$4(m+1)ma_{m+1} + 2(m+1)a_{m+1} - 2ma_m - a_m = 0,$$

gives the recurrence relation as

$$\begin{aligned} a_{m+1} &= \frac{2m+1}{2(m+1)(2m+1)} = \frac{a_m}{2(m+1)}, \\ \Rightarrow a_m &= \frac{a_0}{2^m m!}, \\ \Rightarrow y(x) &= a_0 z^0 \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} = a_0 e^{z/2}. \end{aligned}$$

Substituting this into the original equation yields

$$4a_0 z \frac{e^{z/2}}{4} + 2a_0(1-z) \frac{e^{z/2}}{2} - a_0 e^{z/2} = 0,$$

which is a valid equation and thus verifies the solution.

(b) $\sigma = \frac{1}{2}$

Equating the general coefficient of $z^{m+\frac{1}{2}}$ to zero (with $\sigma = \frac{1}{2}$),

$$4(m+\frac{3}{2})(m+\frac{1}{2})a_{m+1} + 2(m+\frac{3}{2})a_{m+1} - 2(m+\frac{1}{2})a_m - a_m = 0,$$

gives the recurrence relation as

$$\begin{aligned} a_{m+1} &= \frac{2m+2}{(2m+3)(2m+2)} a_m = \frac{a_m}{2m+3}, \\ \Rightarrow a_m &= \frac{a_0}{1 \cdot 3 \cdot 5 \cdots (2m+1)} = \frac{2^m m! a_0}{(2m+1)!}, \end{aligned}$$

and the second series solution as

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{2^n n! z^{n+\frac{1}{2}}}{(2n+1)!}.$$

Problem 6-3 p1

We have the equation

$$36x^2y'' + (5 - 9x^2)y = 0$$

We use the method of Frobenius. We assume a solution of the form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s} \quad (\text{A})$$

This gives

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \quad (\text{B})$$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} \quad (\text{C})$$

We substitute (B) and (C) into (A) to obtain

$$36 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + 5 \sum_{n=0}^{\infty} a_n x^{n+s} - 9 \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$$

Expanding out we have

$$(36s(s-1) + 5)a_0 x^s + (36s(s+1) + 5)a_1 x^{s+1} + [(36(s+2)(s+1) + 5)a_2 - 9a_0]x^{s+2} \\ + \dots + [(36(n+s)(n+s-1) + 5)a_n - 9a_{n-2}]x^{n+s} + \dots = 0$$

Now the total coefficient of each power must be zero. From the coefficient of x^s we get

$$(36s(s-1) + 5)a_0 = 0 \rightarrow 36s^2 - 36s + 5 = 0 \rightarrow s = 1/6, 5/6$$

This equation for s is called the *indicial* equation. From here on we solve two problems, one for each s value. A linear combinations of the two solutions so obtained is the general solution of the problem.

For $s = 1/6$, the coefficient of a_1 gives

$$(36s(s+1) + 5)a_1 = (6(7/6) + 5)a_1 = 12a_1 = 0 \rightarrow a_1 = 0$$

The relationship among the rest ($n \geq 2$) of the coefficients is given by the x^{n+s} term, i.e.,

$$[(36(n+1/6)(n-2+1/6) + 5)a_n - 9a_{n-2}] = 0 \rightarrow a_n = \frac{9}{36(n+1/6)(n-2+1/6) + 5} a_{n-2}$$

Problem 6-3 p2

or

$$a_2 = \frac{9}{36(2 + 1/6)(1 + 1/6) + 5} a_0 = \frac{9}{96} a_0 = \frac{3}{2^5} a_0$$

and similarly

$$a_4 = \frac{3}{5 \cdot 2^5} a_2 = \frac{3^2}{5 \cdot 2^{10}} a_0$$

and so on.

Using the same process for $s = 5/6$, we get

$$b_2 = \frac{3}{2^6} b_0$$

$$b_4 = \frac{3^2}{7 \cdot 2^{11}} b_0$$

and so on. The general solution is then

$$y = Ax^{1/6} \left(1 + \frac{3}{2^5} x^2 + \frac{3^2}{5 \cdot 2^{10}} x^4 + \cdots \right) + Bx^{5/6} \left(1 + \frac{3}{2^6} x^2 + \frac{3^2}{7 \cdot 2^{11}} x^4 + \cdots \right)$$

Problem 6-4 p1

We have the equation

$$x^2 y'' - 6y = 0 \quad (\text{A})$$

We use the method of Frobenius. We assume a solution of the form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s} \quad (\text{A})$$

This gives

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \quad (\text{B})$$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} \quad (\text{C})$$

We substitute (B) and (C) into (A) to obtain

$$\sum_{n=0}^{\infty} [(n+s)(n+s-1) - 6] a_n x^{n+s} = 0$$

Expanding out we have

$$(s(s-1) - 6)a_0 x^s + (s(s+1) - 6)a_1 x^{s+1} + [(s+2)(s+1) - 6]a_2 x^{s+2} + \dots + [(n+s)(n+s-1) - 6]a_n x^{n+s} + \dots = 0$$

Now the total coefficient of each power must be zero. From the coefficient of x^s we get

$$s(s-1) - 6)a_0 = 0 \rightarrow s^2 - s - 6 = 0 \rightarrow s = 3, -2$$

This equation for s is called the *indicial equation*. From here on we solve two problems, one for each s value. A linear combinations of the two solutions so obtained is the general solution of the problem.

For $s = 3$, the coefficient of a_1 gives

$$(s(s+1) - 6)a_1 = 6a_1 = 0 \rightarrow a_1 = 0$$

The relationship among the rest ($n \geq 2$) of the coefficients is given by the x^{n+s} term, i.e.,

$$[(n+s)(n+s-1) - 6]a_n = 0 = [(n^2 + 5n)a_n$$

This says that $a_n = 0$ for all $n \neq 0$. Thus, the solution that corresponds to $s = 3$ is just $a_0 x^3 = a_0 x^3$ or $y = Ax^3$.

Problem 6-4 p2

In a similar manner, for $s = -2$ we also find $a_n = 0$ for all $n \neq 0$, so the corresponding solution is $y = Bx^{-2}$.

The general solution to the differential equations is

$$y = Ax^3 + Bx^{-2}$$

We note that the differential equation is an Euler or Cauchy equations and can also be solved by other methods.

Problem 6-5 p1

We start with $f'' + 2(z - \alpha)f' + 4f = 0$ and set $f(z) = g(x)$ where $x = z - \alpha$. The differential operators are interchanged, i.e.,

$$\frac{d}{dx} = \frac{d}{dz}, \quad \frac{d^2}{dx^2} = \frac{d^2}{dz^2}$$

The resulting equation for $g(x)$ is

$$g'' + 2xg' + 4g = 0$$

We assume a solution of the form

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=0}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

Equating coefficients of x^m to zero then gives

$$a_{m+2} = -\frac{2m+4}{(m+2)(m+1)} a_m = -\frac{2}{m+1} a_m$$

Since this recurrence relation relates indices differing by 2, we can obtain the two linearly independent solutions by taking (i) $a_0 = 1, a_1 = 0$ and (ii) $a_0 = 0, a_1 = 1$.

(i) $a_0 = 1, a_1 = 0$

Here, only even powers of x appear and we can write $m = 2p$. The expansion coefficients are then given by

$$a_{2p} = -\frac{2}{2p-1} a_{2p-2} = \frac{(-1)^p 2^p p!}{(2p)!} a_0$$

and the solution is

$$f_0(z) = g_0(x) = a_0 \sum_{n=0}^{\infty} \frac{(-4)^n n!}{(2n)!} x^{2n} = a_0 \sum_{n=0}^{\infty} \frac{(-4)^n n!}{(2n)!} (z - \alpha)^{2n}$$

(ii) Here only odd powers of x appear and we can write $m = 2p+1$. Expansion coefficients are then given by

$$a_{2p+1} = -\frac{2}{2p} a_{2p-1} = \frac{(-1)^p}{p!} a_1$$

Problem 6-5 p2

and the second solution is

$$f_1(z) = g_1(x) = a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1} = a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-\alpha)^{2n+1} = a_1 (z-\alpha) e^{-(z-\alpha)^2}$$

The general solution is any linear combination of $f_0(z)$ and $f_1(z)$.

Problem 6-6 p1

We have the equation

$$z^2 y'' - \frac{3z}{2} y' + (1+z)y = 0$$

We assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$$

Substituting gives

$$\sum_{n=0}^{\infty} a_n z^{n+\sigma} - \frac{3}{2} \sum_{n=0}^{\infty} a_n z^{n+\sigma} + (1+z) \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0$$

The indicial equation is

$$[\sigma(\sigma-1) - \frac{3}{2}\sigma + 1]a_0 = 0 \rightarrow \sigma^2 - \frac{5}{2}\sigma + 1 = 0 \rightarrow \sigma = 2, 1/2$$

(a) Setting $\sigma = 2$ and equating coefficients of z^{m+2} to zero gives

$$(2+m)(1+m)a_m - \frac{3}{2}(2+m)a_m + a_m + a_{m-1} = 0$$

$$\rightarrow a_m(2+3m+m^2-3-\frac{3}{2}m+1) = -a_{m-1}$$

$$\rightarrow a_m[m(\frac{3}{2}+m)] = -a_{m-1}$$

Repeated application of this relation gives

$$a_m = -\frac{2}{m(2m+3)}a_{m-1} = \frac{(-1)^m 2^m}{m![(2m+3)(2m+1)\cdots 5]}a_0$$

$$a_m = \frac{(-1)^m 2^m 3^{m+1} (m+1)!}{m!(2m+3)!}a_0$$

$$a_m = \frac{6(-1)^m 2^{2m} (m+1)}{(2m+3)!}a_0$$

leading to

$$y_1(z) = Az^2 \sum_{m=0}^{\infty} \frac{6(-1)^m 2^{2m} (m+1)}{(2m+3)!} z^n$$

Problem 6-6 p2

- (b) Setting $\sigma = 1/2$ and equating coefficients of $z^{m+1/2}$ to zero (same procedure as above) gives

$$\left(\frac{1}{2} + m\right)\left(-\frac{1}{2} + m\right)a_m - \frac{3}{2}\left(\frac{1}{2} + m\right)a_m + a_m + a_{m-1} = 0$$

$$a_m(4m^2 - 1 - 3 - 6m + 4) = -4a_{m-1}$$

which reduces to

$$a_m = -\frac{2}{m(2m-3)}a_{m-1}$$

In particular, for $m = 1$,

$$a_1 = -\frac{2}{1(-1)}a_0 = 2a_0$$

For $m \geq 2$,

$$a_m = \frac{(-1)^{m-1}2^{m-1}2^{m-2}(m-2)!}{m!(2m-3)!}a_1 = \frac{(-1)^{m-1}2^{2m}2a_0}{8m(m-1)(2m-3)!}$$

and the second solution is

$$y_2(z) = Bz^{1/2} \left(1 + 2z - \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}2^{2m}}{m(m-1)(2m-3)!} \right)$$

The general solution is

$$y(z) = Ay_1(z) + By_2(z)$$

Problem 6-7 p1

- (a) We have the equation $(1-x)u' + u = 0$. We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} b_n x^n$$

Substituting we get

$$\sum_{n=1}^{\infty} n b_n x^{n-1} - \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=0}^{\infty} b_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} (n+1) b_{n+1} x^n - \sum_{n=0}^{\infty} n (b_n x^n + \sum_{n=0}^{\infty} b_n x^n) = 0$$

or

$$\sum_{n=0}^{\infty} [(n+1) b_{n+1} - (n-1) b_n] x^n = 0$$

This says that

$$b_1 = -b_0, \quad b_2 = 0 \quad \text{and} \quad b_n = 0, n \geq 2$$

Hence,

$$u = b_0(1-x)$$

- (b) We have the equation $(x^2-1)u'' - 4u = 0$. We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} b_n x^n$$

Substituting we get

$$(x^2-1) \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} - 4 \sum_{n=0}^{\infty} b_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) b_n x^n - \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} - 4 \sum_{n=0}^{\infty} b_n x^n = 0$$

Therefore

$$-2b_2 - 4b_0 - (6b_3 + 4b_1)x + \sum_{n=2}^{\infty} [n(n-1)b_n - (n+2)(n+1)b_{n+2} - 4b_n]x^n = 0$$

This says that

$$-2b_2 + b_0 = 0, \quad 6b_3 + 4b_1 = 0$$

Problem 6-7 p2

and

$$b_{n+2} = \frac{n(n-1)-4}{(n+2)(n+1)} b_n \quad n \geq 0$$

Therefore,

$$b_2 = \frac{-4}{2 \cdot 1}, \quad b_4 = \frac{-2}{4 \cdot 3}, \dots, \quad b_{2k} = \frac{2k(2k-1)-4}{2k(2k-1)} b_{2k-2}$$

or

$$b_{2k} = (-4)(-2) \dots [2k(2k-1)-4] \frac{b_0}{(2k)!}$$

For the odd subscripted coefficients we similarly get

$$b_3 = \frac{-4}{3 \cdot 2} b_1, \quad b_5 = \frac{2}{5 \cdot 4} b_3, \dots, \quad b_{2k+1} = \frac{2k(2k+1)-4}{(2k+1)2k} b_{2k-1}$$

or

$$b_{2k+1} = \frac{-4 \cdot 2 \dots [2k(2k+1)-4]}{(2k+1)!} b_1$$

Finally the general solution is

$$u(x) = b_0 \left[1 + \sum_{k=0}^{\infty} \frac{(-4)(-2) \dots [2k(2k-1)-4]}{(2k)!} \right] + b_1 \left[x + \sum_{k=0}^{\infty} \frac{-4 \cdot 2 \dots [2k(2k+1)-4]}{(2k+1)!} x^{2k+1} \right]$$

- (c) We have the equation $(4-x^2)u'' + 2u = 0$ $u(0) = 0, u'(0) = 1$. We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} b_n x^n$$

We note that the initial conditions give

$$b_0 = 0, \quad b_1 = 1$$

Substituting we get

$$(4-x^2) \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} + 2 \sum_{n=0}^{\infty} b_n x^n = 0$$

Problem 6-7 p3

Therefore,

$$\sum_{n=0}^{\infty} [4(n+2)(n+1)b_{n+2} + 2b_n]x^n - \sum_{n=2}^{\infty} n(n-1)b_n x^n = 0$$

Hence,

$$4 \cdot 2b_2 + 2b_0 = 0, \quad 4 \cdot 3 \cdot 2b_3 + 2b_1 = 0$$

and

$$4(n+2)(n+1)b_{n+2} = (n-2)(n+1)b_n \quad n \geq 2$$

Since $b_0 = 0$ we have $b_{2k} = 0$ for all k .

Therefore,

$$b_3 = \frac{-1}{4 \cdot 3} b_1, \quad b_5 = \frac{1}{5 \cdot 4} b_3 \rightarrow b_{2k+1} = \frac{2k-3}{4(2k+1)} b_{2k-1}$$

This implies that (using $b_1 = 1$)

$$b_{2k+1} = \frac{-1 \cdot 1 \cdot 3 \cdot 5 \cdots (2k-3)}{4^k \cdot 1 \cdot 3 \cdot 5 \cdots (2k+1)} = \frac{1}{4^k (2k-1)(2k+1)}$$

Therefore the general solution is

$$u(x) = x - \sum_{k=1}^{\infty} \frac{x^{2k+2}}{4^k (2k-1)(2k+1)}$$

- d) We have the equation $u'' - x^2 u' + \sin x u = 0$ $u(0) = 0, u'(0) = 1$. We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} b_n x^n$$

We use

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

We note that the initial conditions give

$$b_0 = 0, \quad b_1 = 1$$

Then substitution gives

$$\sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n b_n x^n + \left(x - \frac{x^3}{3!} + \cdots \right) \sum_{n=0}^{\infty} b_n x^n = 0$$

Problem 6-7 p4

As in parts (a) - (c) we get

$$2b_2 = 0 \quad , \quad 6b_3 = 0 \quad , \quad 12b_4 + 1 = 0 \quad , \quad 20b_5 - 1 = 0 \cdots$$

and the general solution is

$$u(x) = x - \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots$$

Problem 6-8 p1

We are considering general equations of the form

(a) We have the equation

$$2x(1-x)u'' + u' - u = 0$$

We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

so that

$$u'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting we get

$$2x(1-x) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

or

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + (n+r)] a_n x^{n+r-1} - \sum_{n=0}^{\infty} [2(n+r)(n+r-1) + 1] a_n x^{n+r} = 0$$

Therefore we get

$$\begin{aligned} & [2r(r-1) + r] a_0 x^{r-1} \\ & + \sum_{n=0}^{\infty} [[2(n+r+1)(n+r) + (n+r+1)] a_{n+1} - [2(n+r)(n+r-1) + 1] a_n] x^{r+n} = 0 \end{aligned}$$

which says

$$2r^2 - r = 0 \rightarrow r = 0, 1/2, \dots, \quad a_{n+1} = \frac{2(n+r)(n+r-1) + 1}{2(n+r+1)(n+r) + (n+r+1)} a_n, \quad n \geq 0$$

Now for $r = 0$,

$$a_{n+1} = \frac{2n(n-1) + 1}{(n+1)(2n+1)} a_n, \quad a_0 = 1$$

Problem 6-8 p2

leads to

$$a_1 = 1, \quad a_2 = \frac{1}{6}, \quad a_3 = \frac{1}{18}, \quad \dots$$

and

$$u_1(x) = 1 + x + \frac{1}{6}x^2 + \frac{1}{18}x^3 + \dots$$

For $r = 1/2$,

$$a_{n+1} = \frac{2(2n+1)(2n-1)+4}{(2n+2)(2n+3)}a_n, \quad a_0 = 1$$

leads to

$$a_1 = \frac{1}{3}, \quad a_2 = \frac{1}{2 \cdot 3}, \quad \dots$$

and

$$u_2(x) = \left[1 + \frac{1}{3}x + \frac{1}{6}x^2 + \dots\right]x^{1/2}$$

and

$$u(x) = c_1\left[1 + x + \frac{1}{6}x^2 + \frac{1}{18}x^3 + \dots\right] + c_2\left[1 + \frac{1}{3}x + \frac{1}{6}x^2 + \dots\right]x^{1/2}$$

(b) Same approach as (a). We have the equation

$$4x^2(1-x)u'' - xu' + (1-x)u = 0$$

We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

so that

$$u'(x) = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

Substituting we get

$$(4r(r-1) - r + 1)a_0 x^r$$

$$+ \sum_{n=1}^{\infty} \{[4(n+r)(n+r-1) - (n+r) + 1]a_n + [4(n+r-1)(n+r-2) + 1]a_{n-1}\} = 0$$

Problem 6-8 p3

Now $4r(r-1) - r + 1 = 0 \rightarrow r = 1, 1$

For $r = 1$

$$a_n = \frac{(2n-1)^2}{n(4n+3)} a_{n-1}, \quad a_0 = 1$$

so

$$a_1 = \frac{1}{7}, \quad a_2 = \frac{9}{22} \cdot \frac{1}{7}, \quad \dots$$

and

$$u_1(x) = x \left[1 + \frac{1}{7}x + \frac{9}{154}x^2 + \dots \right]$$

For $r = 1/4$

$$a_n = \frac{(4n-5)^2}{4n(4n-3)} a_{n-1}, \quad a_0 = 0$$

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{9}{8 \cdot 5 \cdot 4}, \quad \dots$$

and

$$u_2(x) x^{1/4} \left[1 + \frac{1}{4}x + \frac{9}{160}x^2 + \dots \right]$$

and

$$u(x) = c_1 x \left[1 + \frac{1}{7}x + \frac{9}{154}x^2 + \dots \right] + c_2 x^{1/4} \left[1 + \frac{1}{4}x + \frac{9}{160}x^2 + \dots \right]$$

c) Same approach as (a). We have the equation

$$2x^2 u'' + x^2 u' + (4/9)u = 0$$

We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

so that

$$u'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Problem 6-8 p4

Substituting we get

$$[2r(r-1) + 4/9]a + 0x^2 + \sum_{n=1}^{\infty} \{[2(n+r)(n+r-1) + 4/9]a_n + (n+r-1)a_{n-1}\} x^{n+r} = 0$$

Therefore

$$2r(r-1) + 4/9 = 0 \rightarrow (3r-2)93r-1 = 0 \rightarrow r = 2/3, 1/3$$

For $r = 2/3$

$$a_n = -\frac{2}{27} \frac{(3n-1)}{(3n)(3n+1)} a_{n-1}, \quad a_0 = 1$$

Then

$$a_1 = -\frac{1}{81}, \quad a_2 = \frac{5}{7 \cdot 81^2}, \quad \dots$$

so

$$u_1(x) = x^{2/3} \left[1 - \frac{1}{81}x + \frac{5}{7 \cdot 81^2}x^2 + \dots \right]$$

For $r = 1/3$

$$a_n = -\frac{2}{27} \frac{(3n-1)}{(3n-2)(3n-1)} a_{n-1}, \quad a_0 = 1$$

Then

$$a_1 = -\frac{1}{81}, \quad a_2 = \frac{4}{5 \cdot 81^2}, \quad \dots$$

so

$$u_2(x) = x^{1/3} \left[1 - \frac{1}{81}x + \frac{4}{5 \cdot 81^2}x^2 + \dots \right]$$

and

$$u(x) = c_1 x^{2/3} \left[1 - \frac{1}{81}x + \frac{5}{7 \cdot 81^2}x^2 + \dots \right] + c_2 x^{1/3} \left[1 - \frac{1}{81}x + \frac{4}{5 \cdot 81^2}x^2 + \dots \right]$$

(d) Same approach as (a). We have the equation

$$3xu'' + 2(1-x)u' - 4u = 0$$

We assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Problem 6-8 p5

so that

$$u'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$u''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting we get

$$\sum_{n=0}^{\infty} [3(n+r)(n+r-1) + 2(n+r)] a_n x^{n+r-1}$$

$$- 2 \sum_{n=0}^{\infty} (n+r+2) a_n x^{n+r} = 0$$

Therefore

$$r(3r-1) = 0 \rightarrow r = 0, 1/3 \quad , \quad a_N = \frac{2(n+r+1)}{(n+r)(3n+3r-1)} a_{n-1}$$

For $r = 0$

$$a_n = \frac{2(n+1)}{n(3n-1)} a_{n-1} \quad , \quad a_0 = 1$$

Then

$$a_1 = -\frac{2 \cdot 2}{1 \cdot 2} \quad , \quad a_2 = \frac{2 \cdot 3}{2 \cdot 5} a_1 \quad , \quad \dots a_k = \frac{2(k+1)}{k(3k-1)} a_{k-1}$$

so

$$a_k = \frac{2^k(k+1)}{2 \cdot 5 \cdot 8 \dots (3k-1)}$$

and

$$u_1(x) = [1 + 2x + \dots + \frac{2^k(k+1)}{2 \cdot 5 \cdot 8 \dots (3k-1)} x^k + \dots]$$

For $r = 1/3$

$$a_n = \frac{2}{3} \frac{3n+4}{n(3n+1)} a_{n-1} \quad , \quad a_0 = 1$$

Then

$$a_1 = \frac{2}{3} \frac{7}{1 \cdot 4} \quad , \quad a_2 = \frac{2}{3} \frac{10}{2 \cdot 7} \quad , \quad a_k = \frac{2}{3} \frac{3k+4}{k(3k+1)} a_{k-1}$$

so

$$a_k = \left(\frac{2}{3}\right)^k \frac{3k+4}{k!}$$

Problem 6-8 p6

and

$$u_2(x) = x^{1/3} \left[1 + \frac{7}{6}x + \dots + \left(\frac{2}{3}\right)^k \frac{3k+4}{k!} x^k + \dots \right]$$

and

$$u(x) = c_1 \left[1 + 2x + \dots + \frac{2^k(k+1)}{2 \cdot 5 \cdot 8 \dots (3k-1)} x^k + \dots \right] + c_2 x^{1/3} \left[1 + \frac{7}{6}x + \dots + \left(\frac{2}{3}\right)^k \frac{3k+4}{k!} x^k + \dots \right]$$

Problem 6-9 p1

We have the equation

$$x^2 u'' - x(1+x)u' + u = 0$$

where one solution is given as $u_1(x) = xe^x$.

The solution method for the second solution will use the Wronskian as follows. An equation of the type

$$x^2 u'' + xp_0(x)u' + p_1(x)u = 0$$

always has a solution in the form

$$u_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0$$

for some r_1 . We know from general principles that there exists an independent solution $u_2(x)$ in some interval $a < x < b$. The Wronskian $W(x)$ of $u_1(x)$ and $u_2(x)$ is defined as

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$$

Now, in general, we have

$$\begin{aligned} W'(x) &= u_1 u_2'' + u_1' u_2' - u_2' u_1' - u_2 u_1'' = u_1 u_2'' - u_2 u_1'' \\ &= u_1 [-p_0 u_2' - p_1 u_2] - u_2 [-p_0 u_1' - p_1 u_1] = -p_0 [u_1 u_2' - u_1' u_2] \\ &= -p_0 W(x) \end{aligned}$$

so that

$$W(x) = K e^{-\int [p_0(x)/x] dx}$$

Setting the two expression for $W(x)$ equal to each other we get

$$\frac{u_1(x)u_2'(x) - u_1'(x)u_2(x)}{u_1^2(x)} = \frac{K}{u_1^2(x)} e^{-\int p_0(x)/x dx} = \frac{d}{dx} \left(\frac{u_2(x)}{u_1(x)} \right)$$

Thus,

$$\frac{u_2(x)}{u_1(x)} = K \int \frac{1}{u_1^2(x)} e^{-\int [p_0(x)/x] dx} dx$$

so that we can obtain a second solution if we know the first solution. We get

$$u_2 = u_1 \int \frac{e^{-2x}}{x^2} e^{\int ((1+x)/x) dx} dx = x e^x \int \frac{e^{-2x}}{x^2} e^{\ln x + x} dx = x e^x \int \frac{e^{-x}}{x} dx$$

Problem 6-9 p2

or

$$u_2 = x\left(1+x+\frac{x^2}{2}+\cdots\right) \int \left(\frac{1}{x} - 1 + \frac{x}{2} + \cdots\right) dx = x\left(1+x+\frac{x^2}{2}+\cdots\right) \left(\ln x - x + \frac{x^2}{4} + \cdots\right)$$

or

$$u_2 = xe^x \ln x - x^2 + \cdots$$

Assignment 7

Problem 7-1 p1

Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Assume

$$y = X(x)T(t)$$

which gives

$$X'' + k^2 X = 0 \quad , \quad \ddot{T} + k^2 v^2 T = 0$$

Basic solutions are

$$y = \begin{cases} \sin kx \sin \omega t \\ \sin kx \cos \omega t \\ \cos kx \sin \omega t \\ \cos kx \cos \omega t \end{cases}$$

where $k = 2\pi/\lambda$, $\omega = 2\pi\nu$, $\lambda = v/\nu$, $\omega = kv$ Since string is fastened at $x = 0$ and $x = L$, we must have $y = 0$ for these values of x and all t . This means that we only want terms with $\sin kx$ and we must also select $kL = n\pi$. The basic solution then become

$$y = \begin{cases} \sin \frac{n\pi x}{L} \sin \frac{n\pi vt}{L} \\ \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L} \end{cases}$$

The particular combination of these solutions that we should take to solve the problem depends on the initial conditions. In this problem we are given that the initial displacement of the string is $y_0 = f(x)$ and the initial velocity of points of the string is $V = (\partial y / \partial t)_{t=0} = 0$. This means we must assume a solution of the form

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}$$

The coefficients b_n are determined so that at $t = 0$ we have $y_0 = f(x)$, that is

$$y_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = f(x)$$

where

$$f(x) = \begin{cases} \frac{8h}{L}x & 0 < x < \frac{L}{8} \\ \frac{8h}{L}\left(\frac{L}{4} - x\right) & \frac{L}{8} < x < \frac{L}{4} \\ 0 & \frac{L}{4} < x < L \end{cases}$$

Problem 7-1 p2

Then

$$\begin{aligned} b_n &= \frac{2}{L} \left[\int_0^{L/8} \frac{8h}{L} x \sin \frac{n\pi x}{L} dx + \int_{L/8}^{L/4} \frac{8h}{L} \left(\frac{L}{4} - x \right) \sin \frac{n\pi x}{L} dx \right] \\ &= \frac{16h}{n^2 \pi^2} \left(2 \sin \frac{n\pi}{8} - \sin \frac{n\pi}{4} \right) \end{aligned}$$

We then have a solution for the vibrating string

$$y = \frac{16h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}$$

Problem 7-2 p1

The wave equation in two-dimensional rectangular coordinates is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2} \quad XYT + \frac{XY''T}{XYT} = \frac{1}{v^2} \frac{XY\ddot{T}}{XYT} \rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{v^2} \frac{\ddot{T}}{T} \quad (X''YT)$$

Each of the three terms is a function of only one variable; therefore each of the terms must be constant. We put

$$\frac{X''}{X} = -k_x^2, \quad \frac{Y''}{Y} = -k_y^2, \quad \frac{\ddot{T}}{T} = -(k_x^2 + k_y^2)v^2$$

The solutions are

$$X = \begin{cases} \sin k_x x \\ \cos k_x x \end{cases} \quad Y = \begin{cases} \sin k_y y \\ \cos k_y y \end{cases} \quad T = \begin{cases} \sin vt\sqrt{k_x^2 + k_y^2} \\ \cos vt\sqrt{k_x^2 + k_y^2} \end{cases}$$

Since the membrane is attached to supports along the sides, we must have

$$X = 0 \text{ at } x = 0 \text{ and } x = a, \quad Y = 0 \text{ at } y = 0 \text{ and } y = b$$

We use the sine solution to make $X = 0$ when $x = 0$. At $x = a$ we want $\sin k_x a = 0$ so $k_x = n\pi/a$, and $X = \sin \frac{n\pi x}{a}$. Similarly $Y = \sin \frac{m\pi y}{b}$, and

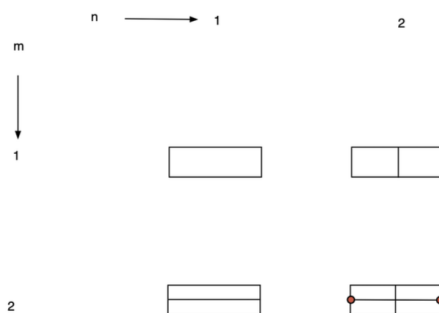
$$T = \begin{cases} \sin \left(vt\pi\sqrt{(n/a)^2 + (m/b)^2} \right) \\ \cos \left(vt\pi\sqrt{(n/a)^2 + (m/b)^2} \right) \end{cases}$$

Thus the frequencies are given by

$$\nu_{nm} = \frac{v}{2} \sqrt{(n/a)^2 + (m/b)^2}$$

The first few normal modes of vibration are:

Problem 7-2 p2

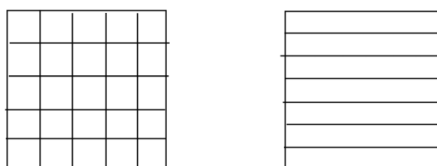


For a square membrane of side a , the frequencies are

$$\nu_{nm} = \frac{v}{2a} \sqrt{n^2 + m^2}$$

The 5,5 and 7,1 modes both have the frequency

$$\nu_{55} = \nu_{71} = \frac{v}{2a} \sqrt{50}$$



Problem 7-3

We know the procedure now from problems 1 and 2 in Assignment 7, therefore we can write.

If $u(0, t) = 0$ and $u(x, 0) = 0$, then

$$u(x, t) = A \sin kx \sin \omega t$$

where $u(L = \pi, 0) = 0$ implies that $kL = n\pi$ or $k = n$ and using $v = 60$, we have $\omega = nv = 60n$, so that

$$u_n(x, t) = A_n \sin nx \sin 60nt$$

and general solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin nx \sin 60nt$$

Now the initial condition is

$$\frac{\partial u}{\partial t}(x, 0) = \begin{cases} 0 & 0 < x < \pi/4 \\ 20 & \pi/4 < x < 3\pi/4 \\ 0 & 3\pi/4 < x < \pi \end{cases}$$

Then we get

$$A_n = \frac{2}{60\pi n} \int_{\pi/4}^{3\pi/4} 20 \sin nx dx = \frac{2 \cdot 20}{60\pi n^2} \left(\cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right)$$

and thus solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{3\pi n^2} \left(\cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right) \sin nx \sin 60nt$$

Problem 7-4 p1

The heat flow or diffusion equation is

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \quad (\text{A})$$

where u is the temperature and α^2 is a constant characteristic of the material. Let us separate (A) in stages. First, we assume

$$u = F(x, y, z)T(t)$$

Substitution into (A) gives

$$T \nabla^2 F = \frac{1}{\alpha^2} F \frac{\partial T}{\partial t}$$

Now dividing by FT gives

$$\frac{1}{F} \nabla^2 F = \frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt}$$

The left hand side is a function only of the space variables x, y, z , and the right-hand side is a function only of time. Therefore both sides are the same constant and we have

$$\frac{1}{F} \nabla^2 F = -k^2 \rightarrow \nabla^2 F + k^2 F = 0$$

and

$$\frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = -k^2 \rightarrow \frac{dT}{dt} = -k^2 \alpha^2 T$$

The time equation can be integrated to give

$$T = e^{-k^2 \alpha^2 t}$$

We can now see a physical reason for choosing the separation constant ($-k^2$) to be negative. With this choice, as t increases the temperature of the body decreases to 0.

The space equation is the Helmholtz equation. We are considering a bar 10 cm long with insulated sides, initially at 100° . Starting at $t = 0$, the ends are held at 0° . Because of the insulated sides, the heat flow is in one direction only (say x). Thus, our boundary conditions are $u(0, t) = 0 = u(L = 10, t)$. We also have $u(x, 0) = 100$.

Problem 7-4 p2

The space equation is then

$$\frac{d^2 F}{dx^2} + k^2 F = 0$$

with possible solutions

$$F(x) = \begin{cases} \sin kx \\ \cos kx \end{cases}$$

and the possible complete solutions are

$$u = \begin{cases} e^{-k^2 \alpha^2 t} \sin kx \\ e^{-k^2 \alpha^2 t} \cos kx \end{cases}$$

We discard the $\cos kx$ solution since $u = 0$ at $x = 0$. We also want $u = 0$ at $x = L = 10$ which implies that $10k_n = n\pi$. Thus the possible solutions are

$$u_n = e^{-(\frac{n\pi\alpha}{10})^2 t} \sin \frac{n\pi x}{10}$$

and the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi\alpha}{10})^2 t} \sin \frac{n\pi x}{10}$$

Finally, we impose the initial condition

$$u(x, 0) = 100 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{10}$$

so that

$$\begin{aligned} A_n &= \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10} dx = 20 \int_0^{10} \sin \frac{n\pi x}{10} dx \\ &= 20 \frac{10}{n\pi} \left(-\cos \frac{n\pi x}{10} \right) \Big|_0^{10} = \frac{200}{n\pi} [(-1)^n - 1] = \begin{cases} \frac{400}{n\pi} & \text{odd } n \\ 0 & \text{even } n \end{cases} \end{aligned}$$

Thus we have a solution

$$u(x, t) = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-(\frac{n\pi\alpha}{10})^2 t} \sin \frac{n\pi x}{10}$$

Problem 7-5 p1

The heat flow or diffusion equation is

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \quad (\text{A})$$

where u is the temperature and α^2 is a constant characteristic of the material. Let us separate (A) in stages. First, we assume

$$u = F(x, y, z)T(t)$$

Substitution into (A) gives

$$T \nabla^2 F = \frac{1}{\alpha^2} F \frac{\partial T}{\partial t}$$

Now dividing by FT gives

$$\frac{1}{F} \nabla^2 F = \frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt}$$

The left hand side is a function only of the space variables x, y, z , and the right-hand side is a function only of time. Therefore both sides are the same constant and we have

$$\frac{1}{F} \nabla^2 F = -k^2 \rightarrow \nabla^2 F + k^2 F = 0$$

and

$$\frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = -k^2 \rightarrow \frac{dT}{dt} = -k^2 \alpha^2 T$$

The time equation can be integrated to give

$$T = e^{-k^2 \alpha^2 t}$$

We are considering an infinite bar of thickness d . Because of infinite size in directions other than x , the heat flow is in the x direction only. Initially at $t = 0$, the end $x = 0$ is held at 0 and the end $x = d$ is at 100. Thus, the initial steady-state temperature distribution u_0 which satisfies Laplace's equation, which in one dimension if

$$\frac{d^2 u_0}{dx^2} = 0$$

This has a solution $u_0 = ax + b$, where a and b must be chosen to satisfy the initial steady-state situation. Thus we have

$$u_0(0, 0) = 0 \rightarrow b = 0 \text{ and } u_0(d, 0) = 100 \rightarrow a = 100/d$$

Problem 7-5 p2

or

$$u_0 = \frac{100}{d}x$$

The boundary conditions are $u(0, t) = 100$ and $u(d, t) = 0$.

The space equation is then

$$\frac{d^2 F}{dx^2} + k^2 F = 0$$

with possible solutions

$$F(x) = \begin{cases} \sin kx & k > 0 \\ \cos kx & k > 0 \\ mx + p & k = 0 \end{cases}$$

and the possible complete solutions are

$$u = \begin{cases} e^{-k^2 \alpha^2 t} \sin kx \\ e^{-k^2 \alpha^2 t} \cos kx \end{cases}$$

and the general solution is

$$u(x, t) = \sum_k e^{-k^2 \alpha^2 t} (b_k \sin kx + a_k \cos kx) + mx + p \quad (\text{B})$$

As $t \rightarrow \infty$ we see from (B) the $u = mx + p$ which must be the final steady state u_f . We solve for u_f in the same way we solved for u_0 , i.e.,

$$\frac{d^2 u_f}{dx^2} = 0$$

This has a solution $u_f = mx + p$, where m and p must be chosen to satisfy the final steady-state situation. Thus we have

$$u_0(0, 0) = 100 \rightarrow p = 100 \text{ and } u_0(d, 0) = 100 \rightarrow m = -100/d$$

or

$$u_f = -\frac{100}{d}x + 100$$

so we have

$$u(x, t) = \sum_k e^{-k^2 \alpha^2 t} (b_k \sin kx + a_k \cos kx) + 100 - \frac{100}{d}x \quad (\text{B})$$

Problem 7-5 p3

Now we must satisfy the conditions $u = 100$ at $x = 0$ and $u = 0$ at $x = d$ for all t . From (B)

$$100 = \sum_k a_k + 100 \rightarrow a_k = 0$$

$$0 = \sum_k b_k \sin kd \rightarrow k = n\pi/d$$

So finally we have

$$u(x, t) = \sum_n e^{-(\frac{n\pi\alpha}{d})^2 t} b_n \sin \frac{n\pi x}{d} + 100 - \frac{100}{d}x \quad (C)$$

Finally, we impose the initial conditions

$$u_0 = \frac{100}{d}x = \sum_n (b_n \sin \frac{n\pi x}{d} + 100 - \frac{100}{d}x)$$

so that

$$\sum_n b_n \sin \frac{n\pi x}{d} = \frac{200}{d}x - 100$$

We then have

$$\begin{aligned} b_n &= \frac{2}{d} \int_0^d 1 \left(\frac{200}{d}x - 100 \right) \sin \frac{n\pi x}{d} dx \\ &= \frac{400}{d^2} \int_0^d x \sin \frac{n\pi x}{d} dx - \frac{200}{d} \int_0^d \sin \frac{n\pi x}{d} dx \\ &= (-1)^n \left(-\frac{400}{n\pi} + \frac{200}{n\pi} \right) - \frac{200}{n\pi} = -((-1)^n + 1) \frac{200}{n\pi} \\ &= -\frac{400}{n\pi} \quad n \text{ even (otherwise } = 0) \end{aligned}$$

Thus we have a solution

$$u(x, t) = 100 - \frac{100}{d}x - \frac{400}{\pi} \sum_{n \text{ even}} \frac{1}{n} e^{-(\frac{n\pi\alpha}{d})^2 t} \sin \frac{n\pi x}{d}$$

Problem 7-6 p1

Initially the temperatures are

$$u(0, y) = 0, \quad u(1, y) = 100, \quad u(2, y) = 0$$

For each slab, the temperature is linear, $u_0 = ax + b$, where a and b must be found for each slab.

For the first slab:

$$u_0 = 0 \text{ when } x = 0 \text{ and } u_0 = 100 \text{ when } x = 1.$$

For the second slab:

$$u_0 = 100 \text{ when } x = 1 \text{ and } u_0 = 0 \text{ when } x = 2.$$

Thus, the initial temperature distribution is

$$u_0 = \begin{cases} 100x & 0 < x < 1 \\ 100(2-x) & 1 < x < 2 \end{cases}$$

The final temperature is $u_f = 100$. The temperature distribution as a function of x and t must be some linear combination of the basic solutions of the heat flow equation (as we have already seen in problems 4 and 5 of this assignment). When $k > 0$ these solutions are

$$u = \begin{cases} e^{-k^2 \alpha^2 t} \sin kx \\ e^{-k^2 \alpha^2 t} \cos kx \end{cases}$$

When $k = 0$, we have the equations

$$\nabla^2 F = 0 \text{ or } \frac{d^2 F}{dx^2} = 0 \rightarrow F = ax + b$$

$$\frac{dT}{dt} = 0 \rightarrow T = \text{constant}$$

Thus, the basic solutions are

$$u = \begin{cases} e^{-k^2 \alpha^2 t} \sin kx & k > 0 \\ e^{-k^2 \alpha^2 t} \cos kx & k > 0 \\ ax + b & k = 0 \end{cases}$$

Problem 7-6 p2

and we can write the solution to the problem in the form

$$u = \sum_k e^{-k^2 \alpha^2 t} (b_k \sin kx + a_k \cos kx) + ax + b \quad (\text{A})$$

As $t \rightarrow \infty$, we can see from (A) that $u = ax + b$; this must be the final steady state u_f , which is $u_f = 100$ here. So we can write (A) as

$$u = \sum_k e^{-k^2 \alpha^2 t} (b_k \sin kx + a_k \cos kx) + 100 \quad (\text{B})$$

Now we must satisfy the conditions $u = 100$ at $x = 0$ and $x = 2$ for all t . From (B) we see that $u = 100$ if the terms in the series are zero. This will be true for all t if we keep only the sine terms and take $k = n\pi/2$. Thus, we write (B) as

$$u = 100 + \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/2)^2 t} \sin \frac{n\pi x}{2} \quad (\text{C})$$

When $t = 0$, (C) becomes

$$u_0 = 100 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

or

$$u_0 - 100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \quad (\text{D})$$

Equation (D) says to expand in a Fourier sine series, the function

$$u_0 - u_f = \begin{cases} 100x - 100 = 100(x - 1) & 0 < x < 1 \\ 100(2 - x) - 100 = -100(x - 1) & 1 < x < 2 \end{cases}$$

We find

$$\begin{aligned} \frac{b_n}{100} &= \frac{2}{2} \left[\int_0^1 (x - 1) \sin \frac{n\pi x}{2} dx - \int_1^2 (x - 1) \sin \frac{n\pi x}{2} dx \right] \\ &+ \left[\left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi x}{2} - \frac{2}{n\pi} (x - 1) \cos \frac{n\pi x}{2} \right]_0^1 - \left[\text{same} \right]_1^2 \\ &= \left(\frac{2}{n\pi} \right)^2 \left(2 \sin \frac{n\pi}{2} + \frac{2}{n\pi} (-1 + \cos n\pi) \right) \end{aligned}$$

and thus

$$b_n = 100 \left\{ \begin{array}{ll} 0, & \text{even } n \\ -\frac{8}{n^2 \pi^2} - \frac{4}{n\pi}, & n = 1 + 4k \\ -\frac{8}{n^2 \pi^2} + \frac{4}{n\pi}, & n = 3 + 4k \end{array} \right\} = 400 \left\{ \begin{array}{ll} 0, & \text{even } n \\ \frac{2}{n^2 \pi^2} - \frac{1}{n\pi}, & n = 1 + 4k \\ -\frac{2}{n^2 \pi^2} - \frac{1}{n\pi}, & n = 3 + 4k \end{array} \right\}$$

and the solution is (C) with these values for the b_n .

Problem 7-7

When the ends of the bar are insulated, we want $\frac{\partial u}{\partial x} = 0$ at the ends. In this case we use the basic solution as outlined in earlier problem solutions containing $\cos kx$, since then $\frac{\partial(\cos kx)}{\partial x} = -k \sin kx$ is zero at $x = 0$ and, with $k = n\pi/L$ it is also zero at $x = L$. When $k = 0$, The equation is $F'' = 0$, so the $k = 0$ solutions are x and constant. Since $\frac{\partial u}{\partial x} = 0$ at the ends, we use only the constant which is already contained in $\cos kx$ with $k = 0$. Thus, the temperature distribution in the bar is given by

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-(n\pi\alpha/L)^2 t} \cos \frac{n\pi x}{L}$$

We are given $u(x, 0) = x$ so the a_n 's are determined by

$$u(x, 0) = x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}$$

This says to expand x on $(0, L)$ in a Fourier cosine series. We find

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx = \frac{2}{L} \left(\frac{L}{n\pi} \right)^2 \left(\cos \frac{n\pi x}{L} - \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \right) \Big|_0^L \\ &= \frac{2L}{n^2 \pi^2} (\cos n\pi - 1) = \begin{cases} 0 & \text{even } n \\ -\frac{4L}{n^2 \pi^2} & \text{odd } n \end{cases} \end{aligned}$$

The constant term is called a_0 in the series instead of $a_0/2$ as in standard Fourier series; this mean that the integral for a_0 is given by'

$$a_0 = \frac{1}{L} \int_0^L x dx = \frac{1}{L} \frac{L^2}{2} = \frac{L}{2}$$

Thus,

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{\text{odd } n} \frac{1}{n^2} e^{-(n\pi\alpha/L)^2 t} \cos \frac{n\pi x}{L}$$

Problem 7-8

The equation governing the heat flow is

$$k \frac{\partial^2 u}{\partial x^2} = s \frac{\partial u}{\partial t},$$

which is the diffusion equation with diffusion constant $\kappa = k/s = 3.8 \times 10^2 / 3.4 \times 10^6 = 1.12 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$.

Making the usual separation of variables substitution shows that the time variation is of the form $T(t) = T(0)e^{-\kappa\lambda^2 t}$ when the spatial solution is a sinusoidal function of λx . The final common temperature is 50°C and we make this explicit by writing the general solution as

$$u(x, t) = 50 + \sum_{\lambda} (A_{\lambda} \sin \lambda x + B_{\lambda} \cos \lambda x) e^{-\kappa\lambda^2 t}.$$

This term having been taken out, the summation must be antisymmetric about $x = 0$ and therefore contain no cosine terms, i.e. $B_{\lambda} = 0$.

The boundary condition is that there is no heat flow at $x = \pm a$; this means that $\partial u / \partial x = 0$ at these points and requires

$$\lambda A_{\lambda} \cos \lambda x|_{x=\pm a} = 0 \Rightarrow \lambda a = (n + \frac{1}{2})\pi \Rightarrow \lambda = \frac{(2n+1)\pi}{2a},$$

where n is an integer. This corresponds to a fundamental Fourier period of $4a$. The solution thus takes the form

$$u(x, t) = 50 + \sum_{n=0}^{\infty} A_n \sin \frac{(2n+1)\pi x}{2a} \exp \left(-\frac{(2n+1)^2 \pi^2 \kappa t}{4a^2} \right).$$

At $t = 0$, the sum must take the values $+50$ for $0 < x < 2a$ and -50 for $-2a < x < 0$. This is (yet) another square-wave function — one that is antisymmetric about $x = 0$ and has amplitude 50. The calculation will not be repeated here but gives $A_n = 200 / [(2n+1)\pi]$, making the complete solution

$$u(x, t) = 50 + \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi x}{2a} \exp \left(-\frac{(2n+1)^2 \pi^2 \kappa t}{4a^2} \right).$$

For a free end, where $x = a$ and $\sin[(2n+1)\pi x/2a] = (-1)^n$, to attain 55°C needs

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp \left(-\frac{(2n+1)^2 \pi^2 1.12 \times 10^{-4}}{4 \times 0.25} t \right) = \frac{5\pi}{200} = 0.0785.$$

In principle this is an insoluble equation but, because the RHS $\ll 1$, the $n = 0$ term alone will give a good approximation to t :

$$\exp(-1.105 \times 10^{-3} t) \approx 0.0785 \Rightarrow t \approx 2300 \text{ s}.$$

Assignment 8

Problem 8-1

We have now solved several diffusion or heat equation problems, so the procedure to use is clearly specified in early problems. In this problem, I will assume that you have read the other problems and just put down the bare minimum necessary to produce a solution here.

The diffusion equation is

$$\begin{aligned}\frac{\partial T}{\partial t} &= k \frac{\partial^2 T}{\partial x^2} \\ T = \Theta(t)X(x) &\rightarrow \frac{1}{k} \frac{\dot{\Theta}}{\Theta} = \frac{X''}{X} = -\beta^2 \\ \dot{\Theta} + k\beta^2\Theta &= 0 \rightarrow \Theta(t) = e^{-k\beta^2 t} \\ X'' + \beta^2 X &= 0 \rightarrow X(x) = A \sin \beta x + B \cos \beta x\end{aligned}$$

Therefore

$$T(x, t) = (A \sin \beta x + B \cos \beta x) e^{-k\beta^2 t}$$

The $\beta = 0$ solution is often of use in such problems. It leads to $\dot{\Theta} = 0$ and $X'' = 0$, resulting in $\Theta =$ and $X = ax + b$. Therefore the most general solution is

$$T(x, t) = ax + b + (A \sin \beta x + B \cos \beta x) e^{-k\beta^2 t}$$

Now

$$\begin{aligned}T(0, t) &= 100 \rightarrow b = 0, B = 0 \rightarrow T(x, t) = ax + A \sin \beta x e^{-k\beta^2 t} \\ T(2, t) &= 0 \rightarrow a = 0, \beta = n\pi/2 \rightarrow T(x, t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/4} \sin \frac{n\pi x}{2} \\ T(x, 0) &= 100(2x - x^2) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} \rightarrow A_n = \int_0^2 100(2x - x^2) \sin \frac{n\pi x}{2} dx\end{aligned}$$

Problem 8-2

This problem involves solving Laplace's equation. We have a semi-infinite plate of width 10cm where the temperature of the bottom edge $[0 \leq x \leq 10, y = 0]$ is $T = f(x) = x$, the temperature of the sides $[x = 0, y \geq 0]$ and $[x = 10, y \geq 0]$ is 0° . The temperature T satisfies Laplace's equation inside the plate (no sources of heat), that is

$$\nabla^2 T = 0 \rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where we have used rectangular coordinates because of the shape of the boundaries. We want $T \rightarrow 0$ as $y \rightarrow \infty$ and $T = 0$ when $x = 0$, so we use the solution

$$e^{-ky} \sin kx$$

We also want $T = 0$ when $x = 10$ so we set

$$\sin 10k = 0 \rightarrow k = \frac{n\pi}{10}$$

Thus our basic solutions are

$$e^{-n\pi y/10} \sin \frac{n\pi x}{10} \quad n = 1, 2, 3, \dots$$

Any linear combination of these solutions is a solution to Laplace's equation satisfying the given boundary conditions on three sides of the semi-infinite plate. We write

$$T = \sum_{n=1}^{\infty} b_n e^{-n\pi y/10} \sin \frac{n\pi x}{10} \quad (\text{A})$$

We then set $T = x$ when $y = 0$

$$T = x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

This means we want to expand x in a Fourier sine series. We find

$$\begin{aligned} b_n &= \frac{2}{10} \int_0^{10} x \sin \frac{n\pi x}{10} dx = \frac{2}{10} \left(\frac{10}{n\pi} \right)^2 \left(\sin \frac{n\pi x}{10} - \frac{n\pi x}{10} \cos \frac{n\pi x}{10} \right) \Big|_0^{10} \\ &= \frac{20}{n^2 \pi^2} (-n\pi \cos n\pi) = \frac{20}{n^2 \pi^2} (-1)^n \end{aligned}$$

The final solution is then

$$T = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/10} \sin \frac{n\pi x}{10}$$

Problem 8-3

This problem involves solving Laplace's equation. We have a semi-infinite plate of width 20cm where the temperature of the bottom edge $[0 < x < 10, y = 0]$ is $T = 0$ and $[10 < x < 20, y = 0]$ is $T = 100$, the temperature of the sides $[x = 0, y \geq 0]$ and $[x = 20, y \geq 0]$ is 0° . The temperature T satisfies Laplace's equation inside the plate (no sources of heat), that is

$$\nabla^2 T = 0 \rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where we have used rectangular coordinates because of the shape of the boundaries. We want $T \rightarrow 0$ as $y \rightarrow \infty$ and $T = 0$ when $x = 0$, so we use the solution

$$e^{-ky} \sin kx$$

We also want $T = 0$ when $x = 20$ so we set

$$\sin 20k = 0 \rightarrow k = \frac{n\pi}{20}$$

Thus our basic solutions are

$$e^{-n\pi y/20} \sin \frac{n\pi x}{20} \quad n = 1, 2, 3, \dots$$

Any linear combination of these solutions is a solution to Laplace's equation satisfying the given boundary conditions on three sides of the semi-infinite plate. We write

$$T = \sum_{n=1}^{\infty} b_n e^{-n\pi y/20} \sin \frac{n\pi x}{20} \quad (\text{A})$$

We then set

$$T = G(x) = \begin{cases} 0 & 0 < x < 10 \\ 100 & 10 < x < 20 \end{cases}$$

when $y = 0$

$$T = G(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

This means we want to expand x in a Fourier sine series. We find

$$\begin{aligned} b_n &= \frac{2}{20} \int_{10}^{20} 100 \sin \frac{n\pi x}{20} dx = \frac{200}{10} \left(\frac{20}{n\pi} \right) \left(-\cos \frac{n\pi x}{20} \right) \Big|_{10}^{20} \\ &= \frac{400}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) = \begin{cases} -\frac{400}{n\pi} & n \text{ even} \\ \frac{200}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

The final solution is then given by (A) with these b_n values.

Problem 8-4

The only change from problem 3 of assignment 8 is we cut off at $y = 10$ and keep the top edge at 0° .

The only change is to replace equation (A) in that problem with

$$T = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi}{20} (10 - y) \sin \frac{n\pi x}{20} \quad (\text{A})$$

so that $T = 0$ at $y = 10$. Everything else remains the same. What has been done here goes as follows: we no longer have any reason to discard the e^{ky} solution since y does not become infinite. So we replace the linear combination e^{-ky} by $ae^{-ky} + be^{ky}$ which is zero at $y = 10$. The most convenient way is to use the combination

$$\frac{1}{2}e^{k(10-y)} - \frac{1}{2}e^{-k(10-y)} \rightarrow a = \frac{1}{2}e^{10k} \text{ and } b = -\frac{1}{2}e^{-10k}$$

which is zero at $y = 10$. This combination is exactly $\sinh k(10 - y)$ which is what we used.

Then at $y = 0$ we want

$$T = G(x) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi}{2} \sin \frac{n\pi x}{20} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20}$$

where $B_n = \sinh \frac{n\pi}{2} b_n$. The Fourier coefficients B_n are the same as the earlier problem. Therefore we can write

$$b_n = \begin{cases} -\frac{400}{n\pi} \frac{1}{\sinh \frac{n\pi}{2}} & n \text{ even} \\ \frac{200}{n\pi} \frac{1}{\sinh \frac{n\pi}{2}} & n \text{ odd} \end{cases}$$

Problem 8-5 p1

We solve this problem by solving two problems(that we can do easily) and adding them together. This works because the sum of solutions of Laplace's equation is also a solution and the sum of these two problems corresponds to the problem we are trying to solve.

We have

Problem X: $T = 0$ for $[x = 0, 0 < y < 30]$ and $[x = 10, 0 < y < 30]$ and $[0 < x < 10, y = 30]$; $T = 100$ for $[0 < x < 10, y = 0]$

Problem Y: $T = 0$ for $[x = 10, 0 < y < 30]$ and $[0 < x < 10, y = 30]$ and $[0 < x < 10, y = 0]$; $T = 100$ for $[x = 0, 0 < y < 30]$

Problem X+Y: $T = 100$ for $[x = 0, 0 < y < 30]$ and $[0 < x < 10, y = 0]$; $T = 0$ for $[0 < x < 10, y = 30]$ and $[x = 10, 0 < y < 30]$

Solve problem X: As indicated in problem 4 in this assignment, we now assume the form

$$(ae^{ky} + be^{-ky})(B \sin kx + A \cos kx)$$

We then have $T = 0$ when $x = 0$ so we drop the cosine terms. We also have $T = 0$ when $x = 10$, so $k = \frac{n\pi}{10}$. We have $T = 0$ when $y = 30$ and this can be handled as earlier by writing

$$ae^{ky} + be^{-ky} = \sinh \frac{n\pi}{10}(30 - y)$$

So we have

$$T = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{10}(30 - y) \sin \frac{n\pi x}{10}$$

The B_n are found from boundary condition at $y = 0$, i.e., $T = 100$

$$100 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

where $b_n = B_n \sinh 3n\pi$. We find, using Fourier series

$$b_n = \frac{400}{n\pi} \quad n \text{ odd}; b_n = 0 \quad n \text{ even}$$

so that

$$T_X = \sum_{n \text{ odd}} \frac{400}{n\pi \sinh 3n\pi} \sinh \frac{n\pi}{10}(30 - y) \sin \frac{n\pi x}{10}$$

Problem 8-5 p2

Solve problem Y: Here the solution are of the form

$$(ae^{kx} + be^{-kx})(B \sin ky + A \cos ky)$$

Since we want $T = 0$ when $y = 0$, we drop the cosine parts. We also want $T = 0$ when $y = 30$, so $\sin 30k = 0 \rightarrow k = n\pi/30$. For the x part of the solution, we need a linear combination of e^{kx} and e^{-kx} which is 0 when $x = 10$. We use

$$\sinh k(10 - x) = \frac{1}{2}(e^{k(10-x)} - e^{-k(10-x)})$$

Then the series

$$T_Y = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{30} (10 - x) \sin \frac{n\pi y}{30}$$

We want $T = 100$ when $x = 0$, that is

$$T = 100 = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{30} \sin \frac{n\pi y}{30} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{30}$$

where

$$B_n = \frac{b_n}{\sinh \frac{n\pi}{30}}$$

We expand in a Fourier sine series on $(0, 30)$ to find

$$\begin{aligned} b_n &= \frac{2}{30} \int_0^{30} 100 \sin \frac{n\pi y}{30} dy = -\frac{200}{30} \frac{1}{n\pi} \cos \frac{n\pi y}{30} \Big|_0^{30} \\ &= \frac{200}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n} & n \text{ odd} \end{cases} \end{aligned}$$

Then for odd n ,

$$B_n = \frac{400}{n\pi \sinh \frac{n\pi}{30}}$$

and thus

$$T_Y = \sum_{n \text{ odd}} \frac{400}{n\pi \sinh \frac{n\pi}{30}} \sinh \frac{n\pi}{30} (10 - x) \sin \frac{n\pi y}{30}$$

Now the final solution (adding X and Y) is

$$\begin{aligned} T(x, y) &= T_X + T_Y \\ &= \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left(\frac{1}{\sinh 3n\pi} \sinh \frac{n\pi}{10} (30 - y) \sin \frac{n\pi x}{10} \right. \\ &\quad \left. + \frac{1}{\sinh \frac{n\pi}{3}} \sinh \frac{n\pi}{30} (10 - x) \sin \frac{n\pi y}{30} \right) \end{aligned}$$

Problem 8-6

From steady-state heat equation we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\beta^2$$

Therefore

$$T(x, y) = c_1 \sin \beta x + c_2 \cos \beta x (c_3 e^{\beta y} + c_4 e^{-\beta y})$$

Now

$$T(0, y) = 0 \rightarrow c_2 = 0$$

$$\frac{\partial T}{\partial y}(x, 0) = 0 \rightarrow c_3 = c_4$$

Let $c_1 c_4 = A$. Then

$$\frac{\partial T}{\partial x}(x, 1) = 0 = A \beta \cos \beta (e^{\beta y} + e^{-\beta y}) \rightarrow \beta = \frac{2n-1}{2} \pi, \quad n = 1, 2, 3, \dots$$

We then have

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{2n-1}{2} \pi x (e^{\frac{2n-1}{2} \pi y} + e^{-\frac{2n-1}{2} \pi y})$$

Now

$$T(x, 1) = 100 = \sum_{n=1}^{\infty} A_n \sin \frac{2n-1}{2} \pi x (e^{\frac{2n-1}{2} \pi} + e^{-\frac{2n-1}{2} \pi})$$

and

$$A_n = \frac{200}{e^{\frac{2n-1}{2} \pi} + e^{-\frac{2n-1}{2} \pi}} \int_0^1 \sin \frac{2n-1}{2} \pi x dx = \frac{400}{(2n-1)\pi (e^{\frac{2n-1}{2} \pi} + e^{-\frac{2n-1}{2} \pi})}$$

Problem 8-7 p1

We have a solid semi-infinite cylinder with boundary temperatures $u = 0$ at $r = 1$ and $u = y = r \sin \theta$ at $z = 0$. To find the steady state temperature distribution we must solve Laplace's equation in cylindrical-polar coordinates

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

To separate the variables, we assume a solution of the form

$$u = R(r)\Theta(\theta)Z(z)$$

Substitution and dividing by $R\Theta Z$ gives

$$\frac{1}{R} \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{1}{r^2} \frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad (\text{A})$$

The last term is a function of z only, while the other two terms do not contain z . Therefore, the last term is a constant and the sum of the first two terms is minus the same constant. Thus we have

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = k^2 \rightarrow Z = \begin{cases} e^{kz} \\ e^{-kz} \end{cases}$$

Since we want the temperature $u \rightarrow 0$ as $z \rightarrow \infty$ (semi-infinite cylinder) we call the separation constant $+k^2$ ($k > 0$) and then only use the e^{-kz} solution.

Next write (A) with the last term replaced by k^2

$$\frac{1}{R} \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{1}{r^2} \frac{d^2 \Theta}{d\theta^2} + k^2 = 0 \quad (\text{B})$$

We can separate the variables by multiplying by r^2

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} + k^2 r^2 = 0 \quad (\text{C})$$

The second term is a function of θ only and the other terms are independent of θ . Thus we have

$$\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -n^2 \rightarrow \Theta = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

Here we must use $-n^2$ as the separation constant and then require n to be an integer. Why? When we locate a point in polar coordinates, we can choose

Problem 8-7 p2

the angle as θ or as $\theta + 2m\pi$ where m is an integer. But remember there is only one temperature at that point and therefore the mathematical formula for the temperature must give the same value for the two points. This says the temperature must be a periodic function of θ with period 2π . They say we must use sines and cosines.

Finally, the r equation is

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - n^2 + k^2 r^2 = 0$$

or

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) + (k^2 r^2 - n^2) R = 0$$

which is Bessel' equation with solutions $J_n(kr)$ and $N_n(kr)$. Since the base of the cylinder we are considering contains the origin we can only use the J_n functions. Hence we have

$$R(r) = J_n(kr)$$

Now since we need a solution containing $\sin \theta$ we let $n = 1$ and use the basic solution

$$J_1(kr) \sin \theta e^{kz}$$

The condition $u = 0$ when $r = 1$ gives $J_1(k) = 0$. Thus, the possible values of k are the zeroes of J_1 ; call these k_m . Then we try to find a solution of the problem in the form

$$u(r, \theta, z) = \sum_{m=1}^{\infty} c_m J_1(k_m r) \sin \theta e^{-k_m z} \quad (\text{A})$$

We want $u = r \sin \theta$ when $z = 0$:

$$u(r, \theta, 0) = r \sin \theta = \sum_{m=1}^{\infty} c_m J_1(k_m r) \sin \theta$$

or cancelling $\sin \theta$ we have

$$r = \sum_{m=1}^{\infty} c_m J_1(k_m r) \quad (\text{B})$$

To expand r in a Bessel series, we multiply both sides of the equation (B) by $r J_1(k_\ell r)$ and integrate from 0 to 1. By orthogonality, all terms drop out

Problem 8-7 p3

except the $m = \ell$ term; using the properties of Bessel functions we get

$$\begin{aligned}\int_0^1 r^2 J_1(k_\ell r) dr &= \sum_{m=1}^{\infty} c_m \int_0^1 r J_1(k_\ell r) J_1(k_m r) dr \\ &= c_\ell \int_0^1 r [J_1(k_\ell r)]^2 dr = c_\ell \cdot \frac{1}{2} J_2^2(k_\ell)\end{aligned}\quad (C)$$

To integrate $r^2 J_1$ we use properties of Bessel function to get

$$\frac{d}{d(k_\ell r)} [(k_\ell r)^2 J_2(k_\ell r)] = (k_\ell r)^2 J_1(k_\ell r)$$

Cancelling k_ℓ^2 and integrating gives

$$\int_0^1 r^2 J_1(k_\ell r) dr = \frac{r^2}{k_\ell} J_2(k_\ell r) \Big|_0^1 = \frac{1}{k_\ell} J_2(k_\ell) \quad (D)$$

Combining equations (C) and (D) we get

$$c_\ell = \frac{2}{J_2^2(k_\ell)} \frac{1}{k_\ell} J_2 = \frac{2}{k_\ell J_2(k_\ell)}$$

Substituting into (A) gives the solution to the problem

$$u(r, \theta, z) = \sum_{m=1}^{\infty} \frac{2}{k_m J_2(k_m)} J_1(k_m r) \sin \theta e^{-k_m z}$$

where k_m = zeroes of J_1

Problem 8-8 p1

We need the basic solutions of the heat flow equation

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

in polar coordinates. We just discussed these in the previous problem. The solutions are of the form $F(r, \theta)T(t)$ where

$$T(t) = e^{-(k\alpha)^2 t}$$

and F satisfies the Helmholtz equation $\nabla^2 F + k^2 F = 0$. As in last problem, this equation in polar coordinates has solutions

$$R(r)\Theta(\theta) = J_n(kr) \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

We consider a circular cross-section of the pipe. The temperature is a function only of r (no θ dependence) so we use the $\cos n\theta$ solution with $n = 0$. Thus the basic solutions are

$$J_0(kr)e^{-(k\alpha)^2 t} \quad (\text{A})$$

These solutions tend to zero as $t \rightarrow \infty$

The problem states there is a final steady state distribution of temperature, namely $u_f = 40$. The solution u_f must be added to any solutions of the form (A).

Then we want solutions of the form (A) to be zero when $r = 1$ for any $t > 0$. Thus we must have $J_0(k) = 0$ so $k = k_m$ are the zeroes of J_0 . We write the solution to the problem in the form

$$u(r, t) = 40 + \sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-(k_m \alpha)^2 t} \quad (\text{B})$$

and find the coefficients c_m so that $u = 100$ when $t = 0$

$$u(r, 0) = 100 = 40 + \sum_{m=1}^{\infty} c_m J_0(k_m r)$$

$$\sum_{m=1}^{\infty} c_m J_0(k_m r) = 60$$

Problem 8-8 p2

We find the coefficients as follows:

$$60 \int_0^1 r J_0(k_\mu r) dr = \sum_{m=1}^{\infty} c_m \int_0^1 r J_0(k_\mu r) J_0(k_m r) dr$$

Because of orthogonality all terms of the series drop out except the term with $m = \mu$ and we have

$$60 \int_0^1 r J_0(k_\mu r) dr = c_\mu \int_0^1 r [J_0(k_\mu r)]^2 dr \quad \mu = 1, 2, 3, \dots \quad (C)$$

Now

$$\int_0^1 r [J_0(k_m r)]^2 dr = \frac{1}{2} J_1^2(k_m)$$

We also have

$$\frac{d}{dx} [x J_1(x)] = x J_0(x)$$

Putting $x = k_m r$ we get

$$\frac{1}{k_m} \frac{d}{dr} [k_m r J_1(k_m r)] = k_m r J_0(k_m r)$$

Cancelling on k_m factor and integrating, we have

$$\int_0^1 r J_0(k_m r) dr = \frac{1}{k_m} r J_0(k_m r) \Big|_0^1 = \frac{1}{k_m} J_1(k_m)$$

Now equation (C) gives

$$c_m = \frac{2 \cdot 60}{k_m J_1(k_m)} = \frac{120}{k_m J_1(k_m)}$$

Now equation (B) gives the solution

$$u(r, t) = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(k_m \alpha)^2 t}$$

Assignment 9

Problem 9-1 p1

We need the solution of Laplace's equation in polar coordinates. Follow work in other problems we write

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = 0$$

We then separate the variables to get the θ equation/solutions

$$\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -n^2 \rightarrow \Theta(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

and the r equation

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = n^2 \quad \text{or} \quad r \frac{d}{dr} \left(r \frac{dR}{dr} \right) = n^2 R$$

This equation is solved by r^n and r^{-n} . We only use the r^n solutions with $n > 0$ since $r = 0$ is a point on the plate. The basic solutions are then

$$\{r^n\} \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

There are also $n = 0$ solutions possible here, i.e., for $n = 0$ we have a possible solution $a \ln r + b$. We cannot use the $\ln r$ solution because it diverges at $r = 0$ but the constant solution is allowed. We now have $T = 0$ when $\theta = 0$ so that we only use the $\sin n\theta$ solutions. We also have $T(a, \pi) = 0$ so that $\sin n\pi = 0$ so that $n = \text{integer}$. Finally we have the general solution for the temperature of the plate

$$T(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta + H$$

Now it is clear that H = average temperature on the circle $r = a$ since the $\sin n\theta$ terms have zero average. Thus $H = 50$. The B_n 's are determined by

$$T(a, \theta) = \begin{cases} 50 & 0 < \theta < \pi \\ -50 & \pi < \theta < 2\pi \end{cases} = \sum_{n=1}^{\infty} B_n a^n \sin n\theta = \sum_{n=1}^{\infty} b_n \sin n\theta$$

where $b_n = B_n a^n$. We now expand this function in a Fourier sine series to get

$$\int_0^{2\pi} b_n \sin m\theta \sin n\theta d\theta = \int_0^{2\pi} \begin{cases} 50 & 0 < \theta < \pi \\ -50 & \pi < \theta < 2\pi \end{cases} \sin m\theta d\theta$$

Problem 9-1 p2

$$\pi b_m = 50 \int_0^\pi \sin m\theta d\theta - 50 \int_\pi^{2\pi} \sin m\theta d\theta$$

$$\pi b_m = \frac{50}{m} \int_0^{m\pi} \sin z dz - \frac{50}{m} \int_\pi^{2\pi} \sin z dz = -\frac{50}{m} (\cos m\pi - 1) + \frac{50}{m} (1 - \cos m\pi)$$

Thus,

$$b_m = \frac{200}{m\pi} \text{ for } m \text{ odd and } 0 \text{ otherwise}$$

The solution is then

$$T(r, \theta) = 50 + \sum_{n=1}^{\infty} \frac{200}{n\pi a^n} r^n \sin n\theta = 50 + \frac{200}{\pi} \sum_{n \text{ odd}} \left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n}$$

Problem 9-2 p1

Laplace's equation in plane polar coordinates is

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

If we separate variables by assuming $u(r, \theta) = R(r)\Theta(\theta)$, see earlier problems, we have

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \quad m = \text{integer}$$

and

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -m^2$$

The solutions are

$$\Theta(\theta) = \begin{cases} \sin m\theta \\ \cos m\theta \end{cases}$$

and

$$R(r) = \begin{cases} r^m, r^{-m} & m > 0 \\ \ln r, \text{ constant} & m = 0 \end{cases}$$

We have a circular annulus of inner radius $r = 1$ and outer radius $r = 2$. Since $r = 0$ is not a point on the annulus we must use all possible solutions. Therefore, basic solutions are then

$$\begin{cases} r^m \\ r^{-m} \end{cases} \begin{cases} \sin m\theta \\ \cos m\theta \end{cases} \text{ for } m > 0 \text{ and } \begin{cases} \text{constant} \\ \ln r \end{cases} \text{ for } m = 0$$

Now for any θ , the temperature is zero when $r = 1$. Thus we use a combination of r solutions which is zero when $r = 1$, namely, $(r^m - r^{-m})$. The combination $(a + b \ln r)$ also must be zero when $r = 1$, so $a = 0$. Thus we write the solution to the problem in the form

$$u(r, \theta) = b \ln r + \sum_{m=1}^{\infty} (r^m - r^{-m}) (a_m \cos m\theta + b_m \sin m\theta)$$

This satisfies $u = 0$ when $r = 1$. When $r = 2$, we have

$$u(2\theta) = b \ln 2 + \sum_{m=1}^{\infty} (2^m - 2^{-m}) (a_m \cos m\theta + b_m \sin m\theta) = \begin{cases} 100 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \text{ or } -\pi < \theta < 0 \end{cases}$$

Problem 9-2 p2

We can simplify the evaluation of the Fourier coefficients by observing that the given function is 50 + and odd function, namely

$$50 + \begin{cases} 50 & 0 < \theta < \pi \\ -50 & -\pi < \theta < 0 \end{cases}$$

Thus,

$$b \ln 2 = 50 \quad , \quad a_M = 0$$

$$\begin{aligned} (2^m - 2^{-m})b_m &= 50 \cdot \frac{2}{\pi} \int_0^\pi \sin m\theta d\theta = -\frac{100 \cos m\theta}{\pi n} \Big|_0^\pi \\ &= \begin{cases} \frac{200}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \end{aligned}$$

Then the temperature in the annular ring is

$$u(r, \theta) = \frac{50 \ln r}{\ln 2} + \frac{200}{\pi} \sum_{n \text{ odd}} \frac{r^m - r^{-m}}{n(2^m - 2^{-m})} \sin m\theta$$

Problem 9-3 p1

Heat transfer in a cylindrical body is governed by the equation

$$\frac{\partial T}{\partial t} = k \nabla^2 T$$

For steady state using cylindrical coordinates this becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

where considering the boundary condition given in the problem, we have assumed that the temperature is independent of θ . We then assume a separated solution of the form

$$T(r, z) = R(r)Z(z)$$

which leads to the equations

$$\frac{1}{R} \left(R'' + \frac{1}{r} R' \right) = -\frac{Z''}{Z} = -\mu^2$$

where a negative sign is chosen on the separation constant since we anticipate an exponential variation with z . We are thus confronted with solving two ordinary differential equations:

$$R'' + \frac{1}{r} R' + \mu^2 R = 0 \quad (\text{A})$$

$$Z'' - \mu^2 Z = 0 \quad (\text{B})$$

The solution to (B) is

$$Z(z) = c_1 e^{\mu z} + c_2 e^{-\mu z} \quad \mu > 0$$

and

$$Z(z) = c'_1 z + c'_2 \quad \mu = 0$$

which may be of use. We note that equation (A) is close to being Bessel's equation

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + (x^2 - \lambda^2) u = 0$$

with $\lambda = 0$. By substituting $x = \mu r$, (A) becomes

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + x^2 R = 0$$

Problem 9-3 p2

which is Bessel's equation with $\lambda = 0$. It possesses the general solution

$$R(x) = c_3 J_0(x) + c_4 Y_0(x)$$

Since $r = 0$ is on the body we must choose $c_4 = 0$ so that the solution is

$$T(r, z) = J_0(\mu r) [A e^{\mu z} + B e^{-\mu z}]$$

The temperature on the surface $z = 0$ is maintained at $T = 0$. This give $B = -A$. The temperature at $r = 1$ is also maintained at $T = 0$: that is,

$$T(r_0, z) = A J_0(\mu) [e^{\mu z} - e^{-\mu z}] = 0 \quad (C)$$

The Bessel function $J_0(\mu)$ has infinitely many roots none of which are zero. Let μ_n be the roots of $J_0(\mu) = 0$. Then we have

$$T(r, z) = \sum_{n=1}^{\infty} T_n(r, z) = \sum_{n=1}^{\infty} J_0(\mu_n r) A_n [e^{\mu_n z} - e^{-\mu_n z}]$$

This solution must allow the initial and final end conditions to be satisfied. It is

$$T(r, 1) = 100 = \sum_{n=1}^{\infty} A_n J_0(\mu_n r) [e^{\mu_n} - e^{-\mu_n}]$$

We assume that the series converges. We use the property that

$$\int_0^b x J_j(\mu_n x) J_j(\mu_m x) dx = \begin{cases} 0 & n \neq m \\ \frac{b^2}{2} J_{j+1}^2(\mu) n b & n = m \end{cases}$$

where as stated earlier the μ_n are the roots of $J_0(\mu r_0) = 0$. The allows us to determine the coefficients

$$A_n = \frac{2(e^{\mu_n} - e^{-\mu_n})^{-1}}{J_1^2(\mu_n)} \int_0^1 100r J_0(\mu_n r) dr$$

The roots of J_0 are

$$\mu_1 = 2.4, \mu_2 = 5.52, \mu_3 = 8.65, \dots$$

and

$$J_1(2.4) = 0.52, J_1(5.52) = -0.339, J_1(8.65) = 0.271, \dots$$

Problem 9-3 p3

Then

$$A_1 = 67.7 \int_0^1 r J_0(2.4r) dr = \frac{67.7}{(2.4)^2} \int_0^{2.4} x J_0(x) dx = \frac{67.7}{(2.4)^2} \cdot 2.4 \cdot J_1(2.4) = 14.7$$

$$A_2 = 6.97 \int_0^1 r J_0(5.52r) dr = \frac{6.97}{(5.52)^2} \int_0^{5.52} x J_0(x) dx = \frac{6.97}{(5.52)^2} \cdot 5.52 \cdot J_1(5.52) = -0.43$$

$$A_3 = 0.48 \int_0^1 r J_0(8.65r) dr = \frac{0.48}{(8.65)^2} \int_0^{8.65} x J_0(x) dx = \frac{0.48}{(8.65)^2} \cdot 8.65 \cdot J_1(8.65) = 0.015$$

Thus,

$$T(r, z) = 29.4 J_0(2.4r) \sinh(2.4z) - 0.86 J_0(5.52r) \sinh(5.52z) + 0.03 J_0(8.65r) \sinh(8.65z) + \dots$$

Problem 9-4 p1

These problem requires Laplace's equation in spherical coordinates.

- (a) The solution is independent of ϕ so $m = 0$ and the solutions of the θ equation are $P_\ell(\cos \theta)$. Since $r = 0$ is in the body and the temperature remain finite, the r solution is r^ℓ (see other problems for details). Thus we are looking for solutions of the form

$$u(r, \theta) = \sum_{\ell=0}^{\infty} c_\ell r^\ell P_\ell(\cos \theta) \quad (\text{A})$$

When $r = 1$ (surface of sphere), (A) becomes

$$u(1, \theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos \theta) = 35(\cos \theta)^4$$

We use orthogonality of the Legendre polynomials to determine the c_ℓ as follows:

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn} \quad x = \cos \theta$$

Then

$$35x^4 = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

or

$$35 \int_{-1}^1 x^4 P_m(x) dx = \sum_{\ell=0}^{\infty} c_\ell \int_{-1}^1 P_m(x) P_\ell(x) dx = c_m \frac{2}{2m+1}$$

or

$$c_m = \frac{35(2m+1)}{2} \int_{-1}^1 x^4 P_m(x) dx$$

This produces

$$c_4 = 8, c_2 = 20, c_0 = 7$$

all others are zero. Therefore solution is

$$u(r, \theta) = 8r^4 P_4(\cos \theta) + 20r^2 P_2(\cos \theta) + 7P_0(\cos \theta)$$

- (b) This is similar to Part (a). Once again the solution is

$$u(r, \theta) = \sum_{\ell=0}^{\infty} c_\ell r^\ell P_\ell(\cos \theta) \quad (\text{B})$$

Problem 9-4 p2

and again when $r = 1$ (surface of sphere), (B) becomes

$$u(1, \theta) = \sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \theta) = \begin{cases} \cos \theta & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < \pi \end{cases}$$

We use orthogonality of the Legendre polynomials to determine the c_{ℓ} as follows:

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{n+1} \delta_{mn} \quad x = \cos \theta$$

Then

$$\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(x) = \begin{cases} x & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$$

or

$$\sum_{\ell=0}^{\infty} c_{\ell} P_m(x) P_{\ell}(x) dx = c_m \frac{2}{2m+1} = \int_0^1 x P_m(x) dx$$

so that

$$c_m = \frac{2m+1}{2} \int_0^1 x P_m(x) dx$$

This produces

$$c_0 = \frac{1}{4}, c_1 = \frac{1}{2}, c_2 = \frac{5}{16}, \dots$$

Therefore solution is

$$u(r, \theta) = \frac{1}{4} P_0(\cos \theta) + \frac{1}{2} P_1(\cos \theta) + \frac{5}{16} P_2(\cos \theta) + \dots$$

Problem 9-5 p1

The equation satisfied by the temperature inside the sphere (potato) is the heat flow equation

$$\nabla^2 u - \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

We want ∇^2 in spherical coordinates. First we separate the space-dependent and time-dependent parts as we have done in several problems. We have

$$u = F(r, \theta, \phi) e^{-k^2 \alpha^2 t}$$

and

$$F = R(r) \Theta(\theta) \Phi(\phi)$$

so that

$$\nabla^2 F + k^2 F = 0$$

As before, we then get

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} + k^2 r^2 = 0$$

where the ϕ dependence has already been taken out (that is where separation constant m^2 came from) such that

$$\Phi(\phi) = A \cos m\phi + B \sin m\phi$$

The first and last terms are functions of r only; the middle terms are functions of θ only. Thus, we can separate the r and θ equations (as we have done before) to get

$$\begin{aligned} \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + k^2 r^2 &= \ell(\ell + 1) \\ \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} + \ell(\ell + 1) &= 0 \end{aligned}$$

where we have called the separation constant $\ell(\ell + 1)$.

The solutions of the θ equation are $P_\ell^m(\cos \theta)$. The solutions of the r equation are $R = r^{-1/2} J_p(kr)$ where $p = \ell + 1/2$. Then

$$u = r^{-1/2} J_p(kr) P_\ell^m(\cos \theta) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} e^{-k^2 \alpha^2 t}$$

Problem 9-5 p2

Since there is no θ or ϕ dependence in this problem, we take $m = 0, \ell = 0, p = 1/2$. Thus,

$$u = r^{-1/2} J_{1/2}(kr) e^{-k^2 \alpha^2 t}$$

or

$$u = j_0(kr) e^{-k^2 \alpha^2 t}$$

or

$$u = r^{-1} \sin kr e^{-k^2 \alpha^2 t}$$

Following the hint, we want $u = 0$ for all $t > 0$ when $r = a$, and $u = -100$ for $t = 0$.

$$j_0(ka) = \frac{1}{ka} \sin ka = 0 \rightarrow k = n\pi/a$$

Therefore

$$\begin{aligned} u &= \sum_n c_n j_0\left(\frac{n\pi r}{a}\right) e^{-(n\pi\alpha/a)^2 t} & \text{or} & \quad u = \frac{1}{r} \sum_n b_n \sin \frac{n\pi r}{a} e^{-(n\pi\alpha/a)^2 t} \\ -100 &= \sum_n c_n j_0\left(\frac{n\pi r}{a}\right) & \text{or} & \quad -100r = \sum_n b_n \sin \frac{n\pi r}{a} \end{aligned} \quad (\text{A})$$

Fourier series solution: Expand $-100r$ in a Fourier sine series on $(0, a)$. You should find

$$b_n = (-1)^n \frac{200a}{n\pi}$$

Substitute these coefficients back into (A) and add back the 100° we subtracted to get the answer as a Fourier series.

Bessel series solution:

$$-100 = \sum_n c_n j_0\left(\frac{n\pi r}{a}\right) = \sum_n c_n \sqrt{\frac{\pi a}{2n\pi r}} J_{1/2}\left(\frac{n\pi r}{a}\right)$$

Let $x = r/a$ to get

$$-100\sqrt{x} = \sum_n \frac{c_n}{\sqrt{2n}} J_{1/2}(n\pi x)$$

Multiply by $x J_{1/2}(m\pi x)$ and integrate from 0 to 1.

$$-100 \int_0^1 x^{3/2} J_{1/2}(m\pi x) dx = \frac{c_m}{\sqrt{2m}} \int_0^1 x J_{1/2}^2(m\pi x) dx \quad (\text{B})$$

Using properties of Bessel functions we have:

Problem 9-5 p3

Right-hand side:

$$\int_0^1 x J_{1/2}^2(m\pi x) dx = \frac{1}{2} J_{3/2}^2(m\pi)$$

Left-hand side:

$$\begin{aligned} \int_0^1 x^{3/2} J_{1/2}(m\pi x) dx &= \int_0^{m\pi} \left(\frac{u}{m\pi}\right)^{3/2} J_{1/2}(u) \frac{du}{m\pi} \\ &= \frac{1}{(m\pi)^{5/2}} u^{3/2} J_{3/2}(u) \Big|_0^{m\pi} \\ &= \frac{1}{m\pi} J_{3/2}(m\pi) \end{aligned}$$

Substituting into (B) we get

$$-100 \frac{1}{m\pi} J_{3/2}(m\pi) = \frac{c_m}{\sqrt{2m}} \frac{1}{2} J_{3/2}^2(m\pi)$$

or

$$c_m = \frac{-200\sqrt{2m}}{m\pi J_{3/2}(m\pi)} \quad (C)$$

Now we have the following relations:

$$\begin{aligned} J_{3/2}(x) &= \sqrt{\frac{2x}{\pi}} x \left(-\frac{1}{x} \frac{d}{dx} \right) \frac{\sin x}{x} = -\sqrt{\frac{2x}{\pi}} \frac{x \cos x - \sin x}{x^2} \\ J_{3/2}(m\pi) &= -\sqrt{\frac{2m\pi}{\pi}} \frac{m\pi \cos m\pi - \sin m\pi}{(m\pi)^2} = -\sqrt{2m} \frac{(-1)^m}{m\pi} \end{aligned}$$

Thus from (C)

$$c_m = 200(-1)^m$$

and (A) plus 100° gives the result as a Bessel series

$$u = 200 \sum_n (-1)^n j_0(n\pi r/a) e^{-(n\pi a/a)^2 t} + 100$$

If we use

$$j_0(x) = \frac{\sin x}{x}$$

i.e.,

$$j_0(n\pi r/a) = \frac{a}{n\pi r} \sin(n\pi r/a)$$

we find

$$u = \frac{200}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi r}{a} e^{-(n\pi a/a)^2 t} + 100$$

which is the Fourier series solution that you should have obtained earlier

Problem 9-6 p1

We consider concentric spherical surfaces each held at a fixed potential. The boundary potentials in this problem depends only on r ; i.e., at $r = 1$ we have $V = 0$ and at $r = 2$ we have $V = 100$. That means there is no θ or ϕ dependence in the solution to the problem.

Let us review the solution of Laplace's equation in 3-dimension with spherical-polar coordinate. We have

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

This potential has no ϕ dependence so we can write

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

Let us assume for the moment that there is some θ dependence in the potential, it is some function $f(\theta)$ instead of a constant as in the actual problem.

We separate variables as usual using $V(r, \theta) = R(r)\Theta(\theta)$ and get

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = - \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\Theta \sin \theta) = \mu$$

or setting $\cos \theta = x$ so that $\Theta = \Theta(x)$ we have

$$r^2 R'' + 2rR' - \mu R = 0$$

$$(1 - x^2)\Theta'' - 2x\Theta' - \mu\Theta = 0$$

The first of these equations is the Cauchy-Euler equation and has solution

$$R(r) = c_1 r^{-1/2 + \sqrt{\mu+1/4}} + c_2 r^{-1/2 - \sqrt{\mu+1/4}}$$

This can be put in a better form by letting $-1/2 + \sqrt{\mu+1/4} = n$. Then

$$R(r) = c_1 r^n + \frac{c_2}{r^{n+1}}$$

Then the equation for Θ becomes Legendre's equation

$$(1 - x^2)\Theta'' - 2x\Theta' + n(n+1)\Theta = 0$$

Problem 9-6 p2

where n must be a positive integer for a proper solution to exist. The general solution to Legendre's equation is

$$\Theta(x) = c_3 P_n(x) + c_4 Q_n(x)$$

Since $Q_n(x) \rightarrow \infty$ as $x \rightarrow 1$ we choose $c_4 = 0$. Thus the solution is

$$V(r, x) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(x)$$

Now

$$V(1, x) = 0 = \sum_{n=0}^{\infty} (A_n + B_n) P_n(x) \rightarrow B_n = -A_n$$

or

$$V(r, x) = \sum_{n=0}^{\infty} A_n (r^n - r^{-(n+1)}) P_n(x)$$

Now

$$V(2, x) = 100 = \sum_{n=0}^{\infty} A_n (2^n - 2^{-(n+1)}) P_n(x)$$

Using the property

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

we get

$$A_n = \frac{100(2n+1)}{2^{n+1} - 2^{-n}} \int_{-1}^1 P_n(x) dx$$

so that

$$A_0 = 200; A_n = 0, n = 1, 2, 3, \dots$$

The solution is then

$$V(r, \theta) = 200 \left(1 - \frac{1}{r} \right)$$

Problem 9-7 p1

We will solve this problem by solving a different problem, namely, having the two halves of the sphere held at $2V$ and 0 instead of V and $-V$ and then subtracting V from the answer.

Choose the top hemisphere to be charged to $2V$ and the bottom hemisphere to be at zero potential, with the plane in which the two hemispheres meet perpendicular to the polar axis. The boundary condition then becomes

$$v(a, \theta, \phi) = \begin{cases} 2V & \text{for } 0 < \theta < \pi/2 \quad (0 < \cos \theta < 1) \\ 0 & \text{for } \pi/2 < \theta < \pi \quad (-1 < \cos \theta < 0) \end{cases} \quad (1)$$

The problem is clearly axially symmetric and so we may set $m = 0$ (no ϕ dependence). Also, we require the solution to be finite on the polar axis and so it cannot contain the $Q_\ell(\cos \theta)$ functions. Therefore the general form of the solution Laplace's equation in spherical polar coordinates is

$$v(r, \theta, \phi) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos \theta) \quad (B)$$

Inside the sphere (for $r < a$) we require the solution to be finite at the origin and so $B_\ell = 0$ for all ℓ in (B). Imposing the boundary conditions at $r = a$, we must then have

$$v(a, \theta, \phi) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos \theta)$$

where $v(a, \theta, \phi)$ is given by (A). Exploiting the mutual orthogonality of the Legendre polynomials, the coefficients in the Legendre polynomial expansion are given by (writing $\mu = \cos \theta$)

$$A_\ell a^\ell = \frac{2\ell+1}{2} \int_{-1}^1 v(a, \theta, \phi) P_\ell(\mu) d\mu = (2\ell+1)V \int_0^1 P_\ell(\mu) d\mu$$

where we have used (A). The integrals of the Legendre polynomials are easily evaluated

$$A_0 = V, \quad A_1 = \frac{2V}{2a}, \quad A_2 = 0, \quad A_3 = -\frac{7V}{8a^3}, \quad \dots$$

so that the required solution inside the sphere is

$$v(r, \theta, \phi) = V \left[1 + \frac{3r}{2a} P_1(\cos \theta) - \frac{7r^3}{8a^3} P_3(\cos \theta) + \dots \right]$$

Problem 9-7 p2

Not needed for original problem but we will show it anyway. Outside the sphere (for $r > a$) we require the solution to be bounded as $r \rightarrow \infty$ and so in (B) we must have $A_\ell = 0$ for all ℓ . In this case, by imposing the boundary condition at $r = a$ we require

$$v(a, \theta, \phi) = \sum_{\ell=0}^{\infty} B_\ell a^{-(\ell+1)} P_\ell(\cos \theta)$$

where $v(a, \theta, \phi)$ is given by (A). In the same way as above, the coefficients in the expansion

$$B_\ell a^{-(\ell+1)} = \frac{2\ell+1}{2} \int_{-1}^1 v(a, \theta, \phi) P_\ell(\mu) d\mu = (2\ell+1)V \int_0^1 P_\ell(\mu) d\mu$$

so that the required solution outside the sphere is

$$v(r, \theta, \phi) = V \frac{a}{r} \left[1 + \frac{3a}{2r} P_1(\cos \theta) - \frac{7a^3}{8r^3} P_3(\cos \theta) + \dots \right]$$

All that remains to get the original problem is to subtract V from the solution inside the sphere. This has the effect of removing the constant term and giving

$$v_{\text{inside}}(r, \theta, \phi) = V \left[\frac{3r}{2a} P_1(\cos \theta) - \frac{7r^3}{8a^3} P_3(\cos \theta) + \dots \right]$$

We also note that the full solution for the coefficients (inside the sphere) is

$$A_{2n+1} a^{2n+1} = \frac{2(2n+1)+1}{2} 2V \frac{(-1)^n (2n)!}{2^{2n+1} n! (n+1)!}$$

giving

$$v_{\text{inside}}(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (4n+3)}{2^{2n+1} n! (n+1)!} \left(\frac{r}{a} \right)^{2n+1} P_{2n+1}(\cos \theta)$$

Assignment 10

Problem 10-1

(a)

$$(-1)^{1/3} = (e^{i\pi})^{1/3} = e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$(-1)^{1/3} = (e^{i3\pi})^{1/3} = e^{i\pi} = \cos \pi + i \sin \pi = -1$$

$$(-1)^{1/3} = (e^{i5\pi})^{1/3} = e^{i5\pi/3} = \cos 5\pi/3 + i \sin 5\pi/3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(b) Since

$$a + bi = re^{i\theta} \quad \text{where} \quad r = \sqrt{a^2 + b^2}, \tan \theta = b/a$$

then

$$2 - i = \sqrt{5}e^{i5.82i}$$

$$(2 - i)^{1/2} = 5^{1/4}e^{i2.91i} = 1.495(\cos 2.91 + i \sin 2.91) = -1.455 + 0.3431i$$

(c)

$$\begin{aligned} (-1 - i)^{-i/2} &= e^{-(i/2) \ln(-1-i)} = e^{-\frac{i}{2}(0.6931 + 3.927i)} = e^{1.964 - 0.5466i} \\ &= e^{1.964}(\cos 0.3466 - i \sin 0.3466) = 6.704 - 2.421i \end{aligned}$$

Problem 10-2

$$f(z) = \ln(1+z) \text{ , } f'(z) = \frac{1}{1+z} \text{ , } f''(z) = \frac{-1}{(1+z)^2} \text{ , } f'''(z) = \frac{2!}{(1+z)^3} \text{ , } \dots$$

Therefore

$$\ln(1+z) = \ln(1) + z - \frac{z^2}{2!} + \frac{2!}{3!}z^3 - \dots = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

Problem 10-3

$$\frac{\sin z}{1-z} = \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)(1 + z + z^2 + \cdots) = z + z^2 + \frac{5}{6}z^3 + \frac{1}{12}z^4 + \cdots$$

Problem 10-4

(a) $a = 1$

$$\begin{aligned}\frac{1}{z} &= \frac{1}{1 = (z-1)} = 1 - (z-1) + (z-1)^2 - \dots \quad 0 \leq |z-1| < 1 \\ \frac{1}{z} &= \frac{1}{z-1} \frac{1}{1 + \frac{1}{z-1}} = \frac{1}{z-1} \left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \dots \right] = \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \dots \quad 1 \leq |z-1|\end{aligned}$$

(b) $a = i$

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{z+i} \frac{1}{z-i} = \frac{1}{z-i} \frac{1}{2i+z-i} = \frac{1}{z-i} \frac{-i/2}{1 + \frac{z-i}{2i}} \\ &= \frac{-i/2}{z-i} \left[1 - \frac{z-i}{2i} + \frac{(z-i)^2}{4} - \dots \right] \\ &= -\frac{i}{2} \left[\frac{1}{z-i} + \frac{i}{2} + \frac{z-i}{4} - \frac{(z-i)^2}{8} + \dots \right] \quad 0 < |z-i| < 2 \\ \frac{1}{z^2+1} &= \frac{1}{(z-i)^2} \frac{1}{1 + \frac{2i}{z-i}} = \frac{1}{(z-i)^2} \left[1 - \frac{2i}{z-i} + \frac{4}{(z-i)^2} - \dots \right] \\ &= \frac{1}{(z-i)^2} - \frac{2i}{(z-i)^3} + \frac{4}{(z-i)^4} - \frac{8}{(z-i)^5} + \dots \quad 2 < |z-i|\end{aligned}$$

(c) $a = 2$

$$\begin{aligned}\frac{1}{(z+1)(z-2)} &= \frac{1}{z-2} \frac{1/3}{1 - \frac{z+1}{3}} = \frac{1}{3(z-2)} \left[1 - \frac{z-2}{3} + \frac{(z-2)^2}{9} - \dots \right] \\ &= \frac{1}{3(z-2)} - \frac{1}{9} + \frac{z-2}{27} - \frac{(z-2)^2}{81} + \dots \quad 0 < |z-2| < 3 \\ \frac{1}{(z+1)(z-2)} &= \frac{1}{z-2} \frac{1}{z-2} \frac{1}{1 - \frac{3}{z-2}} = \frac{1}{(z-2)^2} \left[1 + \frac{3}{z-2} + \frac{9}{(z-2)^2} + \dots \right] \\ &= \frac{1}{(z-2)^2} + \frac{3}{(z-2)^3} + \frac{9}{(z-2)^4} + \dots \quad 3 < |z-2|\end{aligned}$$

Problem 10-5

(a)

$$\frac{1}{z^2} \sin 2z = \frac{1}{z^2} \left(2z - \frac{8}{3} z^3 + \dots \right) \rightarrow b_1 = 2 \text{ at } z = 0$$

(b)

$$\begin{aligned} \frac{\cos z}{z^2 + 2z + 1} &= \frac{1}{(z+1)^2} \cos[(z+1) - 1] = \frac{1}{(z+1)^2} [\cos(z+1) \cos 1 + \sin(z+1) \sin 1] \\ &= \frac{\cos 1}{(z+1)^2} \left[1 - \frac{(z+1)^2}{2} + \dots \right] + \frac{\sin 1}{(z+1)^2} [z+1 - \dots] \rightarrow b_1 = \sin 1 \text{ at } z = -1 \end{aligned}$$

Problem 10-6

(a)

$$\frac{\sin z}{z^3} = \frac{1}{z^3} \left(z - \frac{1}{3!} z^3 + \dots \right) \rightarrow b_1 = 0 \rightarrow \oint \frac{\sin z}{z^3} dz = 0$$

(b)

$$\frac{\sin z}{z^2(z-1)} = \frac{1}{z} \left(z - \frac{1}{6} z^3 + \dots \right) (-1 - z - z^2 - \dots) \rightarrow b_1 = -1 \text{ at } z = 0$$

$$\begin{aligned} \frac{\sin z}{z^2(z-1)} &= \frac{\sin[(z-1)+1]}{[(z-1)+1]^2(z-1)} = \frac{\sin(z-1)\cos 1 + \cos(z-1)\sin 1}{(z-1)[1-(z-1)^2]^2} \\ &= \frac{1}{z-1} [1 - (z-1) + \dots]^2 [0.54[(z-1) - \dots] + 0.841[1 - \frac{(z-1)^2}{2} + \dots]] \\ &\rightarrow b_1 = 0.842 \text{ at } z = 1 \end{aligned}$$

Therefore

$$\oint \frac{\sin z}{z^3 - z^2} dz = 2\pi i(-1 + 0.841) = -i$$

(c)

$$\frac{z^2+1}{z(z+1)^3} = \frac{z^2+1}{z} [1 - z + z^2 - \dots] \rightarrow b_1 = 1 \text{ at } z = 0$$

$$\begin{aligned} \frac{z^2+1}{z(z+1)^3} &= \frac{z^2+1}{(z+1)^3} \frac{-1}{-(z+1)} = \frac{(z+1)^2 - 2(z+1) + 2}{z(z+1)^3} (-1)[1 + (z+1) + (z+1)^2 + \dots] \\ &= \left[-\frac{1}{z+1} + \frac{2}{(z+1)^2} - \frac{2}{(z+1)^3} \right] [1 + (z+1) + (z+1)^2 + \dots] \end{aligned}$$

Thus,

$$\rightarrow b + 1 = -1 + 2 - 2 = -1$$

Therefore

$$\oint \frac{z^2+1}{z(z+1)^3} dz = 2\pi i(1-1) = 0$$

Problem 10-7

(a)

$$\int_0^{2\pi} \frac{d\theta}{(2 + \cos \theta)^2} = \oint \frac{4z^2}{(z^2 + 4z + 1)^2} \frac{dz}{iz} = \oint \frac{-4zidz}{(z + 0.268)^2(z + 3.73)^2}$$

$$\begin{aligned} \frac{z}{(z + 0.268)^2} \frac{1}{[3.46 + (z + 0.268)]^2} &= \frac{(z + 0.268) - 0.268}{(z + 0.268)^2} \frac{1/(3.46)^2}{\left(1 + \frac{z+0.268}{3.46}\right)^2} \\ &= \left[\frac{1}{z + 0.268} - \frac{0.268}{(z + 0.268)^2} \right] \frac{1}{(3.46)^2} \left[1 - \frac{2 + 0.268}{3.46} \right] \end{aligned}$$

Therefore

$$b_1 = \frac{1}{(3.46)^2} \left[1 + \frac{2 \cdot 0.268}{3.46} \right] = 0.0965$$

and

$$\int_0^{2\pi} \frac{d\theta}{(2 + \cos \theta)^2} = -4i(0.0965)2\pi i = 2.43$$

(b)

$$\int_0^{2\pi} \frac{\sin 2\theta d\theta}{5 + 4 \cos \theta} = \oint \frac{\frac{1}{2i} \left(z^2 + \frac{1}{z^2} \right) dz}{5 + \frac{4(z^2+1)}{2z}} \frac{dz}{iz} = \frac{1}{2} \oint \frac{\frac{1}{z^2} - z^2}{2z^2 + 5z + 2} dz = \frac{1}{4} \oint \frac{\frac{1}{z^2} - z^2}{(z+2)(z+1/2)}$$

$$\begin{aligned} \frac{\frac{1}{z^2} - z^2}{(z+1/2)(3/2 + (z+1/2))} &= \frac{1}{z+1/2} \left\{ \frac{1}{[(z+1/2) - 1/2]^2} - [(z+1/2) - 1/2]^2 \right\} \frac{2/3}{1 + \frac{z+1/2}{3/2}} \\ &= \frac{1}{z+1/2} \left\{ 4 \left[1 + \frac{z+1/2}{1/2} + \dots \right]^2 - \frac{1}{4} \left[1 - \frac{z+1/2}{1/2} \right]^2 \right\} \frac{2}{3} \left[1 - \frac{z+1/2}{3/2} + \dots \right] \end{aligned}$$

$$b_1 = (4 - 1/4) \frac{2}{3} = \frac{15}{6}$$

and

$$\int_0^{2\pi} \frac{\sin 2\theta d\theta}{5 + 4 \cos \theta} = \frac{1}{4} \frac{15}{6} 2\pi i = 3.93i$$

Problem 10-8

- (a) $(1+z^3)$ has zeros $e^{\pi i/3}$, $e^{\pi i}$, $e^{5\pi i/3}$. Only $e^{\pi i/3}$ is in the upper half-plane.
There

$$z = \frac{1 + \sqrt{3}i}{2}$$

Therefore

$$b_1 = \left(z - \frac{1 + \sqrt{3}i}{2} \right) \frac{1+z}{1+z^3} \Big|_{z=\frac{1+\sqrt{3}i}{2}} = \frac{1 + \frac{1+\sqrt{3}i}{2}}{\left(\frac{1+\sqrt{3}i}{2} + i \right) \left(\frac{1+\sqrt{3}i}{2} - \frac{1-\sqrt{3}i}{2} \right)} = 0.366(-1-i)$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^3} dx = -0.366(1+i)2\pi i = 2.3(1-i)$$

- (b) $(1+z)^2$ has a zero of order two at $z=i$. Therefore

$$b_1 = \frac{d}{dz} \frac{(z-i)^2 e^{iz}}{(z-i)2(z+i)^2} \Big|_{z=i} = \frac{i(z+i)^3 e^{iz} - e^{iz}(z+i)2}{(z+i)^4} \Big|_{z=i} = \frac{i(2i)^2 e^{-1} - e^{-1}2i(2)}{(2i)^2} = -\frac{i}{2e}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2} = 2\pi i \left(-\frac{i}{2e} \right) = \frac{\pi}{e}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2} = \frac{\pi}{e}$$

- (c) $(1+z^4)$ has zeros at $z = \frac{1+i}{\sqrt{2}}$, $z = \frac{-1+i}{\sqrt{2}}$ in the upper half-plane.

$$b_1 = \left(z - \frac{1+i}{\sqrt{2}} \right) \frac{e^{iz}}{1+z^4} \Big|_{z=\frac{1+i}{\sqrt{2}}} = \frac{e^{\frac{1+i}{\sqrt{2}}}}{2i \left(\frac{1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} \right)} = -0.00958 - 0.123i$$

$$b_1 = \left(z - \frac{-1+i}{\sqrt{2}} \right) \frac{e^{iz}}{1+z^4} \Big|_{z=\frac{-1+i}{\sqrt{2}}} = \frac{e^{\frac{-1+i}{\sqrt{2}}}}{2i \left(\frac{-1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \right)} = 0.00958 - 0.123i$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4} dx = 2\pi i(-0.246i) = 1.546 \rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^4} = 0.773$$

Problem 10-9

$$\int \frac{dz}{z^4-1} = \int \frac{dz}{(z-1)(z+1)(z-i)(z+i)} = \frac{1}{4} \int \frac{dz}{z-1} - \frac{1}{4} \int \frac{dz}{z+1} + \frac{1}{4i} \int \frac{dz}{z+i} - \frac{1}{4i} \int \frac{dz}{z-i}$$

For the first integral let $z-1 = \epsilon e^{i\theta}$. Then

$$\int_{-\pi}^0 \frac{\epsilon i e^{i\theta}}{\epsilon e^{i\theta}} d\theta = -i(-\pi) = \pi i$$

For the first integral let $z+1 = \epsilon e^{i\theta}$. Then

$$\int_{-\pi}^0 \frac{\epsilon i e^{i\theta}}{\epsilon e^{i\theta}} d\theta = -i(-\pi) = \pi i$$

For the integral with a zero in the upper half-plane,

$$b_1 = (z-i) \frac{1}{z^4+1} \Big|_{z=i} = \frac{1}{-2(2i)} = \frac{i}{4}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^4-1} = \pi i + \pi i + 2\pi i \left(\frac{i}{4} \right) = -\frac{\pi}{2} + 2\pi i$$

Assignment 11

Problem 11-1

Consider a rotation of the (unprimed) coordinate axes through an angle θ to give the new (primed) axes. Under this rotation,

$$\begin{aligned}x_1 &\rightarrow x'_1 = x_1 \cos \theta + x_2 \sin \theta, \\x_2 &\rightarrow x'_2 = -x_1 \sin \theta + x_2 \cos \theta, \\x_3 &\rightarrow x'_3 = x_3,\end{aligned}$$

and the transformation matrix L_{ij} is given by

$$L = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Denoting $\cos \theta$ and $\sin \theta$ by c and s , respectively, we compare

$$u'_1 = x'_1 \cos \phi - x'_2 \sin \phi = cx_1 \cos \phi + sx_2 \cos \phi + sx_1 \sin \phi - cx_2 \sin \phi$$

with

$$u'_1 = cu_1 + su_2 = cx_1 \cos \phi - cx_2 \sin \phi + sx_1 \sin \phi + sx_2 \cos \phi.$$

These two are equal, showing that the first component transforms correctly. However, this alone is not sufficient; for (u_1, u_2) to be the components of a first-order tensor, *all* components must transform correctly. We therefore also compare the remaining transformed component:

$$u'_2 = x'_1 \sin \phi + x'_2 \cos \phi = cx_1 \sin \phi + sx_2 \sin \phi - sx_1 \cos \phi + cx_2 \cos \phi$$

is to be compared with

$$u'_2 = -su_1 + cu_2 = -sx_1 \cos \phi + sx_2 \sin \phi + cx_1 \sin \phi + cx_2 \cos \phi.$$

These two are also equal, showing that both components do transform correctly and that (u_1, u_2) are indeed the components of a first-order tensor.

We note, in passing, that $u_1 + iu_2$ is the complex vector obtained by rotating the 'base vector' $x_1 + ix_2$ through an angle ϕ in the complex plane:

$$\begin{aligned}u_1 + iu_2 &= e^{i\phi}(x_1 + ix_2) \\&= (\cos \phi + i \sin \phi)(x_1 + ix_2) \\&= (x_1 \cos \phi - x_2 \sin \phi) + i(x_1 \sin \phi + x_2 \cos \phi).\end{aligned}$$

In view of this observation, and of the definition of a first-order tensor as a set of objects 'that transform in the same way as a position vector', it is perhaps not surprising to find that the given expressions form the components of a tensor.

(b) Consider the transform of the first element $u_{11} = x_1^2$. This becomes

$$u'_{11} = (x'_1)^2 = (-sx_1 + cx_2)^2 = s^2x_1^2 - 2scx_1x_2 + c^2x_2^2.$$

If it transforms as a component of a tensor, then it must also be the case that

$$u'_{11} = L_{1k}L_{1l}u_{kl} = c^2x_2^2 + cscx_1x_2 + scx_1x_2 + s^2x_1^2.$$

But, these two RHSs are not equal, and it follows that the given set of expressions cannot form the components of a tensor of order 2. It is not necessary to consider any more u_{ij} ; failure of any one element to transform correctly rules out the possibility of the set being a tensor.

Problem 11-2

Since we must have $x'_i = L_{ij}x_j$ and **A** and **B** have their components transformed into the given values, **L** must have the form

$$\mathbf{L} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix}.$$

We determine a , b and c by requiring that **L** is orthogonal and has $|\mathbf{L}| = +1$.

$$\mathbf{L}\mathbf{L}^T = \frac{1}{4} \begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ a & b & c \end{pmatrix} = \mathbf{I},$$

giving $a = 0$, $b = \pm 2$ and $c = 0$. The determinant of **L** is $\frac{1}{8}(-3b - b + 0)$, thus requiring that $b = -2$. Hence the required orthogonal matrix **L** is

$$\mathbf{L} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}.$$

The third column of **L** could have been obtained by considering $\mathbf{A}' \times \mathbf{B}'$. The matrix product $\mathbf{T}' = \mathbf{L}\mathbf{T}\mathbf{L}^T$ is given by

$$\begin{aligned} \mathbf{T}' &= \frac{1}{4} \begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ 0 & -2 & 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 5 \\ -1 & 0 & 5\sqrt{3} \\ 0 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \end{aligned}$$

As a check, we note that $\text{Tr } \mathbf{T}' = 1 + 2 + 5 = 2 + 4 + 2 = \text{Tr } \mathbf{T}$.

In this new coordinate system **T** is diagonal – and therefore very convenient for calculating the following invariants (tensors of order 0); their values are independent of the frame in which they are calculated.

$$T_{ij}T_{ji} = T'_{ij}T'_{ji} = 1 + 4 + 25 = 30.$$

$$T_{ki}T_{jk}T_{ij} = T'_{ij}T'_{jk}T'_{ki} = 1 + 8 + 125 = 134.$$

$$T_{ik}T_{mn}T_{ni}T_{km} = T'_{ik}T'_{km}T'_{mn}T'_{ni} = 1 + 16 + 625 = 642.$$

Problem 11-3

To test whether the array is a second-order tensor we need to contract it with an arbitrary known second-order tensor. By 'arbitrary tensor' we mean a tensor in which any one component can be made to be the only non-zero component.

Since any second-order tensor can always be written as the sum of a symmetric and an anti-symmetric tensor, and all operations are linear, it will be sufficient to prove the result for one known symmetric tensor and one known antisymmetric tensor. The simplest symmetric second-order tensor S_{ij} is the (symmetric) outer product of the (by definition) first-order tensor (x, y, z) with itself, i.e. $S_{ij} = x_i x_j$. Denoting the given array by B_{ij} , we consider

$$\begin{aligned} B_{ij}S_{ij} &= B_{ij}x_i x_j \\ &= x^2(y^2 + z^2 - x^2) + y^2(x^2 + z^2 - y^2) + z^2(x^2 + y^2 - z^2) \\ &\quad + 2[xy(-2xy) + xz(-2xz) + yz(-2yz)] \\ &= -2x^2y^2 - 2x^2z^2 - 2y^2z^2 - x^4 - y^4 - z^4 \\ &= -(x^2 + y^2 + z^2)^2 = -|\mathbf{x}|^4. \end{aligned}$$

The term in parentheses in the last line is formally $x_i x_i$, i.e. the contracted product of a first-order tensor with itself, and therefore an invariant zero-order tensor. Squaring an invariant or multiplying it by a constant (-1) leaves it as an invariant, leading to the conclusion that $B_{ij}S_{ij}$ is a zero-order tensor.

We now turn to an antisymmetric tensor, where a suitable second-order tensor A_{ij} is the contraction of the third-order tensor ϵ_{ijk} with the first-order tensor x_i . Thus A_{ij} has the form

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

and the contracted tensor is

$$B_{ij}A_{ij} = 0 - 2xyz + 2yxz + 2zyx + 0 - 2xyz - 2yzx + 2xzy + 0 = 0.$$

Now, 0 is an even more obvious invariant than $|\mathbf{x}|^2$ and so $B_{ij}A_{ij}$ is also a zero-order tensor.

Taking the results of the last two paragraphs together, it follows from the quotient law that B_{ij} is a second-order tensor.

Problem 11-4

(a) The matrix equation $\mathbf{A}\mathbf{b} = \mathbf{c}$ will have the explicit form

$$\mathbf{c} = \mathbf{A}\mathbf{b} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix} = \mathbf{a} \times \mathbf{b},$$

i.e. $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

(b) The characteristic equation for $\mathbf{A}$ is

$$\begin{aligned} 0 = |\mathbf{A} - \lambda \mathbf{I}| &= \begin{vmatrix} -\lambda & -a_z & a_y \\ a_z & -\lambda & -a_x \\ -a_y & a_x & -\lambda \end{vmatrix} \\ &= -\lambda^3 - a_x^2 \lambda - a_z(a_x a_y + \lambda a_z) + a_y(a_z a_x - \lambda a_y). \end{aligned}$$

Thus $\lambda = 0$ or $-\lambda^2 - a_x^2 - a_z^2 - a_y^2 = 0$. The second possibility gives $\lambda = \pm i(a_x^2 + a_y^2 + a_z^2)^{1/2} = \pm i|\mathbf{a}|$.

Now consider $\mathbf{A}\mathbf{a}$. From part (a) this is

$$\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} -a_z a_y + a_y a_z \\ a_z a_x - a_x a_z \\ -a_y a_x + a_x a_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\mathbf{a}.$$

Thus $\mathbf{a}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue 0. That this must be so also follows from the general conclusion of part (a) that if $\mathbf{c} = \mathbf{A}\mathbf{b}$ then $\mathbf{c} = \mathbf{a} \times \mathbf{b}$; if $\mathbf{b} = \mathbf{a}$, then $\mathbf{c} = \mathbf{a} \times \mathbf{a} = \mathbf{0} = 0\mathbf{a}$.

Problem 11-5

For the covariant derivative of the covariant components of a vector, we have

$$v_{i;j} = \frac{\partial v_i}{\partial u^j} - \Gamma_{ij}^k v_k,$$

where Γ_{ij}^k is a Christoffel symbol of the second kind. Hence,

$$\begin{aligned} v_{i;jk} &\equiv (v_{i;j})_{;k} \\ &= \left(\frac{\partial v_i}{\partial u^j} - \Gamma_{ij}^l v_l \right)_{;k} \\ &= \frac{\partial}{\partial u^k} \left(\frac{\partial v_i}{\partial u^j} - \Gamma_{ij}^l v_l \right) - \Gamma_{ik}^m \left(\frac{\partial v_m}{\partial u^j} - \Gamma_{mj}^l v_l \right) \\ &= \frac{\partial^2 v_i}{\partial u^k \partial u^j} - \Gamma_{ij}^l \frac{\partial v_l}{\partial u^k} - v_l \frac{\partial \Gamma_{ij}^l}{\partial u^k} - \Gamma_{ik}^m \frac{\partial v_m}{\partial u^j} + \Gamma_{ik}^m \Gamma_{mj}^l v_l. \end{aligned}$$

Interchanging subscripts j and k ,

$$v_{i;kj} = \frac{\partial^2 v_i}{\partial u^j \partial u^k} - \Gamma_{ik}^l \frac{\partial v_l}{\partial u^j} - v_l \frac{\partial \Gamma_{ik}^l}{\partial u^j} - \Gamma_{ij}^m \frac{\partial v_m}{\partial u^k} + \Gamma_{ij}^m \Gamma_{mk}^l v_l.$$

When these two expressions are subtracted to define the Riemann tensor, the first, second and fourth terms (the second of one with the fourth of the other and vice versa) on the two RHSs cancel in pairs to yield

$$R_{ijk}^l v_l \equiv v_{i;jk} - v_{i;kj} = \left(\frac{\partial \Gamma_{ik}^l}{\partial u^j} - \frac{\partial \Gamma_{ij}^l}{\partial u^k} + \Gamma_{ik}^m \Gamma_{mj}^l - \Gamma_{ij}^m \Gamma_{mk}^l \right) v_l.$$

Now, in three-dimensional Euclidean space, one possible coordinate system is the Cartesian one. In this system $g = 1$ and all of its derivatives are zero. Thus all Christoffel symbols and their derivatives are zero, as are all components of

the Riemann tensor. As *all* the components vanish in this Cartesian coordinate system, they must do so in *any* coordinate system in this space.

Problem 11-6

Starting from Christoffel symbols of the second kind, we have

$$\begin{aligned}\Gamma_{ijk} &= g_{il}\Gamma_{jk}^l \\ &= \frac{1}{2}g_{il}g^{ln}\left(\frac{\partial g_{kn}}{\partial u^j} + \frac{\partial g_{nj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^n}\right) \\ &= \frac{1}{2}\delta_i^n\left(\frac{\partial g_{kn}}{\partial u^j} + \frac{\partial g_{nj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^n}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right).\end{aligned}$$

Next, forming the symmetric sum of two Christoffel symbols:

$$\begin{aligned}\Gamma_{ijk} + \Gamma_{jik} &= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right) + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ji}}{\partial u^k}\right) + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ik}}{\partial u^j}\right) \\ &\quad + \frac{1}{2}\left(-\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{kj}}{\partial u^i}\right) \\ &= \frac{\partial g_{ij}}{\partial u^k} + 0 + 0.\end{aligned}$$

To obtain the last line we have used the fact that the metric tensor is symmetric, $g_{ij} = g_{ji}$.

Further, since $\Gamma_{jk}^l = \Gamma_{kj}^l$, and therefore $g_{il}\Gamma_{jk}^l = g_{il}\Gamma_{kj}^l$, we have that $\Gamma_{ijk} = \Gamma_{ikj}$, i.e. Christoffel symbols of the first kind are symmetric under the interchange of the last two indices.

Finally, forming the covariant derivative of g_{ij} :

$$\begin{aligned}g_{ij;k} &= \frac{\partial}{\partial u^k}(g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j) \\ &= \frac{\partial g_{ij}}{\partial u^k}\mathbf{e}^i \otimes \mathbf{e}^j + g_{ij}\frac{\partial \mathbf{e}^i}{\partial u^k} \otimes \mathbf{e}^j + g_{ij}\mathbf{e}^i \otimes \frac{\partial \mathbf{e}^j}{\partial u^k} \\ &= \frac{\partial g_{ij}}{\partial u^k}\mathbf{e}^i \otimes \mathbf{e}^j + g_{ij}(-\Gamma_{ik}^i\mathbf{e}^i) \otimes \mathbf{e}^j + g_{ij}\mathbf{e}^i \otimes (-\Gamma_{mk}^j\mathbf{e}^m) \\ &= \frac{\partial g_{ij}}{\partial u^k}\mathbf{e}^i \otimes \mathbf{e}^j - \Gamma_{jk}^i\mathbf{e}^i \otimes \mathbf{e}^j - \Gamma_{mk}^j\mathbf{e}^i \otimes \mathbf{e}^m, \text{ since } g_{ij} = g_{ji}, \\ &= \left(\frac{\partial g_{ij}}{\partial u^k} - \Gamma_{jk}^i - \Gamma_{ik}^j\right)\mathbf{e}^i \otimes \mathbf{e}^j, \text{ renaming dummy suffices,} \\ &= 0(\mathbf{e}^i \otimes \mathbf{e}^j), \text{ from the previous result.}\end{aligned}$$

Thus the covariant derivative of the metric tensor is identically zero in all coordinate systems.

Problem 11-7 p1

To find a surface of constant u we eliminate v from the given relationships:

$$x^2 + y^2 = u^2 v^2 \quad \Rightarrow \quad 2z = u^2 - \frac{x^2 + y^2}{u^2}.$$

This is an inverted paraboloid of revolution about the z -axis. The range of z is $-\infty < z \leq \frac{1}{2}u^2$.

Similarly, the surface of constant v is given by

$$2z = \frac{x^2 + y^2}{v^2} - v^2.$$

This is also a paraboloid of revolution about the z -axis, but this time it is not inverted. The range of z is $-\frac{1}{2}v^2 \leq z < \infty$.

Since every constant- u paraboloid has some part of its surface in the region $z > 0$ and every constant- v paraboloid has some part of its surface in the region $z < 0$, it follows that every member of the first set intersects each member of the second, and vice-versa.

The surfaces of constant ϕ , $y = x \tan \phi$, are clearly (half-) planes containing the z -axis; each cuts the members of the other two sets in parabolic lines.

We now determine (the Cartesian components of) the tangential vectors and test their orthogonality:

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial u} = (v \cos \phi, v \sin \phi, u), \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial v} = (u \cos \phi, u \sin \phi, -v), \\ \mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial \phi} = (-uv \sin \phi, uv \cos \phi, 0), \\ \mathbf{e}_1 \cdot \mathbf{e}_2 &= uv(\cos \phi \cos \phi + \sin \phi \sin \phi) - uv = 0, \\ \mathbf{e}_2 \cdot \mathbf{e}_3 &= u^2 v(-\cos \phi \sin \phi + \sin \phi \cos \phi) = 0, \\ \mathbf{e}_1 \cdot \mathbf{e}_3 &= uv^2(-\cos \phi \sin \phi + \sin \phi \cos \phi) = 0. \end{aligned}$$

This shows that all pairs of tangential vectors are orthogonal and therefore that the coordinate system is an orthogonal one. Its scale factors are given by the magnitudes of these tangential vectors:

$$\begin{aligned} h_u^2 &= |\mathbf{e}_1|^2 = (v \cos \phi)^2 + (v \sin \phi)^2 + u^2 = u^2 + v^2, \\ h_v^2 &= |\mathbf{e}_2|^2 = (u \cos \phi)^2 + (u \sin \phi)^2 + v^2 = u^2 + v^2, \\ h_\phi^2 &= |\mathbf{e}_3|^2 = (uv \sin \phi)^2 + (uv \cos \phi)^2 = u^2 v^2. \end{aligned}$$

Problem 11-7 p2

Thus

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\phi = uv.$$

The u -component of $\nabla \times \mathbf{a}$ is given by

$$\begin{aligned} [\nabla \times \mathbf{a}]_u &= \frac{h_u}{h_u h_v h_\phi} \left[\frac{\partial}{\partial v} (h_\phi a_\phi) - \frac{\partial}{\partial \phi} (h_v a_v) \right] \\ &= \frac{1}{uv \sqrt{u^2 + v^2}} \left[\frac{\partial}{\partial v} (uv a_\phi) - \frac{\partial}{\partial \phi} (\sqrt{u^2 + v^2} a_v) \right] \\ &= \frac{1}{\sqrt{u^2 + v^2}} \left(\frac{a_\phi}{v} + \frac{\partial a_\phi}{\partial v} \right) - \frac{1}{uv} \frac{\partial a_v}{\partial \phi}, \end{aligned}$$

as stated in the question.

Problem 11-8 p1

As K_{mn} is to be a second-order tensor, we need to construct such a tensor from T_{ijkl} . Since the latter is of fourth order, it needs to be contracted n times with a tensor of order $2n-2$ for some positive integer n . In view of the final stated expression for T_{ijkl} , involving $\epsilon_{ijm}\epsilon_{kln}$, i.e. a sixth-order tensor, we try $n = 4$ and, starting from $T_{ijkl} = \epsilon_{ijm}\epsilon_{kln}K_{mn}$ consider

$$\begin{aligned}\epsilon_{pij}\epsilon_{qkl}T_{ijkl} &= \epsilon_{pij}\epsilon_{qkl}\epsilon_{ijm}\epsilon_{kln}K_{mn} \\ &= (\delta_{pj}\delta_{qm} - \delta_{jm}\delta_{pq})(\delta_{il}\delta_{qn} - \delta_{ln}\delta_{qi})K_{mn} \\ &= (3\delta_{pm} - \delta_{pm})(3\delta_{qn} - \delta_{qn})K_{mn} \\ &= 4K_{pq}\end{aligned}$$

Clearly, $K_{mn} = \frac{1}{4}\epsilon_{mij}\epsilon_{nkl}T_{ijkl}$ has the required property.

- (a) Given that T_{ijkl} is isotropic, and noting that ϵ_{mij} and ϵ_{nkl} are also isotropic, we conclude that K_{mn} must itself be isotropic. It must therefore be some multiple of δ_{mn} (as this is the most general isotropic second-order tensor), i.e., $K_{mn} = \lambda\delta_{mn}$ for one or more values of λ . Thus,

$$T_{ijkl} = \epsilon_{ijm}\epsilon_{kln}\lambda\delta_{mn} = \lambda\epsilon_{ijm}\epsilon_{klm} = \lambda(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Now, since $T_{ijji} = 1$,

$$\begin{aligned}1 &= \lambda(\delta_{ij}\delta_{ji} - \delta_{ii}\delta_{jj}) \\ &= \lambda[\delta_{ii} - (\delta_{ii})^2] = \lambda[3 - 9] \rightarrow \lambda = -\frac{1}{6}\end{aligned}$$

We conclude that λ , and therefore also T_{ijkl} , is unique with $T_{ijkl} = \frac{1}{6}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$.

- (b) To examine the implications of the antisymmetry indicated by $T_{klji} = -T_{ijkl}$ we interchange the pair of dummy suffices $\{i, j\}$ with the pair $\{k, l\}$ to obtain the third line below - and then switch them back again

Problem 11-8 p2

in the fourth line using the antisymmetry:

$$\begin{aligned}
 K_{mn} &= \frac{1}{4} \epsilon m i j \epsilon_{nkl} T_{ijkl} \\
 K_{nm} &= \frac{1}{4} \epsilon n i j l \epsilon_{mkl} T_{ijkl} \\
 &= \frac{1}{4} \epsilon_{nkl} \epsilon m i j T_{kl ij} \\
 &= -\frac{1}{4} \epsilon_{nkl} \epsilon m i j T_{ijkl} \\
 &= -K_{mn}
 \end{aligned}$$

Thus K_{mn} is antisymmetric. It therefore has zeros on its leading diagonal and only three linearly independent components as non-diagonal elements. Since T_{ijkl} is uniquely defined in terms of K_{mn} , it too has only three linearly independent components.

Now consider

$$\begin{aligned}
 \epsilon_{jkl} T_{ijkl} &= \\
 -4V_i &= (\delta_{km} \delta_{li} - \delta_{ki} \delta_{lm}) \epsilon_{kl n} K_{mn} \\
 &= (\epsilon_{min} - \epsilon_{imn}) K_{mn} \\
 &= 2\epsilon_{min} K_{mn}
 \end{aligned}$$

To 'invert' this relationship, consider

$$\begin{aligned}
 \epsilon_{irs} V_i &= -\frac{1}{2} \epsilon_{irs} \epsilon_{min} K_{mn} \\
 &= -\frac{1}{2} (\delta_{rn} \delta_{sm} - \delta_{rm} \delta_{sn}) K_{mn} \\
 &= -\frac{1}{2} (K_{sr} - K_{rs}) \\
 &= K_{rs} \rightarrow K_{mn} = \epsilon_{pmn} V_p
 \end{aligned}$$

Finally, expressing T_{ijkl} as given in the question, explicitly in terms of the vector V_i , using the result obtained above, we have

$$\begin{aligned}
 T_{ijkl} &= \epsilon_{ijm} \epsilon_{kln} \epsilon_{pmn} V_p \\
 &= (\delta_{in} \delta_{jp} - \delta_{ip} \delta_{jn}) \epsilon_{kl n} V_p \\
 &= \epsilon_{kli} V_j - \epsilon_{klj} V_i
 \end{aligned}$$

Problem 11-9

$$ds^2 = \sum_{ij} g_{ij} dx_i dx_j = \sum_{kl} g'_{kl} dx'_k dx'_l$$

We know that dx is a contravariant vector, that is

$$dx_i = \sum_k \frac{\partial x_i}{\partial x'_k} dx'_k$$

so

$$\begin{aligned} \sum_{kl} g'_{kl} dx'_k dx'_l &\equiv \sum_{ij} g_{ij} \sum_k \frac{\partial x_i}{\partial x'_k} dx'_k \sum_l \frac{\partial x_j}{\partial x'_l} dx'_l \\ &= \sum_{kl} \left(\sum_{ij} g_{ij} \frac{\partial x_i}{\partial x'_k} \frac{\partial x_j}{\partial x'_l} \right) dx'_k dx'_l \end{aligned}$$

Because this is an identity, the coefficients of $dx'_k dx'_l$ on the two sides must be equal. hence

$$g'_{kl} = \sum_{ij} g_{ij} \frac{\partial x_i}{\partial x'_k} \frac{\partial x_j}{\partial x'_l}$$

This is the transformation equation for a second-order covariant tensor!

Assignment 12

Problem 12-1

We first calculate the probabilities $p(x)$ and the corresponding gains $g(x) = p(x)x^2 - [1 - p(x)]x$ for each value of the total x . Expressing both in units of $1/16$, they are as follows:

x	2	3	4	5	6	7	8
$p(x)$	1	2	3	4	3	2	1
$g(x)$	-26	-24	-4	40	30	0	-56

(a) If B 's guess is random in the range $2 \leq x \leq 8$ then his expected return is

$$\frac{1}{16} \frac{1}{7} (-26 - 24 - 4 + 40 + 30 + 0 - 56) = -\frac{40}{112} = -0.36 \text{ euros.}$$

(b) If he picks by throwing his own dice then his distribution of guesses is the same as that of $p(x)$ and his expected return is

$$\begin{aligned} &\frac{1}{16} \frac{1}{16} [1(-26) + 2(-24) + 3(-4) + 4(40) + 3(30) + 2(0) + 1(-56)] \\ &= \frac{108}{256} = 0.42 \text{ euros.} \end{aligned}$$

(c) As is clear from the tabulation, the best return of $40/16 = 2.5$ euros is expected if B always chooses '5' as his guess. Of course, you should not advise him but offer to take his place!

Problem 12-2

Each shot has only two possible outcomes, a hit or a miss. P_1 is the probability that A will win when it is his turn to fire the next shot, and he is still able to do so (event W). There are three possible outcomes of the first two shots: C_1 , A hits with his shot; C_2 , A misses but B hits; C_3 , both miss. Thus

$$\begin{aligned} P_1 &= \sum_i Pr(C_i)Pr(W|C_i) \\ &= [\alpha \times 1] + [(1 - \alpha)\beta \times 0] + [(1 - \alpha)(1 - \beta) \times P_1] \end{aligned}$$

Therefore

$$P_1 = \frac{\alpha}{\alpha + \beta - \alpha\beta}$$

When B fires first but misses, the situation is the one just considered. But if B hits with his first shot then clearly A's chances of winning are zero. Since these are the only two possible outcomes of B's first shot, we can write

$$P_2 = [\beta \times 0] + [(1 - \beta) \times P_1] \rightarrow P_2 = \frac{(1 - \beta)\alpha}{\alpha + \beta - \alpha\beta}$$

When both fire at the same time there are four possible outcomes D_i to the first round: D_1 , A hits and B misses; D_2 , B hits but A misses; D_3 , they both hit; D_4 , they both miss. If getting hit, even if you manage to hit your opponent, does not count as a win, then

$$\begin{aligned} P_3 &= \sum_i Pr(D_i)Pr(W|D_i) \\ &= [\alpha(1 - \beta) \times 1] + [(1 - \alpha)\beta \times 0] + [\alpha\beta \times 0] + [(1 - \alpha)(1 - \beta) \times P_3] \end{aligned}$$

This can be rearranged as

$$P_3 = \frac{\alpha(1 - \beta)}{\alpha + \beta - \alpha\beta} = P_2$$

Thus the result is the same as if B had fired first. However, we also note that if all that matters to A is that B is hit, whether or not he is hit himself, then the third bracket takes the value $\alpha\beta \times 1$ and P_3 takes the same value as P_1 .

Problem 12-3

Since the number of tests failed by a defective chip are mutually exclusive outcomes (0, 1 or ≥ 2), a chip supplied by X has a probability of failing just one test given by $0.02(1 - 0.1 - 0.4) = 0.010$. The corresponding probabilities for chips supplied by Y and Z are $0.04(1 - 0.1 - 0.6) = 0.012$ and $0.04(1 - 0.1 - 0.8) = 0.004$, respectively.

Using '1' to denote failing a single test, Bayes' theorem gives the probability that the chip was supplied by X as

$$\begin{aligned}\Pr(X|1) &= \frac{\Pr(1|X) \Pr(X)}{\Pr(1|X) \Pr(X) + \Pr(1|Y) \Pr(Y) + \Pr(1|Z) \Pr(Z)} \\ &= \frac{0.010 \times 0.5}{0.010 \times 0.5 + 0.012 \times 0.3 + 0.004 \times 0.2} = \frac{50}{94}.\end{aligned}$$

Problem 12-4

(a) Out of the 36 equally likely outcomes for a single roll of each die, there are 6 in which the scores are equal and result in a win for A . The other 30 cases, in which the scores are unequal, provide 15 wins for A and 15 wins for B . Thus A wins in $6 + 15 = 21$ of the 36 cases and B in only 15 cases, giving A favourable odds of 7 to 5.

(b) The probability distribution $p(T)$ for the total T from two rolls of a die is

T	2	3	4	5	6	7	8	9	10	11	12
$36p(T)$	1	2	3	4	5	6	5	4	3	2	1

The probability that $T_D > T_C$ could be calculated for each T_D by adding all the probabilities for T_C up to $T_D - 1$ and the overall probability found by then weighting each sum by the probability of T_D . However, it is simpler to add up the probabilities for the two totals being equal, subtract this from unity and then take $\Pr(T_D > T_C)$ as one-half of what is left, i.e.

$$\begin{aligned}\Pr(T_D > T_C) &= \frac{1}{2} \left\{ 1 - \frac{1}{(36)^2} [2(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)] \right\} \\ &= \frac{1}{2} \left[1 - \frac{146}{(36)^2} \right] = 0.4437.\end{aligned}$$

This is the chance that D wins and so C 's expected return for a £1.10 stake is £2.10(1 - 0.4437) = £1.168, i.e. a profit of about 7 pence per game.

A straightforward calculation shows that C should stake just over £1.25 to make the game 'fair'.

Problem 12-5

This result is trivially true if either of x or y is zero, as no kittens need to be removed. We therefore consider $x, y \geq 1$. It is also clear that $N(a, b)$ must be equal to $N(b, a)$, as this is only a matter of labelling the litters.

Let the cage contain x kittens from the first litter and $y + 1$ from the second and consider the removal of one randomly chosen kitten. With probability $(y +$

$1)/(x + y + 1)$ it will be from the second litter and the expected final number of kittens will be $N(x, y)$. Correspondingly, with probability $x/(x + y + 1)$ it will be from the first litter and the expected final number of kittens will be $N(x - 1, y + 1)$. Thus we have the recurrence relation

$$N(x, y + 1) = \frac{y + 1}{x + y + 1} N(x, y) + \frac{x}{x + y + 1} N(x - 1, y + 1).$$

Now suppose that

$$N(x, y) = \frac{x}{y + 1} + \frac{y}{x + 1} \quad (*)$$

for all x, y such that $x + y = n$ and apply the assumption to the RHS of the above equation.

$$\begin{aligned} N(x, y + 1) &= \frac{y + 1}{x + y + 1} \left(\frac{x}{y + 1} + \frac{y}{x + 1} \right) \\ &\quad + \frac{x}{x + y + 1} \left(\frac{x - 1}{y + 2} + \frac{y + 1}{x} \right) \\ &= \frac{1}{x + y + 1} \left(x + \frac{y^2 + y}{x + 1} + \frac{x^2 - x}{y + 2} + y + 1 \right) \\ &= \frac{1}{x + y + 1} \left[\left(x + \frac{x^2 - x}{y + 2} \right) + \left(y + 1 + \frac{y^2 + y}{x + 1} \right) \right] \\ &= \frac{1}{x + y + 1} \left(\frac{xy + x + x^2}{y + 2} + \frac{xy + x + 2y + 1 + y^2}{x + 1} \right) \\ &= \frac{x}{y + 2} + \frac{yx + y^2 + y + x + y + 1}{(x + y + 1)(x + 1)} \\ &= \frac{x}{y + 2} + \frac{y + 1}{x + 1} \quad (**) \\ &= \frac{x}{(y + 1) + 1} + \frac{(y + 1)}{x + 1}. \end{aligned}$$

Either by interchanging x and y in $(**)$ to obtain $N(y, x + 1)$ and then using $N(a, b) = N(b, a)$, or by a similar calculation to the above, we can also show that

$$N(x + 1, y) = \frac{(x + 1)}{y + 1} + \frac{y}{(x + 1) + 1}.$$

Combining the two results then shows that the assumption that $(*)$ is valid for $x + y = n$ implies that it is valid for $x + y = n + 1$.

However, it is valid by direct inspection for $n = 2$, since $N(2, 0) = 2 = N(0, 2)$ (no kitten need be removed) and $N(1, 1) = 1$ (one kitten must be removed). Thus $(*)$ is valid for all n and hence for all x and y . It is formally valid, by inspection, for $n = 1$, but the proof given then involves negative arguments of N , albeit with N multiplied by zero.

Problem 12-6

The number of winners x will follow a Poisson distribution; let its mean be μ . I will keep my job provided the number of winners is 1 or 2. The probability $y(\mu)$ of this is

$$y(\mu) = \frac{\mu}{1!} e^{-\mu} + \frac{\mu^2}{2!} e^{-\mu}.$$

This is maximal when μ is chosen to satisfy

$$0 = \frac{dy}{d\mu} = -(\tfrac{1}{2}\mu^2 + \mu)e^{-\mu} + (1 + \mu)e^{-\mu} \Rightarrow \mu = \sqrt{2}.$$

The corresponding value of p is $\sqrt{2}/(5 \times 10^6)$ and the chance that I keep my job is

$$y(\sqrt{2}) = (\sqrt{2} + 1)e^{-\sqrt{2}} = 0.587,$$

i.e. a 41% chance that I will be clearing my desk.

Problem 12-7

The 'experiment' will end after n 'trials' if the previous $n-1$ trials have produced either $n-2$ boys and 1 girl or $n-2$ girls and one boy, and the n th trial produces the girl or boy (respectively) needed to complete the desired family. These two situations have respective probabilities

$$({}^{n-1}C_1 pq^{n-2})p \quad \text{and} \quad ({}^{n-1}C_1 qp^{n-2})q.$$

Thus the probability that the size of the family is n is

$${}^{n-1}C_1 p^2 q^{n-2} + {}^{n-1}C_1 q^2 p^{n-2} = (n-1)(p^2 q^{n-2} + q^2 p^{n-2}).$$

Averaging this over all possible values of n (≥ 4) gives the expected size as

$$\langle n \rangle = \sum_{n=4}^{\infty} n(n-1)(p^2 q^{n-2} + q^2 p^{n-2}).$$

Now, from the formula for the sum of a geometric series, we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Differentiating this (twice) with respect to r gives

$$\begin{aligned} \sum_{n=1}^{\infty} nr^{n-1} &= \frac{1}{(1-r)^2}, \\ \sum_{n=2}^{\infty} n(n-1)r^{n-2} &= \frac{2}{(1-r)^3}. \end{aligned}$$

So, as the minimum size of their family is 4,

$$\begin{aligned} \langle n \rangle &= \sum_{n=4}^{\infty} n(n-1)(p^2 q^{n-2} + q^2 p^{n-2}) \\ &= p^2 \left[\frac{2}{(1-q)^3} - (2)(1)q^0 - (3)(2)q^1 \right] \\ &\quad + q^2 \left[\frac{2}{(1-p)^3} - (2)(1)p^0 - (3)(2)p^1 \right] \\ &= p^2 \left[\frac{2}{p^3} - 2 - 6q \right] + q^2 \left[\frac{2}{q^3} - 2 - 6p \right] \\ &= 2 \left[\frac{1}{p} + \frac{1}{q} - p^2 - q^2 - 3pq(p+q) \right] \\ &= 2 \left[\frac{p+q}{pq} - (1^2 - 2pq) - 3pq(1) \right] \\ &= 2 \left[\frac{1}{pq} - 1 - pq \right], \end{aligned}$$

as given in the question.

Problem 12-8

With O as the centre of the unit circle and Q as the point $(1, 0)$, let OP make an angle θ with the x-axis OQ . The random variable X then has the value $2 \sin(\theta/2)$ with θ uniformly distributed on $(0, 2\pi)$, i.e.

$$f(x) dx = \frac{1}{2\pi} d\theta.$$

The mean of X is given straightforwardly by

$$\langle X \rangle = \int_0^{2\pi} X f(x) dx = \int_0^{2\pi} 2 \sin\left(\frac{\theta}{2}\right) \frac{1}{2\pi} d\theta = \frac{1}{\pi} \left[-2 \cos \frac{\theta}{2} \right]_0^{2\pi} = \frac{4}{\pi}.$$

For the variance we have

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2 = \int_0^{2\pi} 4 \sin^2\left(\frac{\theta}{2}\right) \frac{1}{2\pi} d\theta - \frac{16}{\pi^2} = \frac{4}{2\pi} \int_0^{2\pi} \sin^2\left(\frac{\theta}{2}\right) d\theta - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2}.$$

When $X = \langle X \rangle = 4/\pi$, the angle $\theta = 2 \sin^{-1}(2/\pi)$ and so

$$\Pr(X > \langle X \rangle) = \frac{2\pi - 4 \sin^{-1} \frac{2}{\pi}}{2\pi} = 0.561.$$

Problem 12-9

Let S mean "spade" and N mean "not spade". Then the symbol $N_1S_2S_3$ means: First card drawn is not a spade, second card drawn is a spade, third card drawn is a spade. In this notation, the possible ways of discarding two cards and then drawing a spade are:

$$S_1S_2S_3, S_1N_2S_3, N_1S_2S_3, N_1N_2S_3 \quad (A)$$

These events are mutually exclusive so by the rule

$$P(A + B) = P(A) + P(B) \quad A \text{ and } B \text{ mutually exclusive}$$

we add their probabilities. Now to compute the probability of, say, $S_1N_2S_3$ we use the equation

$$P(AB) = P(B) \cdot P(A) \quad A \text{ and } B \text{ independent}$$

repeatedly. The probability of S_1 is $13/52$. Then the probability of N_2 is $39/51$ since there are now 51 cards left and 39 of them are not spades. Now the probability of S_3 is $12/50$ since there are 12 spades left and 50 cards left. Thus the probability of $S_1N_2S_3$ is

$$\frac{13}{52} \cdot \frac{39}{51} \cdot \frac{12}{50}$$

Similarly, we compute all four of the probabilities in (A) (verify the other three) and add them to get

$$\begin{aligned} & \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} + \frac{13}{52} \cdot \frac{39}{51} \cdot \frac{12}{50} + \frac{39}{52} \cdot \frac{13}{51} \cdot \frac{12}{50} + \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50} \\ &= \frac{13}{52} \left[\frac{12(11+39) + 39(12+38)}{51 \cdot 50} \right] = \frac{1}{4} \end{aligned}$$

Problem 12-10

Let A mean "nickel" and B mean "right pocket". Then we want the conditional probability $P_A(B) = P(A|B)$. From Baye's formula,

$$P_A(B) = \frac{P(AB)}{P(A)} \quad (\text{B})$$

Let us find $P(A)$ and $P(AB)$. $P(AB)$ is the probability that you select a nickel from your right pocket. This is

$$\begin{aligned} P(AB) &= \text{probability of selecting the right pocket} \\ &\quad \text{and then selecting a nickel from it} \\ &= \frac{1}{2} \cdot \frac{3}{7} = \frac{3}{14} \end{aligned}$$

since there are two equally likely pockets and then 3 nickels out of 7 coins in your right pocket. To find $P(A)$ we add the probabilities of the mutually exclusive events of selecting a nickel from the right pocket or the left pocket:

$$P(A) = \frac{1}{2} \cdot \frac{3}{7} + \frac{1}{2} \cdot \frac{2}{3} = \frac{23}{42}$$

Then by (B)

$$P_A(B) = \frac{3}{14} \cdot \frac{23}{42} = \frac{9}{23}$$

Problem 12-11

The average number of misprints per page is $\mu = 100/40 = 2.5$. The Poisson probability for getting exactly n counts is

$$P_n = \frac{\mu^n}{n!} e^{-\mu}$$

so the probability of no misprints is then given by

$$P_0 = \frac{\mu^0}{0!} e^{-\mu} = e^{-2.5} = 0.082$$

The expected number of pages with no misprints is

$$(\text{total number of pages}) P_0 = 40(0.082) = 3.3$$

Similarly, the expected number of pages with two misprints is

$$40P_2 = 40 \frac{(2.5)^2 e^{-2.5}}{2!} = 10.3$$

and the expected number of pages with five misprints is

$$40P_5 = 40 \frac{(2.5)^5 e^{-2.5}}{5!} = 2.7$$

Remember that "expected" is a technical term in probability meaning "average". Although a number of pages must be an integer, an average number may not be.

Assignment 13

Problem 13-1

To demonstrate the associativity we need to show that $x \bullet (y \bullet z)$ is the same thing as $(x \bullet y) \bullet z$. So consider

$$\begin{aligned} x \bullet (y \bullet z) &= x + (y \bullet z) + rx(y \bullet z) \\ &= x + y + z + ryz + rx(y + z + ryz) \\ &= x + y + z + r(yz + xy + xz) + r^2xyz \end{aligned}$$

and

$$\begin{aligned} (x \bullet y) \bullet z &= (x \bullet y) + z + r(x \bullet y)z \\ &= x + y + rxy + z + r(x + y + rxy)z \\ &= x + y + z + r(xy + xz + yz) + r^2xyz. \end{aligned}$$

The two RHSs are equal, showing that the operation $\bullet$ is associative.

Firstly, suppose that $x = -r^{-1}$. Then

$$x \bullet y = -\frac{1}{r} + y + r\left(-\frac{1}{r}\right)y = -\frac{1}{r} + y - y = -\frac{1}{r}.$$

Similarly $y = -r^{-1} \Rightarrow x \bullet y = -r^{-1}$.

Secondly, suppose that $x \bullet y = -r^{-1}$. Then

$$\begin{aligned} x + y + rxy &= -r^{-1}, \\ rx + ry + r^2xy + 1 &= 0, \\ (rx + 1)(ry + 1) &= 0, \\ \Rightarrow \text{either } x &= -r^{-1} \text{ or } y = -r^{-1}. \end{aligned}$$

Thus $x \bullet y = -r^{-1} \iff (x = -r^{-1} \text{ or } y = -r^{-1})$.

If $S = \{\text{real numbers} \neq -r^{-1}\}$, then

- (i) Associativity under $\bullet$ has already been shown.
- (ii) If x and y belong to S , then $x \bullet y$ is a real number and, in view of the second result above, is $\neq -r^{-1}$. Thus $x \bullet y$ belongs to S and the set is closed under the operation $\bullet$.
- (iii) For any x belonging to S , $x \bullet 0 = x + 0 + rx0 = x$. Thus 0 is an identity element.
- (iv) An inverse x^{-1} of x must satisfy $x \bullet x^{-1} = 0$, i.e.

$$x + x^{-1} + rxx^{-1} = 0 \Rightarrow x^{-1} = -\frac{x}{1 + rx}.$$

This is a real (finite) number since $x \neq -r^{-1}$ and, further, $x^{-1} \neq -r^{-1}$, since if it were we could deduce that $1 = 0$. Thus the set S contains an inverse for each of its elements.

These four results together show that S is a group under the operation $\bullet$.

Problem 13-2

The condition $wz - xy = 1$ is the same as $|A| = 1$; it follows that the set contains an identity element (with $w = z = 1$ and $x = y = 0$). Moreover, each matrix in $\mathcal{S}$ has an inverse and, since $|A^{-1}| |A| = |I| = 1$ implies that $|A^{-1}| = 1$, the inverses also belong to the set.

If A and B belong to $\mathcal{S}$ then, since $|AB| = |A| |B| = 1 \times 1 = 1$, their product also belongs to $\mathcal{S}$, i.e. the set is closed.

These observations, together with the associativity of matrix multiplication establish that the set $\mathcal{S}$ is, in fact, a group under this operation.

If A is to have order 2 then

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{i.e. } w^2 + xy = 1, \quad x(w+z) = 0, \quad y(w+z) = 0, \quad xy + z^2 = 1.$$

These imply that $w^2 = z^2$ and that either $z = -w$ or $x = y = 0$. If $z = -w$, then both

$$\begin{aligned} w^2 + xy &= 1, & \text{from the above condition,} \\ \text{and } -w^2 - xy &= 1, & \text{from } wz - xy = 1. \end{aligned}$$

This is not possible and so we must have $x = y = 0$, implying that w and z are either both $+1$ or both -1 . The former gives the identity (of order 1), and so the matrix given by the latter, $A = -I$, is the only element in $\mathcal{S}$ of order 2.

If $w + z + 1 = 0$ (as well as $xy = wz - 1$), A^2 can be written as

$$\begin{aligned} A^2 &= \begin{pmatrix} w^2 + xy & x(w+z) \\ y(w+z) & xy + z^2 \end{pmatrix} \\ &= \begin{pmatrix} w^2 + wz - 1 & -x \\ -y & wz - 1 + z^2 \end{pmatrix} \\ &= \begin{pmatrix} -w - 1 & -x \\ -y & -z - 1 \end{pmatrix}. \end{aligned}$$

Multiplying again by A gives

$$\begin{aligned} A^3 &= \begin{pmatrix} -w - 1 & -x \\ -y & -z - 1 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ &= - \begin{pmatrix} w(w+1) + xy & (w+1)x + xz \\ wy + y(z+1) & xy + z(z+1) \end{pmatrix} \\ &= - \begin{pmatrix} w(w+1) + wz - 1 & x \times 0 \\ y \times 0 & wz - 1 + z(z+1) \end{pmatrix} \\ &= - \begin{pmatrix} (w \times 0) - 1 & 0 \\ 0 & (z \times 0) - 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

- Thus A has order 3. -

Problem 13-3

From the given combination law

$$e = ac + bdp \quad \text{and} \quad f = ad + bc.$$

As a , b , c and d are all rational numbers, so are e and f . The set of rational number pairs is therefore closed. Associativity and commutativity are obvious and the number pair $I = (1, 0)$ clearly has the property that $IX = X$ for any rational number pair X . The existence of inverses is the only requirement remaining to be established in order to complete the proof that the set forms an Abelian group under the given combination law.

Now,

$$(c, d) \bullet (a, b) = (1, 0) \Rightarrow c + d\sqrt{p} = \frac{1}{a + b\sqrt{p}} = \frac{a - b\sqrt{p}}{a^2 - b^2p}.$$

Further, since a and b are rational (and $\sqrt{p}$ is not), $a^2 \neq b^2p$. Although it is not zero, $a^2 - b^2p$ is rational, and so, therefore are c and d . To summarise,

$$(a, b)^{-1} = (c, d) = \left(\frac{a}{a^2 - b^2p}, \frac{-b}{a^2 - b^2p} \right),$$

a rational number pair that is included in the set. The proof is now complete.

For the mapping $(a, b) \rightarrow (a, -b)$

$$[(a, b) \bullet (c, d)]' = (ac + bdp, ad + bc)' = (ac + bdp, -ad - bc),$$

whilst

$$(a, b)' \bullet (c, d)' = (a, -b) \bullet (c, -d) = (ac + bdp, -bc - ad).$$

Hence

$$[(a, b) \bullet (c, d)]' = (a, b)' \bullet (c, d)',$$

showing that the mapping is a homomorphism. The mapping is clearly one-to-one, making it an isomorphism and, finally, since the object and image sets are the same, it is an automorphism.