The following set of lectures is designed to introduce you to the ideas of Quantum Field Theory.

The detailed derivations involve mathematical operations that are will beyond almost all of you.

Therefore, I shall, for the most part, simply quote the end results of derivations and describe them
in detailed text sections.

All important aspects of the subject material will be covered and I hope my explanations will be enough to allow you to understand how the derivation proceeded and what is the fundamental meaning of all the equations and results.

You should come out of this class with a good understanding of what quantum field theory entails, what the fundamental equations are and their physical meanings.

If you want to go deeper than this coverage, then you have to pick up a full-blown textbook, do all the mathematical derivations in detail, and also do all the problems to make sure you understanding is deep.

So let us proceed……It will be an adventure.

Quantum Field Theory

Part I - Spin Zero

1 Attempts at relativistic quantum mechanics

In order to combine quantum mechanics and relativity, we must first understand what we mean by "quantum mechanics" and "relativity".

Let us begin with quantum mechanics.

Somewhere in most textbooks on the subject, one can find a list of the "axioms of quantum mechanics".

These include statements along the lines of

> *The state of the system is represented by a vector in Hilbert space.*
>
> *Observables are represented by hermitian operators.*
>
> *The measurement of an observable yields one of its eigenvalues as the result*

And so on.

We do not need to review these closely here.

The axiom we need to focus on is the one that says that the time evolution of the state of the system is governed by the Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \ , \tag{1.1}$$

where H is the hamiltonian operator, representing the total energy.

Let us consider a very simple system: a spinless, nonrelativistic particle with no forces acting on it.

In this case, the hamiltonian is

$$H = \frac{1}{2m}\mathbf{P}^2 \ , \tag{1.2}$$

where m is the particle's mass, and $\boldsymbol{P}$ is the momentum operator.

In the position basis, eq. (1.1) becomes

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) \ , \tag{1.3}$$

where $\psi(\mathbf{x},t) = \langle \mathbf{x}|\psi,t\rangle$ is the position-space wave function.

We would like to generalize this to relativistic motion.

The obvious way to proceed is to take

$$H = +\sqrt{\mathbf{P}^2c^2 + m^2c^4}\ , \tag{1.4}$$

which yields the correct relativistic energy-momentum relation.

If we formally expand this hamiltonian in inverse powers of the speed of light c, we get

$$H = mc^2 + \frac{1}{2m}\mathbf{P}^2 + \dots\ . \tag{1.5}$$

This is simply a constant (the rest energy), plus the usual nonrelativistic hamiltonian, eq. (1.2), plus higher-order corrections.

With the hamiltonian given by eq. (1.4), the Schrodinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = +\sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\ \psi(\mathbf{x},t)\ . \tag{1.6}$$

Unfortunately, this equation presents us with a number of difficulties.

One is that it apparently treats space and time on a different footing: the time derivative appears only on the left, outside the square root, and the space derivatives appear only on the right, under the square root.

This asymmetry between space and time is not what we would expect of a relativistic theory.

Furthermore, if we expand the square root in powers of ∇^2, we get an infinite number of spatial derivatives acting on $\psi(\mathbf{x},t)$; this implies that eq. (1.6) is not local in space.

We can alleviate these problems by squaring the differential operators on each side of eq. (1.6) before applying them to the wave function.

Then we get

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right)\psi(\mathbf{x},t) \ . \tag{1.7}$$

This is the Klein-Gordon equation, and it looks a lot nicer than eq. (1.6). It is second-order in both space and time derivatives, and they appear in a symmetric fashion.

To better understand the Klein-Gordon equation, let us consider in more detail what we mean by "relativity".

Special relativity tells us that physics looks the same in all inertial frames.

To explain what this means, we first suppose that a certain spacetime coordinate system (ct,$\mathbf{x}$) represents (by fiat) an inertial frame.

Let us define $x_0 = ct$, and write x^μ, where $\mu = 0,1,2,3$, in place of (ct,$\mathbf{x}$).

It is also convenient (for reasons not at all obvious at this point) to define $x_0 = -x^0$ and $x_i = x^i$, where $i = 1, 2, 3$.

This can be expressed more elegantly if we first introduce the *Minkowski metric*,

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \tag{1.8}$$

where blank entries are zero.

We then have $x_\mu = g_{\mu\nu}x^\nu$, where a repeated index is summed.

To invert this formula, we introduce the inverse of g, which is confusingly also called g, except with both indices up:

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{1.9}$$

We then have $g^{\mu\nu}g_{\nu\varrho} = \delta^\mu{}_\varrho$, where $\delta^\mu{}_\varrho$ is the Kronecker delta (equal to one if its two indices take on the same value, zero otherwise).

Now we can also write $x^\mu = g^{\mu\nu}x_\nu$.

It is a general rule that any pair of repeated (and therefore summed) indices must consist of one superscript and one subscript; these indices are said to be *contracted*.

Also, any unrepeated (and therefore unsummed) indices must match (in both name and height) on the left- and right-hand sides of any valid equation.

Now we are ready to specify what we mean by an inertial frame.

If the coordinates x^μ represent an inertial frame (which they do, by assumption), then so do any other coordinates $\bar{x}^\mu$ that are related by

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \ , \tag{1.10}$$

where $\Lambda^\mu{}_\nu$ is a *Lorentz transformation matrix* and a^μ is a translation vector.

Both $\Lambda^\mu{}_\nu$ and a^μ are constant (that is, independent of x^μ).

Furthermore, $\Lambda^\mu{}_\nu$ must obey

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma} \ . \tag{1.11}$$

Eq. (1.11) ensures that the interval between two different spacetime points that are labeled by x^μ and x'^μ in one inertial frame, and by $\bar{x}^\mu$ and $\bar{x}'^\mu$ in another, is the same.

This interval is defined to be

$$\begin{aligned} (x-x')^2 &\equiv g_{\mu\nu}(x-x')^\mu(x-x')^\nu \\ &= (\mathbf{x}-\mathbf{x}')^2 - c^2(t-t')^2 \ . \end{aligned} \tag{1.12}$$

In the other frame, we have

$$\begin{aligned}(\bar{x}-\bar{x}')^2 &= g_{\mu\nu}(\bar{x}-\bar{x}')^\mu(\bar{x}-\bar{x}')^\nu \\ &= g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma(x-x')^\rho(x-x')^\sigma \\ &= g_{\rho\sigma}(x-x')^\rho(x-x')^\sigma \\ &= (x-x')^2\,,\end{aligned} \tag{1.13}$$

as desired.

When we say that physics looks the same, we mean that two observers (Alice and Bob, say) using two different sets of coordinates (representing two different inertial frames) should agree on the predicted results of all possible experiments.

In the case of quantum mechanics, this requires Alice and Bob to agree on the value of the wave function at a particular spacetime point, a point that is called x by Alice and $\bar{x}$ by Bob.

Thus if Alice's predicted wave function is ψ(x), and Bob's is $\bar{\psi}(\bar{x})$, then we should have ψ(x) = $\bar{\psi}(\bar{x})$.

Furthermore, in order to maintain ψ(x) = $\bar{\psi}(\bar{x})$ throughout spacetime, ψ(x) and $\bar{\psi}(\bar{x})$ should obey identical equations of motion.

Thus a candidate wave equation should take the same form in any inertial frame.

Let us see if this is true of the Klein-Gordon equation.

We first introduce some useful notation for spacetime derivatives:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(+\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \tag{1.14}$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right). \tag{1.15}$$

Note that

$$\partial^\mu x^\nu = g^{\mu\nu}\ , \tag{1.16}$$

so that our matching-index-height rule is satisfied.

If $\bar{x}$ and x are related by eq. (1.10), then $\bar{\psi}$ and ∂ are related by

$$\bar{\partial}^\mu = \Lambda^\mu{}_\nu \partial^\nu\ . \tag{1.17}$$

To check this, we note that

$$\bar{\partial}^\rho \bar{x}^\sigma = (\Lambda^\rho{}_\mu \partial^\mu)(\Lambda^\sigma{}_\nu x^\nu + a^\mu) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu (\partial^\mu x^\nu) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}\ , \tag{1.18}$$

as expected.

The last equality in eq. (1.18) is another form of eq. (1.11); see section 2.

We can now write eq. (1.7) as

$$-\hbar^2 c^2 \partial_0^2 \psi(x) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\psi(x) \ . \tag{1.19}$$

After rearranging and identifying $\partial^2 \equiv \partial^\mu \partial_\mu = -\partial^2_0 + \nabla^2$, we have

$$(-\partial^2 + m^2 c^2/\hbar^2)\psi(x) = 0 \ . \tag{1.20}$$

This is Alice's form of the equation.

Bob would write

$$(-\bar{\partial}^2 + m^2 c^2/\hbar^2)\bar{\psi}(\bar{x}) = 0 \ . \tag{1.21}$$

Is Bob's equation equivalent to Alice's equation?

To see that it is, we set $\bar{\psi}(\bar{x}) = \psi(x)$, and note that

$$\bar{\partial}^2 = g_{\mu\nu}\bar{\partial}^\mu \bar{\partial}^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\mu{}_\sigma \partial^\rho \partial^\sigma = \partial^2 \ . \tag{1.22}$$

Thus, eq. (1.21) is indeed equivalent to eq. (1.20).

The Klein-Gordon equation is therefore manifestly consistent with relativity: it takes the same form in every inertial frame.

This is the good news.

The bad news is that the Klein-Gordon equation violates one of the axioms of quantum mechanics: eq. (1.1), the Schrodinger equation in its abstract form.

The abstract Schrodinger equation has the fundamental property of being first order in the time derivative, whereas the Klein-Gordon equation is second order.

This may not seem too important, but in fact it has drastic consequences.

One of these is that the norm of a state,

$$\langle \psi, t | \psi, t \rangle = \int d^3x \, \langle \psi, t | \mathbf{x} \rangle \langle \mathbf{x} | \psi, t \rangle = \int d^3x \, \psi^*(x) \psi(x), \tag{1.23}$$

is not in general time independent.

Thus probability is not conserved.

The Klein-Gordon equation obeys relativity, but not quantum mechanics.

Dirac attempted to solve this problem (for spin-one-half particles) by introducing an extra discrete label on the wave function, to account for spin: $\psi_a(x)$, a = 1, 2.

He then tried a Schrodinger equation of the form

$$i\hbar\frac{\partial}{\partial t}\psi_a(x) = \left(-i\hbar c(\alpha^j)_{ab}\partial_j + mc^2(\beta)_{ab}\right)\psi_b(x)\ , \tag{1.24}$$

where all repeated indices are summed, and α^j and β are matrices in spin-space.

This equation, the Dirac equation, is consistent with the abstract Schrodinger equation.

The state $|\psi,a,t\rangle$ carries a spin label a, and the hamiltonian is

$$H_{ab} = cP_j(\alpha^j)_{ab} + mc^2(\beta)_{ab}\ , \tag{1.25}$$

where P_j is a component of the momentum operator.

Since the Dirac equation is linear in both time and space derivatives, it has a chance to be consistent with relativity.

Note that squaring the hamiltonian yields

$$(H^2)_{ab} = c^2P_jP_k(\alpha^j\alpha^k)_{ab} + mc^3P_j(\alpha^j\beta + \beta\alpha^j)_{ab} + (mc^2)^2(\beta^2)_{ab}\ . \tag{1.26}$$

Since $P_j P_k$ is symmetric on exchange of j and k, we can replace $\alpha^j\alpha^k$ by its symmetric part, $\frac{1}{2}\{\alpha^j,\alpha^k\}$, where $\{A,B\} = AB + BA$ is the anticommutator.

Then, if we choose matrices such that

$$\{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab} \ , \quad \{\alpha^j, \beta\}_{ab} = 0 \ , \quad (\beta^2)_{ab} = \delta_{ab} \ , \tag{1.27}$$

we will get

$$(H^2)_{ab} = (\mathbf{P}^2c^2 + m^2c^4)\delta_{ab} \ . \tag{1.28}$$

Thus, the eigenstates of H^2 are momentum eigenstates, with H^2 eigenvalue $\mathbf{p}^2c^2 + m^2c^4$.

This is, of course, the correct relativistic energy-momentum relation.

While it is outside the scope of this section(comes later) to demonstrate it, it turns out that the Dirac equation is fully consistent with relativity provided the Dirac matrices obey eq. (1.27).

So we have apparently succeeded in constructing a quantum mechanical, relativistic theory!

There are, however, some problems.

We would like the Dirac matrices to be 2×2, in order to account for electron spin.

However, they must in fact be larger.

To see this, note that the 2×2 Pauli matrices obey $\{\sigma^i,\sigma^j\} = 2\delta^{ij,}$ and are thus candidates for the Dirac α^i matrices.

However, there is no fourth matrix that anticommutes with these three (easily proven by writing down the most general 2×2 matrix and working out the three anticommutators explicitly).

Also, we can show that the Dirac matrices must be even dimensional.

Thus their minimum size is 4×4, and it remains for us to interpret the two extra possible "spin" states.

However, these extra states cause a more severe problem than a mere overcounting.

Acting on a momentum eigenstate, H becomes the matrix $c\alpha\cdot\mathbf{p} + mc^2\beta$.

One can show that the trace of this matrix is zero.

Thus the four eigenvalues must be $+E(\mathbf{p})$, $+E(\mathbf{p})$,$-E(\mathbf{p})$,$-E(\mathbf{p})$, where $E(\mathbf{p}) = +(\mathbf{p}^2c^2 + m^2c^4)^{1/2}$.

The negative eigenvalues are the problem: they indicate that there is no ground state.

In a more elaborate theory that included interactions with photons, there seems to be no reason why a positive energy electron could not emit a photon and drop down into a negative energy state.

This downward cascade could continue forever.

(The same problem also arises in attempts to interpret the Klein-Gordon equation as a modified form of quantum mechanics.)

Dirac made a wildly brilliant attempt to fix this problem of negative energy states.

His solution is based on an empirical fact about electrons: they obey the Pauli exclusion principle.

It is impossible to put more than one of them in the same quantum state.

What if, Dirac speculated, all the negative energy states were *already occupied*?

In this case, a positive energy electron could not drop into one of these states, by Pauli exclusion.

Many questions immediately arise.

Why don't we see the negative electric charge of this *Dirac sea* of electrons?

Dirac's answer: because we're used to it. (More precisely, the physical effects of a uniform charge density depend on the boundary conditions at infinity that we impose on Maxwell's equations, and there is a choice that renders such a uniform charge density
invisible.)

However, Dirac noted, if one of these negative energy electrons were excited into a positive energy state (by, say, a sufficiently energetic photon), it would leave behind a *hole* in the sea of negative energy electrons.

This hole would appear to have positive charge, and positive energy.

Dirac therefore predicted (in 1927) the existence of the *positron*, a particle with the same mass as the electron, but opposite charge.

The positron was found experimentally five years later.

However, we have now jumped from an attempt at a quantum description of a single relativistic particle to a theory that apparently requires an infinite number of particles.

Even if we accept this, we still have not solved the problem of how to describe particles like photons or pions or alpha nuclei that do not obey Pauli exclusion.

At this point, it is worthwhile to stop and reflect on why it has proven to be so hard to find an acceptable relativistic wave equation for a single quantum particle.

Perhaps there is something wrong with our basic approach.

And there is.

Recall the axiom of quantum mechanics that says that "Observables are represented by hermitian operators."

This is not entirely true. There is one observable in quantum mechanics that is *not* represented by a hermitian operator: time.

Time enters into quantum mechanics only when we announce that the "state of the system" depends on an extra parameter t.

This parameter is not the eigenvalue of any operator.

This is in sharp contrast to the particle's position x, which is the eigenvalue of an operator.

Thus, space and time are treated very differently, a fact that is obscured by writing the Schrodinger equation in terms of the position-space wave function $\psi(\mathbf{x},t)$.

Since space and time are treated asymmetrically, it is not surprising that we are having trouble incorporating a symmetry that mixes them up.

So, what are we to do?

In principle, the problem could be an intractable one: it might be impossible to combine quantum mechanics and relativity.

In this case, there would have to be some meta-theory, one that reduces in the nonrelativistic limit to quantum mechanics, and in the classical limit to relativistic particle dynamics, but is actually neither.

This, however, turns out not to be the case.

We can solve our problem, but we must put space and time on an equal footing at the outset.

There are two ways to do this.

One is to demote position from its status as an operator, and render it as an extra label, like time.

The other is to promote time to an operator.

Let us discuss the second option first.

If time becomes an operator, what do we use as the time parameter in the Schrodinger equation?

Happily, in relativistic theories, there is more than one notion of time.

We can use the proper time τ of the particle (the time measured by a clock that moves with it) as the time parameter.

The coordinate time T (the time measured by a stationary clock in an inertial frame) is then promoted to an operator.

In the Heisenberg picture (where the state of the system is fixed, but the operators are functions of time that obey the classical equations of motion), we would have operators $X^\mu(\tau)$, where $X^0 = T$.

Relativistic quantum mechanics can indeed be developed along these lines, but it is surprisingly complicated to do so. (The many times are the problem; any monotonic function of τ is just as good a candidate as τ itself for the proper time, and this infinite redundancy of descriptions must be understood and accounted for.)

One of the advantages of considering different formalisms is that they may suggest different directions for generalizations.

For example, once we have $X^\mu(\tau)$, why not consider adding some more parameters?

Then we would have, for example, $X^\mu(\sigma,\tau)$.

Classically, this would give us a continuous family of worldlines, what we might call a worldsheet, and so $X^\mu(\sigma,\tau)$ would describe a propagating string.

This is indeed the starting point for string theory.

Thus, promoting time to an operator is a viable option, but is complicated in practice.

Since we now know that "string theory" does not work.

We now know not to go down that path!

Let us then turn to the other option, demoting position to a label.

The first question is, label on what?

The answer is, *on operators*.

Thus, consider assigning an operator to each point **x** in space; call these operators $\varphi(\mathbf{x})$.

A set of operators like this is called a *quantum field*.

In the Heisenberg picture, the operators are also time dependent:

$$\varphi(\mathbf{x},t) = e^{iHt/\hbar}\varphi(\mathbf{x},0)e^{-iHt/\hbar}\,. \tag{1.29}$$

Thus, both position and (in the Heisenberg picture) time are now labels on operators; neither is itself the eigenvalue of an operator.

So, now we have two different approaches to relativistic quantum theory, approaches that might, in principle, yield different results.

This, however, is not the case: it turns out that any relativistic quantum physics that can be treated in one formalism can also be treated in the other.

Which we use is a matter of convenience and taste.

And, *quantum field theory*, the formalism in which position and time are both labels on operators, is much more convenient and efficient for most problems.

There is another useful equivalence: ordinary nonrelativistic quantum mechanics, for a fixed number of particles, can be rewritten as a quantum field theory.

This is an informative exercise, since the corresponding physics is already familiar.

Let us carry it out.

Begin with the position-basis Schrodinger equation for n particles, all with the same mass m, moving in an external potential $U(\mathbf{x})$, and interacting with each other via an interparticle potential $V(\mathbf{x}_1 - \mathbf{x}_2)$:

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1} V(\mathbf{x}_j - \mathbf{x}_k)\right]\psi \ , \tag{1.30}$$

where $\psi = \psi(\mathbf{x}_1,...,\mathbf{x}_n; t)$ is the position-space wave function.

The quantum mechanics of this system can be rewritten in the abstract form of eq. (1.1) by first introducing (in, for now, the Schrodinger picture) a quantum field $a(\mathbf{x})$ and its hermitian conjugate $a^\dagger(\mathbf{x})$.

We take these operators to have the commutation relations

$$\begin{aligned}
[a(\mathbf{x}), a(\mathbf{x}')] &= 0 \ , \\
[a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] &= 0 \ , \\
[a(\mathbf{x}), a^\dagger(\mathbf{x}')] &= \delta^3(\mathbf{x} - \mathbf{x}') \ ,
\end{aligned} \tag{1.31}$$

where $\delta^3(\mathbf{x})$ is the three-dimensional Dirac delta function.

Thus, $a^\dagger(\mathbf{x})$ and $a(\mathbf{x})$ behave like harmonic-oscillator creation and annihilation operators that are labeled by a continuous index.

In terms of them, we introduce the hamiltonian operator of our quantum field theory,

$$H = \int d^3x\; a^\dagger(\mathbf{x})\Big(-\tfrac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\Big)a(\mathbf{x}) + \tfrac{1}{2}\int d^3x\, d^3y\; V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\ . \tag{1.32}$$

Now consider a time-dependent state of the form

$$|\psi, t\rangle = \int d^3x_1 \ldots d^3x_n\; \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n; t) a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle\ , \tag{1.33}$$

where $\psi(\mathbf{x}_1,...,\mathbf{x}_n; t)$ is some function of the n particle positions and time, and $|0\rangle$ is the vacuum state, the state that is annihilated by all the a's,

$$a(\mathbf{x})|0\rangle = 0\ . \tag{1.34}$$

It is now straightforward (though tedious) to verify that eq. (1.1), the abstract Schrodinger equation, is obeyed if and only if the function ψ satisfies eq. (1.30).

Thus we can interpret the state $|0\rangle$ as a state of "no particles", the state $a^\dagger(\mathbf{x}_1)|0\rangle$ as a state with one particle at position $\mathbf{x}_1$, the state $a^\dagger(\mathbf{x}_1)a^\dagger(\mathbf{x}_2)|0\rangle$ as a state with one particle at position $\mathbf{x}_1$ and another at position $\mathbf{x}_2$, and so on.

The operator

$$N = \int d^3x \; a^\dagger(\mathbf{x}) a(\mathbf{x}) \tag{1.35}$$

counts the total number of particles.

It commutes with the hamiltonian, as is easily checked; thus, if we start with a state of n particles, we remain with a state of n particles at all times.

However, we can imagine generalizations of this version of the theory (generalizations that would not be possible without the field formalism) in which the number of particles is not conserved.

For example, we could try adding to H a term like

$$\Delta H \propto \int d^3x \left[a^\dagger(\mathbf{x}) a^2(\mathbf{x}) + \text{h.c.} \right] \; . \tag{1.36}$$

This term does not commute with N, and so the number of particles would not be conserved with this addition to H.

Theories in which the number of particles can change as time evolves are a good thing: they are needed for correct phenomenology.

We are already familiar with the notion that atoms can emit and absorb photons, and so we had better have a formalism that can incorporate this phenomenon.

We are less familiar with emission and absorption (that is to say, creation and annihilation) of electrons, but this process also occurs in nature; it is less common because it must be accompanied by the emission or absorption of a positron, antiparticle to the electron.

There are not a lot of positrons around to facilitate electron annihilation, while e^+e^- pair creation requires us to have on hand at least $2mc^2$ of energy available for the rest-mass energy of these two particles.

The photon, on the other hand, is its own antiparticle, and has zero rest mass; thus photons are easily and copiously produced and destroyed.

There is another important aspect of the quantum theory specified by eqs. (1.32) and (1.33).

Because the creation operators commute with each other, only the completely symmetric part of ψ survives the integration in eq. (1.33).

Therefore, without loss of generality, we can restrict our attention to ψ's of this type:

$$\psi(\ldots \mathbf{x}_i \ldots \mathbf{x}_j \ldots ; t) = +\psi(\ldots \mathbf{x}_j \ldots \mathbf{x}_i \ldots ; t) \ . \tag{1.37}$$

This means that we have a theory of *bosons*, particles that (like photons or pions or alpha nuclei) obey Bose-Einstein statistics.

If we want Fermi-Dirac statistics instead, we must replace eq. (1.31) with

$$
\begin{aligned}
\{a(\mathbf{x}), a(\mathbf{x}')\} &= 0 \ , \\
\{a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')\} &= 0 \ , \\
\{a(\mathbf{x}), a^\dagger(\mathbf{x}')\} &= \delta^3(\mathbf{x} - \mathbf{x}') \ ,
\end{aligned}
\tag{1.38}
$$

where again $\{A,B\} = AB + BA$ is the anticommutator.

Now only the fully antisymmetric part of ψ survives the integration in eq. (1.33), and so we can restrict our attention to

$$
\psi(\ldots \mathbf{x}_i \ldots \mathbf{x}_j \ldots ; t) = -\psi(\ldots \mathbf{x}_j \ldots \mathbf{x}_i \ldots ; t) \ .
\tag{1.39}
$$

Thus we have a theory of *fermions*.

It is straightforward to check that the abstract Schrodinger equation, eq. (1.1), still implies that ψ obeys the differential equation (1.30).

Now, however, the ordering of the a and $a^\dagger$ operators in the last term of eq. (1.32) becomes significant, and must be as written.

Interestingly, there is no simple way to write down a quantum field theory with particles that obey Boltzmann statistics, corresponding to a wave function with no particular symmetry.

This is a hint of the *spin-statistics* theorem, which applies to *relativistic* quantum field theory.

It says that interacting particles with integer spin must be bosons, and interacting particles with half-integer spin must be fermions.

In our nonrelativistic example, the interacting particles clearly have spin zero (because their creation operators carry no labels that could be interpreted as corresponding to different spin states), but can be either bosons or fermions, as we have seen.

Now that we have seen how to rewrite the nonrelativistic quantum mechanics of multiple bosons or fermions as a quantum field theory, it is time to try to construct a relativistic version.

2 Lorentz Invariance

Prerequisite: 1

A *Lorentz transformation* is a linear, homogeneous change of coordinates from x^μ to $\bar{x}^\mu$,

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu \ , \tag{2.1}$$

that preserves the *interval* x^2 between x^μ and the origin, where

$$x^2 \equiv x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = \mathbf{x}^2 - c^2 t^2 \ . \tag{2.2}$$

This means that the matrix $\Lambda^\mu{}_\nu$ must obey

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \ , \tag{2.3}$$

where

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} . \tag{2.4}$$

is the Minkowski metric.

Note that this set of transformations includes ordinary spatial rotations: take $\Lambda^0{}_0 = 1$, $\Lambda^0{}_i = \Lambda^i{}_0 = 0$, and $\Lambda^i{}_j = R^i{}_j$ where R is an orthogonal rotation matrix

The set of all Lorentz transformations forms a group: the product of any two Lorentz transformations is another Lorentz transformation; the product is associative; there is an identity transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$; and every Lorentz transformation has an inverse.

It is easy to demonstrate these statements explicitly.

For example, to find the inverse transformation $(\Lambda^{-1})^\mu{}_\nu$, note that the left-hand side of eq. (2.3) can be written as $\Lambda_{\nu\varrho}\Lambda^\nu{}_\sigma$, and that we can raise the ϱ index on both sides to get $\Lambda_\nu{}^\varrho\Lambda^\nu{}_\sigma = \delta^\varrho{}_\sigma$.

On the other hand, by definition, $(\Lambda^{-1})^\varrho{}_\nu\Lambda^\nu{}_\sigma = \delta^\varrho{}_\sigma$.

Therefore

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho \ . \tag{2.5}$$

Another useful version of eq. (2.3) is

$$g^{\mu\nu}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = g^{\rho\sigma} \ . \tag{2.6}$$

To get eq. (2.6), start with eq. (2.3), but with the inverse transformations $(\Lambda^{-1})^{\mu}{}_{\varrho}$ and $(\Lambda^{-1})^{\nu}{}_{\sigma}$.

Then use eq. (2.5), raise all down indices, and lower all up indices.

The result is eq. (2.6).

For an infinitesimal Lorentz transformation, we can write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} \ . \tag{2.7}$$

Eq. (2.3) can be used to show that $\delta\omega$ with both indices down (or up) is antisymmetric:

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho} \ . \tag{2.8}$$

Thus there are six independent infinitesimal Lorentz transformations (in four spacetime dimensions).

These can be divided into three rotations ($\delta\omega_{ij} = -\varepsilon_{ijk}\,\hat{n}_k\,\delta\theta$ for a rotation by angle $\delta\theta$ about the unit vector $\hat{\mathbf{n}}$) and three boosts ($\delta\omega_{i0} = \hat{n}_i\,\delta\eta$ for a boost in the direction $\hat{\mathbf{n}}$ by rapidity $\delta\eta$).

Not all Lorentz transformations can be reached by compounding infinitesimal ones.

If we take the determinant of eq. (2.5), we get $(\det \Lambda)^{-1} = \det \Lambda$, which implies $\det \Lambda = \pm 1$.

Transformations with $\det \Lambda = +1$ are *proper*, and transformations with $\det \Lambda = -1$ are *improper*.

Note that the product of any two proper Lorentz transformations is proper, and that infinitesimal transformations of the form $\Lambda = 1 + \delta\omega$ are proper.

Therefore, any transformation that can be reached by compounding infinitesimal ones is proper.

The proper transformations form a *subgroup* of the Lorentz group.

Another subgroup is that of the **orthochronous** Lorentz transformations: those for which $\Lambda^0{}_0 \geq +1$.

Note that eq. (2.3) implies $(\Lambda^0{}_0)2 - \Lambda^i{}_0 \, \Lambda^i{}_0 = 1$; thus, either $\Lambda \, \Lambda^0{}_0 \geq +1$ or $\Lambda \, \Lambda^0{}_0 \leq -1$.

An infinitesimal transformation is clearly orthochronous, and it is straightforward to show that the product of two orthochronous transformations is also orthochronous.

Thus, the Lorentz transformations that can be reached by compounding infinitesimal ones are both proper and orthochronous, and they form a subgroup.

We can introduce two discrete transformations that take us out of this subgroup: *parity* and *time reversal*.

The parity transformation is

$$\mathcal{P}^{\mu}{}_{\nu} = (\mathcal{P}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{2.9}$$

It is orthochronous, but improper.

The time-reversal transformation is

$$\mathcal{T}^{\mu}{}_{\nu} = (\mathcal{T}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{2.10}$$

It is nonorthochronous and improper.

Generally, when a theory is said to be Lorentz invariant, this means under the proper orthochronous subgroup only.

Parity and time reversal are treated separately.

It is possible for a quantum field theory to be invariant under the proper orthochronous subgroup, but not under parity and/or time-reversal.

From here on, in this section, we will treat the proper orthochronous subgroup only.

Parity and time reversal will be treated in section 23.

In quantum theory, symmetries are represented by unitary (or antiunitary) operators.

This means that we associate a unitary operator $U(\Lambda)$ to each proper, orthochronous Lorentz transformation Λ.

These operators must obey the composition rule

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda) \ . \tag{2.11}$$

For an infinitesimal transformation, we can write

$$U(1+\delta\omega) = I + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu} \ , \tag{2.12}$$

where $M^{\mu\nu} = -M^{\nu\mu}$ is a set of hermitian operators called the *generators* of the Lorentz group.

If we start with $U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$, let $\Lambda' = 1 + \delta\omega'$, and expand both sides to linear order in $\delta\omega$, we get

$$\delta\omega_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \delta\omega_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \ . \tag{2.13}$$

We see that each vector index on $M^{\mu\nu}$ undergoes its own Lorentz transformation.

This is a general result: any operator carrying one or more vector indices should behave similarly.

For example, consider the energy-momentum four-vector P^μ, where P^0 is the hamiltonian H and P^i are the components of the total three-momentum operator.

We expect

$$U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \ . \tag{2.15}$$

If we now let $\Lambda = 1 + \delta\omega$ in eq. (2.14), expand to linear order in $\delta\omega$, and equate the antisymmetric part of the coefficients of $\delta\omega_{\mu\nu}$, we get the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\Big(g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)\Big) - (\rho \leftrightarrow \sigma) \ . \tag{2.16}$$

These commutation relations specify the Lie algebra of the Lorentz group.

We can identify the components of the angular momentum operator $\mathbf{J}$ as $J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk}$, and the components of the boost operator $\mathbf{K}$ as $K_i \equiv M^{i0}$.

We then find from eq. (2.16) that

$$\begin{aligned} [J_i, J_j] &= i\hbar\varepsilon_{ijk} J_k \ , \\ [J_i, K_j] &= i\hbar\varepsilon_{ijk} K_k \ , \\ [K_i, K_j] &= -i\hbar\varepsilon_{ijk} J_k \ . \end{aligned} \tag{2.17}$$

The first of these is the usual set of commutators for angular momentum, and the second says that $\mathbf{K}$ transforms as a three-vector under rotations.

The third implies that a series of boosts can be equivalent to a rotation.

Similarly, we can let $\Lambda = 1 + \delta\omega$ in eq. (2.15) to get

$$[P^\mu, M^{\rho\sigma}] = i\hbar\Big(g^{\mu\sigma}P^\rho - (\rho\leftrightarrow\sigma)\Big) \ , \tag{2.18}$$

which becomes

$$\begin{aligned}
[J_i, H] &= 0 \ , \\
[J_i, P_j] &= i\hbar\varepsilon_{ijk}P_k \ , \\
[K_i, H] &= i\hbar P_i \ , \\
[K_i, P_j] &= i\hbar\delta_{ij}H \ ,
\end{aligned} \tag{2.19}$$

Also, the components of P^μ should commute with each other:

$$\begin{aligned}
[P_i, P_j] &= 0 \ , \\
[P_i, H] &= 0 \ .
\end{aligned} \tag{2.20}$$

Together, eqs. (2.17), (2.19), and (2.20) form the Lie algebra of the Poincare group.

Let us now consider what should happen to a quantum scalar field φ(x) under a Lorentz transformation.

We begin by recalling how time evolution works in the Heisenberg picture:

$$e^{+iHt/\hbar}\varphi(\mathbf{x},0)e^{-iHt/\hbar} = \varphi(\mathbf{x},t) \ . \tag{2.21}$$

Obviously, this should have a relativistic generalization,

$$e^{-iPx/\hbar}\varphi(0)e^{+iPx/\hbar} = \varphi(x) \ , \tag{2.22}$$

where $P_x = P^\mu x_\mu = \mathbf{P}\cdot\mathbf{x} - Hct$.

We can make this a little fancier by defining the unitary *spacetime translation operator*

$$T(a) \equiv \exp(-iP^\mu a_\mu/\hbar) \ . \tag{2.23}$$

Then we have

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a) \ . \tag{2.24}$$

For an infinitesimal translation,

$$T(\delta a) = I - \tfrac{i}{\hbar}\delta a_\mu P^\mu \ . \tag{2.25}$$

Comparing eqs. (2.12) and (2.25), we see that eq. (2.24) leads us to expect

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \ . \tag{2.26}$$

Derivatives of φ then carry vector indices that transform in the appropriate way, e.g.,

$$U(\Lambda)^{-1}\partial^{\mu}\varphi(x)U(\Lambda) = \Lambda^{\mu}{}_{\rho}\bar{\partial}^{\rho}\varphi(\Lambda^{-1}x) \ , \tag{2.27}$$

where the bar on a derivative means that it is with respect to the argument $\bar{x} = \Lambda^{-1}$x.

Eq. (2.27) also implies

$$U(\Lambda)^{-1}\partial^{2}\varphi(x)U(\Lambda) = \bar{\partial}^{2}\varphi(\Lambda^{-1}x) \ , \tag{2.28}$$

so that the Klein-Gordon equation, $(-\partial^2 + m^2/\hbar^2c^2)\varphi = 0$, is Lorentz invariant, as we saw in section 1.

3 Canonical Quantization of Scalar Fields

Prerequisite: 2

Let us go back and drastically simplify the hamiltonian we constructed in section 1, reducing it to the hamiltonian for free particles:

$$\begin{aligned} H &= \int d^3x \ a^{\dagger}(\mathbf{x})\Big(-\tfrac{1}{2m}\nabla^2\Big)a(\mathbf{x}) \\ &= \int d^3p \ \tfrac{1}{2m}\mathbf{p}^2 \ \widetilde{a}^{\dagger}(\mathbf{p})\widetilde{a}(\mathbf{p}) \ , \end{aligned} \tag{3.1}$$

where

$$\widetilde{a}(\mathbf{p}) = \int \frac{d^3x}{(2\pi)^{3/2}} \ e^{-i\mathbf{p}\cdot\mathbf{x}} \ a(\mathbf{x}) \ . \tag{3.2}$$

Here we have simplified our notation by setting

$$\hbar = 1 \ . \tag{3.3}$$

The appropriate factors of $\hbar$ can always be restored in any of our formulas via dimensional analysis.

The commutation (or anticommutation) relations of the $\tilde{a}(\mathbf{p})$ and $\tilde{a}^\dagger(\mathbf{p})$ operators are

$$\begin{aligned} [\widetilde{a}(\mathbf{p}), \widetilde{a}(\mathbf{p}')]_\mp &= 0 \ , \\ [\widetilde{a}^\dagger(\mathbf{p}), \widetilde{a}^\dagger(\mathbf{p}')]_\mp &= 0 \ , \\ [\widetilde{a}(\mathbf{p}), \widetilde{a}^\dagger(\mathbf{p}')]_\mp &= \delta^3(\mathbf{p} - \mathbf{p}') \ , \end{aligned} \tag{3.4}$$

where [A,B]$_\mp$ is either the commutator (if we want a theory of bosons) or the anticommutator (if we want a theory of fermions).

Thus $\tilde{a}^\dagger(\mathbf{p})$ can be interpreted as creating a state of definite momentum $\mathbf{p}$, and eq. (3.1) describes a theory of free particles.

The ground state is the vacuum $|0\rangle$; it is annihilated by $\tilde{a}(\mathbf{p})$

$$\widetilde{a}(\mathbf{p})|0\rangle = 0 \ , \tag{3.5}$$

and so its energy eigenvalue is zero.

The other eigenstates of *H* are all of the form $\tilde{a}^\dagger(\mathbf{p}_1)\cdots\tilde{a}^\dagger(\mathbf{p}_n)\,|0\rangle$, and the corresponding energy eigenvalue is E($\mathbf{p}_1$) +...+ E($\mathbf{p}_n$), where E($\mathbf{p}$) = $\frac{1}{2}$ m$\mathbf{p}^2$.

It is easy to see how to generalize this theory to a relativistic one; all we need to do is use the relativistic energy formula E($\mathbf{p}$) = +($\mathbf{p}^2c^2$ + m^2c^4)$^{1/2}$:

$$H = \int d^3p\,(\mathbf{p}^2c^2 + m^2c^4)^{1/2}\,\tilde{a}^\dagger(\mathbf{p})\tilde{a}(\mathbf{p})\ . \tag{3.6}$$

Now we have a theory of free relativistic spin-zero particles, and they can be either bosons or fermions.

Is this theory really Lorentz invariant?

We will answer this question (in the affirmative) in a very roundabout way: by constructing it again, from a rather different point of view, a point of view that emphasizes Lorentz invariance from the beginning.

We will start with the *classical* physics of a *real scalar field* φ(x).

Real means that φ(x) assigns a real number to every point in spacetime.

Scalar means that Alice [who uses coordinates x^μ and calls the field φ(x)] and Bob [who uses coordinates $\bar{x}^\mu$, related to Alice's coordinates by $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu$, and calls the field $\bar{\varphi}(\bar{x})$], agree on the numerical value of the field: $\varphi(x) = \bar{\varphi}(\bar{x})$.

This then implies that the equation of motion for φ(x) must be the same as that for $\bar{\varphi}(\bar{x})$.

We have already met an equation of this type: the Klein-Gordon equation,

$$(-\partial^2 + m^2)\varphi(x) = 0 \; . \tag{3.7}$$

Here we have simplified our notation by setting

$$c = 1 \tag{3.8}$$

in addition to $\hbar = 1$.

As with $\hbar$, factors of c can restored, if desired, by dimensional analysis.

We will adopt eq. (3.7) as the equation of motion that we would like φ(x) to obey.

It should be emphasized at this point that we are doing classical physics of a real scalar field.

We are not to think of φ(x) as a quantum wave function.

Thus, there should not be any factors of $\hbar$ in this version of the Klein-Gordon equation.

This means that the parameter m must have dimensions of inverse length; m is *not* (yet) to be thought of as a mass.

The equation of motion can be derived from variation of an action $S = \int L\, dt$, where L is the lagrangian.

Since the Klein-Gordon equation is local, we expect that the lagrangian can be written as the space integral of a lagrangian density $\mathcal{L}$: $L = \int d^3x\, \mathcal{L}$.

Thus, $S = \int d^4x\, \mathcal{L}$.

The integration measure d^4x is Lorentz invariant: if we change to coordinates $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$, we have $d^4\bar{x} = |\det \Lambda|\, d^4x = d^4x$.

Thus, for the action to be Lorentz invariant, the lagrangian density must be a Lorentz scalar: $\mathcal{L}(x) = \mathcal{L}(\bar{x})$.

Then we have $\bar{S} = \int d^4\bar{x}\bar{\mathcal{L}}(\bar{x}) = \int d^4x\mathcal{L}(x) = S$

Any simple function of φ is a Lorentz scalar, and so are products of derivatives with all indices contracted, such as $\partial^\mu\varphi\partial_\mu\varphi$.

We will take for $\mathcal{L}$

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 + \Omega_0 \ , \tag{3.9}$$

where Ω_0 is an arbitrary constant.

We find the equation motion (also known as the Euler-Lagrange equation) by making an infinitesimal variation $\delta\varphi(x)$ in $\varphi(x)$, and requiring the corresponding variation of the action to vanish:

$$\begin{aligned} 0 &= \delta S \\ &= \int d^4x \left[-\tfrac{1}{2}\partial^\mu \delta\varphi \partial_\mu \varphi - \tfrac{1}{2}\partial^\mu \varphi \partial_\mu \delta\varphi - m^2 \varphi\, \delta\varphi \right] \\ &= \int d^4x \left[+\partial^\mu \partial_\mu \varphi - m^2 \varphi \right] \delta\varphi \ . \end{aligned} \tag{3.10}$$

In the last line, we have integrated by parts in each of the first two terms, putting both derivatives on φ.

We assume $\delta\varphi(x)$ vanishes at infinity in any direction (spatial or temporal), so that there is no surface term.

Since $\delta\varphi$ has an arbitrary x dependence, eq. (3.10) can be true if and only if $(-\partial^2 + m^2)\varphi = 0$.

One solution of the Klein-Gordon equation is a plane wave of the form $\exp(i\mathbf{k}\cdot\mathbf{x} \pm i\omega t)$, where $\mathbf{k}$ is an arbitrary real wave-vector, and

$$\omega = +(\mathbf{k}^2 + m^2)^{1/2} \ . \tag{3.11}$$

The general solution (assuming boundary conditions that require φ to remain finite at spatial infinity) is then

$$\varphi(\mathbf{x}, t) = \int \frac{d^3k}{f(k)} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} + i\omega t} \right] \ , \tag{3.12}$$

where a(**k**) and b(**k**) are arbitrary functions of the wave vector **k**, and f(k) is a redundant function of the magnitude of **k** which we have inserted for later convenience.

Note that, if we were attempting to interpret φ(x) as a quantum wave function (which we most definitely are not), then the second term would constitute the "negative energy" contributions to the wave function.

This is because a plane-wave solution of the nonrelativistic Schrodinger equation for a single particle looks like exp(i**p** · **x**– iE(**p**)t), with E(p) = $\frac{1}{2}$ m**p**^{2}; there is a minus sign in front of the positive energy.

We are trying to interpret eq. (3.12) as a real classical field, but this formula does not generically result in φbeing real.

We must impose φ^*(x) = φ(x), where

$$\begin{aligned}\varphi^*(\mathbf{x},t) &= \int \frac{d^3k}{f(k)}\left[a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t} + b^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right] \\ &= \int \frac{d^3k}{f(k)}\left[a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t} + b^*(-\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right] \ .\end{aligned} \tag{3.13}$$

In the second term on the second line, we have changed the dummy integration variable from **k** to **–k**.

Comparing eqs. (3.12) and (3.13), we see that φ^*(x) = φ(x) requires b*(**–k**) = a(**k**).

Imposing this condition, we can rewrite φ as

$$\begin{aligned}\varphi(\mathbf{x},t) &= \int \frac{d^3k}{f(k)}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^*(-\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right] \\ &= \int \frac{d^3k}{f(k)}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right] \\ &= \int \frac{d^3k}{f(k)}\left[a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right] \ ,\end{aligned} \tag{3.14}$$

where $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$ is the Lorentz-invariant product of the four-vectors $x^\mu = (t, \mathbf{x})$ and $k^\mu = (\omega, \mathbf{k})$: $kx = k^\mu x_\mu = g_{\mu\nu}k^\mu x^\nu$.

Note that

$$k^2 = k^\mu k_\mu = \mathbf{k}^2 - \omega^2 = -m^2 \ . \tag{3.15}$$

A four-momentum k^μ that obeys $k^2 = -m^2$ is said to be *on the mass shell*, or *on shell* for short.

It is now convenient to choose $f(k)$ so that $d^3k/f(k)$ is Lorentz invariant.

An integration measure that is manifestly invariant under orthochronous Lorentz transformations is $d^4k\delta(k^2+m^2)\,\theta(k^0)$, where $\delta(x)$ is the Dirac delta function, $\theta(x)$ is the unit step function, and k^0 is treated as an independent integration variable.

We then have

$$\int_{-\infty}^{+\infty} dk^0\, \delta(k^2+m^2)\, \theta(k^0) = \frac{1}{2\omega}\ . \tag{3.16}$$

Here we have used the rule

$$\int_{-\infty}^{+\infty} dx\ \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\ , \tag{3.17}$$

where g(x) is any smooth function of x with simple zeros at x= x_i ; in our case, the only zero is at $k^0 = \omega$.

Thus we see that if we take $f(k)\ \alpha\ \omega$, then $d^3k/f(k)$ will be Lorentz invariant.

We will take $f(k) = (2\pi)^3 2\omega$.

It is then convenient to give the corresponding Lorentz-invariant differential its own name:

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega}\ . \tag{3.18}$$

Thus we finally have

$$\varphi(x) = \int \widetilde{dk}\ \left[a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right]\ . \tag{3.19}$$

We can also invert this formula to get $a(\mathbf{k})$ in terms of φ(x).

We have

$$\int d^3x\, e^{-ikx}\varphi(x) = \tfrac{1}{2\omega}a(\mathbf{k}) + \tfrac{1}{2\omega}e^{2i\omega t}a^*(-\mathbf{k})\ ,$$
$$\int d^3x\, e^{-ikx}\partial_0\varphi(x) = -\tfrac{i}{2}a(\mathbf{k}) + \tfrac{i}{2}e^{2i\omega t}a^*(-\mathbf{k})\ . \qquad (3.20)$$

We can combine these to get

$$a(\mathbf{k}) = \int d^3x\, e^{-ikx}\Big[i\partial_0\varphi(x) + \omega\varphi(x)\Big]$$
$$= i\int d^3x\, e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)\ , \qquad (3.21)$$

where $f\overleftrightarrow{\partial}g \equiv f(\partial_\mu g) - (\partial_\mu f)g$, and $\partial_0\varphi = \partial\varphi/\partial t = \dot{\varphi}$.

Note that a(k) is time independent.

Now that we have the lagrangian, we can construct the hamiltonian by the usual rules.

Recall that, given a lagrangian $L(q_i, \dot{q}_i)$ as a function of some coordinates q_i and their time derivatives $\dot{q}_i$, the conjugate momenta are given by $p_i = \partial L/\partial\dot{q}_i$, and the hamiltonian by $H = ip_i\dot{q}_i - L$.

In our case, the role of $q_i(t)$ is played by $\varphi(\mathbf{x},t)$, with $\mathbf{x}$ playing the role of a (continuous) index.

The appropriate generalizations are then

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \tag{3.22}$$

and

$$\mathcal{H} = \Pi\dot{\varphi} - \mathcal{L}\ , \tag{3.23}$$

where $\mathscr{H}$ is the hamiltonian density, and the hamiltonian itself is $H = \int d^3x\ \mathscr{H}$.

In our case, we have

$$\Pi(x) = \dot{\varphi}(x) \tag{3.24}$$

and

$$\mathcal{H} = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 - \Omega_0\ . \tag{3.25}$$

Using eq. (3.19), we can write H in terms of the $a(\mathbf{k})$ and $a^*(\mathbf{k})$ coefficients

$$\begin{aligned} H = -\Omega_0 V + \tfrac{1}{2}\int \widetilde{dk}\ \widetilde{dk}'\ d^3x\ \Big[& \\ & \left(-i\omega\, a(\mathbf{k})e^{ikx} + i\omega\, a^*(\mathbf{k})e^{-ikx}\right)\left(-i\omega'\, a(\mathbf{k}')e^{ik'x} + i\omega'\, a^*(\mathbf{k}')e^{-ik'x}\right) \\ & + \left(+i\mathbf{k}\, a(\mathbf{k})e^{ikx} - i\mathbf{k}\, a^*(\mathbf{k})e^{-ikx}\right)\cdot\left(+i\mathbf{k}'\, a(\mathbf{k}')e^{ik'x} - i\mathbf{k}'\, a^*(\mathbf{k}')e^{-ik'x}\right) \\ & + m^2\left(a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right)\left(a(\mathbf{k}')e^{ik'x} + a^*(\mathbf{k}')e^{-ik'x}\right)\Big] \end{aligned}$$

$$
\begin{aligned}
&= -\Omega_0 V + \tfrac{1}{2}(2\pi)^3 \int \widetilde{dk}\; \widetilde{dk}'\; \Big[\\
&\qquad \delta^3(\mathbf{k}-\mathbf{k}')(+\omega\omega' + \mathbf{k}\cdot\mathbf{k}' + m^2) \\
&\qquad\quad \times \Big(a^*(\mathbf{k}) a(\mathbf{k}') e^{-i(\omega-\omega')t} + a(\mathbf{k}) a^*(\mathbf{k}') e^{+i(\omega-\omega')t} \Big) \\
&\quad + \delta^3(\mathbf{k}+\mathbf{k}')(-\omega\omega' - \mathbf{k}\cdot\mathbf{k}' + m^2) \\
&\qquad\quad \times \Big(a(\mathbf{k}) a(\mathbf{k}') e^{-i(\omega+\omega')t} + a^*(\mathbf{k}) a^*(\mathbf{k}') e^{+i(\omega+\omega')t} \Big) \\
&= -\Omega_0 V + \tfrac{1}{2} \int \widetilde{dk}\; \tfrac{1}{2\omega} \Big[\\
&\qquad (+\omega^2 + \mathbf{k}^2 + m^2) \Big(a^*(\mathbf{k}) a(\mathbf{k}) + a(\mathbf{k}) a^*(\mathbf{k}) \Big) \\
&\qquad + (-\omega^2 + \mathbf{k}^2 + m^2) \Big(a(\mathbf{k}) a(-\mathbf{k}) e^{-2i\omega t} + a^*(\mathbf{k}) a^*(-\mathbf{k}) e^{+2i\omega t} \Big) \Big] \\
&= -\Omega_0 V + \tfrac{1}{2} \int \widetilde{dk}\; \omega \Big(a^*(\mathbf{k}) a(\mathbf{k}) + a(\mathbf{k}) a^*(\mathbf{k}) \Big) \;,
\end{aligned}
\tag{3.26}
$$

where V is the volume of space.

To get the second equality, we used

$$
\int d^3x\; e^{i\mathbf{q}\cdot x} = (2\pi)^3 \delta^3(\mathbf{q}) \;. \tag{3.27}
$$

To get the third equality, we integrated over $\mathbf{k}'$, using $\widetilde{dk}' = d^3k'/(2\pi)^3 2\omega'$.

The last equality then follows from $\omega = (\mathbf{k}^2+m^2)^{1/2}$.

Also, we were careful to keep the ordering of $a(\mathbf{k})$ and $a^*(\mathbf{k})$ unchanged throughout, in anticipation of passing to the quantum theory where these classical functions will become operators that may not commute.

Let us take up the quantum theory now.

We can go from classical to quantum mechanics via *canonical quantization*.

This means that we promote q_i and p_i to operators, with commutation relations $[q_i , q_j] = 0$, $[p_i , p_j] = 0$, and $[q_i , p_j] = i\hbar\delta_{ij}$.

In the Heisenberg picture, these operators should be taken at equal times.

In our case, where the "index" is continuous (and we have set $\hbar = 1$), we have

$$\begin{aligned} [\varphi(\mathbf{x},t),\varphi(\mathbf{x}',t)] &= 0 \ , \\ [\Pi(\mathbf{x},t),\Pi(\mathbf{x}',t)] &= 0 \ , \\ [\varphi(\mathbf{x},t),\Pi(\mathbf{x}',t)] &= i\delta^3(\mathbf{x}-\mathbf{x}') \ . \end{aligned} \tag{3.28}$$

From these *canonical commutation relations*, and from eqs. (3.21) and (3.24), we can deduce

$$\begin{aligned} [a(\mathbf{k}), a(\mathbf{k}')] &= 0 \ , \\ [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] &= 0 \ , \\ [a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= (2\pi)^3 2\omega \, \delta^3(\mathbf{k} - \mathbf{k}') \ . \end{aligned} \tag{3.29}$$

We are now denoting $a^*(\mathbf{k})$ as $a^\dagger(\mathbf{k})$, since $a^\dagger(\mathbf{k})$ is now the hermitian conjugate (rather than the complex conjugate) of the operator $a(\mathbf{k})$.

We can now rewrite the hamiltonian as

$$H = \int \widetilde{dk} \, \omega \, a^\dagger(\mathbf{k}) a(\mathbf{k}) + (\mathcal{E}_0 - \Omega_0) V \ , \tag{3.30}$$

where

$$\mathcal{E}_0 = \tfrac{1}{2}(2\pi)^{-3} \int d^3k \, \omega \tag{3.31}$$

is the total zero-point energy of all the oscillators per unit volume, and, using eq. (3.27), we have interpreted $(2\pi)^3\delta^3(0)$ as the volume of space V.

If we integrate in eq. (3.31) over the whole range of $\mathbf{k}$, the value of $\boldsymbol{\mathcal{E}}_0$ is infinite.

If we integrate only up to a maximum value of $\boldsymbol{\Lambda}$, a number known as the *ultraviolet cutoff*, we find

$$\mathcal{E}_0 = \frac{\Lambda^4}{16\pi^2} \ , \tag{3.32}$$

where we have assumed $\Lambda \gg m$.

This is physically justified if, in the real world, the formalism of quantum field theory breaks down at some large energy scale.

For now, we simply note that the value of Ω_0 is arbitrary, and so we are free to choose $\Omega_0 = \mathcal{E}_0$.

With this choice, the ground state has energy eigenvalue zero.

Now, if we like, we can take the limit $\Lambda \to \infty$, with no further consequences.

We will meet more of these *ultraviolet divergences* after we introduce interactions.

The hamiltonian of eq. (3.30) is now the same as that of eq. (3.6), with $a(\mathbf{k}) = [(2\pi)^3 2\omega]^{1/2}\ \tilde{a}\,(\mathbf{k})$.

The commutation relations (3.4) and (3.29) are also equivalent, if we choose commutators (rather than anticommutators) in eq. (3.4).

Thus, *we have re-derived the hamiltonian of free relativistic bosons by quantization of a scalar field whose equation of motion is the Klein-Gordon equation*.

The parameter m in the lagrangian is now seen to be the mass of the particle in the quantum theory.

More precisely, since m has dimensions of inverse length, the particle mass is $\hbar c m$.)

What if we want fermions?

Then we should use anticommutators in eqs. (3.28) and (3.29).

There is a problem, though; eq. (3.26) does not then become eq. (3.30).

Instead, we get $H = -\Omega_0 V$, a simple constant.

Clearly there is something wrong with using anticommutators.

This is another hint of the spin-statistics theorem, which we will take up in section 4.

Next, we would like to add Lorentz-invariant interactions to our theory.

With the formalism we have developed, this is easy to do.

Any local function of $\varphi(x)$ is a Lorentz scalar, and so if we add a term like φ^3 or φ^4 to the lagrangian density $\mathcal{L}$, the resulting action will still be Lorentz invariant.

Now, however, we will have interactions among the particles.

Our next task is to deduce the consequences of these interactions.

However, we already have enough tools at our disposal to prove the spin-statistics theorem for spin-zero particles, and that is what we turn to next.

4 The Spin-Statistics Theorem

Prerequisite: 3

Let us consider a theory of free, spin-zero particles specified by the hamiltonian

$$H_0 = \int \widetilde{dk}\, \omega\, a^\dagger(\mathbf{k}) a(\mathbf{k}) \ , \tag{4.1}$$

where $\omega = (\mathbf{k}^2 + m^2)^{1/2}$, and either the commutation or anticommutation relations

$$\begin{aligned} [a(\mathbf{k}), a(\mathbf{k}')]_\mp &= 0 \ , \\ [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp &= 0 \ , \\ [a(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp &= (2\pi)^3 2\omega\, \delta^3(\mathbf{k} - \mathbf{k}') \ . \end{aligned} \tag{4.2}$$

Of course, if we want a theory of bosons, we should use commutators, and if we want fermions, we should use anticommutators.

Now let us consider adding terms to the hamiltonian that will result in local, *Lorentz invariant* interactions.

In order to do this, it is convenient to define a nonhermitian field,

$$\varphi^+(\mathbf{x},0) \equiv \int \widetilde{dk}\; e^{i\mathbf{k}\cdot\mathbf{x}}\; a(\mathbf{k})\;, \tag{4.3}$$

and its hermitian conjugate

$$\varphi^+(\mathbf{x},0) \equiv \int \widetilde{dk}\; e^{i\mathbf{k}\cdot\mathbf{x}}\; a(\mathbf{k})\;, \tag{4.4}$$

These are then time-evolved with H_0:

$$\begin{aligned}
\varphi^+(\mathbf{x},t) &= e^{iH_0t}\varphi^+(\mathbf{x},0)e^{-iH_0t} = \int \widetilde{dk}\; e^{ikx}\; a(\mathbf{k})\;, \\
\varphi^-(\mathbf{x},t) &= e^{iH_0t}\varphi^-(\mathbf{x},0)e^{-iH_0t} = \int \widetilde{dk}\; e^{-ikx}\; a^\dagger(\mathbf{k})\;.
\end{aligned} \tag{4.5}$$

Note that the usual hermitian free field $\varphi(x)$ is just the sum of these: $\varphi(x) = \varphi^+(x) + \varphi^-(x)$.

For a proper orthochronous Lorentz transformation Λ, we have

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)\;. \tag{4.6}$$

This implies that the particle creation and annihilation operators transform as

$$\begin{aligned}
U(\Lambda)^{-1}a(\mathbf{k})U(\Lambda) &= a(\Lambda^{-1}\mathbf{k})\;, \\
U(\Lambda)^{-1}a^\dagger(\mathbf{k})U(\Lambda) &= a^\dagger(\Lambda^{-1}\mathbf{k})\;.
\end{aligned} \tag{4.7}$$

This, in turn, implies that $\varphi^+(x)$ and $\varphi^-(x)$ are Lorentz scalars:

$$U(\Lambda)^{-1}\varphi^{\pm}(x)U(\Lambda) = \varphi^{\pm}(\Lambda^{-1}x) \ . \tag{4.8}$$

We will then have local, Lorentz invariant interactions if we take the interaction lagrangian density $\mathscr{L}_1$ to be a hermitian function of $\varphi^+(x)$ and $\varphi^-(x)$.

To proceed we need to recall some facts about time-dependent perturbation theory in quantum mechanics.

The transition amplitude $\mathcal{T}_{f\leftarrow i}$ to start with an initial state $|i\rangle$ at time $t = -\infty$ and end with a final state $|f\rangle$ at time $t = +\infty$ is

$$\mathcal{T}_{f\leftarrow i} = \langle f|\,\mathrm{T}\exp\left[-i\int_{-\infty}^{+\infty} dt\ H_I(t)\right]|i\rangle \ , \tag{4.9}$$

where $H_I(t)$ is the perturbing hamiltonian in the interaction picture,

$$H_I(t) = \exp(+iH_0t)\,H_1\exp(-iH_0t) \ , \tag{4.10}$$

H_1 is the perturbing hamiltonian in the Schrodinger picture, and T is the time ordering symbol: a product of operators to its right is to be ordered, not as written, but with operators at later times to the left of those at earlier times.

We write $H_1 = \int d^3x\ \mathcal{H}_1(\mathbf{x},0)$, and specify $\mathcal{H}_1(\mathbf{x},0)$ as a hermitian function of $\varphi^+(\mathbf{x},0)$ and $\varphi^-(\mathbf{x},0)$.

Then, using eqs. (4.5) and (4.10), we can see that, in the interaction picture, the perturbing hamiltonian density $\mathcal{H}_I(\mathbf{x},t)$ is simply given by the same function of $\varphi^+(\mathbf{x},t)$ and $\varphi^-(\mathbf{x},t)$.

Now we come to the key point: for the transition amplitude $\mathcal{T}_{f\leftarrow i}$ to be Lorentz invariant, the time ordering must be *frame independent*.

The time ordering of two spacetime points x and x′ is frame independent if their separation is timelike; this means that the interval between them is negative, $(x-x')^2 < 0$.

Two spacetime points whose separation is spacelike, $(x- x')^2 > 0$, can have different temporal ordering in different frames.

In order to avoid $\mathcal{T}_{f\leftarrow i}$ being different in different frames, we must then require

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \quad \text{whenever} \quad (x-x')^2 > 0\ . \tag{4.11}$$

Obviously, $[\varphi^+(x), \varphi^+(x')]_\mp = [\varphi^-(x), \varphi^-(x')]_\mp = 0$.

However,

$$
\begin{aligned}
[\varphi^+(x), \varphi^-(x')]_\mp &= \int \widetilde{dk}\ \widetilde{dk}'\ e^{i(kx-k'x')}[a(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp \\
&= \int \widetilde{dk}\ e^{ik(x-x')} \\
&= \frac{m}{4\pi^2 r} K_1(mr) \\
&\equiv C(r)\ .
\end{aligned}
\tag{4.12}
$$

In the next-to-last line, we have taken $(x - x')^2 = r^2 > 0$, and $K_1(z)$ is a modified Bessel function.

This Lorentz-invariant integral is most easily evaluated in the frame where $t' = t$.)

The function C(r) is not zero for any $r > 0$.

Not even when $m = 0$; in this case, $C(r) = 1/4\pi^2 r^2$.

On the other hand, $\mathscr{H}_I(x)$ must involve both $\varphi^+(x)$ and $\varphi^-(x)$, by hermiticity.

Thus, generically, we will not be able to satisfy eq. (4.11).

To resolve this problem, let us try using only particular linear combinations of $\varphi^+(x)$ and $\varphi^-(x)$.

Define

$$\begin{aligned}\varphi_\lambda(x) &\equiv \varphi^+(x) + \lambda\varphi^-(x)\ ,\\ \varphi^\dagger_\lambda(x) &\equiv \varphi^-(x) + \lambda^*\varphi^+(x)\ ,\end{aligned} \tag{4.13}$$

where λ is an arbitrary complex number.

We then have

$$\begin{aligned}[\varphi_\lambda(x), \varphi^\dagger_\lambda(x')]_\mp &= [\varphi^+(x), \varphi^-(x')]_\mp + |\lambda|^2[\varphi^-(x), \varphi^+(x')]_\mp\\ &= (1 \mp |\lambda|^2)\, C(r)\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}[\varphi_\lambda(x), \varphi_\lambda(x')]_\mp &= \lambda[\varphi^+(x), \varphi^-(x')]_\mp + \lambda[\varphi^-(x), \varphi^+(x')]_\mp\\ &= \lambda(1 \mp 1)\, C(r)\ .\end{aligned} \tag{4.15}$$

Thus, if we want $\varphi_\lambda(x)$ to either commute or anticommute with both $\varphi_\lambda(x')$ and $\varphi^\dagger_\lambda(x')$ at spacelike separations, we must choose $|\lambda| = 1$, and we must choose commutators.

Then (and only then), we can build a suitable $\mathcal{H}_I(x)$ by making it a hermitian function of $\varphi_\lambda(x)$.

But this has simply returned us to the theory of a real scalar field, because, for $\lambda = e^{i\alpha}$, $e^{-i\alpha/2}\varphi_\lambda(x)$ is hermitian.

In fact, if we make the replacements $a(\mathbf{k}) \longrightarrow e^{+i\alpha/2}a(\mathbf{k})$ and $a^\dagger(\mathbf{k}) \longrightarrow e^{-i\alpha/2}a\dagger(\mathbf{k})$, then the commutation relations of eq. (4.2) are unchanged, and $e^{-i\alpha/2}\varphi_\lambda(x) = \varphi(x) = \varphi^+(x) + \varphi^-(x)$.

Thus, our attempt to start with the creation and annihilation operators $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$ as the fundamental objects has simply led us back to the real, commuting, scalar field $\varphi(x)$ as the fundamental object.

Let us return to thinking of $\varphi(x)$ as fundamental, with a lagrangian density given by some function of the Lorentz scalars $\varphi(x)$ and $\partial^\mu\varphi(x)\partial_\mu\varphi(x)$.

Then, quantization will result in $[\varphi(x),\varphi(x')]_\mp = 0$ for $t = t'$.

If we choose anticommutators, then $[\varphi(x)]^2 = 0$ and $[\partial_\mu\varphi(x)]^2 = 0$, resulting in a trivial $\mathcal{L}$ that is at most linear in φ, and independent of $\dot{\varphi}$.

This clearly does not lead to the correct physics.

This situation turns out to generalize to fields of higher spin, in any number of spacetime dimensions.

One choice of quantization (commutators or anticommutators) always leads to a trivial $\mathcal{L}$, and so this choice is disallowed.

Furthermore, the allowed choice is always commutators for fields of integer spin, and anticommutators for fields of half-integer spin.

If we try treating the particle creation and annihilation operators as fundamental, rather than the fields, we find a situation similar to that of the spin-zero case, and are led to the reconstruction of a field that must obey the appropriate quantization scheme.

5 The LSZ Reduction Formula

Prerequisite: 3

Let us now consider how to construct appropriate initial and final states for scattering experiments.

In the free theory, we can create a state of one particle by acting on the vacuum state with a creation operator

$$|k\rangle = a^\dagger(\mathbf{k})|0\rangle \ , \tag{5.1}$$

where

$$a^\dagger(\mathbf{k}) = -i \int d^3x \ e^{ikx} \overleftrightarrow{\partial_0} \varphi(x) \ . \tag{5.2}$$

The vacuum state $|0\rangle$ is annihilated by every $a(\mathbf{k})$,

$$a(\mathbf{k})|0\rangle = 0 \ , \tag{5.3}$$

and has unit norm,

$$\langle 0|0\rangle = 1 \ . \tag{5.4}$$

The one-particle state $|k\rangle$ then has the Lorentz-invariant normalization

$$\langle k|k'\rangle = (2\pi)^3\, 2\omega\, \delta^3(\mathbf{k}-\mathbf{k}')\ , \tag{5.5}$$

where $\omega = (k^2 + m^2)^{1/2}$.

Next, let us define a time-independent operator that (in the free theory) creates a particle localized in momentum space near $\mathbf{k}_1$, and localized in position space near the origin:

$$a_1^\dagger \equiv \int d^3k\ f_1(\mathbf{k}) a^\dagger(\mathbf{k})\ , \tag{5.6}$$

where

$$f_1(\mathbf{k}) \propto \exp[-(\mathbf{k}-\mathbf{k}_1)^2/4\sigma^2] \tag{5.7}$$

is an appropriate wave packet, and σ is its width in momentum space.

Consider the state $a_1^\dagger|0\rangle$.

If we time evolve this state in the Schrodinger picture, the wave packet will propagate (and spread out).

The particle is thus localized far from the origin as $t \to \pm\infty$.

If we consider instead a state of the form $a_1^\dagger a_2^\dagger|0\rangle$, where $\mathbf{k}_1 \neq \mathbf{k}_2$, then the two particles are widely separated in the far past.

Let us guess that this still works in the interacting theory.

One complication is that $a^\dagger(\mathbf{k})$ will no longer be time independent, and so $a^\dagger{}_1$, eq. (5.6), becomes time dependent as well.

Our guess for a suitable initial state of a scattering experiment is then

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)|0\rangle \ . \tag{5.8}$$

By appropriately normalizing the wave packets, we can make $\langle \mathrm{i} \mid \mathrm{i}\rangle = 1$, and we will assume that this is the case.

Similarly, we can consider a final state

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)|0\rangle \ , \tag{5.9}$$

where $\mathbf{k}'_1 \neq \mathbf{k}'_2$, and $\langle f \mid f\rangle = 1$.

This describes two widely separated particles in the far future.

We could also consider acting with more creation operators, if we are interested in the production of some extra particles in the collision of two.

Now the scattering amplitude is simply given by $\langle f \mid \mathrm{i}\rangle$.

We need to find a more useful expression for $\langle f \,|\, \mathrm{i} \rangle$.

To this end, let us note that

$$
\begin{aligned}
a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{+\infty} dt\; \partial_0 a_1^\dagger(t) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; \partial_0 \left(e^{ikx} \overset{\leftrightarrow}{\partial_0} \varphi(x) \right) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; e^{ikx} (\partial_0^2 + \omega^2) \varphi(x) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; e^{ikx} (\partial_0^2 + \mathbf{k}^2 + m^2) \varphi(x) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; e^{ikx} (\partial_0^2 - \overset{\leftarrow}{\nabla}^2 + m^2) \varphi(x) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; e^{ikx} (\partial_0^2 - \overset{\rightarrow}{\nabla}^2 + m^2) \varphi(x) \\
&= -i \int d^3k\; f_1(\mathbf{k}) \int d^4x\; e^{ikx} (-\partial^2 + m^2) \varphi(x) \; .
\end{aligned}
\tag{5.10}
$$

The first equality is just the fundamental theorem of calculus.

To get the second, we substituted the definition of $a^\dagger_1(\mathrm{t})$, and combined the $\mathrm{d}^3\mathrm{x}$ from this definition with the dt to get $\mathrm{d}^4\mathrm{x}$.

The third comes from straightforward evaluation of the time derivatives.

The fourth uses $\omega^2 = \mathbf{k}^2 + m^2$.

The fifth writes $\mathbf{k}^2$ as $-\nabla^2$ acting on $e^{i\mathbf{k}\cdot\mathbf{x}}$.

The sixth uses integration by parts to move the ∇^2 onto the field $\phi(x)$; here the wave packet is needed to avoid a surface term.

The seventh simply identifies $\partial^2_0 - \nabla^2$ as $-\partial^2$.

In free-field theory, the right-hand side of eq. (5.10) is zero, since $\varphi(x)$ obeys the Klein-Gordon equation.

In an interacting theory, with (say) $\mathscr{L}_1 = \frac{1}{6} g\varphi^3$, we have instead $(-\partial^2 + m^2)\varphi = \frac{1}{2} g\varphi^2$.

Thus the right-hand side of eq. (5.10) is not zero in an interacting theory.

Rearranging eq. (5.10), we have

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i \int d^3k\ f_1(\mathbf{k}) \int d^4x\ e^{ikx}(-\partial^2 + m^2)\varphi(x)\ . \tag{5.11}$$

We will also need the hermitian conjugate of this formula, which (after a little more rearranging) reads

$$a_1(+\infty) = a_1(-\infty) + i \int d^3k\ f_1(\mathbf{k}) \int d^4x\ e^{-ikx}(-\partial^2 + m^2)\varphi(x)\ . \tag{5.12}$$

Let us return to the scattering amplitude,

$$\langle f|i\rangle = \langle 0|a_{1'}(+\infty)a_{2'}(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|0\rangle \ . \tag{5.13}$$

Note that the operators are in time order.

Thus, if we feel like it, we can put in a time-ordering symbol without changing anything:

$$\langle f|i\rangle = \langle 0|\mathrm{T}a_{1'}(+\infty)a_{2'}(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|0\rangle \ . \tag{5.14}$$

The symbol T means the product of operators to its right is to be ordered, as written, but with operators at later times to the left of those at earlier times.

Now let us use eqs. (5.11) and (5.12) in eq. (5.14).

The time-ordering symbol automatically moves all $a_{i'}(-\infty)$'s to the right, where they annihilate $|0\rangle$.

Similarly, all $a^\dagger_i(+\infty)$'s move to the left, where they annihilate $\langle 0|$.

The wave packets no longer play a key role, and we can take the $\sigma \to 0$ limit in eq. (5.7), so that $f_1(\mathbf{k}) = \delta^3(\mathbf{k} - \mathbf{k}_1)$.

The initial and final states now have a delta-function normalization, the multiparticle generalization of eq. (5.5).

We are left with

$$\begin{aligned}\langle f|i\rangle = i^{n+n'} \int d^4x_1\, e^{ik_1x_1}(-\partial_1^2 + m^2)\ldots \\ d^4x_1'\, e^{-ik_1'x_1'}(-\partial_{1'}^2 + m^2)\ldots \\ \times \langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi(x_1')\ldots|0\rangle\ .\end{aligned} \tag{5.15}$$

This formula has been written to apply to the more general case of n incoming particles and n' outgoing particles; the ellipses stand for similar factors for each of the other incoming and outgoing particles.

Eq. (5.15) is the *Lehmann-Symanzik-Zimmermann* reduction formula, or LSZ formula for short.

It is one of the key equations of quantum field theory.

However, our derivation of the LSZ formula relied on the supposition that the creation operators of free field theory would work comparably in the interacting theory.

This is a rather suspect assumption, and so we must review it.

Let us consider what we can deduce about the energy and momentum eigenstates of the interacting theory on physical grounds.

First, we assume that there is a unique ground state $|0\rangle$, with zero energy and momentum.

The first excited state is a state of a single particle with mass m.

This state can have an arbitrary three-momentum **k**; its energy is then $E = \omega = (\mathbf{k}^2 + m^2)^{1/2}$.

The next excited state is that of two particles.

These two particles could form a bound state with energy less than 2m (like the hydrogen atom in quantum electrodynamics), but, to keep things simple, let us assume that there are no such bound states.

Then the lowest possible energy of a two-particle state is 2m.

However, a two-particle state with zero total three-momentum can have any energy above 2m, because the two particles could have some relative momentum that contributes to their total energy.

Thus we are led to a picture of the states of theory as shown in fig. (5.1).

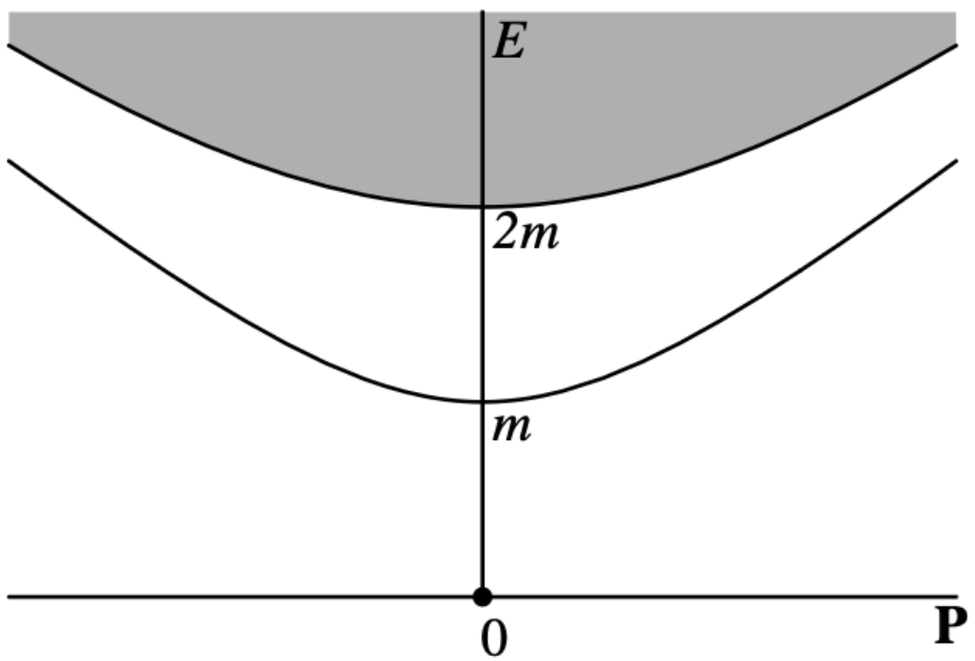


Figure 5.1: The exact energy eigenstates in the (**P**,E) plane. The ground state is isolated at (0,0), the one-particle states form an isolated hyperbola that passes through (0,m), and the multi-particle continuum lies at and above the hyperbola that passes through (0,2m).

Now let us consider what happens when we act on the ground state with the field operator $\varphi(x)$.

To this end, it is helpful to write

$$\varphi(x) = \exp(-iP^\mu x_\mu)\varphi(0)\exp(+iP^\mu x_\mu) \ , \tag{5.16}$$

where P^μ is the energy-momentum four-vector.

This equation, introduced in section 2, is just the relativistic generalization of the Heisenberg equation.

Now let us sandwich φ(x) between the ground state (on the right), and other possible states (on the left).

For example, let us put the ground state on the left as well.

Then we have

$$\begin{aligned} \langle 0|\varphi(x)|0\rangle &= \langle 0|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle \\ &= \langle 0|\varphi(0)|0\rangle \ . \end{aligned} \tag{5.17}$$

To get the second line, we used $P^\mu|0\rangle = 0$.

The final expression is just a Lorentz-invariant number.

Since $|0\rangle$ is the exact ground state of the interacting theory, we have (in general) no idea what this number is.

We would like $\langle 0|\varphi(0)|0\rangle$ to be zero.

This is because we would like $a^\dagger{}_1(\pm\infty)$, when acting on $|0\rangle$, to create a single particle state.

We do not want $a^\dagger{}_1(\pm\infty)$ to create a linear combination of a single particle state and the ground state.

But this is precisely what will happen if $\langle 0|\varphi(0)|0\rangle$ is not zero.

So, if $v \equiv \langle 0|\varphi(0)|0\rangle$ is not zero, we will shift the field $\varphi(x)$ by the constant v.

This means that we go back to the lagrangian, and replace $\varphi(x)$ everywhere by $\varphi(x) + v$.

This is just a change of the name of the operator of interest, and does not affect the physics.

However, the shifted $\varphi(x)$ obeys, by construction, $\langle 0|\varphi(x)|0\rangle = 0$.

Let us now consider $\langle p|\varphi(x)|0\rangle$, where $|p\rangle$ is a one-particle state with four-momentum p, normalized according to eq. (5.5).

Again using eq. (5.16), we have

$$\begin{aligned} \langle p|\varphi(x)|0\rangle &= \langle p|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle \\ &= e^{-ipx}\langle p|\varphi(0)|0\rangle \ , \end{aligned} \tag{5.18}$$

where $\langle p|\varphi(0)|0\rangle$ is a Lorentz-invariant number.

It is a function of p, but the only Lorentz-invariant functions of p are functions of p^2, and p^2 is just the constant $-m^2$.

So $\langle p|\varphi(0)|0\rangle$ is just some number that depends on m and (presumably) the other parameters in the lagrangian.

We would like $\langle p|\varphi(0)|0\rangle$ to be one.

That is what it is in free-field theory, and we know that, in free-field theory, $a^\dagger_1(\pm\infty)$ creates a correctly normalized one-particle state.

Thus, for $a^\dagger_1(\pm\infty)$ to create a correctly normalized one-particle state in the interacting theory, we must have $\langle p|\varphi(0)|0\rangle = 1$.

So, if $\langle p|\varphi(0)|0\rangle$ is not equal to one, we will rescale (or, one might say, renormalize) $\varphi(x)$ by a multiplicative constant.

This is just a change of the name of the operator of interest, and does not affect the physics.

However, the rescaled $\varphi(x)$ obeys, by construction, $\langle p|\varphi(0)|0\rangle = 1$.

Finally, consider $\langle p,n|\varphi(x)|0\rangle$, where $|p,n\rangle$ is a multiparticle state with total four-momentum p, and n is short for all other labels (such as relative momenta) needed to specify this state.

We have

$$\begin{aligned}\langle p,n|\varphi(x)|0\rangle &= \langle p,n|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle \\ &= e^{-ipx}\langle p,n|\varphi(0)|0\rangle \\ &= e^{-ipx}A_n(\mathbf{p})\ ,\end{aligned} \tag{5.19}$$

where $A_n(p)$ is a function of Lorentz invariant products of the various (relative and total) four-momenta needed to specify the state.

Note that, from fig. (5.1), $p^0 = (\mathbf{p}^2 + M^2)^{1/2}$ with $M \geq 2m$.

The invariant mass M is one of the parameters included in the set n.

We would like $\langle p,n|\varphi(x)|0\rangle$ to be zero, because we would like $a^\dagger_1(\pm\infty)$, when acting on $|0\rangle$, to create a single particle state.

We do not want $a^\dagger_1(\pm\infty)$ to create any multiparticle states.

But this is precisely what may happen if $\langle p,n|\varphi(x)|0\rangle$ is not zero.

Actually, we are being a little too strict.

We really need $\langle p,n|\, a^\dagger_1(\pm\infty)\, |0\rangle$to be zero, and perhaps it will be zero even if $\langle p,n|\varphi(x)|0\rangle$ is not.

Also, we really should test $a^\dagger_1(\pm\infty)|0\rangle$ only against normalizable states.

Mathematically, non-normalizable states cause all sorts of trouble; mathematicians don't consider them to be states at all.

In physics, this usually doesn't bother us, but here we must be especially careful.

So let us write

$$|\psi\rangle = \sum_n \int d^3p\, \psi_n(\mathbf{p})|p,n\rangle\ , \tag{5.20}$$

where the $\psi_n(p)$'s are wave packets for the total three-momentum $\mathbf{p}$.

Note that eq. (5.20) is highly schematic; the sum over n includes integrals over continuous parameters like relative momenta.

Now we want to examine

$$\langle\psi|a_1^\dagger(t)|0\rangle = -i\sum_n \int d^3p\, \psi_n^*(\mathbf{p}) \int d^3k\ f_1(\mathbf{k}) \int d^3x\ e^{ikx} \overleftrightarrow{\partial_0} \langle p,n|\varphi(x)|0\rangle \tag{5.21}$$

We will take the limit $t \to \pm\infty$ in a moment.

Using eq. (5.19), eq. (5.21) becomes

$$\begin{aligned}\langle\psi|a_1^\dagger(t)|0\rangle &= -i\sum_n \int d^3p\,\psi_n^*(\mathbf{p})\int d^3k\; f_1(\mathbf{k})\int d^3x\Big(e^{ikx}\overset{\leftrightarrow}{\partial_0}e^{-ipx}\Big)A_n(\mathbf{p}) \\ &= \sum_n \int d^3p\,\psi_n^*(\mathbf{p})\int d^3k\; f_1(\mathbf{k})\int d^3x\,(p^0+k^0)e^{i(k-p)x}A_n(\mathbf{p})\;.\end{aligned} \tag{5.22}$$

Next we use $\int d^3x\, e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}} = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})$ to get

$$\langle\psi|a_1^\dagger(t)|0\rangle = \sum_n \int d^3p\,(2\pi)^3(p^0+k^0)\psi_n^*(\mathbf{p})f_1(\mathbf{p})A_n(\mathbf{p})e^{i(p^0-k^0)t}\;, \tag{5.23}$$

where $p^0 = (p^2 + M^2)^{1/2}$ and $k^0 = (p^2 + m^2)^{1/2}$.

Now comes the key point.

Note that p^0 is strictly greater than k^0, because $M \geq 2m > m$.

Thus the integrand of eq. (5.23) contains a phase factor that oscillates more and more rapidly as $t \to \pm\infty$.

Therefore, by the *Riemann-Lebesgue lemma*, the right-hand side of eq. (5.23) vanishes as $t\to \pm\infty$

Physically, this means that a one-particle wave packet spreads out differently than a multiparticle wave packet, and the overlap between them goes to zero as the elapsed time goes to infinity.

Thus, even though our operator $a^\dagger{}_1(t)$ creates some multiparticle states that we don't want, we can "follow" the one-particle state that we do want by using an appropriate wave packet.

By waiting long enough, we can make the multiparticle contribution to the scattering amplitude as small as we like.

Let us recap.

The basic formula for a scattering amplitude in terms of the fields of an interacting quantum field theory is the LSZ formula, which is worth writing down again:

$$\begin{aligned}\langle f|i\rangle = i^{n+n'} \int & d^4x_1\, e^{ik_1x_1}(-\partial_1^2+m^2)\ldots \\ & d^4x_{1'}\, e^{-ik'_1x'_1}(-\partial_{1'}^2+m^2)\ldots \\ & \times\langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi(x'_1)\ldots|0\rangle\ .\end{aligned} \tag{5.24}$$

The LSZ formula is valid *provided* that the field obeys

$$\langle 0|\varphi(x)|0\rangle = 0 \qquad \text{and} \qquad \langle k|\varphi(x)|0\rangle = e^{-ikx}\ . \tag{5.25}$$

These normalization conditions may conflict with our original choice of field and parameter normalization in the lagrangian.

Consider, for example, a lagrangian originally specified as

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{6}g\varphi^3 \ . \tag{5.26}$$

After shifting and rescaling (and renaming some parameters), we will have instead

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}Z_m m^2\varphi^2 + \tfrac{1}{6}Z_g g\varphi^3 + Y\varphi \ . \tag{5.27}$$

Here the three Z's and Y are as yet unknown constants.

They must be chosen to ensure the validity of eq. (5.25); this gives us two conditions in four unknowns.

We fix the parameter m by requiring it to be equal to the actual mass of the particle (equivalently, the energy of the first excited state relative to the ground state), and we fix the parameter g by requiring some particular scattering cross section to depend on g in some particular way.

For example, in quantum electrodynamics, the parameter analogous to g is the electron charge e.

The low-energy Coulomb scattering cross section is proportional to e^4, with a definite constant of proportionality and no higher-order corrections; this relationship defines e.

Thus we have four conditions in four unknowns, and it is possible to calculate Y and the three Z's order by order in powers of g.

Next, we must develop the tools needed to compute the correlation functions $\langle 0|T\varphi(x_1)...|0\rangle$ in an interacting quantum field theory.

6 Path Integrals in Quantum Mechanics

Prerequisite: none

Consider the nonrelativistic quantum mechanics of one particle in one dimension; the hamiltonian is

$$H(P,Q) = \frac{1}{2m}P^2 + V(Q)\ , \tag{6.1}$$

where P and Q are operators obeying [Q,P] = i.

We set $\hbar = 1$ for notational convenience.

We wish to evaluate the probability amplitude for the particle to start at position q′ at time t′, and end at position q″ at time t″.

This amplitude is $\langle q''|e^{-iH(t''-t')}|q'\rangle$, where $|q'\rangle$ and $|q''\rangle$ are eigenstates of the position operator Q.

We can also formulate this question in the Heisenberg picture, where operators are time dependent and the state of the system is time independent, as opposed to the more familiar Schrodinger picture.

In the Heisenberg picture, we write $Q(t) = e^{iHt}Qe^{-iHt}$.

We can then define an instantaneous eigenstate of Q(t) via $Q(t)|q,t\rangle = q|q,t\rangle$.

These instantaneous eigenstates can be expressed explicitly as $|q,t\rangle = e^{+iHt}|q\rangle$, where $Q|q\rangle = q|q\rangle$.

Then our transition amplitude can be written as $\langle q'', t''|q', t'\rangle$ in the Heisenberg picture.

To evaluate $\langle q'',t''|q',t'\rangle$, we begin by dividing the time interval $T \equiv t'' - t'$ into $N + 1$ equal pieces of duration $\delta t = T/(N+1)$.

Then introduce N complete sets of position eigenstates to get

$$\langle q'', t''|q', t'\rangle = \int \prod_{j=1}^{N} dq_j \, \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \ldots \langle q_1|e^{-iH\delta t}|q'\rangle \; . \tag{6.2}$$

The integrals over the q's all run from $-\infty$ to $+\infty$.

Now consider $\langle q_2|e^{-iH\delta t}|q_1\rangle$.

We can use the Campbell-Baker-Hausdorf formula

$$\exp(A+B) = \exp(A)\exp(B)\exp(-\tfrac{1}{2}[A,B] + \ldots) \tag{6.3}$$

to write

$$\exp(-iH\delta t) = \exp[-i(\delta t/2m)P^2]\exp[-i\delta t V(Q)]\exp[O(\delta t^2)] \; . \tag{6.4}$$

Then, in the limit of small δt, we should be able to ignore the final exponential.

Inserting a complete set of momentum states then gives

$$\begin{aligned}
\langle q_2|e^{-iH\delta t}|q_1\rangle &= \int dp_1\ \langle q_2|e^{-i(\delta t/2m)P^2}|p_1\rangle\langle p_1|e^{-i\delta t V(Q)}|q_1\rangle \\
&= \int dp_1\ e^{-i(\delta t/2m)p_1^2}\ e^{-i\delta t V(q_1)}\ \langle q_2|p_1\rangle\langle p_1|q_1\rangle \\
&= \int \frac{dp_1}{2\pi}\ e^{-i(\delta t/2m)p_1^2}\ e^{-i\delta t V(q_1)}\ e^{ip_1(q_2-q_1)}\ . \\
&= \int \frac{dp_1}{2\pi}\ e^{-iH(p_1,q_1)\delta t}\ e^{ip_1(q_2-q_1)}\ .
\end{aligned} \tag{6.5}$$

To get the third line, we used $\langle q|p\rangle = (2\pi)^{-1/2}\exp(ipq)$.

If we happen to be interested in more general hamiltonians than eq. (6.1), then we must worry about the ordering of the P and Q operators in any term that contains both.

If we adopt *Weyl ordering*, where the quantum hamiltonian H(P,Q) is given in terms of the classical hamiltonian H(p,q) by

$$H(P,Q) \equiv \int \frac{dx}{2\pi}\,\frac{dk}{2\pi}\,e^{ixP+ikQ}\int dp\,dq\,e^{-ixp-ikq}\,H(p,q)\ , \tag{6.6}$$

then eq. (6.5) is not quite correct; in the last line, $H(p_1,q_1)$ should be replaced with $H(p_1,\bar{q}_1)$, where $\bar{q}_1 = \frac{1}{2}(q_1+q_2)$.

For the hamiltonian of eq. (6.1), which is Weyl ordered, this replacement makes no difference in the limit $\delta t \to 0$.

Adopting Weyl ordering for the general case, we now have

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} \, e^{ip_j(q_{j+1}-q_j)} \, e^{-iH(p_j,\bar{q}_j)\delta t} \,, \tag{6.7}$$

where $\bar{q}_j = \frac{1}{2}(q_j + q_{j+1})$, $q_0 = q'$, and $q_{N+1} = q''$.

If we now define $\dot{q}_j \equiv (q_{j+1} - q_j)/\delta t$, and take the formal limit of $\delta t \to 0$, we get

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \, \mathcal{D}p \, \exp\left[i \int_{t'}^{t''} dt \Big(p(t)\dot{q}(t) - H(p(t), q(t)) \Big) \right]. \tag{6.8}$$

The integration is to be understood as over all paths in phase space that start at $q(t') = q'$ (with an arbitrary value of the initial momentum) and end at $q(t'') = q''$(with an arbitrary value of the final momentum).

If H(p,q) is no more than quadratic in the momenta [as is the case for eq. (6.1)], then the integral over p is gaussian, and can be done in closed form.

If the term that is quadratic in p is independent of q [as is the case for eq. (6.1)], then the prefactors generated by the gaussian integrals are all constants, and can be absorbed into the definition of $\mathcal{D}$q.

The result of integrating out p is then

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q \, \exp\left[i \int_{t'}^{t''} dt \, L(\dot{q}(t), q(t))\right], \tag{6.9}$$

where L($\dot{q}$, q) is computed by first finding the stationary point of the p integral by solving

$$0 = \frac{\partial}{\partial p}\Big(p\dot{q} - H(p,q)\Big) = \dot{q} - \frac{\partial H(p,q)}{\partial p} \tag{6.10}$$

for p in terms of $\dot{q}$ and q, and then plugging this solution back into p $\dot{q}$ − H to get L.

We recognize this procedure from classical mechanics: we are passing from the hamiltonian formulation to the lagrangian formulation.

Now that we have eqs. (6.8) and (6.9), what are we going to do with them?

Let us begin by considering some generalizations; let us examine, for example, ⟨q″,t″|Q(t_1)|q′,t′⟩, where $t' < t_1 < t''$.

This is given by

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \langle q''|e^{-iH(t''-t_1)}Qe^{-iH(t_1-t')}|q'\rangle \ . \tag{6.11}$$

In the path integral formula, the extra operator Q inserted at time t_1 will simply result in an extra factor of q(t_1).

Thus

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)\, e^{iS} \;, \tag{6.12}$$

where $S = \int_{t'}^{t''} dt\,(p\dot{q} - H)$.

Now let us go in the other direction; consider $\int \mathcal{D}p\,\mathcal{D}q\, q(t_1)q(t_2)e^{iS}$.

This clearly requires the operators Q(t_1) and Q(t_2) but their order depends on whether $t_1 < t_2$ or $t_2 < t_1$.

Thus we have

$$\int \mathcal{D}p\,\mathcal{D}q\; q(t_1)q(t_2)\, e^{iS} = \langle q'', t''|\mathrm{T}Q(t_1)Q(t_2)|q', t'\rangle \;. \tag{6.13}$$

where T is the time ordering symbol: a product of operators to its right is to be ordered, not as written, but with operators at later times to the left of those at earlier times.

This is significant, because time-ordered products enter into the LSZ formula for scattering amplitudes.

To further develop these methods, we need another trick: functional derivatives.

We define the functional derivative $\delta/\delta f(t)$ via

$$\frac{\delta}{\delta f(t_1)}\, f(t_2) = \delta(t_1 - t_2) \;, \tag{6.14}$$

where $\delta(t)$ is the Dirac delta function.

Also, functional derivatives are defined to satisfy all the usual rules of derivatives (product rule, chain rule, etc).

Eq. (6.14) can be thought of as the continuous generalization of $(\partial/\partial x_i)x_j = \delta_{ij}$.

Now, consider modifying the lagrangian of our theory by including external forces acting on the particle:

$$H(p,q) \to H(p,q) - f(t)q(t) - h(t)p(t) \ , \tag{6.15}$$

where f(t) and h(t) are specified functions.

In this case we will write

$$\langle q'', t''|q', t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\ \exp\left[i\int_{t'}^{t''} dt\left(p\dot{q} - H + fq + hp\right)\right] . \tag{6.16}$$

where H is the original hamiltonian.

Then we have

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)\, e^{i\int dt\,[p\dot{q}-H+fq+hp]}\,,$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)q(t_2)\, e^{i\int dt\,[p\dot{q}-H+fq+hp]}\,, \tag{6.17}$$

$$\frac{1}{i}\frac{\delta}{\delta h(t_1)}\langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; p(t_1)\, e^{i\int dt\,[p\dot{q}-H+fq+hp]}\,,$$

and so on.

After we are done bringing down as many factors of $q(t_i)$ or $p(t_i)$ as we like, we can set $f(t) = h(t) = 0$, and return to the original hamiltonian.

Thus,

$$\begin{aligned}&\langle q'',t''|\mathrm{T}Q(t_1)\dots P(t_n)\dots|q',t'\rangle \\ &\quad = \frac{1}{i}\frac{\delta}{\delta f(t_1)}\cdots\frac{1}{i}\frac{\delta}{\delta h(t_n)}\cdots\langle q'',t''|q',t'\rangle_{f,h}\Big|_{f=h=0}\,.\end{aligned} \tag{6.18}$$

Suppose we are also interested in initial and final states other than position eigenstates.

Then we must multiply by the wave functions for these states, and integrate.

We will be interested, in particular, in the ground state as both the initial and final state.

Also, we will take the limits $t'\to -\infty$ and $t''\to +\infty$.

The object of our attention is then

$$\langle 0|0\rangle_{f,h} = \lim_{\substack{t'\to-\infty \\ t''\to+\infty}} \int dq''\, dq'\, \psi_0^*(q'')\, \langle q'', t''|q', t'\rangle_{f,h}\, \psi_0(q')\ , \tag{6.19}$$

where $\psi_0(q) = \langle q|0\rangle$ is the ground-state wave function.

Eq. (6.19) is a rather cumbersome formula, however.

We will, therefore, employ a trick to simplify it.

Let $|n\rangle$ denote an eigenstate of H with eigenvalue E_n.

We will suppose that $E_0 = 0$; if this is not the case, we will shift H by an appropriate constant.

Next we write

$$\begin{aligned} |q', t'\rangle &= e^{iHt'}|q'\rangle \\ &= \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle \\ &= \sum_{n=0}^{\infty} \psi_n^*(q') e^{iE_n t'}|n\rangle\ , \end{aligned} \tag{6.20}$$

where $\psi_n(q) = \langle q|n\rangle$ is the wave function of the n^{th} eigenstate.

Now, replace H with $(1-i\epsilon)H$ in eq. (6.20), where ϵ is a small positive infinitesimal.

Then, take the limit $t' \to -\infty$ of eq. (6.20) with ϵ held fixed.

Every state except the ground state is then multiplied by a vanishing exponential factor, and so the limit is simply $\psi^*_0(q')|0\rangle$.

Next, multiply by an arbitrary function $\chi(q')$, and integrate over q'.

The only requirement is that $\langle 0|\chi\rangle \neq 0$.

We then have a constant times $|0\rangle$, and this constant can be absorbed into the normalization of the path integral.

A similar analysis of $\langle q'',t''| = \langle q''|e^{-iHt''}$ shows that the replacement $H \to (1-i\epsilon)H$ also picks out the ground state as the final state in the $t'' \to +\infty$ limit.

What all this means is that if we use $(1-i\epsilon)H$ instead of H, we can be cavalier about the boundary conditions on the endpoints of the path.

Any reasonable boundary conditions will result in the ground state as both the initial and final state.

Thus we have

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\ \exp\left[i\int_{-\infty}^{+\infty} dt\left(p\dot{q} - (1-i\epsilon)H + fq + hp\right)\right]. \tag{6.21}$$

Now let us suppose that $H = H_0 + H_1$, where we can solve for the eigenstates and eigenvalues of H_0, and H_1 can be treated as a perturbation.

Suppressing the $i\epsilon$, eq. (6.21) can be written as

$$\begin{aligned}
\langle 0|0\rangle_{f,h} &= \int \mathcal{D}p\,\mathcal{D}q\ \exp\left[i\int_{-\infty}^{+\infty} dt\left(p\dot{q} - H_0(p,q) - H_1(p,q) + fq + hp\right)\right] \\
&= \exp\left[-i\int_{-\infty}^{+\infty} dt\, H_1\left(\frac{1}{i}\frac{\delta}{\delta h(t)}, \frac{1}{i}\frac{\delta}{\delta f(t)}\right)\right] \\
&\quad \times \int \mathcal{D}p\,\mathcal{D}q\ \exp\left[i\int_{-\infty}^{+\infty} dt\left(p\dot{q} - H_0(p,q) + fq + hp\right)\right] .
\end{aligned} \tag{6.22}$$

To understand the second line of this equation, take the exponential prefactor inside the path integral.

Then the functional derivatives (that appear as the arguments of H_1) just pull out appropriate factors of p(t) and q(t), generating the right-hand side of the first line.

We assume that we can compute the functional integral in the second line, since it involves only the solvable hamiltonian H_0.

The exponential prefactor can then be expanded in powers of H_1 to generate a perturbation series.

If H_1 depends only on q (and not on p), and if we are only interested in time-ordered products of Q's (and not P's), and if H is no more than quadratic in P, and if the term quadratic in P does not involve Q, then eq. (6.22) can be simplified to

$$\begin{aligned}\langle 0|0\rangle_f \;=\; \exp\left[i\int_{-\infty}^{+\infty} dt\, L_1\!\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\right)\right] \\ \times\int \mathcal{D}q\;\exp\left[i\int_{-\infty}^{+\infty} dt\,\Big(L_0(\dot q,q)+fq\Big)\right]\,.\end{aligned} \tag{6.23}$$

where $L_1(q) = -H_1(q)$.

7 The Path Integral for the Harmonic Oscillator

Prerequisite: 6

Consider a harmonic oscillator with hamiltonian

$$H(P,Q) = \tfrac{1}{2m}P^2 + \tfrac{1}{2}m\omega^2 Q^2\;. \tag{7.1}$$

We begin with the formula from section 6 for the ground state to ground state transition amplitude in the presence of an external force, specialized to the case of a harmonic oscillator:

$$\langle 0|0\rangle_f = \int \mathcal{D}p\,Dq\;\exp i\int_{-\infty}^{+\infty} dt\Big[p\dot q - (1-i\epsilon)H + fq\Big]\;. \tag{7.2}$$

Looking at eq. (7.1), we see that multiplying H by $1-i\epsilon$ is equivalent to the replacements $m^{-1} \to (1-i\epsilon)m^{-1}$ [or, equivalently, $m \to (1+i\epsilon)m$] and $m\omega^2 \to (1-i\epsilon)m\omega^2$.

Passing to the lagrangian formulation then gives

$$\langle 0|0\rangle_f = \int Dq \, \exp i \int_{-\infty}^{+\infty} dt \Big[\tfrac{1}{2}(1+i\epsilon) m\dot{q}^2 - \tfrac{1}{2}(1-i\epsilon) m\omega^2 q^2 + fq \Big] \, . \tag{7.3}$$

From now on, we will simplify the notation by setting m = 1.

Next, let us use Fourier-transformed variables,

$$\widetilde{q}(E) = \int_{-\infty}^{+\infty} dt \, e^{iEt} \, q(t) \, , \qquad q(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \, e^{-iEt} \, \widetilde{q}(E) \, . \tag{7.4}$$

The expression in square brackets in eq. (7.3) becomes

$$\begin{aligned} \Big[\cdots \Big] = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{dE'}{2\pi} \, e^{-i(E+E')t} \Big[\Big(-(1+i\epsilon)EE' - (1-i\epsilon)\omega^2 \Big) \widetilde{q}(E) \widetilde{q}(E') \\ + \widetilde{f}(E) \widetilde{q}(E') + \widetilde{f}(E') \widetilde{q}(E) \Big] . \end{aligned} \tag{7.5}$$

Note that the only t dependence is now in the prefactor.

Integrating over t then generates a factor of $2\pi\delta(E+ E')$.

Then we can easily integrate over E′ to get

$$
\begin{aligned}
S &= \int_{-\infty}^{+\infty} dt \left[\cdots\right] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \Big[\Big((1+i\epsilon)E^2 - (1-i\epsilon)\omega^2\Big)\widetilde{q}(E)\widetilde{q}(-E) \\
&\qquad\qquad + \widetilde{f}(E)\widetilde{q}(-E) + \widetilde{f}(-E)\widetilde{q}(E)\Big].
\end{aligned}
\tag{7.6}
$$

The factor in large parentheses is equal to $E^2 - \omega^2 + i(E^2 + \omega^2)\epsilon$, and we can absorb the positive coefficient into ϵ to get $E^2 - \omega^2 + i\epsilon$.

Now it is convenient to change integration variables to

$$
\widetilde{x}(E) = \widetilde{q}(E) + \frac{\widetilde{f}(E)}{E^2 - \omega^2 + i\epsilon}\,. \tag{7.7}
$$

Then we get

$$
S = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \left[\widetilde{x}(E)(E^2 - \omega^2 + i\epsilon)\widetilde{x}(-E) - \frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2 - \omega^2 + i\epsilon}\right]. \tag{7.8}
$$

Furthermore, because eq. (7.7) is just a shift by a constant, $\mathcal{D}$q= $\mathcal{D}$x.

Now we have

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right] \times \int \mathcal{D}x\,\exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,\widetilde{x}(E)(E^2-\omega^2+i\epsilon)\widetilde{x}(-E)\right]. \tag{7.9}$$

Now comes the key point.

The path integral on the second line of eq. (7.9) is what we get for $\langle 0|0\rangle_f$ in the case $f = 0$.

On the other hand, if there is no external force, a system in its ground state will remain in its ground state, and so $\langle 0|0\rangle_{f=0} = 1$.

Thus $\langle 0|0\rangle_f$ is given by the first line of eq. (7.9),

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right]. \tag{7.10}$$

We can also rewrite $\langle 0|0\rangle_f$ in terms of time-domain variables as

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dt\,dt'\,f(t)G(t-t')f(t')\right], \tag{7.11}$$

where

$$G(t-t') = \int_{-\infty}^{+\infty}\frac{dE}{2\pi}\,\frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon}\,. \tag{7.12}$$

Note that G(t−t′) is a Green's function for the oscillator equation of motion:

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G(t-t')=\delta(t-t')\ . \tag{7.13}$$

This can be seen directly by plugging eq. (7.12) into eq. (7.13) and then taking the $\epsilon\to 0$ limit.

We can also evaluate G(t− t′) explicitly by treating the integral over E on the right-hand side of eq. (7.12) as a contour integral in the complex E plane, and then evaluating it via the residue theorem.

The result is

$$G(t-t')=\frac{i}{2\omega}\exp\Big(-i\omega|t-t'|\Big)\ . \tag{7.14}$$

Consider now the formula from section 6 for the time-ordered product of operators.

In the case of initial and final ground states, it becomes

$$\langle 0|\mathrm{T}Q(t_1)\dots|0\rangle=\frac{1}{i}\,\frac{\delta}{\delta f(t_1)}\dots\langle 0|0\rangle_f\Big|_{f=0}\ . \tag{7.15}$$

Using our explicit formula, eq. (7.11), we have

the

$$
\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f\Big|_{f=0} \\
&= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty} dt'\, G(t_2-t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \left[\tfrac{1}{i}G(t_2-t_1) + (\text{term with } f\text{'s})\right]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \tfrac{1}{i}G(t_2-t_1)\,.
\end{aligned}
\tag{7.16}
$$

We can continue in this way to compute the ground-state expectation value of the time-ordered product of more Q(t)'s.

If the number of Q(t)'s is odd, then there is always a left-over *f*(t) in the prefactor, and so the result is zero.

If the number of Q(t)'s is even, then we must pair up the functional derivatives in an appropriate way to get a nonzero result.

Thus, for example,

$$
\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\Big[&G(t_1{-}t_2)G(t_3{-}t_4) \\
&+ G(t_1{-}t_3)G(t_2{-}t_4) \\
&+ G(t_1{-}t_4)G(t_2{-}t_3)\Big].
\end{aligned}
\tag{7.17}
$$

More generally,

$$\langle 0|\mathrm{T}Q(t_1)\dots Q(t_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i_1}-t_{i_2})\dots G(t_{i_{2n-1}}-t_{i_{2n}})\ . \tag{7.18}$$

8 The Path Integral for Free Field Theory

Prerequisite: 3, 7

Our results for the harmonic oscillator can be straightforwardly generalized to a free field theory with hamiltonian density

$$\mathcal{H}_0 = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2\ . \tag{8.1}$$

The dictionary we need is

$$\begin{aligned} q(t) &\longrightarrow \varphi(\mathbf{x},t) \quad \text{(classical field)} \\ Q(t) &\longrightarrow \varphi(\mathbf{x},t) \quad \text{(operator field)} \\ f(t) &\longrightarrow J(\mathbf{x},t) \quad \text{(classical } \textit{source}\text{)} \end{aligned} \tag{8.2}$$

The distinction between the classical field φ(x) and the corresponding operator field should be clear from context.

To employ the ϵ trick, we multiply H_0 by $1-i\epsilon$.

The results are equivalent to replacing m^2 in H_0 with $m^2 - i\epsilon$.

From now on, for notational simplicity, we will write m^2 when we really mean $m^2 - i\epsilon$.

Let us write down the path integral (also called the *functional integral*)for our free field theory:

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi \; e^{i\int d^4x[\mathcal{L}_0 + J\varphi]} \; , \tag{8.3}$$

where

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 \tag{8.4}$$

is the lagrangian density, and

$$\mathcal{D}\varphi \propto \prod_x d\varphi(x) \tag{8.5}$$

is the *functional measure*.

Note that when we say path integral , we now mean a path in the space of field configurations.

We can evaluate $Z_0(J)$ by mimicking what we did for the harmonic oscillator in section 7.

We introduce four-dimensional Fourier transforms,

$$\widetilde{\varphi}(k) = \int d^4x \, e^{-ikx} \; \varphi(x) \; , \qquad \varphi(x) = \int \frac{d^4k}{(2\pi)^4} \; e^{ikx} \; \widetilde{\varphi}(k) \; , \tag{8.6}$$

where $kx = -k_0t + \mathbf{k}\cdot\mathbf{x}$, and k_0 is an integration variable.

Then, starting with $S_0 = \int d^4x\,[\mathcal{L}_0 + J\varphi]$, we get

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{\varphi}(k)(k^2+m^2)\widetilde{\varphi}(-k) + \widetilde{J}(k)\widetilde{\varphi}(-k) + \widetilde{J}(-k)\widetilde{\varphi}(k)\Big], \tag{8.7}$$

where $k^2 = \mathbf{k}^2 - (k_0)\mathbf{2}$.

We now change path integration variables to

$$\widetilde{\chi}(k) = \widetilde{\varphi}(k) - \frac{\widetilde{J}(k)}{k^2+m^2}\ . \tag{8.8}$$

Since this is merely a shift by a constant, we have $\mathcal{D}\phi = \mathcal{D}\chi$.

The action becomes

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2} - \widetilde{\chi}(k)(k^2+m^2)\widetilde{\chi}(-k)\right]. \tag{8.9}$$

Just as for the harmonic oscillator, the integral over χvsimply yields a factor of $Z_0(0) = \langle 0|0\rangle_{J=0} = 1$.

Therefore

$$
\begin{aligned}
Z_0(J) &= \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2-i\epsilon}\right] \\
&= \exp\left[\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')\right] .
\end{aligned}
\tag{8.10}
$$

Here we have defined the *Feynman propagator,*

$$
\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \ . \tag{8.11}
$$

The Feynman propagator is a Green's function for the Klein-Gordon equation,

$$
(-\partial_x^2+m^2)\Delta(x-x') = \delta^4(x-x') \ . \tag{8.12}
$$

This can be seen directly by plugging eq. (8.11) into eq. (8.12) and then taking the $\epsilon \to 0$ limit.

We can also evaluate $\Delta(x-x')$ explicitly by treating the k_0 integral on the right-hand side of eq. (8.11) as a contour integral in the complex k_0 plane, and then evaluating it via the residue theorem.

The result is

$$
\begin{aligned}
\Delta(x-x') &= i\int \widetilde{dk}\; e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega|t-t'|} \\
&= i\theta(t-t')\int \widetilde{dk}\; e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk}\, e^{-ik(x-x')} \ ,
\end{aligned}
\tag{8.13}
$$

where θ(t) is the unit step function. The integral over $\widetilde{dk}$ can also be performed in terms of Bessel functions; see section 4.

Now, by analogy with the formula for the ground-state expectation value of a time-ordered product of operators for the harmonic oscillator, we have

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\ldots Z_0(J)\Big|_{J=0} . \tag{8.14}$$

Using our explicit formula, eq. (8.10), we have

$$\begin{aligned}
\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)} Z_0(J)\Big|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left[\int d^4x'\,\Delta(x_2-x')J(x')\right] Z_0(J)\Big|_{J=0} \\
&= \left[\tfrac{1}{i}\Delta(x_2-x_1) + (\text{term with } J\text{'s})\right] Z_0(J)\Big|_{J=0} \\
&= \tfrac{1}{i}\Delta(x_2-x_1) .
\end{aligned} \tag{8.15}$$

We can continue in this way to compute the ground-state expectation value of the time-ordered product of more φ's.

If the number of φ's is odd, then there is always a left-over J in the prefactor, and so the result is zero.

If the number of φ's is even, then we must pair up the functional derivatives in an appropriate way to get a nonzero result.

Thus, for example,

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \frac{1}{i^2}\Big[&\Delta(x_1-x_2)\Delta(x_3-x_4)\\ &+\Delta(x_1-x_3)\Delta(x_2-x_4)\\ &+\Delta(x_1-x_4)\Delta(x_2-x_3)\Big].\end{aligned} \tag{8.16}$$

More generally,

$$\langle 0|\mathrm{T}\varphi(x_1)\dots\varphi(x_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}}\Delta(x_{i_1}-x_{i_2})\dots\Delta(x_{i_{2n-1}}-x_{i_{2n}})\,. \tag{8.17}$$

This result is known as *Wick's theorem*.

9 The Path Integral for Interacting Field Theory

Prerequisite: 8

Let us consider an interacting quantum field theory specified by a lagrangian of the form

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}Z_m m^2\varphi^2 + \tfrac{1}{6}Z_g g\varphi^3 + Y\varphi\,. \tag{8.18}$$

As we discussed at the end of section 5, we fix the parameter m by requiring it to be equal to the actual mass of the particle (equivalently, the energy of the first excited state relative to the ground state), and we fix the parameter g by requiring some particular scattering cross section to depend on g in some particular way.

We will have more to say about this after we have learned to calculate cross sections.

We also assume that the field is normalized by

$$\langle 0|\varphi(x)|0\rangle = 0 \qquad \text{and} \qquad \langle k|\varphi(x)|0\rangle = e^{-ikx} \ . \tag{9.2}$$

Here $|0\rangle$ is the ground state, normalized via $\langle 0|0\rangle = 1$, and $|k\rangle$ is a state of one particle with four-momentum k^μ, where $k^2 = k^\mu k_\mu = -m^2$, normalized via

$$\langle k'|k\rangle = (2\pi)^3 2k^0 \delta^3(\mathbf{k}' - \mathbf{k}) \ . \tag{9.3}$$

Thus we have four conditions (the specified values of m, g, $\langle 0|\varphi|0\rangle$, and $\langle k|\varphi|0\rangle$), and we will use these four conditions to determine the values of the four remaining parameters (Y and the three Z's) that appear in $\mathcal{L}$.

Before going further, we should note that this theory (known as φ^3 theory, pronounced "phi-cubed") actually has a fatal flaw.

The hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2} Z_\varphi^{-1} \Pi^2 - Y\varphi + \tfrac{1}{2} Z_m m^2 \varphi^2 - \tfrac{1}{6} Z_g g \varphi^3 \ . \tag{9.4}$$

Classically, we can make this arbitrarily negative by choosing an arbitrarily large value for φ.

Quantum mechanically, this means that this hamiltonian has no ground state.

If we start off near $\varphi = 0$, we can tunnel through the potential barrier to large φ, and then "roll down the hill".

However, this process is invisible in perturbation theory in g.

The situation is exactly analogous to the problem of a harmonic oscillator perturbed by a q^3 term.

This system also has no ground state, but perturbation theory (both time dependent and time independent) does not "know" this.

We will be interested in eq. (9.1) only as an example of how to do perturbation expansions in a simple context, and so we will overlook this problem.

We would like to evaluate the path integral for this theory,

$$Z(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi \; e^{i\int d^4x[\mathcal{L}_0+\mathcal{L}_1+J\varphi]} \; . \tag{9.5}$$

We can evaluate Z(J) by mimicking what we did for quantum mechanics at the end of section 6.

Specifically, we can rewrite eq. (9.5) as

$$\begin{aligned} Z(J) &= e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\varphi \; e^{i\int d^4x[\mathcal{L}_0+J\varphi]} \; . \\ &\propto e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \; Z_0(J) \; , \end{aligned} \tag{9.6}$$

where $Z_0(J)$ is the result in free-field theory,

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')\right]. \tag{9.7}$$

We have written Z(J) as proportional to (rather than equal to) the right-hand side of eq. (9.6) because the ϵ trick does not give us the correct overall normalization; instead, we must require Z(0) = 1, and enforce this by hand.

Note that, in eq. (9.7), we have implicitly assumed that

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2\ , \tag{9.8}$$

since this is the $\mathcal{L}_0$ that gives us eq. (9.7).

Therefore, the rest of $\mathcal{L}$ must be included in $\mathcal{L}_1$.

We write

$$\begin{aligned}\mathcal{L}_1 &= \tfrac{1}{6}Z_g g\varphi^3 + \mathcal{L}_{\rm ct}\ ,\\ \mathcal{L}_{\rm ct} &= -\tfrac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi\ ,\end{aligned} \tag{9.9}$$

where $\mathcal{L}_{\rm ct}$ is called the *counterterm* lagrangian.

We expect that, as g$\rightarrow$ 0, Y $\rightarrow$ 0 and $Z_i \rightarrow 1$.

In fact, as we will see, Y = O(g) and $Z_i = 1 + O(g^2)$.

In order to make use of eq. (9.7), we will have to compute lots and lots of functional derivatives of $Z_0(J)$.

Let us begin by ignoring the counterterms.

We define

$$Z_1(J) \propto \exp\left[\frac{i}{6} Z_g g \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right] Z_0(J)\ , \tag{9.10}$$

where the constant of proportionality is fixed by $Z_1(0) = 1$.

We now make a dual Taylor expansion in powers of g and J to get

$$\begin{aligned} Z_1(J) \propto & \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{iZ_g g}{6}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V \\ & \times \sum_{P=0}^{\infty} \frac{1}{P!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y-z)J(z)\right]^P . \end{aligned} \tag{9.11}$$

If we focus on a term in eq. (9.11) with particular values of V and P, then the number of surviving sources (after we take all the functional derivatives) is E = 2P– 3V.

Here E stands for *external*, a terminology that should become clear by the end of the next section; V stands for *vertex* and P for *propagator* .

The overall phase factor of such a term is then $i^V(1/i)^{3V}i^P = i^{V+E-P}$, and the 3V functional derivatives can act on the 2P sources in (2P)!/(2P−3V)! different combinations.

However, many of the resulting expressions are algebraically identical.

To organize them, we introduce Feynman diagrams.

In these diagrams, a line segment (straight or curved) stands for a propagator $\frac{1}{i}\,\Delta(x-y)$, a filled circle at one end of a line segment for a source $i\int d^4x J(x)$, and a vertex joining three line segments for $iZ_g g\int d^4x$.

Sets of diagrams with different values of E and V are shown in figs. (9.1–9.11).

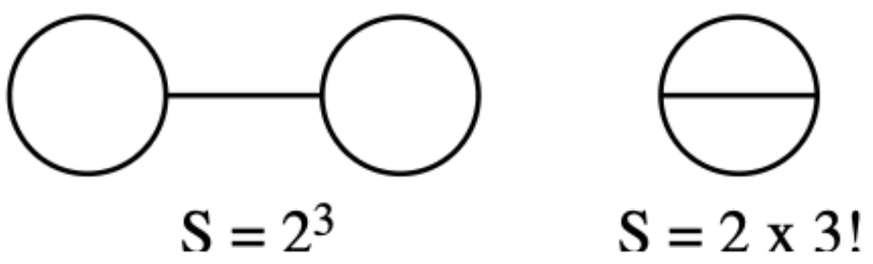


Figure 9.1: All connected diagrams with E = 0 and V = 2.

To count the number of terms on the right-hand side of eq. (9.11) that result in a particular diagram, we first note that, in each diagram, the number of lines is P and the number of vertices is V.

We can rearrange the three functional derivatives from a particular vertex without changing the resulting diagram; this yields a counting factor of 3! for each vertex.

Also, we can rearrange the vertices themselves; this yields a counting factor of V!.

Similarly, we can rearrange the two sources at the ends of a particular propagator without changing the resulting diagram; this yields a counting factor of 2! for each propagator.

Also, we can rearrange the propagators themselves; this yields a counting factor of P!.

All together, these counting factors neatly cancel the numbers from the dual Taylor expansions in eq. (9.11).

However, this procedure generally results in an overcounting of the number of terms that give identical results.

This happens when some rearrangement of derivatives gives the *same* match-up to sources as some rearrangement of sources.

This possibility is always connected to some symmetry property of the diagram, and so the factor by which we have overcounted is called the *symmetry factor*.

The figures show the symmetry factor S of each diagram.

Consider, for example, the second diagram of fig. (9.1).

The three propagators can be rearranged in 3! ways, and all these rearrangements can be duplicated by exchanging the derivatives at the vertices.

Furthermore the endpoints of each propagator can be simultaneously swapped, and the effect duplicated by swapping the two vertices.

Thus, $S = 2 \times 3! = 12$.

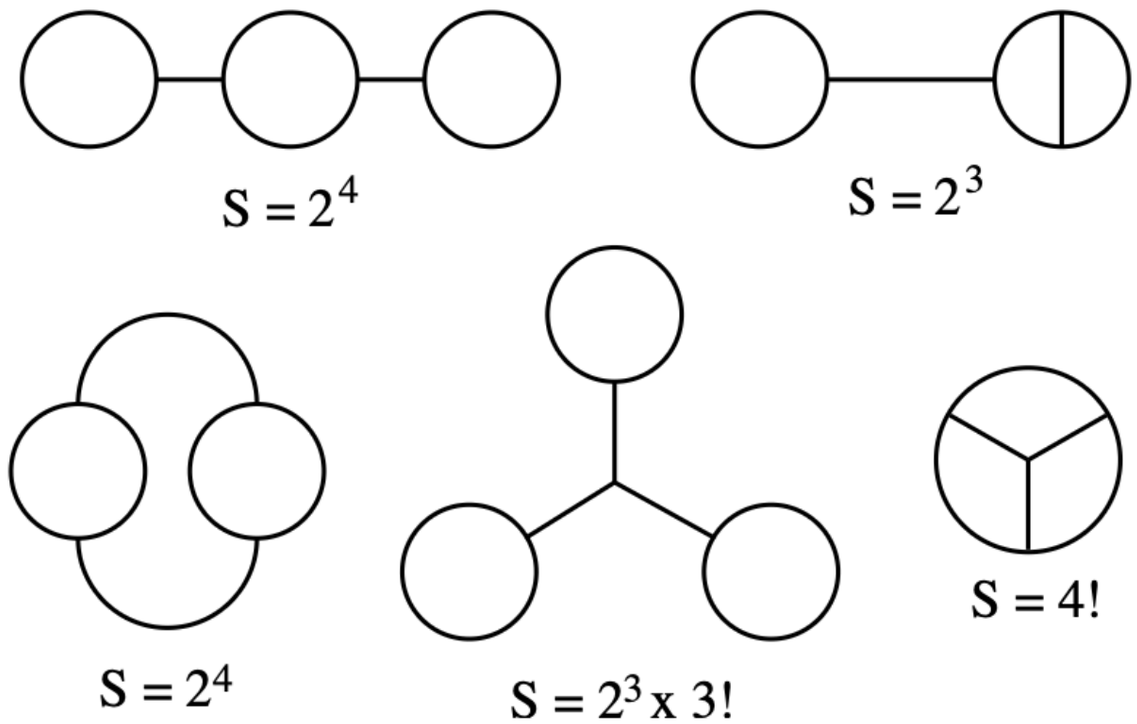


Figure 9.2: All connected diagrams with E = 0 and V = 4.

Let us consider two more examples.

In the first diagram of fig. (9.6), the exchange of the two external propagators (along with their attached sources) can be duplicated by exchanging all the derivatives at one vertex for those at the other, and simultaneously swapping the endpoints of each semicircular propagator.

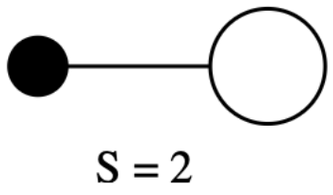


Figure 9.3: All connected diagrams with E = 1 and V = 1.

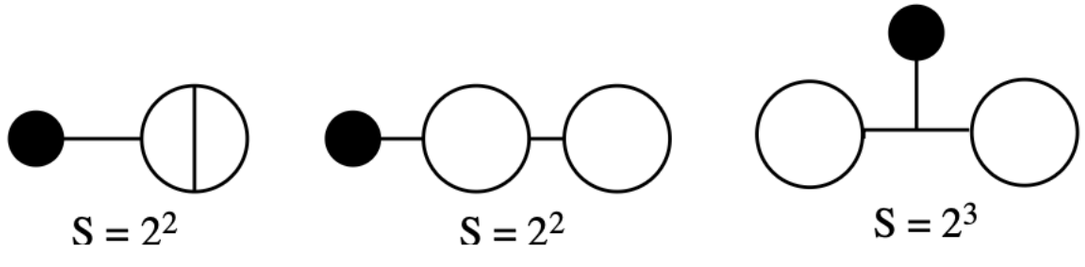


Figure 9.4: All connected diagrams with E = 1 and V = 3.

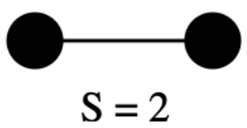


Figure 9.5: All connected diagrams with E = 2 and V = 0.

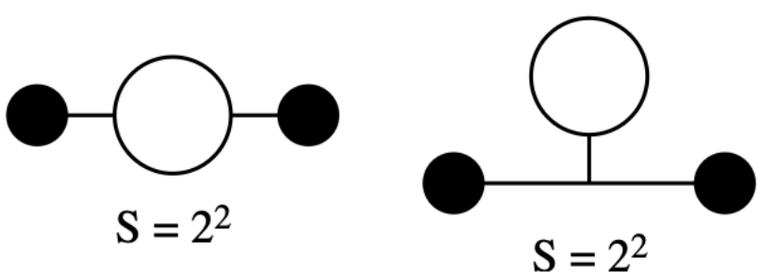


Figure 9.6: All connected diagrams with E = 2 and V = 2.

Also, the effect of swapping the top and bottom semicircular propagators can be duplicated by swapping the corresponding derivatives at each vertex.

Thus, the symmetry factor is $S = 2 \times 2 = 4$.

In the diagram of fig. (9.10), we can exchange derivatives to match swaps of the top and bottom external propagators on the left, or the top and bottom external propagators on the right, or the set of external propagators on the left with the set of external propagators on the right.

Thus, the symmetry factor is $S = 2 \times 2 \times 2 = 8$.

The diagrams in figs. (9.1–9.11) are all connected: we can trace a path through the diagram between any two points on it. However, these are not the only contributions to Z(J).

The most general diagram consists of a product of several connected diagrams.

Let C_I stand for a particular connected diagram, including its symmetry factor.

A general diagram D can then be expressed as

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I} \ , \tag{9.12}$$

where n_I is an integer that counts the number of C_I's in D, and S_D is the additional symmetry factor for D (that is, the part of the symmetry factor that is not already accounted for by the symmetry factors already included in each of the connected diagrams).

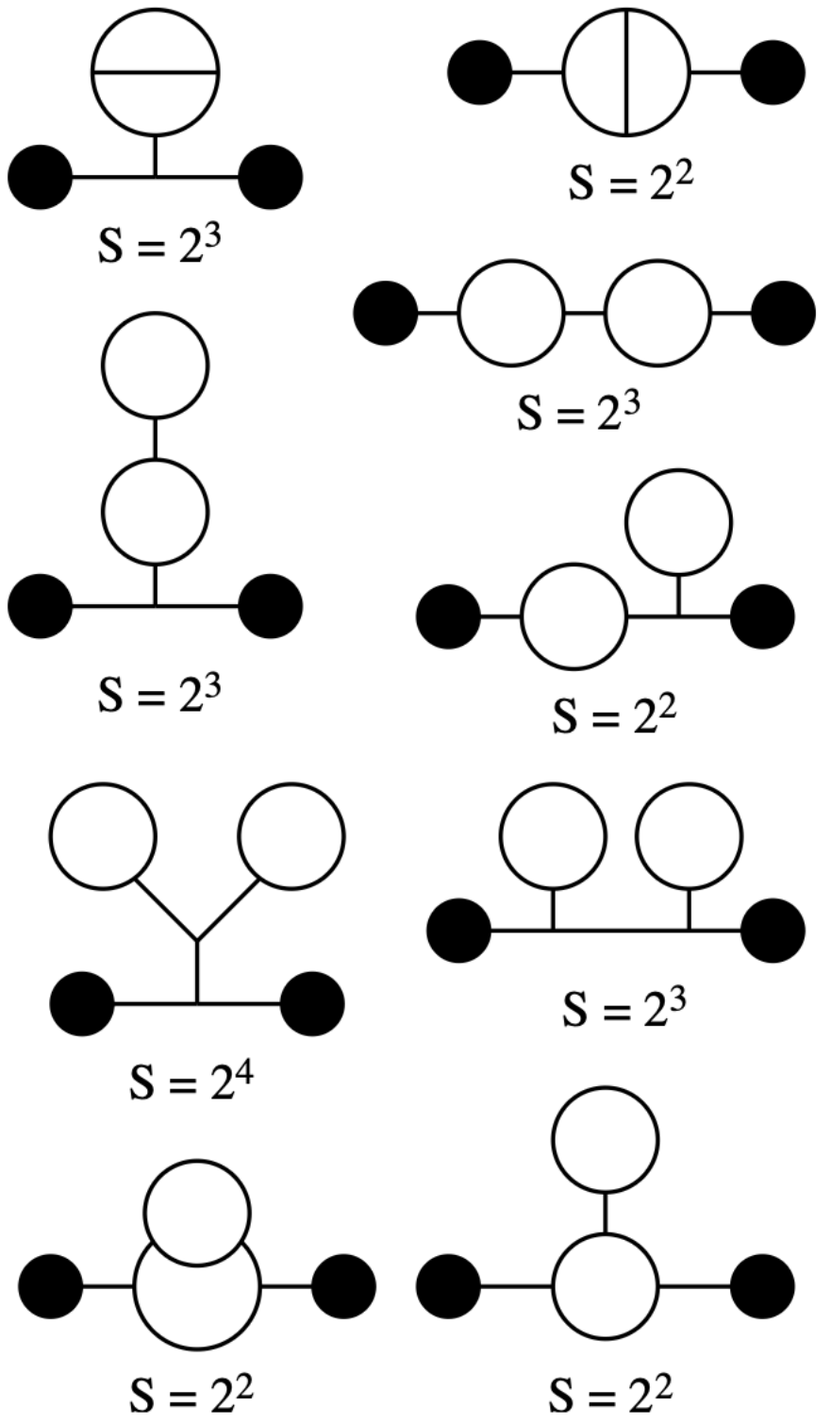


Figure 9.7: All connected diagrams with E = 2 and V = 4.

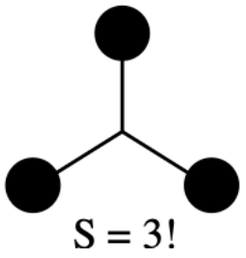


Figure 9.8: All connected diagrams with E = 3 and V = 1.

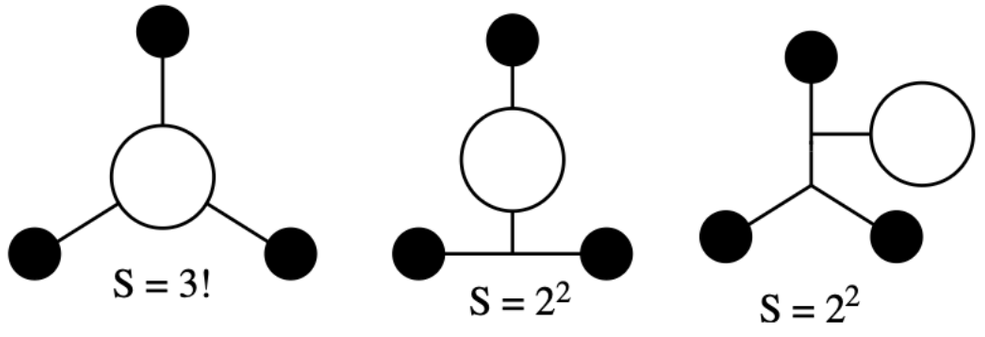


Figure 9.9: All connected diagrams with E = 3 and V = 3.

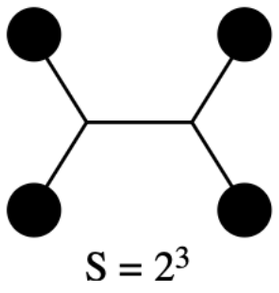


Figure 9.10: All connected diagrams with E = 4 and V = 2.

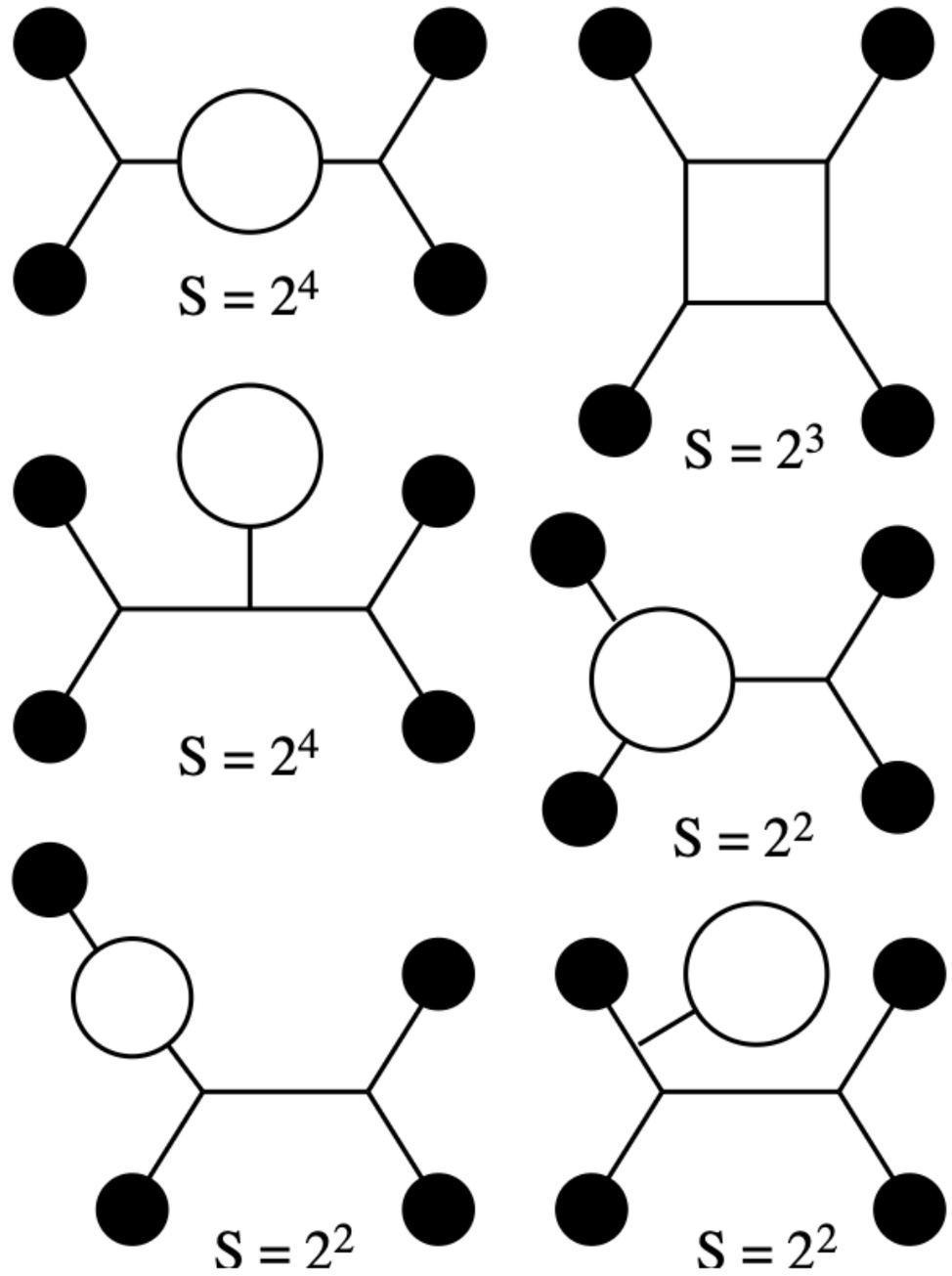


Figure 9.11: All connected diagrams with E = 4 and V = 4.

We now need to determine S_D.

Since we have already accounted for propagator and vertex rearrangements *within* each C_I, we need to consider only exchanges of propagators and vertices among *different* connected diagrams.

These can leave the total diagram D unchanged only if (1) the exchanges are made among different but *identical* connected diagrams, and only if (2) the exchanges involve all of the propagators and vertices in a given connected diagram.

If there are n_I factors of C_I in D, there are $n_I!$ ways to make these rearrangements.

Overall, then, we have

$$S_D = \prod_I n_I! \ . \tag{9.13}$$

Now Z1(J) is given (up to an overall normalization) by summing all diagrams D, and each D is labeled by the integers n_I.

Therefore

$$\begin{aligned} Z_1(J) &\propto \sum_{\{n_I\}} D \\ &\propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} \\ &\propto \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} \\ &\propto \prod_I \exp(C_I) \\ &\propto \exp\left(\textstyle\sum_I C_I\right) \ . \end{aligned} \tag{9.14}$$

Thus we have a remarkable result: $Z_1(J)$ is given by the exponential of the sum of connected diagrams.

This makes it easy to impose the normalization $Z_1(0) = 1$: we simply omit the vacuum diagrams (those with no sources), like those of figs. (9.1) and (9.2).

We then have

$$Z_1(J) = \exp[iW_1(J)] \ , \tag{9.15}$$

where we have defined

$$iW_1(J) \equiv \sum_{I \neq \{0\}} C_I \ , \tag{9.16}$$

and the notation $I \neq \{0\}$ means that the vacuum diagrams are omitted from the sum, so that $W_1(0) = 0$.

We have included a factor of i on the left-hand side of eq. (9.16) because then $W_1(J)$ is real in free-field theory, which can be easily checked.

Were it not for the counterterms in $\mathcal{L}1$, we would have $Z(J) = Z_1(J)$.

Let us see what we would get if this was, in fact, the case.

In particular, let us compute the vacuum expectation value of the field $\varphi(x)$, which is given by

$$\begin{aligned}\langle 0|\varphi(x)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x)} Z_1(J)\Big|_{J=0} \\ &= \frac{\delta}{\delta J(x)} W_1(J)\Big|_{J=0} .\end{aligned} \tag{9.17}$$

This expression is then the sum of all diagrams [such as those in figs. (9.3) and (9.4)] that have a single source, with the source removed:

$$\langle 0|\varphi(x)|0\rangle = \tfrac{1}{2} ig \int d^4y \, \tfrac{1}{i}\Delta(x-y)\tfrac{1}{i}\Delta(y-y) + O(g^3) \; . \tag{9.18}$$

Here we have set $Z_g = 1$ in the first term, since $Z_g = 1 + O(g^2)$.

We see the vacuum-expectation value of $\varphi(x)$ is not zero, as is required for the validity of the LSZ formula.

To fix this, we must introduce the counterterm $Y\varphi$.

Including this term in the interaction lagrangian $\mathcal{L}_1$ introduces a new kind of vertex, one where a single line segment ends; the corresponding vertex factor is iY d4 y.

The simplest diagrams including this new vertex are shown in fig. (9.12), with a cross symbolizing the vertex.

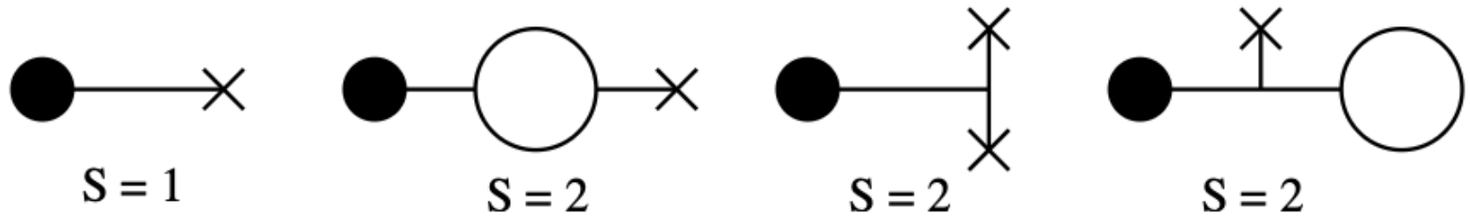


Figure 9.12: All connected diagrams with E = 1, X ≥ 1 (where X is the number of one-point vertices from the linear counterterm), and V + X ≤ 3.

Assuming Y= O(g), only the first diagram in fig. (9.12) contributes at O(g), and we have

$$\langle 0|\varphi(x)|0\rangle = \left(iY + \tfrac{1}{2}(ig)\tfrac{1}{i}\Delta(0)\right)\int d^4y\, \tfrac{1}{i}\Delta(x-y) + O(g^3)\ . \tag{9.19}$$

Thus, in order to have $\langle 0|\varphi(x)|0\rangle = 0$, we should choose

$$Y = \tfrac{1}{2}ig\Delta(0) + O(g^3)\ . \tag{9.20}$$

The factor of i is disturbing, because Y must be a real number: it is the coefficient of a hermitian operator in the hamiltonian, as seen in eq. (9.4).

Therefore, $\Delta(0)$ must be purely imaginary, or we are in trouble.

We have

$$\Delta(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon} \,. \tag{9.21}$$

From eq. (9.21), it is not immediately obvious whether or not $\Delta(0)$ is purely imaginary, but eq. (9.21) does reveal another problem: the integral diverges at large k.

This is another example of an ultraviolet divergence, similar to the one we encountered in section 3 when we computed the zero-point energy of the field.

To make some progress, we introduce an ultraviolet cutoff Λ, which we assume is much larger than m and any other energy of physical interest.

Modifications to the propagator above some cutoff may be well justified physically; for example, quantum fluctuations in spacetime itself should become important above the Planck scale, which is given by the inverse square root of Newton's constant, and has the numerical value of 10^{19} GeV (compared to, say, the proton mass, which is 1 GeV).

In order to retain the Lorentz-transformation properties of the propagator, we implement the ultraviolet cutoff in a more subtle way than we did in section 3; specifically, we make the replacement

$$\Delta(x-y) \to \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \left(\frac{\Lambda^2}{k^2 + \Lambda^2 - i\epsilon} \right)^2 . \tag{9.22}$$

The integral is now convergent, and we can evaluate the modified $\Delta(0)$ with the methods of section 14; for $\Lambda \gg m$, the result is

$$\Delta(0) = \frac{i}{16\pi^2}\,\Lambda^2\ . \tag{9.23}$$

Thus Y is real, as required.

If we like, we can now formally take the limit $\Lambda \to \infty$.

The parameter Y becomes infinite, but $\langle 0|\varphi(x)|0\rangle$ remains zero, at least to this order in g.

It may be disturbing to have a parameter in the lagrangian that is formally infinite.

However, such parameters are not directly measurable, and so need not obey our preconceptions about their magnitudes.

Also, it is important to remember that Y includes a factor of g; this means that we can expand in powers of Y as part of our general expansion in powers of g.

When we compute something measurable (like a scattering cross section), all the formally infinite numbers will cancel in a well-defined way, leaving behind finite coefficients for the various powers of g.

We will see how this works in detail in sections 14–20.

As we go to higher orders in g, things become more complicated, but in principle the procedure is the same.

Thus, at O(g3), we sum up the diagrams of figs. (9.4) and (9.12), and then add to Y whatever O(g3) term is needed to maintain $\langle 0|\varphi(x)|0\rangle = 0$.

In this way we can determine the value of Y order by order in powers of g.

Once this is done, there is a remarkable simplification.

Our adjustment of Y to keep $\langle 0|\varphi(x)|0\rangle = 0$ means that the sum of all connected diagrams with a single source is zero.

Consider now that same infinite set of diagrams, but replace the single source in each of them with some other subdiagram.

Here is the point: no matter what this replacement subdiagram is, the sum of all these diagrams is still zero.

Therefore, we need not bother to compute any of them!

The rule is this: ignore any diagram that, when a single line is cut, falls into two parts, one of which has no sources.

All of these diagrams (known as tadpoles) are canceled by the Y counterterm, no matter what subdiagram they are attached to.

The diagrams that remain (and need to be computed!) are shown in fig. (9.13).

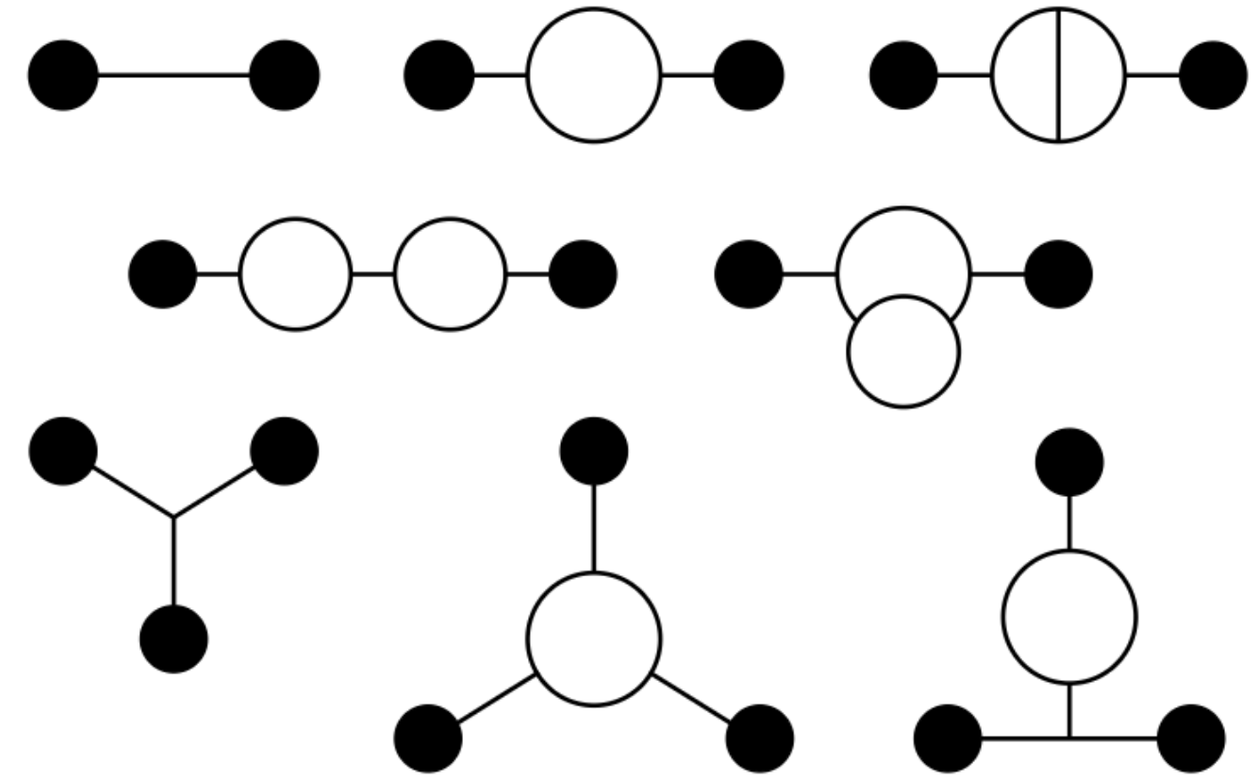


Figure 9.13: All connected diagrams without tadpoles with $E \leq 4$ and $V \leq 4$.

We turn next to the remaining two counterterms.

For notational simplicity we define

$$A = Z_\varphi - 1 \,, \qquad B = Z_m - 1 \,, \tag{9.24}$$

and recall that we expect each of these to be O(g^2).

We now have

$$Z(J) = \exp\left[-\frac{i}{2}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(-A\partial_x^2 + Bm^2\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_1(J) \,. \tag{9.25}$$

We have integrated by parts to put both ∂_x's onto one $\delta/\delta J(x)$.

Note that the time derivatives in this interaction should really be treated by including an extra source term for the conjugate momentum $\Pi = \dot{\varphi}$.

However, the space derivatives are correctly treated, and then the time derivatives must work out comparably by Lorentz invariance.

Eq. (9.25) results in a new vertex at which two lines meet.

The corresponding vertex factor is $(-i)\int d^4x(-A\partial^2_x + Bm^2)$; the ∂^2_x acts on the x in one or the other (but not both) propagators.

Which one does not matter, and can be changed via integration by parts.

Diagramatically, all we need do is sprinkle these new vertices onto the propagators in our existing diagrams.

How many of these vertices we need to add depends on the order in g we are working to achieve.

This completes our calculation of Z(J) in φ^3 theory.

We express it as

$$Z(J) = \exp[iW(J)] \ , \tag{9.26}$$

where W(J) is given by the sum of all connected diagrams with no tadpoles and at least two sources, and including the counterterm vertices just discussed.

Now that we have Z(J), we must find out what we can do with it.

10 Scattering Amplitudes and the Feynman Rules

Prerequisite: 5, 9

Now that we have an expression for Z(J) = exp iW(J), we can take functional derivatives to compute vacuum expectation values of time-ordered products of fields.

Consider the case of two fields; we define the exact propagator via

$$\tfrac{1}{i}\boldsymbol{\Delta}(x_1 - x_2) \equiv \langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle \ . \tag{10.1}$$

For notational simplicity let us define

$$\delta_j \equiv \frac{1}{i}\,\frac{\delta}{\delta J(x_j)} \ . \tag{10.2}$$

Then we have

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle \ &= \ \delta_1\delta_2 Z(J)\Big|_{J=0} \\ &= \ \delta_1\delta_2 iW(J)\Big|_{J=0} - \delta_1 iW(J)\Big|_{J=0}\,\delta_2 iW(J)\Big|_{J=0} \\ &= \ \delta_1\delta_2 iW(J)\Big|_{J=0} \ .\end{aligned} \tag{10.3}$$

To get the last line we used $\delta j W(J)|_{J=0} = \langle 0|\varphi(x_j)|0\rangle = 0$.

Diagramatically, δ_1 removes a source, and labels the propagator endpoint x_1.

Thus $\frac{1}{i}\,\Delta(x_1 - x_2)$ is given by the sum of diagrams with two sources, with those sources removed and the endpoints labeled x_1 and x_2.

The labels must be applied in both ways.

If the diagram was originally symmetric on exchange of the two sources, the associated symmetry factor of 2 is then canceled by the double labeling.)

At lowest order, the only contribution is the "barbell" diagram of fig. (9.5) with the sources removed.

Thus we recover the obvious fact that $\frac{1}{i}\,\mathbf{\Delta}(x_1-x_2) = \frac{1}{i}\,\Delta(x_1-x_2) + O(g^2)$.

We will take up the subject of the $O(g^2)$ corrections in section 14.

For now, let us go on to compute

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle &= \delta_1\delta_2\delta_3\delta_4 Z(J) \\ &= \Big[\,\delta_1\delta_2\delta_3\delta_4 iW \\ &\quad + (\delta_1\delta_2 iW)(\delta_3\delta_4 iW) \\ &\quad + (\delta_1\delta_3 iW)(\delta_2\delta_4 iW) \\ &\quad + (\delta_1\delta_4 iW)(\delta_2\delta_3 iW)\,\Big]_{J=0}\,. \end{aligned} \tag{10.4}$$

We have dropped terms that contain a factor of $\langle 0|\varphi(x)|0\rangle = 0$.

According to eq. (10.3), the last three terms in eq. (10.4) simply give products of the exact propagators.

Let us see what happens when these terms are inserted into the LSZ formula for two incoming and two outgoing particles,

$$\begin{aligned}\langle f|i\rangle \;=\; i^4 \int d^4x_1\, d^4x_2\, d^4x'_1\, d^4x'_2\, e^{i(k_1x_1+k_2x_2-k'_1x'_1-k'_2x'_2)} \\ \times(-\partial_1^2+m^2)(-\partial_2^2+m^2)(-\partial_{1'}^2+m^2)(-\partial_{2'}^2+m^2) \\ \times\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x'_1)\varphi(x'_2)|0\rangle\ .\end{aligned} \tag{10.5}$$

If we consider, for example, $\frac{1}{i}\mathbf{\Delta}(x_1-x'_1)\ \frac{1}{i}\mathbf{\Delta}(x_2-x'_2)$ as one term in the correlation function in eq. (10.5), we get from this term

$$\begin{aligned}\int d^4x_1\, d^4x_2\, d^4x'_1\, d^4x'_2\, e^{i(k_1x_1+k_2x_2-k'_1x'_1-k'_2x'_2)} F(x_{11'})F(x_{22'}) \\ = (2\pi)^4\delta^4(k_1-k'_1)\,(2\pi)^4\delta^4(k_2-k'_2)\,\widetilde{F}(\bar{k}_{11'})\,\widetilde{F}(\bar{k}_{22'})\ ,\end{aligned} \tag{10.6}$$

where $F(x_{ij}) \equiv (-\partial^2_i+m^2)(-\partial^2_j+m^2)\mathbf{\Delta}(x_{ij})$, $\tilde{F}(k)$ is its Fourier transform, $x_{ij}' \equiv x_i-x'_j$, and $k_{ij}' \equiv (k_i+k'_j)/2$.

The important point is the two delta functions: these tell us that the four-momenta of the two outgoing particles ($1'$ and $2'$) are equal to the four-momenta of the two incoming particles (1 and 2).

In other words, no scattering has occurred.

This is not the event whose probability we wish to compute!

The other two similar terms in eq. (10.4) either contribute to "no scattering" events, or vanish due to factors like $\delta^4(k_1+k_2)$ (which is zero because $k^0_1 + k^0_2 \geq 2m > 0$).

In general, the diagrams that contribute to the scattering process of interest are only those that are fully connected: every endpoint can be reached from every other endpoint by tracing through the diagram.

These are the diagrams that arise from all the δ's acting on a single factor of W.

Therefore, from here on, we restrict our attention to those diagrams alone.

We define the *connected* correlation functions via

$$\langle 0|\mathrm{T}\varphi(x_1)\dots\varphi(x_E)|0\rangle_{\mathrm{C}} \equiv \delta_1\dots\delta_E iW(J)\Big|_{J=0}\,, \tag{10.7}$$

and use these instead of $\langle 0|\mathrm{T}\varphi(x_1)\dots\varphi(x_E)|0\rangle$ in the LSZ formula.

Returning to eq. (10.4), we have

$$\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x'_1)\varphi(x'_2)|0\rangle_{\mathrm{C}} = \delta_1\delta_2\delta_{1'}\delta_{2'} iW\Big|_{J=0}\,. \tag{10.8}$$

The lowest-order (in g) nonzero contribution to this comes from the diagram of fig. (9.10), which has four sources and two vertices.

The four δ's remove the four sources; there are 4! ways of matching up the δ's to the sources.

These 24 diagrams can then be collected into 3 groups of 8 diagrams each; the 8 diagrams in each group are identical.

The 3 distinct diagrams are shown in fig. (10.1).

Note that the factor of 8 neatly cancels the symmetry factor S = 8 of the diagram with sources.

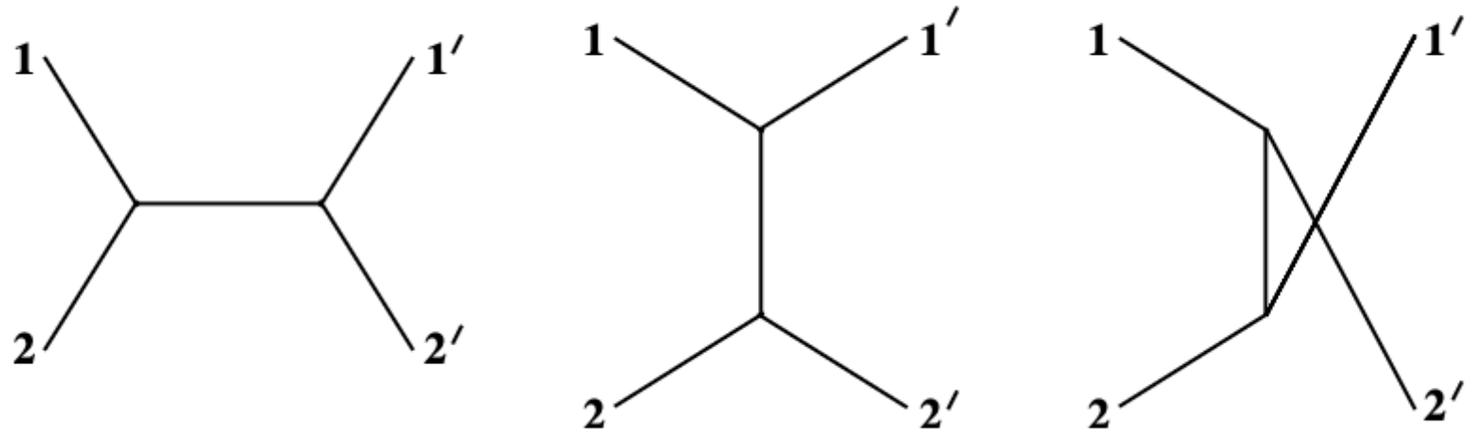


Figure 10.1: The three tree-level Feynman diagrams that contribute to the connected correlation function $\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x'_1)\varphi(x'_2)|0\rangle_C$.

This is a general result for tree diagrams (those with no closed loops): once the sources have been stripped off and the endpoints labeled, each diagram with a distinct endpoint labeling has an overall symmetry factor of one.

The tree diagrams for a given process represent the lowest-order (in g) nonzero contribution to that process.

We now have

$$
\begin{aligned}
&\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_{\mathrm{C}} \\
&\qquad = (ig)^2\left(\tfrac{1}{i}\right)^5 \int d^4y\, d^4z\, \Delta(y-z) \\
&\qquad\quad \times\Big[\, \Delta(x_1-y)\Delta(x_2-y)\Delta(x_1'-z)\Delta(x_2'-z) \\
&\qquad\quad + \Delta(x_1-y)\Delta(x_1'-y)\Delta(x_2-z)\Delta(x_2'-z) \\
&\qquad\quad + \Delta(x_1-y)\Delta(x_2'-y)\Delta(x_2-z)\Delta(x_1'-z)\,\Big] \\
&\qquad\qquad + O(g^4)\ .
\end{aligned}
\tag{10.9}
$$

Next, we use eq. (10.9) in the LSZ formula, eq. (10.5).

Each Klein-Gordon wave operator acts on a propagator to give

$$
(-\partial_i^2 + m^2)\Delta(x_i - y) = \delta^4(x_i - y)\ . \tag{10.10}
$$

The integrals over the external spacetime labels $x_{1,2,1',2'}$ are then trivial, and we get

$$
\begin{aligned}
\langle f|i\rangle = (ig)^2\left(\tfrac{1}{i}\right)\int d^4y\, d^4z\, \Delta(y-z)\Big[\ & e^{i(k_1y+k_2y-k_1'z-k_2'z)} \\
& + e^{i(k_1y+k_2z-k_1'y-k_2'z)} \\
& + e^{i(k_1y+k_2z-k_1'z-k_2'y)}\,\Big] + O(g^4)\ .
\end{aligned}
\tag{10.11}
$$

This can be simplified by substituting

$$\Delta(y-z) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(y-z)}}{k^2+m^2-i\epsilon} \tag{10.12}$$

into eq. (10.9).

Then the spacetime arguments appear only in phase factors, and we can integrate them to get delta functions:

$$\begin{aligned}
\langle f|i\rangle &= ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2-i\epsilon} \\
&\quad \times \Big[(2\pi)^4\delta^4(k_1+k_2+k)\,(2\pi)^4\delta^4(k_1'+k_2'+k) \\
&\quad + (2\pi)^4\delta^4(k_1-k_1'+k)\,(2\pi)^4\delta^4(k_2'-k_2+k) \\
&\quad + (2\pi)^4\delta^4(k_1-k_2'+k)\,(2\pi)^4\delta^4(k_1'-k_2+k) \Big] + O(g^4) \qquad (10.13) \\
&= ig^2\,(2\pi)^4\delta^4(k_1+k_2-k_1'-k_2') \\
&\quad \times \left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_1-k_2')^2+m^2} \right] \\
&\quad + O(g^4)\ .
\end{aligned}$$

the the

In eq. (10.13), we have left out the iϵ's for notational convenience only; m^2 is really $m^2 - i\epsilon$.

The overall delta function in eq. (10.13) tells that that four-momentum is conserved in the scattering process, which we should, of course, expect.

For a general scattering process, it is then convenient to define a scattering matrix element $\mathcal{T}$ via

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_{\text{in}} - k_{\text{out}})i\mathcal{T} \ , \tag{10.14}$$

where k_{in} and k_{out} are the total four-momenta of the incoming and outgoing particles, respectively.

Examining the calculation which led to eq. (10.13), we can take away some universal features that lead to a simple set of Feynman rules for computing contributions to i$\mathcal{T}$ for a given scattering process.

The Feynman rules are:

1. Draw lines (called *external lines*) for each incoming and each outgoing particle.

2. Leave one end of each external line free, and attach the other to a vertex at which exactly three lines meet. Include extra *internal lines* in order to do this. In this way, draw all possible diagrams that are *topologically inequivalent.*

3. On each incoming line, draw an arrow pointing towards the vertex. On each outgoing line, draw an arrow pointing away from the vertex. On each internal line, draw an arrow with an arbitrary direction.

4. Assign each line its own four-momentum. The four-momentum of an external line should be the four-momentum of the corresponding particle.

5. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

6. The value of a diagram consists of the following factors:

   for each external line, 1;

   for each internal line with momentum k, $-i/(k^2 + m^2 - i\epsilon)$;

   for each vertex, $iZ_g g$.

7. A diagram with L closed loops will have L internal momenta that are not fixed by rule #5. Integrate over each of these momenta ℓ_i with measure $d^4\ell_i/(2\pi)^4$.

8. A loop diagram may have some leftover symmetry factors if there are exchanges of *internal* propagators and vertices that leave the diagram unchanged; in this case, divide the value of the diagram by the symmetry factor associated with exchanges of internal propagators and vertices.

9. Include diagrams with the *counterterm vertex* that connects two propagators, each with the same four-momentum k. The value of this vertex is $-i(Ak^2 + Bm^2)$, where $A = Z_\varphi - 1$ and $B = Z_m - 1$, and each is $O(g^2)$.

10. The value of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

For the two-particle scattering process, the tree diagrams resulting from these rules are shown in fig. (10.2).

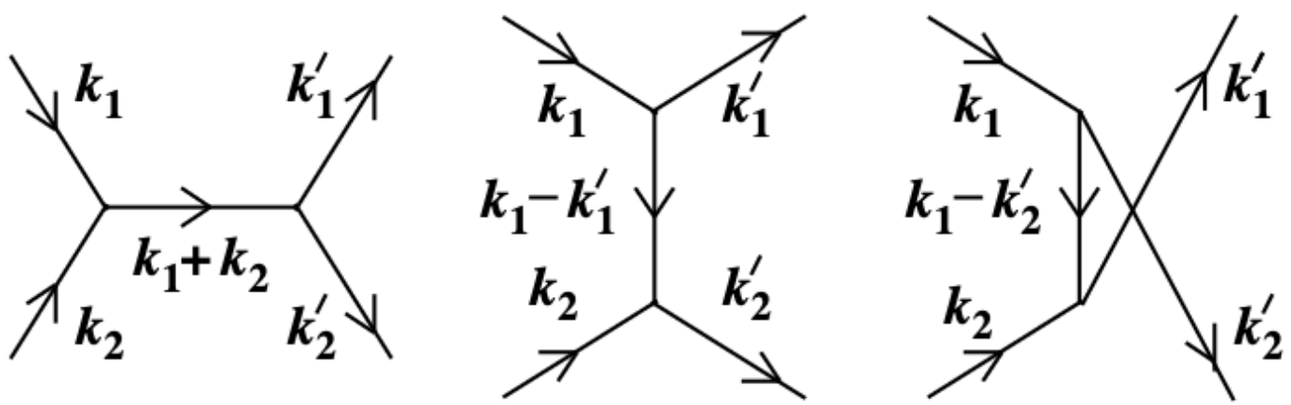


Figure 10.2: The tree-level s-, t-, and u-channel diagrams contributing to i$\mathcal{T}$ for two particle scattering.

Now that we have our procedure for computing the scattering amplitude $\mathcal{T}$, we must see how to relate it to a measurable cross section.

11 Cross Sections and Decay Rates

Prerequisite: 10

Now that we have a method for computing the scattering amplitude $\mathcal{T}$, we must convert it into something that could be measured in an experiment.

In practice, we are almost always concerned with one of two generic cases: one incoming particle, for which we compute a decay rate, or two incoming particles, for which we compute a cross section.

We begin with the latter.

Let us also specialize, for now, to the case of two outgoing particles as well as two incoming particles.

In φ^3 theory, we found in section 10 that in this case we have

$$\mathcal{T} = g^2\left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_1-k_2')^2+m^2}\right] + O(g^4)\,, \tag{11.1}$$

where k_1 and k_2 are the four-momenta of the two incoming particles, k'_1 and k'_2 are the four-momenta of the two outgoing particles, and $k_1 + k_2 = k'_1 + k'_2$.

Also, these particles are all *on shell* : $k^2_i = -m^2_i$.

Here, for later use, we allow for the possibility that the particles all have different masses.

Let us think about the kinematics of this process.

In the *center-of-mass frame*, or *CM frame* for short, we take $k_1 + k_2 = 0$, and choose k_1 to be in the +z direction.

Now the only variable left to specify about the initial state is the magnitude of k_1.

Equivalently, we could specify the total energy in the CM frame, $E_1 + E_2$.

However, it is even more convenient to define a Lorentz scalar $s \equiv -(k_1 + k_2)^2$.

In the CM frame, s reduces to $(E_1 + E_2)^2$; s is therefore called the *center-of-mass energy squared*.

Then, since $E_1 = (k^2_1 + m^2_1)^{1/2}$ and $E^2 = (k^2_1 + m^2_2)^{1/2}$, we can solve for $|\mathbf{k}_1|$ in terms of s, with the result

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \quad \text{(CM frame)} \; . \tag{11.2}$$

Now consider the two outgoing particles.

Since momentum is conserved, we must have $\mathbf{k}'_1 + \mathbf{k}'_2 = 0$, and since energy is conserved, we must also have $(E'_1 + E'_2)^2 = s$.

Then we find

$$|\mathbf{k}'_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2} \quad \text{(CM frame)} \ . \tag{11.3}$$

Now the only variable left to specify about the final state is the angle θ between $\mathbf{k}_1$ and $\mathbf{k}'_1$.

However, it is often more convenient to work with theLorentz scalar $t \equiv -(k_1 - k'_1)^2$, which is related to θ by

$$t = m_1^2 + m_{1'}^2 - 2E_1E'_1 + 2|\mathbf{k}_1||\mathbf{k}'_1|\cos\theta \ . \tag{11.4}$$

This formula is valid in any frame.

The Lorentz scalars sand tare two of the three Mandelstam variables, defined as

$$\begin{aligned} s &\equiv -(k_1+k_2)^2 = -(k'_1+k'_2)^2 \ , \\ t &\equiv -(k_1-k'_1)^2 = -(k_2-k'_2)^2 \ , \\ u &\equiv -(k_1-k'_2)^2 = -(k_2-k'_1)^2 \ . \end{aligned} \tag{11.5}$$

The three Mandelstam variables are not independent; they satisfy the linear relation

$$s + t + u = m_1^2 + m_2^2 + m_{1'}^2 + m_{2'}^2 \ . \tag{11.6}$$

In terms of s, t, and u, we can rewrite eq. (11.1) as

$$\mathcal{T} = g^2\left[\frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u}\right] + O(g^4) \ , \tag{11.7}$$

which demonstrates the notational utility of the Mandelstam variables.

Now let us consider a different frame, the *fixed target or FT frame* (also sometimes called the *lab frame*), in which particle #2 is initially at rest: $\mathbf{k}_2 = 0$.

In this case we have

$$|\mathbf{k}_1| = \frac{1}{2m_2}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \quad \text{(FT frame)} \; . \tag{11.8}$$

Note that, from eqs. (11.8) and (11.2),

$$m_2 |\mathbf{k}_1|_{\rm FT} = \sqrt{s}\, |\mathbf{k}_1|_{\rm CM} \; . \tag{11.9}$$

This will be useful later.

We would now like to derive a formula for the differential scattering cross section.

In order to do so, we assume that the whole experiment is taking place in a big box of volume V, and lasts for a large time T.

We should really think about wave packets coming together, but we will use some simple shortcuts instead.

Also, to get a more general answer, we will let the number of outgoing particles be arbitrary.

Recall from section 10 that the overlap between the initial and final states is given by

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})i\mathcal{T}\ . \tag{11.10}$$

To get a probability, we must square $\langle f\,|\,i\rangle$, and divide by the norms of the initial and final states:

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\ . \tag{11.11}$$

The numerator of this expression is

$$|\langle f|i\rangle|^2 = [(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})]^2\,|\mathcal{T}|^2\ . \tag{11.12}$$

We write the square of the delta function as

$$[(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})]^2 = (2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})\times(2\pi)^4\delta^4(0)\ , \tag{11.13}$$

and note that

$$(2\pi)^4\delta^4(0) = \int d^4x\, e^{i0\cdot x} = VT\ . \tag{11.14}$$

Also, the norm of a single particle state is given by

$$\langle k|k\rangle = (2\pi)^3 2k^0\delta^3(\mathbf{0}) = 2k^0V\ . \tag{11.15}$$

Thus we have

$$\langle i|i\rangle = 4E_1E_2V^2 \ , \tag{11.16}$$

$$\langle f|f\rangle = \prod_{j=1}^{n'} 2{k'_j}^0 V \ , \tag{11.17}$$

where n′ is the number of outgoing particles.

If we now divide eq. (11.11) by the elapsed time T, we get a probability per unit time

$$\dot{P} = \frac{(2\pi)^4\delta^4(k_{\text{in}}-k_{\text{out}})\, V\, |\mathcal{T}|^2}{4E_1E_2V^2\prod_{j=1}^{n'} 2{k'_j}^0 V} \ . \tag{11.18}$$

This is the probability per unit time to scatter into a set of outgoing particles with precise momenta.

To get something measurable, we should sum each outgoing three-momentum $\mathbf{k}'_j$ over some small range.

Due to the box, all three-momenta are quantized: $\mathbf{k}'_j = (2\pi/L)\mathbf{n}'_j$, where $V = L^3$, and $\mathbf{n}'_j$ is a three-vector with integer entries.

Here we have assumed periodic boundary conditions, but this choice does not affect the final result.

In the limit of large L, we have

$$\sum_{\mathbf{n}'_j} \to \frac{V}{(2\pi)^3} \int d^3k'_j \ . \tag{11.19}$$

Thus we should multiply $\dot{P}$ by a factor of $Vd^3k'_j/(2\pi)^3$ for each outgoing particle.

The we get

$$\dot{P} = \frac{(2\pi)^4 \delta^4(k_{\text{in}} - k_{\text{out}})}{4E_1 E_2 V} \, |\mathcal{T}|^2 \prod_{j=1}^{n'} \widetilde{dk}'_j \ , \tag{11.20}$$

where we have identified the Lorentz-invariant phase-space differential

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2k^0} \tag{11.21}$$

that we first introduced in section 3.

To convert $\dot{P}$ to a differential cross section dσ, we must divide by the incident flux.

Let us see how this works in the FT frame, where particle #2 is at rest.

The incident flux is the number of particles per unit volume that are striking the target particle (#2), times their speed.

We have one incident particle (#1) in a volume V with speed $v = |\mathbf{k}_1|/E_1$, and so the incident flux is $|\mathbf{k}_1|/E_1V$.

Dividing eq. (11.20) by this flux cancels the last factor of V, and replaces E_1 in the denominator with $|\mathbf{k}_1|$.

We also set $E_2 = m^2$ and note that eq. (11.8) gives $|\mathbf{k}_1|m^2$ as a function of s; dσ will be Lorentz invariant if, in other frames, we simply use this function as the value of $|\mathbf{k}_1|m^2$.

Adopting this convention, and using eq. (11.9), we have

$$d\sigma = \frac{1}{4|\mathbf{k}_1|_{\rm CM}\sqrt{s}}\,|\mathcal{T}|^2\,d{\rm LIPS}_{n'}(k_1{+}k_2)\ , \tag{11.22}$$

where $|\mathbf{k}_1|_{\rm CM}$ is given as a function of s by eq. (11.2), and we have defined the n′-body Lorentz-invariant phase-space measure

$$d{\rm LIPS}_{n'}(k) \equiv (2\pi)^4\delta^4(k{-}\textstyle\sum_{j=1}^{n'} k'_i)\displaystyle\prod_{j=1}^{n'}\widetilde{dk}'_j\ . \tag{11.23}$$

Eq. (11.22) is our final result for the differential cross section for the scattering of two incoming particles into n outgoing particles.

Let us now specialize to the case of two outgoing particles.

We need to evaluate

$$d{\rm LIPS}_2(k) = (2\pi)^4\delta^4(k{-}k'_1{-}k'_2)\,\widetilde{dk}'_1\widetilde{dk}'_2\ , \tag{11.24}$$

where $k = k_1 + k_2$.

Since dLIPS2(k) is Lorentz invariant, we can compute it in any convenient frame.

Let us work in the CM frame, where $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 = 0$ and $k^0 = E_1 + E_2 = \sqrt{s}$; then we have

$$d\mathrm{LIPS}_2(k) = \frac{1}{4(2\pi)^2 E_1' E_2'}\,\delta(E_1'+E_2'-\sqrt{s}\,)\,\delta^3(\mathbf{k}_1'+\mathbf{k}_2')\,d^3k_1' d^3k_2' \ . \tag{11.25}$$

We can use the spatial part of the delta function to integrate over $d^3k'_2$, with the result

$$d\mathrm{LIPS}_2(k) = \frac{1}{4(2\pi)^2 E_1' E_2'}\,\delta(E_1'+E_2'-\sqrt{s}\,)\,d^3k_1' \ , \tag{11.26}$$

where now

$$E_1' = \sqrt{\mathbf{k}_1'^{\,2} + m_{1'}^2} \quad \text{and} \quad E_2' = \sqrt{\mathbf{k}_1'^{\,2} + m_{2'}^2} \ . \tag{11.27}$$

Next, let us write

$$d^3k_1' = |\mathbf{k}_1'|^2\, d|\mathbf{k}_1'|\, d\Omega_{\mathrm{CM}} \ , \tag{11.28}$$

where $d\Omega_{CM} = \sin\theta d\theta d\phi$ is the differential solid angle, and θ is the angle between $\mathbf{k}_1$ and $\mathbf{k}'_1$ in the CM frame.

We can carry out the integral over the magnitude of $\mathbf{k}'_1$ in eq. (11.26) using $\int dx \delta(f(x)) = \sum_i |f'(x_i)|^{-1}$, where x_i satisfies $f(x_i) = 0$.

In our case, the argument of the delta function vanishes at just one value of $|\mathbf{k}'_1|$, the value given by eq. (11.3).

Also, the derivative of that argument with respect to $|\mathbf{k}'_1|$ is

$$\frac{\partial}{\partial|\mathbf{k}_1'|}\left(E_1' + E_2' - \sqrt{s}\right) = \frac{|\mathbf{k}_1'|}{E_1'} + \frac{|\mathbf{k}_1'|}{E_2'}$$
$$= |\mathbf{k}_1'|\left(\frac{E_1' + E_2'}{E_1'\,E_2'}\right) \tag{11.29}$$
$$= \frac{|\mathbf{k}_1'|\sqrt{s}}{E_1' E_2'}\ .$$

Putting all of this together, we get

$$d\mathrm{LIPS}_2(k) = \frac{|\mathbf{k}_1'|}{16\pi^2\sqrt{s}}\, d\Omega_{\mathrm{CM}}\ . \tag{11.30}$$

Combining this with eq. (11.22), we have

$$\frac{d\sigma}{d\Omega_{\mathrm{CM}}} = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{k}_1'|}{|\mathbf{k}_1|}\,|\mathcal{T}|^2\ , \tag{11.31}$$

where $|\mathbf{k}_1|$ and $|\mathbf{k}'_1|$ are the functions of s given by eqs. (11.2) and (11.3), and $d\Omega_{CM}$ is the differential solid angle in the CM frame.

The differential cross section can also be expressed in a frame-independent manner by noting that, in the CM frame, we can take the differential of eq. (11.4) at fixed s to get

$$dt = 2\,|\mathbf{k}_1|\,|\mathbf{k}'_1|\,d\cos\theta \tag{11.32}$$

$$= 2\,|\mathbf{k}_1|\,|\mathbf{k}'_1|\,\frac{d\Omega_{\rm CM}}{2\pi}\ . \tag{11.33}$$

Now we can rewrite eq. (11.31) as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\mathbf{k}_1|^2}\,|\mathcal{T}|^2\ , \tag{11.34}$$

where $|\mathbf{k}_1|$ is given as a function of s by eq. (11.2).

We can now transform dσ/dt into dσ/dΩ in any frame we might like (such as the FT frame) by taking the differential of eq. (11.4) in that frame.

In general, though, $|\mathbf{k}'_1|$ depends on θ as well as s, so the result is more complicated than it is in eq. (11.32) for the CM frame.

Returning to the general case of n′ outgoing particles, we can define a Lorentz invariant total cross section by integrating completely over all the outgoing momenta, and dividing by an appropriate *symmetry factor* S.

If there are n'_i identical outgoing particles of type i, then

$$S = \prod_i n'_i!\ , \tag{11.35}$$

and

$$\sigma = \frac{1}{S} \int d\sigma \ , \tag{11.36}$$

where dσ is given by eq. (11.22).

We need the symmetry factor because merely integrating over all the outgoing momenta in $dLIPS_{n'}$ treats the final state as being labeled by an ordered list of these momenta.

But if some outgoing particles are identical, this is not correct; the momenta of the identical particles should be specified by an unordered list (because, for example, the state $a^\dagger_1 a^\dagger_2 |0\rangle$ is identical to the state $a^\dagger_2 a^\dagger_1 |0\rangle$).

The symmetry factor provides the appropriate correction.

In the case of two outgoing particles, eq. (11.36) becomes

$$\sigma = \frac{1}{S} \int d\Omega_{\rm CM} \frac{d\sigma}{d\Omega_{\rm CM}} \tag{11.37}$$

$$= \frac{2\pi}{S} \int_{-1}^{+1} d\cos\theta \frac{d\sigma}{d\Omega_{\rm CM}} \ , \tag{11.38}$$

where S = 2 if the two outgoing particles are identical, and S = 1 if they are distinguishable.

Equivalently, we can compute σ from eq. (11.34) via

$$\sigma = \frac{1}{S} \int_{t_{\min}}^{t_{\max}} dt \, \frac{d\sigma}{dt} \ , \tag{11.39}$$

where t_{min} and t_{max} are given by eq. (11.4) in the CM frame with cos θ = −1 and +1, respectively.

To compute σ with eq. (11.38), we should first express t and u in terms of s and θ via eqs. (11.4) and (11.6), and then integrate over θ at fixed s.

To compute σ with eq. (11.39), we should first express u in terms of s and t via eq. (11.6), and then integrate over t at fixed s.

Let us see how all this works for the scattering amplitude of φ^3 theory, eq. (11.7).

In this case, all the masses are equal, and so, in the CM frame, E $= \frac{1}{2}\sqrt{s}$ for all four particles, and $|\mathbf{k}'_1| = |\mathbf{k}_1|$ = eq.(11.4) becomes $\frac{1}{2}$ $(s - 4m^2)^{1/2}$.

Then eq. (11.4) becomes

$$t = -\tfrac{1}{2}(s - 4m^2)(1 - \cos\theta) \ . \tag{11.40}$$

From eq. (11.6), we also have

$$u = -\tfrac{1}{2}(s - 4m^2)(1 + \cos\theta) \ . \tag{11.41}$$

Thus $|\mathcal{T}|^2$ is quite a complicated function of s and θ.

In the nonrelativistic limit, $|\mathbf{k}_1| \ll m$ or equivalently $s - 4m^2 \ll m^2$, we have

$$\mathcal{T} = \frac{5g^2}{3m^2}\left[1 - \frac{8}{15}\left(\frac{s-4m^2}{m^2}\right) + \frac{5}{18}\left(1 + \frac{27}{25}\cos^2\theta\right)\left(\frac{s-4m^2}{m^2}\right)^2 + \ldots\right] \tag{11.42}$$

$$+ O(g^4)\ .$$

Thus the differential cross section is nearly isotropic.

In the extreme relativistic limit, $|\mathbf{k}_1| \gg m$ or equivalently $s \gg m^2$, we have

$$\mathcal{T} = \frac{g^2}{s\sin^2\theta}\left[3 + \cos^2\theta - \left(\frac{(3+\cos^2\theta)^2}{\sin^2\theta} - 16\right)\frac{m^2}{s} + \ldots\right] \tag{11.43}$$

$$+ O(g^4)\ .$$

Now the differential cross section is sharply peaked in the forward ($\theta = 0$) and backward ($\theta = \pi$) directions.

We can compute the total cross section σ from eq. (11.39).

We have in this case $t_{min} = -(s - 4m^2)$ and $t_{max} = 0$.

Since the two outgoing particles are identical, the symmetry factor is $S = 2$.

Then setting u = $4m^2 - s - t$, and performing the integral in eq. (11.39) over t at fixed s, we get

$$\sigma = \frac{g^4}{32\pi s(s-4m^2)}\left[\frac{2}{m^2} + \frac{s-4m^2}{(s-m^2)^2} - \frac{2}{s-3m^2} + \frac{4m^2}{(s-m^2)(s-2m^2)}\ln\left(\frac{s-3m^2}{m^2}\right)\right] + O(g^6)\ . \tag{11.44}$$

In the nonrelativistic limit, this becomes

$$\sigma = \frac{25g^4}{1152\pi m^6}\left[1 - \frac{79}{60}\left(\frac{s-4m^2}{m^2}\right) + \ldots\right] + O(g^6)\ . \tag{11.45}$$

In the extreme relativistic limit, we get

$$\sigma = \frac{g^4}{16\pi m^2 s^2}\left[1 + \frac{7}{2}\frac{m^2}{s} + \ldots\right] + O(g^6)\ . \tag{11.46}$$

These results illustrate how even a very simple quantum field theory can yield specific predictions for cross sections that could be tested experimentally.

Let us now turn to the other basic problem mentioned at the beginning of this section: the case of a single incoming particle that decays to n other particles.

We have an immediate conceptual problem.

According to our development of the LSZ formula in section 5, each incoming and outgoing particle should correspond to a single-particle state that is an exact eigenstate of the exact hamiltonian.

This is clearly not the case for a particle that can decay.

Referring to fig. (5.1), the hyperbola of such a particle must lie above the continuum threshold.

Strictly speaking, then, the LSZ formula is not applicable.

A proper understanding of this issue requires a study of loop corrections that we will undertake in section 25.

For now, we will simply assume that the LSZ formula continues to hold for a single incoming particle.

Then we can retrace the steps from eq. (11.11) to eq. (11.20); the only change is that the norm of the initial state is now

$$\langle i|i\rangle = 2E_1 V \tag{11.47}$$

instead of eq. (11.16).

Identifying the differential decay rate dΓ with $\dot{P}$ then gives

$$d\Gamma = \frac{1}{2E_1}\,|\mathcal{T}|^2\, d\mathrm{LIPS}_{n'}(k_1)\ , \tag{11.48}$$

where now $s = -k^2_1 = m^2_1$.

In the CM frame (which is now the rest frame of the initial particle), we have $E_1 = m_1$; in other frames, the relative factor of E_1/m_1 in $d\Gamma$ accounts for relativistic time dilation of the decay rate.

We can also define a total decay rate by integrating over all the outgoing momenta, and dividing by the symmetry factor of eq. (11.35):

$$\Gamma = \frac{1}{S} \int d\Gamma \ . \tag{11.49}$$

This enables us yo compute a decay rate, more later.

12 Dimensional Analysis with $\hbar = c = 1$

Prerequisite: 3

We have set $\hbar = c = 1$.

This allows us to convert a time T to a length L via $T = c^{-1}L$, and a length L to an inverse mass M^{-1} via $L = \hbar\, c^{-1}M^{-1}$.

Thus any quantity A can be thought of as having units of mass to some power (positive, negative, or zero) that we will call [A].

For example,

$$[m] = +1 \,, \tag{12.1}$$

$$[x^\mu] = -1 \,, \tag{12.2}$$

$$[\partial^\mu] = +1 \,, \tag{12.3}$$

$$[d^d x] = -d \,. \tag{12.4}$$

In the last line, we have generalized our considerations to theories in d spacetime dimensions.

Let us now consider a scalar field in d spacetime dimensions with lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \sum_{n=3}^{N} \tfrac{1}{n!} g_n \varphi^n \,. \tag{12.5}$$

The action is

$$S = \int d^d x \, \mathcal{L} \,, \tag{12.6}$$

and the path integral is

$$Z(J) = \int \mathcal{D}\varphi \, \exp\left[i \int d^d x \, (\mathcal{L} + J\varphi) \right] \,. \tag{12.7}$$

From eq. (12.7), we see that the action S must be dimensionless, because it appears as the argument of the exponential function.

Therefore

$$[S] = 0 \ . \tag{12.8}$$

Combining eqs. (12.4) and (12.8) yields

$$[\mathcal{L}] = d \ . \tag{12.9}$$

Then, from eqs. (12.9) and (12.3), and the fact that $\partial^\mu \varphi \partial_\mu \varphi$ is a term in $\mathcal{L}$, we see that we must have

$$[\varphi] = \tfrac{1}{2}(d-2) \ . \tag{12.10}$$

Then, since $g_n \varphi^n$ is also a term in $\mathcal{L}$, we must have

$$[g_n] = d - \tfrac{1}{2} n (d-2) \ . \tag{12.11}$$

In particular, for the φ^3 theory we have been working with, we have

$$[g_3] = \tfrac{1}{2}(6-d) \ . \tag{12.12}$$

Thus we see that the coupling constant of φ^3 theory is dimensionless in d = 6 spacetime dimensions.

Theories with dimensionless couplings tend to be more interesting than theories with dimensionful couplings.

This is because any nontrivial dependence of a scattering amplitude on a coupling must be expressed as a function of a dimensionless parameter.

If the coupling is itself dimensionful, this parameter must be the ratio of the coupling to the appropriate power of either the particle mass m (if it isn't zero) or, in the high-energy regime $s \gg m^2$, the Mandelstam variable s.

Thus the relevant parameter is $gs^{-[g]/2}$.

If [g] is negative [and it usually is: see eq. (12.11)], then $gs^{-[g]/2}$ blows up at high energies, and the perturbative expansion breaks down.

This behavior is connected to the nonrenormalizability of theories with couplings with negative mass dimension, a subject we will take up in section 18.

It turns out that such theories require an infinite number of input parameters to make sense; see section 29.

In the opposite case, [g] positive, the theory becomes trivial at high energy, because $gs^{-[g]/2}$ goes rapidly to zero.

Thus the case of $[g] = 0$ is just right: scattering amplitudes can have a nontrivial dependence on g at all energies.

Therefore, from here on, we will be primarily interested in φ^3 theory in $d = 6$ spacetime dimensions, where $[g^3] = 0$.

13 The Lehmann-Kallen Form of the Exact Propagator

Prerequisite: 9

Before turning to the subject of loop corrections to scattering amplitudes, it will be helpful to consider what we can learn about the exact propagator $\mathbf{\Delta}(x-y)$ from general principles.

We define the exact propagator via

$$\mathbf{\Delta}(x-y) \equiv i\langle 0|\mathrm{T}\varphi(x)\varphi(y)|0\rangle \ . \tag{13.1}$$

We take the field $\varphi(x)$ to be normalized so that

$$\langle 0|\varphi(x)|0\rangle = 0 \qquad \text{and} \qquad \langle k|\varphi(x)|0\rangle = e^{-ikx} \ . \tag{13.2}$$

In d spacetime dimensions, the one-particle state $|k\rangle$ has the normalization

$$\langle k|k'\rangle = (2\pi)^{d-1}\, 2\omega\, \delta^{d-1}(\mathbf{k}-\mathbf{k}') \ , \tag{13.3}$$

with $\omega = (\mathbf{k}^2 + m^2)^{1/2}$.

The corresponding completeness statement is

$$\int \widetilde{dk}\ |k\rangle\langle k| = I_1 \ , \tag{13.4}$$

where I_1 is the identity operator in the one-particle subspace, and

$$\widetilde{dk} \equiv \frac{d^{d-1}k}{(2\pi)^{d-1}2\omega} \tag{13.5}$$

is the Lorentz invariant phase-space differential.

We also define the exact momentum-space propagator $\tilde{\mathbf{\Delta}}(k^2)$ via

$$\mathbf{\Delta}(x-y) \equiv \int \frac{d^dk}{(2\pi)^d}\, e^{ik(x-y)}\, \tilde{\mathbf{\Delta}}(k^2)\ . \tag{13.6}$$

In free-field theory, the momentum-space propagator is

$$\tilde{\Delta}(k^2) = \frac{1}{k^2+m^2-i\epsilon}\ . \tag{13.7}$$

It has an isolated pole at $k^2 = -m^2$ with residue one; m is the actual, physical mass of the particle, the mass that enters into the energy-momentum relation.

We begin our analysis with eq. (13.1).

We take $x_0 > y_0$, and insert a complete set of energy eigenstates between the two fields.

Recall from section 5 that there are three general classes of energy eigenstates:

1. The ground state or vacuum $|0\rangle$, which is a single state with zero energy and momentum.

2. The one particle states $|k\rangle$, specified by a three-momentum $\mathbf{k}$ and with energy $\omega = (\mathbf{k}^2 + m^2)^{1/2}$.

3. States in the multiparticle continuum $|k, n\rangle$, specified by a three-momentum $\mathbf{k}$ and other parameters (such as relative momenta among the different particles) that we will collectively denote as n. The energy of one of these states is $\omega = (\mathbf{k}^2 + M^2)^{1/2}$, where $M \geq 2m$; M is one of the parameters in the set n.

Thus we get

$$\begin{aligned}\langle 0|\varphi(x)\varphi(y)|0\rangle \;=\; & \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle \\ & + \int \widetilde{dk}\, \langle 0|\varphi(x)|k\rangle\langle k|\varphi(y)|0\rangle \\ & + \sum_n \int \widetilde{dk}\, \langle 0|\varphi(x)|k,n\rangle\langle k,n|\varphi(y)|0\rangle \;\; .\end{aligned} \tag{13.8}$$

The sum over n is schematic, and includes integrals over continuous parameters like relative momenta.

The first two terms in eq. (13.8) can be simplified via eq. (13.2).

Also, writing the field as $\phi(x) = \exp(-iP^\mu x_\mu)\varphi(0)\exp(+iP^\mu x_\mu)$, where P^μ is the energy-momentum operator, gives us

$$\langle k,n|\varphi(x)|0\rangle = e^{-ikx}\langle k,n|\varphi(0)|0\rangle \ , \tag{13.9}$$

where $k^0 = (\mathbf{k}^2 + M^2)^{1/2}$.

We now have

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \widetilde{dk}\, e^{ik(x-y)} + \sum_n \int \widetilde{dk}\, e^{ik(x-y)} |\langle k,n|\varphi(0)|0\rangle|^2 \ . \tag{13.10}$$

Next, we define the *spectral density*

$$\rho(s) \equiv \sum_n |\langle k,n|\varphi(0)|0\rangle|^2 \, \delta(s - M^2) \ . \tag{13.11}$$

Obviously, $\rho(s) \geq 0$ for $s \geq 4m^2$, and $\rho(s) = 0$ for $s < 4m^2$.

Now we have

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \widetilde{dk}\, e^{ik(x-y)} + \int_{4m^2}^{\infty} ds\, \rho(s) \int \widetilde{dk}\, e^{ik(x-y)} \ . \tag{13.12}$$

In the first term, $k^0 = (\mathbf{k}^2 + M^2)^{1/2}$, and in the second term, $k^0 = (\mathbf{k}^2 + s)^{1/2}$.

Clearly we can also swap x and y to get

$$\langle 0|\varphi(y)\varphi(x)|0\rangle = \int \widetilde{dk}\, e^{-ik(x-y)} + \int_{4m^2}^{\infty} ds\, \rho(s) \int \widetilde{dk}\, e^{-ik(x-y)} \tag{13.13}$$

as well.

We can then combine eqs. (13.12) and (13.13) into a formula for the time-ordered product

$$\langle 0|\mathrm{T}\varphi(x)\varphi(y)|0\rangle = \theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle, \tag{13.14}$$

where θ(t) is the unit step function, by means of the identity

$$\begin{aligned} \int \frac{d^dk}{(2\pi)^d}\, \frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon} &= i\theta(x^0-y^0) \int \widetilde{dk}\, e^{ik(x-y)} \\ &\quad + i\theta(y^0-x^0) \int \widetilde{dk}\, e^{-ik(x-y)}\ ; \end{aligned} \tag{13.15}$$

the derivation of eq. (13.15) was sketched in section 8.

Combining eqs. (13.12–13.15), we get

$$\begin{aligned} i\langle 0|\mathrm{T}\varphi(x)\varphi(y)|0\rangle = \int \frac{d^dk}{(2\pi)^d}\, e^{ik(x-y)} \Bigg[&\frac{1}{k^2+m^2-i\epsilon} \\ &+ \int_{4m^2}^{\infty} ds\, \rho(s)\, \frac{1}{k^2+\ s\ -i\epsilon} \Bigg]. \end{aligned} \tag{13.16}$$

Comparing eqs. (13.1), (13.6), and (13.16), we see that

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} ds\, \rho(s)\, \frac{1}{k^2 + s - i\epsilon}\,. \tag{13.17}$$

This is the *Lehmann-Kallen form* of the exact momentum-space propagator $\tilde{\mathbf{\Delta}}(k^2)$.

We note in particular that $\tilde{\mathbf{\Delta}}(k^2)$ *has an isolated pole at* $k^2 = -m^2$ *with residue one*, just like the propagator in free-field theory.

14 Loop Corrections to the Propagator

Prerequisite: 10, 12, 13

In section 10, we wrote the exact propagator as

$$\tfrac{1}{i}\mathbf{\Delta}(x_1 - x_2) \equiv \langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle = \delta_1\delta_2 iW(J)\Big|_{J=0}\,, \tag{14.1}$$

where iW(J) is the sum of connected diagrams, and δ_i acts to remove a source from a diagram and label the corresponding propagator endpoint x_i.

In φ^3 theory, the $O(g^2)$ corrections to $\frac{1}{i}\tilde{\mathbf{\Delta}}(x_1 - x_2)$ come from the diagrams of fig. (14.1).

To compute them, it is simplest to work directly in momentum space, following the Feynman rules of section 10.

An appropriate assignment of momenta to the lines is shown in fig. (14.1).

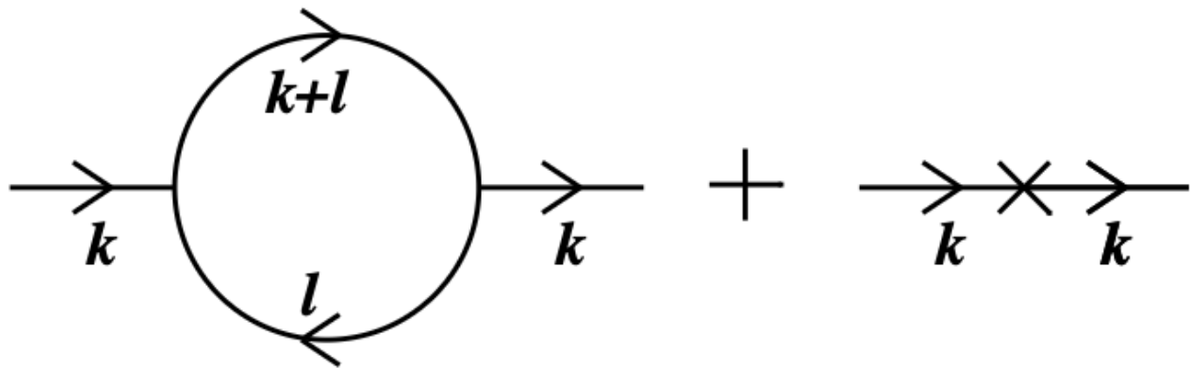


Figure 14.1: The O(g^2) corrections to the propagator.

We then have

$$\tfrac{1}{i}\tilde{\mathbf{\Delta}}(k^2) = \tfrac{1}{i}\tilde{\Delta}(k^2) + \tfrac{1}{i}\tilde{\Delta}(k^2)\left[i\Pi(k^2)\right]\tfrac{1}{i}\tilde{\Delta}(k^2) + O(g^4)\ , \tag{14.2}$$

where

$$\tilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} \tag{14.3}$$

is the free-field propagator, and

$$\begin{aligned} i\Pi(k^2) = \tfrac{1}{2}(ig)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^d\ell}{(2\pi)^d}\, \tilde{\Delta}((\ell+k)^2)\tilde{\Delta}(\ell^2) \\ - i(Ak^2 + Bm^2) + O(g^4) \end{aligned} \tag{14.4}$$

is the self-energy.

Here we have written the integral appropriate for d spacetime dimensions; for now we will leave d arbitrary, but later we will want to focus on d= 6, where the coupling g is dimensionless.

In the first term in eq. (14.4), the factor of one-half is the symmetry factor associated with exchanging the top and bottom semicircular propagators.

Also, we have written the vertex factor as ig rather than $iZ_g\, g$ because we expect $Z_g = 1 + O(g^2)$, and so the $Z_g - 1$ contribution can be lumped into the $O(g^4)$ term.

In the second term, $A = Z_\varphi - 1$ and $B = Z_m - 1$ are both expected to be $O(g^2)$.

It will prove convenient to define $\Pi(k^2)$ to all orders via the geometric series

$$\begin{aligned}\tfrac{1}{i}\tilde{\mathbf{\Delta}}(k^2) = \tfrac{1}{i}\tilde{\Delta}(k^2) &+ \tfrac{1}{i}\tilde{\Delta}(k^2)\Big[i\Pi(k^2)\Big]\tfrac{1}{i}\tilde{\Delta}(k^2) \\ &+ \tfrac{1}{i}\tilde{\Delta}(k^2)\Big[i\Pi(k^2)\Big]\tfrac{1}{i}\tilde{\Delta}(k^2)\Big[i\Pi(k^2)\Big]\tfrac{1}{i}\tilde{\Delta}(k^2) \\ &+ \ldots .\end{aligned} \tag{14.5}$$

This is illustrated in fig. (14.2).

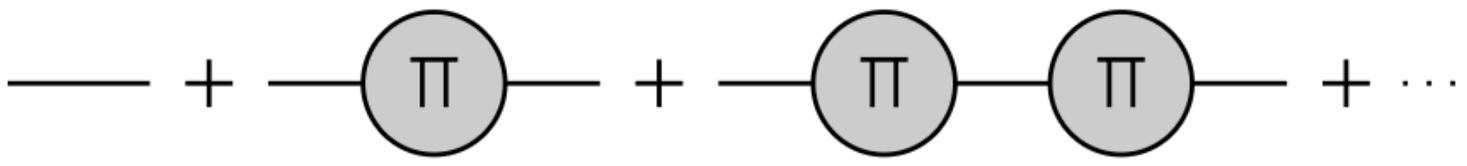


Figure 14.2: The geometric series for the exact propagator.

The sum in eq. (14.5) will include all the diagrams that contribute to $\tilde{\mathbf{\Delta}}(k^2)$ if we take $i\Pi(k^2)$ to be given by the sum of all diagrams that are one-particle irreducible, or 1PI for short.

A diagram is 1PI if it is still connected after any one line is cut.

The 1PI diagrams that make an $O(g^4)$ contribution to $i\Pi(k^2)$ are shown in fig. (14.3).

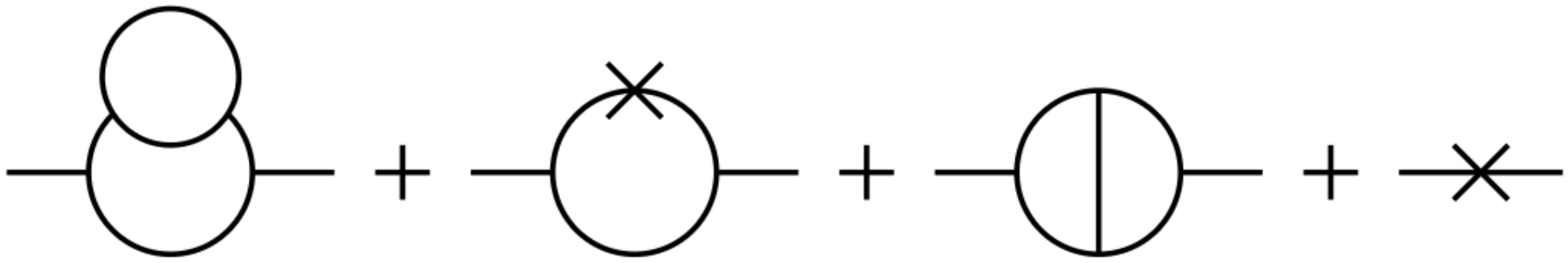


Figure 14.3: The $O(g^4)$ contributions to $i\Pi(k^2)$.

When writing down the value of one of these diagrams, we omit the two external propagators.

If we sum up the series in eq. (14.5), we get

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)} \ . \tag{14.6}$$

In section 13, we learned that the exact propagator has a pole at $k^2 = -m^2$ with residue one.

This is consistent with eq. (14.6) if and only if

$$\Pi(-m^2) = 0 \ , \tag{14.7}$$

$$\Pi'(-m^2) = 0 \ , \tag{14.8}$$

where the prime denotes a derivative with respect to k^2.

We will use eqs. (14.7) and (14.8) to fix the values of A and B.

Next we turn to the evaluation of the $O(g^2)$ contribution to $i\Pi(k^2)$ in eq. (14.4).

We have the immediate problem that the integral on the right-hand side diverges at large ℓ for $d \geq 4$.

We faced a similar situation in section 9 when we evaluated the lowest-order tadpole diagram.

There we˜introduced an ultraviolet cutoff Λ that modified the behavior of $\tilde{\mathbf{\Delta}}(\ell^2)$ at large ℓ^2.

Here, for now, we will simply restrict our attention to $d < 4$, where the integral in eq. (14.4) is finite.

Later we will see what we can say about larger values of d.

We will evaluate the integral in eq. (14.4) with a series of tricks.

We first use *Feynman's formula* to combine denominators,

$$\frac{1}{A_1 \dots A_n} = \int dF_n \, (x_1 A_1 + \dots + x_n A_n)^{-n} \, , \tag{14.9}$$

where the integration measure over the Feynman parameters x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \dots dx_n \, \delta(x_1 + \dots + x_n - 1) \; . \tag{14.10}$$

This measure is normalized so that

$$\int dF_n \, 1 = 1 \; . \tag{14.11}$$

In the case at hand, we have

$$\begin{aligned}
\tilde{\Delta}((k+\ell)^2)\tilde{\Delta}(\ell^2) &= \frac{1}{(\ell^2 + m^2)((\ell+k)^2 + m^2)} \\
&= \int_0^1 dx \left[x((\ell+k)^2 + m^2) + (1-x)(\ell^2 + m^2) \right]^{-2} \\
&= \int_0^1 dx \left[\ell^2 + 2x\ell\cdot k + xk^2 + m^2 \right]^{-2} \\
&= \int_0^1 dx \left[(\ell + xk)^2 + x(1-x)k^2 + m^2 \right]^{-2} \\
&= \int_0^1 dx \left[q^2 + D \right]^{-2} ,
\end{aligned} \tag{14.12}$$

where we have suppressed the $i\epsilon$'s for notational convenience; they can be restored via the replacement $m^2 \to m^2 - i\epsilon$.

In the last line we have defined

$$q \equiv \ell + xk \tag{14.13}$$

and

$$D \equiv x(1-x)k^2 + m^2 \ . \tag{14.14}$$

We then change the integration variable in eq. (14.4) from ℓ to q; the jacobian is trivial, and we have $d^d\ell = d^dq$.

Next, think of the integral over q^0 from $-\infty$ to $+\infty$ as a contour integral in the complex q^0 plane.

If the integrand vanishes fast enough as $|q^0| \to \infty$, we can rotate this contour clockwise by 90°, as shown in fig. (14.4), so that it runs from $-i\infty$ to $+i\infty$.

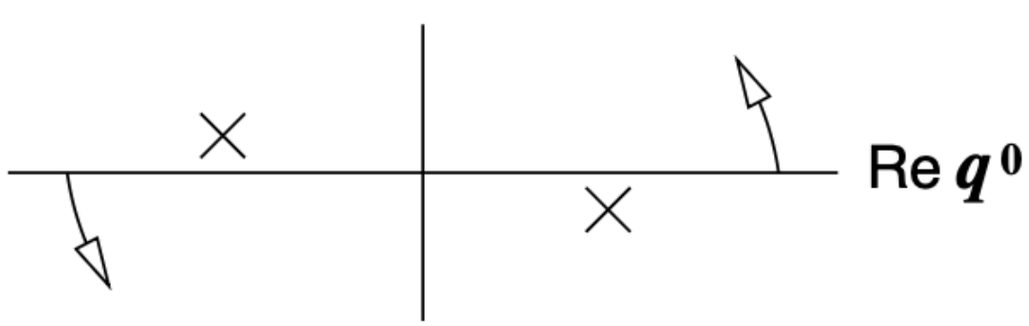


Figure 14.4: The q^0 integration contour along the real axis can be rotated to the imaginary axis without passing through the poles at $q^0 = -\omega + i\epsilon$ and $q^0 = +\omega - i\epsilon$.

In making this *Wick rotation*, the contour does not pass over any poles.

The $i\epsilon$'s are needed to make this statement unambiguous.

Thus the value of the integral is unchanged.

It is now convenient to define a euclidean d-dimensional vector $\bar{q}$ via $q^0 = i\bar{q}_d$ and $q_j = \bar{q}_j$; then $q^2 = \bar{q}^2$, where

$$\bar{q}^2 = \bar{q}_1^2 + \ldots + \bar{q}_d^2 \ . \tag{14.15}$$

Also, $d^dq = id^d\bar{q}$.

Therefore, in general,

$$\int d^dq \ f(q^2 - i\epsilon) = i \int d^d\bar{q} \ f(\bar{q}^2) \tag{14.16}$$

as long as $f(\bar{q}^2) \to 0$ faster than $1/\bar{q}^d$ as $\bar{q} \to \infty$.

Now we can write

$$\Pi(k^2) = \tfrac{1}{2}g^2 I(k^2) - Ak^2 - Bm^2 + O(g^4) \ , \tag{14.17}$$

where

$$I(k^2) \equiv \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} \ . \tag{14.18}$$

It is now straightforward to evaluate the d-dimensional integral over $\vec{q}$ in spherical coordinates.

Before we perform this calculation, however, let us introduce another trick, one that can simplify the task of fixing A and B through the imposition of eqs. (14.7) and (14.8).

Here is the trick: differentiate $\Pi(k^2)$ twice with respect to k^2 to get

$$\Pi''(k^2) = \tfrac{1}{2} g^2 I''(k^2) + O(g^4) \ , \tag{14.19}$$

where, from eqs. (14.18) and (14.14),

$$I''(k^2) = \int_0^1 dx \; 6x^2(1-x)^2 \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^4} \ . \tag{14.20}$$

Then, after we evaluate these integrals, we can get $\Pi(k^2)$ by integrating with respect to k^2, subject to the boundary conditions of eqs. (14.7) and (14.8).

In this way we can construct $\Pi(k^2)$ without ever explicitly computing A and B.

Notice that this trick does something else for us as well.

The integral over $\bar{q}$ in eq. (14.20) is finite for any $d < 8$, whereas the original integral in eq. (14.18) is finite only for $d < 4$.

This expanded range of d now includes the value of greatest interest, $d = 6$.

How did this happen?

We can gain some insight by making a Taylor expansion of $\Pi(k^2)$ about $k^2 = -m^2$:

$$
\begin{aligned}
\Pi(k^2) = & \left[\tfrac{1}{2} g^2 I(-m^2) + (A - B)m^2 \right] \\
& + \left[\tfrac{1}{2} g^2 I'(-m^2) + A \right] (k^2 + m^2) \\
& + \tfrac{1}{2!} \left[\tfrac{1}{2} g^2 I''(-m^2) \right] (k^2 + m^2)^2 + \ldots \\
& + O(g^4) \, .
\end{aligned}
\tag{14.21}
$$

From eqs. (14.18) and (14.14), it is straightforward to see that $I(-m^2)$ is divergent for $d \geq 4$, $I'(-m^2)$ is divergent for $d \geq 6$, and, in general, $I^{(n)}(-m^2)$ is divergent for $d \geq 4 + 2n$.

We can use the $O(g^2)$ terms in A and B to cancel off the $\frac{1}{2}\ g^2 I(-m^2)$ and $\frac{1}{2}\ g^2 I'(-m^2)$ terms in $\Pi(k^2)$, whether or not they are divergent.

But if we are to end up with a finite $\Pi(k^2)$, all of the remaining terms must be finite, since we have no more free parameters left to adjust.

This is the case for $d < 8$.

Of course, for $4 \leq d < 8$, the values of A and B (and hence the lagrangian coeffcients $Z = 1 + A$ and $Z_m = 1 + B$) are formally infinite, and this may be disturbing.

However, these coefficients are not directly measurable, and so need not obey our preconceptions about their magnitudes.

Also, it is important to remember that A and B each includes a factor of g^2; this means that we can expand in powers of A and B as part of our general expansion in powers of g.

When we compute $\Pi(k^2)$ (which enters into observable cross sections), all the formally infinite numbers cancel in a well-defined way, provided $d < 8$.

For $d \geq 8$, this procedure breaks down, and we do not obtain a finite expression for $\Pi(k^2)$.

In this case, we say that the theory is nonrenormalizable.

We will discuss the criteria for renormalizability of a theory in detail in section 18.

It turns out that φ^3 theory is renormalizable for $d \leq 6$. (The problem with $6 < d < 8$ arises from higher-order corrections, as we will see in section 18.)

Now let us return to the calculation of $\Pi(k^2)$.

Rather than using the trick of first computing $\Pi''(k^2)$, we will instead evaluate $\Pi(k^2)$ directly from eq. (14.18) as a function of d for $d < 4$.

Then we will analytically continue the result to arbitrary d.

This procedure is known as *dimensional regularization*.

Then we will fix A and B by imposing eqs. (14.7) and (14.8), and finally take the limit $d \to 6$.

We could just as well use the method of section 9.

Making the replacement

$$\tilde{\Delta}(p^2) \to \frac{1}{p^2 + m^2 - i\epsilon} \, \frac{\Lambda^2}{p^2 + \Lambda^2 - i\epsilon} \,, \tag{14.22}$$

where Λ is the ultraviolet cutoff, renders the $O(g^2)$ term in $\Pi(k^2)$ finite for $d < 8$; This procedure is known as *Pauli–Villars regularization*.

We then evaluate $\Pi(k^2)$ as a function of Λ, fix A and B by imposing eqs. (14.7) and (14.8), and take the $\Lambda \to \infty$ limit.

Calculations with Pauli-Villars regularization are generally much more cumbersome than they are with dimensional regularization.

However, the final result for $\Pi(k^2)$ is the same.

Eq. (14.21) demonstrates that any regularization scheme will give the same result for $d < 8$, at least as long as it preserves the Lorentz invariance of the integrals.

We therefore turn to the evaluation of $I(k^2)$, eq. (14.18).

The angular part of the integral over $\tilde{q}$ yields the area Ω_d of the unit sphere in d dimensions, which is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{1}{2}d)} \; ; \tag{14.23}$$

this is most easily verified by computing the gaussian integral $\int \mathrm{d}^d \tilde{q}\; \mathrm{e}^{-q2}$ in both cartesian and spherical coordinates.

Here $\Gamma(x)$ is the Euler gamma function; for a nonnegative integer n and small x,

$$\Gamma(n+1) = n! \; , \tag{14.24}$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{n!2^n}\sqrt{\pi} \; , \tag{14.25}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum\nolimits_{k=1}^{n} k^{-1} + O(x)\right] \; , \tag{14.26}$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant.

The radial part of the $\tilde{q}$ integral can also be evaluated in terms of gamma functions.

The overall result (generalized slightly) is

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)} \; . \tag{14.27}$$

We will make frequent use of this formula throughout ese notes.

In the case of interest, eq. (14.18), we have a = 0 and b = 2.

There is one more complication to deal with.

Recall that we want to focus on d= 6 because in that case g is dimensionless.

However, for general d, g has mass dimension ε/2, where

$$\varepsilon \equiv 6 - d \ . \tag{14.28}$$

To account for this, we introduce a new parameter $\tilde{\mu}$ with dimensions of mass, and make the replacement

$$g \to g\tilde{\mu}^{\varepsilon/2} \ . \tag{14.29}$$

In this way g remains dimensionless for all ε.

Of course, $\tilde{\mu}$ is not an actual parameter of the d = 6 theory.

Therefore, nothing measurable (like a cross section) can depend on it.

This seemingly innocuous statement is actually quite powerful, and will eventually serve as the foundation of the renormalization group.

We now return to eq. (14.18), use eq. (14.26), and set d = 6 – ε; we get

$$I(k^2) = \frac{\Gamma(-1+\frac{\varepsilon}{2})}{(4\pi)^3} \int_0^1 dx\, D \left(\frac{4\pi}{D}\right)^{\varepsilon/2} \quad . \tag{14.30}$$

Hence, with the substitution of eq. (14.29), and defining

$$\alpha \equiv \frac{g^2}{(4\pi)^3} \tag{14.31}$$

for notational convenience, we have

$$\begin{aligned} \Pi(k^2) = {} & \tfrac{1}{2}\alpha\, \Gamma(-1+\tfrac{\varepsilon}{2}) \int_0^1 dx\, D \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon/2} \\ & - Ak^2 - Bm^2 + O(\alpha^2) \ . \end{aligned} \tag{14.32}$$

Now we can take the ε⟶ 0 limit, using eq. (14.26) and

$$A^{\varepsilon/2} = 1 + \tfrac{\varepsilon}{2} \ln A + O(\varepsilon^2) \ . \tag{14.33}$$

The result is

$$\begin{aligned} \Pi(k^2) = {} & -\tfrac{1}{2}\alpha \left[\left(\tfrac{2}{\varepsilon} + 1\right)\left(\tfrac{1}{6}k^2 + m^2\right) + \int_0^1 dx\, D \ln\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right)\right] \\ & - Ak^2 - Bm^2 + O(\alpha^2) \ . \end{aligned} \tag{14.34}$$

Here we have used $\int_0^1 dx\, D = 16\, k^2 + m^2$.

It is now convenient to define

$$\mu \equiv \sqrt{4\pi}\, e^{-\gamma/2}\, \tilde{\mu}\ , \tag{14.35}$$

and rearrange things to get

$$\begin{aligned}\Pi(k^2) = {} & \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/m^2) \\ & - \left\{ \tfrac{1}{6}\alpha \left[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{2} \right] + A \right\} k^2 \\ & - \left\{ \ \alpha \left[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{2} \right] + B \right\} m^2 + O(\alpha^2)\ .\end{aligned} \tag{14.36}$$

If we take A and B to have the form

$$A = -\tfrac{1}{6}\alpha \left[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{2} + \kappa_A \right] + O(\alpha^2)\ , \tag{14.37}$$

$$B = -\ \alpha \left[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \tfrac{1}{2} + \kappa_B \right] + O(\alpha^2)\ , \tag{14.38}$$

where κ_A and κ_B are purely numerical constants, then we get

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/m^2) + \alpha \left(\tfrac{1}{6}\kappa_A k^2 + \kappa_B m^2 \right) + O(\alpha^2)\ . \tag{14.39}$$

Thus this choice of A and B renders $\Pi(k^2)$ finite and independent of μ, as required.

To fix κ_A and κ_B, we must still impose the conditions $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$.

The easiest way to do this is to first note that, schematically,

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln D + \text{linear in } k^2 \text{ and } m^2 + O(\alpha^2)\ . \tag{14.40}$$

We can then impose $\Pi(-m^2) = 0$ via

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/D_0) + \text{linear in } (k^2 + m^2) + O(\alpha^2)\ . \tag{14.41}$$

where

$$D_0 \equiv D\Big|_{k^2=-m^2} = [1-x(1-x)]m^2\ . \tag{14.42}$$

Now it is straightforward to differentiate eq. (14.41) with respect to k^2, and find that $\Pi'(-m^2)$ vanishes for

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/D_0) - \tfrac{1}{12}\alpha(k^2 + m^2) + O(\alpha^2)\ . \tag{14.43}$$

The integral over x can be done in closed form; the result is

$$\Pi(k^2) = \tfrac{1}{12}\alpha\Big[c_1 k^2 + c_2 m^2 + 2k^2 f(r)\Big] + O(\alpha^2)\ , \tag{14.44}$$

where $c_1 = 3-\pi\sqrt{3}$, $c_2 = 3-2\pi\sqrt{3}$, and

$$f(r) = r^3 \tanh^{-1}(1/r) \,, \tag{14.45}$$

$$r = (1 + 4m^2/k^2)^{1/2} \,. \tag{14.46}$$

There is a branch point at $k^2 = -4m^2$, and $\Pi(k^2)$ acquires an imaginary part for $k^2 < -4m^2$; we will discuss this further in the next section.

We can write the exact propagator as

$$\tilde{\mathbf{\Delta}}(k^2) = \left(\frac{1}{1 - \Pi(k^2)/(k^2 + m^2)} \right) \frac{1}{k^2 + m^2 - i\epsilon} \,. \tag{14.47}$$

In fig. (14.5), we plot the real and imaginary parts of $\Pi(k^2)/(k^2 + m^2)$ in units of α.

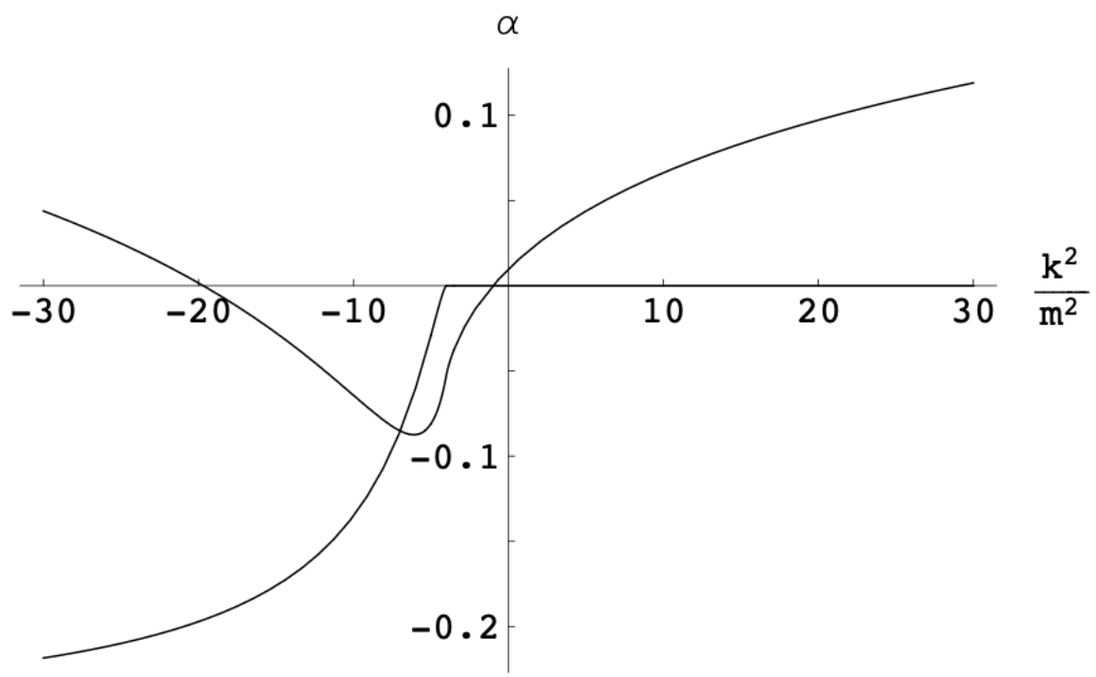


Figure 14.5: The real and imaginary parts of $\Pi(k^2)/(k^2 + m^2)$ in units of α.

We see that its values are quite modest for the plotted range. For much larger values of $|k^2|$, we have

$$\frac{\Pi(k^2)}{k^2+m^2} \simeq \tfrac{1}{12}\alpha\Big[\ln(k^2/m^2)+c_1\Big]+O(\alpha^2)\ . \tag{14.48}$$

If we had kept track of the $i\epsilon$'s, k^2 would be $k^2 - i\epsilon$; when k^2 is negative, we have
$\ln(k^2 - i\epsilon) = \ln |k^2| - i\pi$.

The imaginary part of $\Pi(k^2)/(k^2 + m^2)$ therefore approaches the asymptotic value of
$-\frac{1}{12}\pi\alpha + O(\alpha^2)$ when k^2 is large and negative.

The real part of $\Pi(k^2)/(k^2 + m^2)$, however, continues to increase logarithmically with $|k^2|$ when $|k^2|$ is large.

We will begin to address the meaning of this in section 26.

15 The One-Loop Correction in Lehmann-Kallen Form

Prerequisite: 14

In section 13, we found that the exact propagator could be written in Lehmann-Kallen form as

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} ds\,\rho(s)\,\frac{1}{k^2+s-i\epsilon}\ , \tag{15.1}$$

where the *spectral density* $\rho(s)$ is real and nonnegative.

In section 14, on the other hand, we found that the exact propagator could be written as

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)} \ , \tag{15.2}$$

and that, to $O(g^2)$ in φ^3 theory in six dimensions,

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/D_0) - \tfrac{1}{12}\alpha(k^2 + m^2) + O(\alpha^2) \ , \tag{15.3}$$

where

$$\alpha \equiv g^2/(4\pi)^3 \ , \tag{15.4}$$

$$D = x(1-x)k^2 + m^2 - i\epsilon \ , \tag{15.5}$$

$$D_0 = [1-x(1-x)]m^2 \ . \tag{15.6}$$

In this section, we will attempt to reconcile eqs. (15.2) and (15.3) with eq. (15.1).

Let us begin by considering the imaginary part of the propagator.

We will always take k^2 and m^2 to be real, and explicitly include the appropriate factors of $i\epsilon$ whenever they are needed.

We can use eq. (15.1) and the identity

$$\frac{1}{x - i\epsilon} = \frac{x}{x^2 + \epsilon^2} + \frac{i\epsilon}{x^2 + \epsilon^2}$$
$$= P\,\frac{1}{x} + i\pi\delta(x)\ , \tag{15.7}$$

where P means the principal part, to write

$$\mathrm{Im}\,\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2) + \int_{4m^2}^{\infty} ds\,\rho(s)\,\pi\delta(k^2 + s)$$
$$= \pi\delta(k^2 + m^2) + \pi\rho(-k^2)\ , \tag{15.8}$$

where $\rho(s) \equiv 0$ for $s < 4m^2$.

Thus we have

$$\pi\rho(s) = \mathrm{Im}\,\tilde{\mathbf{\Delta}}(-s) \quad \text{for} \quad s \geq 4m^2\ . \tag{15.9}$$

Let us now suppose that Im $\Pi(k^2) = 0$ for some range of k^2.

In section 14, we saw that the $O(\alpha)$ contribution to $\Pi(k^2)$ is purely real for $k^2 > -4m^2$.

Then, from eqs. (15.2) and (15.7), we get

$$\mathrm{Im}\,\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2 - \Pi(k^2)) \quad \text{for} \quad \mathrm{Im}\,\Pi(k^2) = 0\ . \tag{15.10}$$

From $\Pi(-m^2) = 0$, we know that the argument of the delta function vanishes at $k^2 = -m^2$, and from $\Pi'(-m^2) = 0$, we know that the derivative of this argument with respect to k^2 equals one at $k^2 = -m^2$.

Therefore

$$\mathrm{Im}\,\tilde{\mathbf{\Delta}}(k^2) = \pi\delta(k^2 + m^2) \quad \text{for} \quad \mathrm{Im}\,\Pi(k^2) = 0\ . \tag{15.11}$$

Comparing this with eq. (15.8), we see that $\rho(-k^2) = 0$ if $\mathrm{Im}\Pi(k^2) = 0$.

Now suppose $\mathrm{Im}\Pi(k^2)$ is not zero for some range of k^2.

In section 14, we saw that the $O(\alpha)$ contribution to $\Pi(k^2)$ has a nonzero imaginary part for $k^2 < -4m^2$.

Then we can ignore the $i\epsilon$ in eq. (15.2), and

$$\mathrm{Im}\,\tilde{\mathbf{\Delta}}(k^2) = \frac{\mathrm{Im}\,\Pi(k^2)}{(k^2 + m^2 + \mathrm{Re}\,\Pi(k^2))^2 + (\mathrm{Im}\,\Pi(k^2))^2} \quad \text{for} \quad \mathrm{Im}\,\Pi(k^2) \neq 0\ . \tag{15.12}$$

Comparing this with eq. (15.8) we see that

$$\pi\rho(s) = \frac{\mathrm{Im}\,\Pi(-s)}{(-s + m^2 + \mathrm{Re}\,\Pi(-s))^2 + (\mathrm{Im}\,\Pi(-s))^2}\ . \tag{15.13}$$

Since we know $\rho(s) = 0$ for $s < 4m^2$, this tells us that we must also have $\mathrm{Im}\,\Pi(-s) = 0$ for $s < 4m^2$, or equivalently $\mathrm{Im}\,\Pi(k^2) = 0$ for $k^2 > -4m^2$.

This is just what we found for the $O(\alpha)$ contribution to $\Pi(k^2)$ in section 14.

We can also see this directly from eq. (15.3), without doing the integral over x.

The integrand in this formula is real as long as the argument of the logarithm is real and positive.

From eq. (15.5), we see that D is real and positive if and only if $x(1-x)k^2 > -m^2$.

The maximum value of $x(1-x)$ is 1/4, and so the argument of the logarithm is real and positive for the whole integration range $0 \le x \le 1$ if and only if $k^2 > -4m^2$.

In this regime, $\mathrm{Im}\Pi(k^2) = 0$.

On the other hand, for $k^2 < -4m^2$, the argument of the logarithm becomes negative for some of the integration range, and so $\mathrm{Im}\Pi(k^2) \neq 0$ for $k^2 < -4m^2$.

This is exactly what we need to reconcile eqs. (15.2) and (15.3) with eq. (15.1).

16 Loop Corrections to the Vertex

Prerequisite: 14

Consider the $O(g^3)$ diagram of fig. (16.1), which corrects the ϕ^3 vertex.

In this section we will evaluate this diagram.

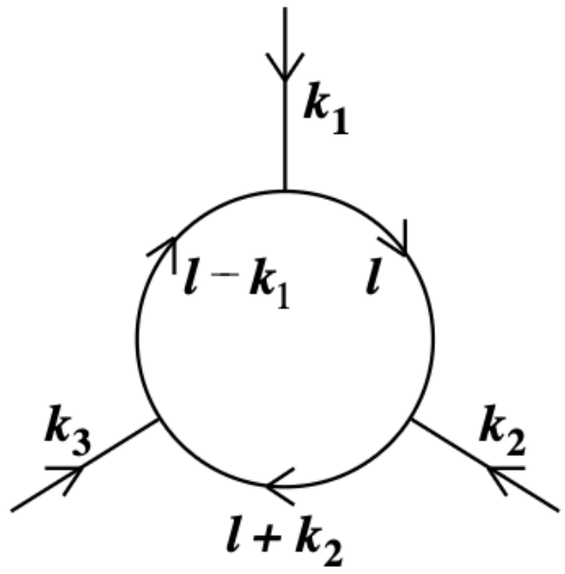


Figure 16.1: The $O(g^3)$ correction to the vertex $i\mathbf{V}_3(k_1, k_2, k_3)$.

We can define an exact three-point vertex function $i\mathbf{V}_3(k_1, k_2, k_3)$ as the sum of one-particle irreducible diagrams with three external lines carrying momenta k_1, k_2 and k_3, all incoming, with $k_1 + k_2 + k_3 = 0$ by momentum conservation.

In adopting this convention, we allow k^0_i to have either sign; if k_i is the momentum of an external particle, then the sign of k^0_i is positive if the particle is incoming, and negative if it is outgoing.

The original vertex $iZ_g g$ is the first term in this sum, and the diagram of fig. (16.1) is the second. Thus we have

$$i\mathbf{V}_3(k_1,k_2,k_3) = iZ_g g + (ig)^3 \left(\tfrac{1}{i}\right)^3 \int \frac{d^d\ell}{(2\pi)^d}\, \tilde{\Delta}((\ell-k_1)^2)\tilde{\Delta}((\ell+k_2)^2)\tilde{\Delta}(\ell^2) \qquad (16.1)$$

$$+ O(g^5)\ .$$

where

$$\int dF_3 = 2\int_0^1 dx_1\, dx_2\, dx_3\; \delta(x_1{+}x_2{+}x_3{-}1)\;. \tag{16.3}$$

We manipulate the right-hand side of eq. (16.2) to get

$$\begin{aligned}
&\tilde{\Delta}((\ell{-}k_1)^2)\tilde{\Delta}((\ell{+}k_2)^2)\tilde{\Delta}(\ell^2) \\
&\quad = \int dF_3 \Big[\, \ell^2 - 2\ell{\cdot}(x_1k_1 - x_2k_2) + x_1k_1^2 + x_2k_2^2 + m^2 \,\Big]^{-3} \\
&\quad = \int dF_3 \Big[\, (\ell - x_1k_1 + x_2k_2)^2 + x_1(1{-}x_1)k_1^2 + x_2(1{-}x_2)k_2^2 \\
&\qquad\qquad + 2x_1x_2k_1{\cdot}k_2 + m^2 \,\Big]^{-3} \\
&\quad = \int dF_3 \Big[\, q^2 + D \,\Big]^{-3}\;.
\end{aligned} \tag{16.4}$$

In the last line, we have defined $q \equiv \ell - x_1k_1 + x_2k_2$, and

$$\begin{aligned}
D &\equiv x_1(1{-}x_1)k_1^2 + x_2(1{-}x_2)k_2^2 + 2x_1x_2k_1{\cdot}k_2 + m^2 \\
&= x_3x_1k_1^2 + x_3x_2k_2^2 + x_1x_2k_3^2 + m^2\;,
\end{aligned} \tag{16.5}$$

where we used $k^2_3 = (k_1 + k_2)^2$ and $x_1 + x_2 + x_3 = 1$ to simplify the second line.

After making a Wick rotation of the q^0 contour, we have

$$\mathbf{V}_3(k_1,k_2,k_3)/g = Z_g + g^2 \int dF_3 \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} + O(g^4)\ , \tag{16.6}$$

where $\bar{q}$ is a euclidean vector.

This integral diverges for $d \geq 6$.

We therefore evaluate it for $d < 6$, using the general formula from section 14; the result is

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} = \frac{\Gamma(3-\frac{1}{2}d)}{2(4\pi)^{d/2}} D^{-(3-d/2)}\ . \tag{16.7}$$

Now we set $d = 6-\varepsilon$.

To keep g dimensionless, we make the replacement $g \to g\,\tilde{\mu}^{\varepsilon/2}$.

Then we have

$$\mathbf{V}_3(k_1,k_2,k_3)/g = Z_g + \tfrac{1}{2}\alpha\,\Gamma(\tfrac{\varepsilon}{2}) \int dF_3 \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon/2} + O(\alpha^2)\ , \tag{16.8}$$

where $\alpha = g^2/(4\pi)^3$.

Now we can take the $\varepsilon \to 0$ limit.

The result is

$$\mathbf{V}_3(k_1,k_2,k_3)/g = Z_g + \tfrac{1}{2}\alpha\left[\frac{2}{\varepsilon} + \int dF_3\ \ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma}D}\right)\right] + O(\alpha^2)\ , \tag{16.9}$$

where we have used $\int dF_3 = 1$.

We now let $\mu^2 = 4\pi\, e^{-\gamma}\, \tilde{\mu}^2$, set

$$Z_g = 1 + C\ , \tag{16.10}$$

and rearrange to get

$$\begin{aligned}\mathbf{V}_3(k_1,k_2,k_3)/g = 1 &+ \left\{\alpha\left[\tfrac{1}{\varepsilon} + \ln(\mu/m)\right] + C\right\} \\ &- \tfrac{1}{2}\alpha\int dF_3\ \ln(D/m^2) \\ &+ O(\alpha^2)\ .\end{aligned} \tag{16.11}$$

If we take C to have the form

$$C = -\alpha\left[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \kappa_C\right] + O(\alpha^2)\ , \tag{16.12}$$

where κ_C is a purely numerical constant, we get

$$\mathbf{V}_3(k_1,k_2,k_3)/g = 1 - \tfrac{1}{2}\alpha\int dF_3\ \ln(D/m^2) - \kappa_C\alpha + O(\alpha^2)\ . \tag{16.13}$$

Thus this choice of C renders $\mathbf{V}_3(k_1, k_2, k_3)$ finite and independent of μ, as required.

We now need a condition, analogous to $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$, to fix the value of κ_C.

These conditions on $\Pi(k^2)$ were mandated by known properties of the exact propagator, but there is nothing directly comparable for the vertex.

Different choices of κ_C correspond to different definitions of the coupling g.

This is because, in order to measure g, we would measure a cross section that depends on g; these cross sections also depend on κ_C.

Thus we can use any value for κ_C that we might fancy, as long as we all agree on that value when we compare our calculations with experimental measurements.

It is then most convenient to simply set $\kappa_C = 0$.

This corresponds to the condition

$$\mathbf{V}_3(0, 0, 0) = g \ . \tag{16.14}$$

This condition can then also be used to fix the higher-order (in g) terms in Z_g.

The integrals over the Feynman parameters in eq. (16.13) cannot be done in closed form, but it is easy to see that if (for example) $|k^2_1| \gg m^2$, then

$$\mathbf{V}_3(k_1,k_2,k_3)/g \simeq 1 - \tfrac{1}{2}\alpha\Big[\ln(k_1^2/m^2) + O(1)\Big] + O(\alpha^2)\ . \tag{16.15}$$

Thus the magnitude of the one-loop correction to the vertex function increases logarithmically with $|k^2_i|$ when $|k^2_i| \gg m^2$.

This is the same behavior that we found for $\Pi(k^2)/(k^2 + m^2)$ in section 14.

17 Other 1PI Vertices

Prerequisite: 16

In section 16, we defined the three-point vertex function $i\mathbf{V}_3(k_1, k_2, k_3)$ as the sum of all one-particle irreducible diagrams with three external lines, with the external propagators removed.

We can extend this definition to the n-point vertex $i\mathbf{V}_n(k_1, \ldots. k_n)$.

There are two key differences between $\mathbf{V}_{n>3}$ and $\mathbf{V}_3$ in φ^3 theory.

The first is that there is no tree-level contribution to $\mathbf{V}_{n>3}$.

The second is that the one-loop contribution to $\mathbf{V}_{n>3}$ is finite for $d < 2n$.

In particular, the one-loop contribution to $\mathbf{V}_{n>3}$ is finite for $d = 6$.

Let us see how this works for the case n = 4.

We treat all the external momenta as incoming, so that $k_1 + k_2 + k_3 + k_4 = 0$.

One of the three contributing one-loop diagrams is shown in fig. (17.1); in this diagram, the k_3 vertex is opposite to the k_1 vertex.

Two other inequivalent diagrams are then obtained by swapping $k_3 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$.

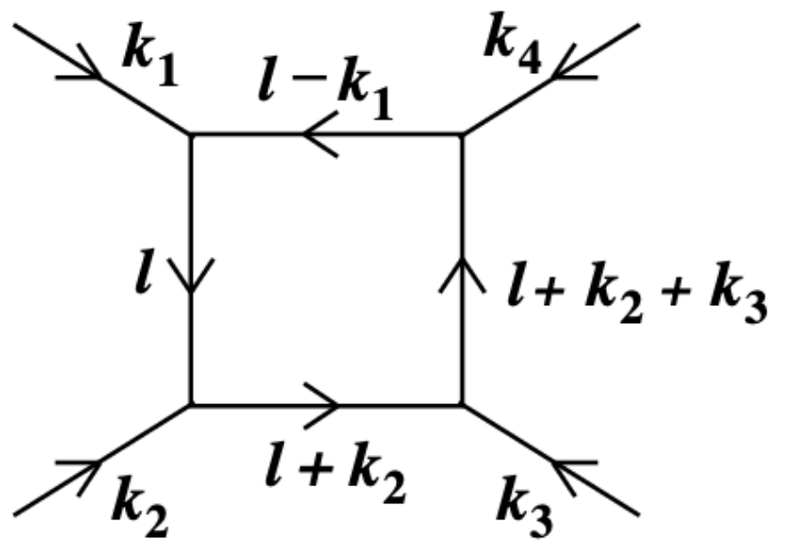


Figure 17.1: One of the three one-loop Feynman diagrams contributing to the four-point vertex $i\mathbf{V}_4(k_1, k_2, k_3, k_4)$; the other two are obtained by swapping $k_3 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$.

We then have

$$
\begin{aligned}
i\mathbf{V}_4 = g^4 \int \frac{d^6\ell}{(2\pi)^6} \tilde{\Delta}((\ell - k_1)^2)\tilde{\Delta}((\ell + k_2)^2)\tilde{\Delta}((\ell + k_2 + k_3)^2)\tilde{\Delta}(\ell^2) \\
+ (k_3 \leftrightarrow k_2) + (k_3 \leftrightarrow k_4) \\
+ O(g^6) \, .
\end{aligned}
\tag{17.1}
$$

Feynman's formula gives

$$\begin{aligned}
&\tilde{\Delta}((\ell-k_1)^2)\tilde{\Delta}((\ell+k_2)^2)\tilde{\Delta}((\ell+k_2+k_3)^2)\tilde{\Delta}(\ell^2) \\
&\quad = \int dF_4 \left[x_1(\ell-k_1)^2 + x_2(\ell+k_2)^2 + x_3(\ell+k_2+k_3)^2 + x_4\ell^2 + m^2 \right]^{-4} \\
&\quad = \int dF_4 \left[q^2 + D_{1234} \right]^{-4} ,
\end{aligned} \tag{17.2}$$

where $q = \ell - x_1k_1 + x_2k_2 + x_3(k_2+k_3)$ and, after making repeated use of $x_1+x_2+x_3+x_4 = 1$ and $k_1+k_2+k_3+k_4 = 0$,

$$\begin{aligned}
D_{1234} = {} & x_1x_4k_1^2 + x_2x_4k_2^2 + x_2x_3k_3^2 + x_1x_3k_4^2 \\
& + x_1x_2(k_1+k_2)^2 + x_3x_4(k_2+k_3)^2 + m^2 \ .
\end{aligned} \tag{17.3}$$

We see that the integral over q is finite for $d < 8$, and in particular for $d = 6$.

After a Wick rotation of the q^0 contour and applying the general formula of section 14, we find

$$\int \frac{d^6q}{(2\pi)^6} \frac{1}{(q^2+D)^4} = \frac{i}{6(4\pi)^3 D} \ . \tag{17.4}$$

Thus we get

$$\mathbf{V}_4 = \frac{g^4}{6(4\pi)^3} \int dF_4 \left(\frac{1}{D_{1234}} + \frac{1}{D_{1324}} + \frac{1}{D_{1243}} \right) + O(g^6) \; . \tag{17.5}$$

This expression is finite and well-defined; the same is true for the one-loop contribution to $\mathbf{V}_n$ for all $n > 3$.

18 Higher-Order Corrections and Renormalizability

Prerequisite: 17

In sections 14–17, we computed the one-loop diagrams with two, three, and four external lines for φ^3 theory in six dimensions.

We found that the first two involved divergent momentum integrals, but that these divergences could be absorbed into the coefficients of terms in the lagrangian.

If this is true for all higher-order (in g) contributions to the propagator and to the one-particle irreducible vertex functions (with $n \geq 3$ external lines), then we say that the theory is renormalizable.

If this is not the case, and further divergences arise, it may be possible to absorb them by adding some new terms to the lagrangian.

If a finite number of such new terms is required, the theory is still said to be *renormalizable*.

However, if an infinite number of new terms is required, then the theory is said to be *nonrenormalizable*.

Despite the infinite number of parameters needed to specify it, a nonrenormalizable theory is generally able to make useful predictions at energies below some ultraviolet cutoff Λ; we will discuss this in section 29.

In this section, we will deduce the necessary conditions for renormalizability.

As an example, we will analyze a scalar field theory in d spacetime dimensions of the form

$$\mathcal{L} = -\tfrac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} Z_m m^2 \varphi^2 - \sum_{n=3}^{\infty} \tfrac{1}{n!} Z_n g_n \varphi^n \ . \tag{18.1}$$

Consider a Feynman diagram with E external lines, I internal lines, L closed loops, and V_n vertices that connect n lines.

Here V_n is just a number, not to be confused with the vertex function $\mathbf{V}_n$.

Do the momentum integrals associated with this diagram diverge?

We begin by noting that each closed loop gives a factor of $d^d\ell_i$, and each internal propagator gives a factor of $1/(p^2 + m^2)$, where p is some linear combination of external momenta k_i and loop momenta ℓ_i.

The diagram would then appear to have an ultraviolet divergence at large ℓ_i if there are more ℓ's in the numerator than there are in the denominator.

The number of ℓ's in the numerator minus the number of ℓ's in the denominator is the diagram's *superficial degree of divergence*.

$$D \equiv dL - 2I \ , \tag{18.2}$$

and the diagram appears to be divergent if

$$D \geq 0 \ . \tag{18.3}$$

Next we derive a more useful formula for D.

The diagram has E external lines, so another contributing diagram is the tree diagram where all the lines are joined by a single vertex, with vertex factor $-iZ_E g_E$; this is, in fact, the value of this entire diagram, which then has mass dimension $[g_E]$.

The Z's are all dimensionless, by definition.

Therefore, the original diagram also has mass dimension $[g_E]$, since both are contributions to the same scattering amplitude:

$$[\text{diagram}] = [g_E] \ . \tag{18.4}$$

On the other hand, the mass dimension of any diagram is given by the sum of the mass dimensions of its components, namely

$$[\text{diagram}] = dL - 2I + \sum_{n=3}^{\infty} V_n [g_n] \ . \tag{18.5}$$

From eqs. (18.2), (18.4), and (18.5), we get

$$D = [g_E] - \sum_{n=3}^{\infty} V_n [g_n] \ . \tag{18.6}$$

This is the formula we need.

From eq. (18.6), it is immediately clear that if any $[g_n] < 0$, we expect uncontrollable divergences, since D increases with every added vertex of this type.

Therefore, *a theory with any* $[g_n] < 0$ *is nonrenormalizable*.

According to our results in section 12, the coupling constants have mass dimension

$$[g_n] = d - \tfrac{1}{2} n(d-2) \ , \tag{18.7}$$

and so we have

$$[g_n] < 0 \quad \text{if} \quad n > \frac{2d}{d-2} \ . \tag{18.8}$$

Thus we are limited to powers no higher than φ^4 in four dimensions, and no higher than φ^3 in six dimensions.

The same criterion applies to more complicated theories as well: *a theory is nonrenormalizable if any coefficient of any term in the lagrangian has negative mass dimension*.

What about theories with couplings with only positive or zero mass dimension?

We see from eq. (18.6) that the only dangerous diagrams (those with $D \geq 0$) are those for which $[g_E] \geq 0$.

But in this case, we can absorb the divergence simply by adjusting the value of Z_E.

This discussion also applies to the propagator; we can think of $\Pi(k^2)$ as representing the loop-corrected counterterm vertex $Ak^2 + Bm^2$, with A and Bm^2 playing the roles of two couplings.

We have $[A] = 0$ and $[Bm^2] = 2$, so the contributing diagrams are expected to be divergent (as we have already seen in detail), and the divergences must be absorbed into A and Bm^2.

D is called the *superficial* degree of divergence because a diagram might diverge even if $D < 0$, or might be finite even if $D \geq 0$.

The latter can happen if there are cancellations among ℓ's in the numerator.

Quantum electrodynamics provides an example of this phenomenon that we will encounter in Part III.

For now we turn our attention to the case of diagrams with $D < 0$ that nevertheless diverge.

Consider, for example, the diagrams of figs. (18.1) and (18.2).

The one-loop diagram of fig. (18.1) with $E = 4$ is finite, but the two-loop correction from the first diagram of fig. (18.2) is not: the bubble on the upper propagator diverges.

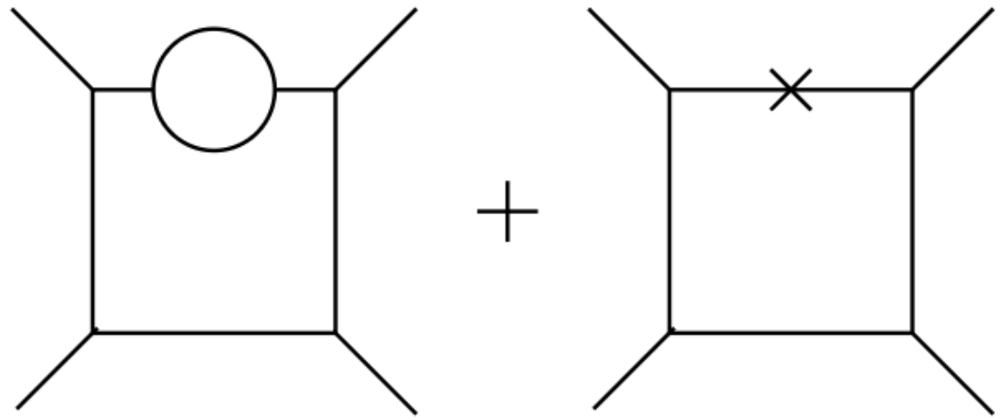


Figure 18.2: A two-loop contribution to $\mathbf{V}_4$, and the corresponding counterterm insertion.

This is an example of a divergent subdiagram.

However, this is not a problem in this case, because this divergence is canceled by the second diagram of fig. (18.2), which has a counterterm vertex in place of the bubble.

This is the generic situation: divergent subdiagrams are diagrams that, considered in isolation, have $D \geq 0$.

These are precisely the diagrams whose divergences can be canceled by adjusting the Z factor of the corresponding tree diagram (in theories where $[g_n] \geq 0$ for all nonzero g_n).

Thus, we expect that theories with couplings whose mass dimensions are all positive or zero will be renormalizable.

A detailed study of the properties of the momentum integrals in Feynman diagrams is necessary to give a complete proof of this.

Theories with spin-one fields are renormalizable for $d = 4$ if and only if these spin-one fields are associated with a gauge symmetry.

We will study this in Part III.

Theories of fields with spin greater than one are never renormalizable for $d \geq 4$.

19 Perturbation Theory to All Orders

Prerequisite: 18

In section 18, we found that, generally, a theory is renormalizable if all of its lagrangian coefficients have positive or zero mass dimension.

In this section, using φ^3 theory in six dimensions as our example, we will see how to construct a finite expression for a scattering amplitude to arbitrarily high order in the φ^3 coupling g.

We begin by summing all one-particle irreducible diagrams with two external lines; this gives us the self-energy $\Pi(k^2)$.

We next sum all 1PI diagrams with three external lines; this gives us the three-point vertex function $\mathbf{V}_3(k_1,k_2,k_3)$.

Order by order in g, we must adjust the value of the lagrangian coefficients Z_φ, Z_m, and Z_g to maintain the conditions $\Pi(-m^2) = 0$, $\Pi'(-m^2) = 0$, and $\mathbf{V}_3(0,0,0) = g$.

Next we will construct the n-point vertex functions $\mathbf{V}_n(k_1,...,k_n)$ with $4 \le n \le E$, where E is the number of external lines in the process of interest.

We compute these using a *skeleton expansion*.

This means that we draw all the contributing 1PI diagrams, but omit diagrams that include either propagator or three-point vertex corrections.

That is, we include only diagrams that are not only 1PI, but also 2PI and 3PI: they remain connected when any one, two, or three lines are cut.

Cutting three lines may isolate a single tree-level vertex, but nothing more complicated.

Then we take the propagators and vertices in these diagrams to be given by the *exact* propagator $\widetilde{\Delta}(k^2) = (k^2 + m^2 - \Pi(k^2))^{-1}$ and vertex $\mathbf{V}_3(k_1,k_2,k_3)$, rather than by the tree-level propagator $\Delta(k^2) = (k^2 + m^2)^{-1}$ and vertex g.

We then sum these *skeleton diagrams* to get $\mathbf{V}_n$ for $4 \le n \le E$.

Order by order in g, this procedure is equivalent to computing $\mathbf{V}_n$ by summing the usual set of contributing 1PI diagrams.

Next we draw all tree-level diagrams that contribute to the process of interest (which has E external lines), including not only three-point vertices, but also n-point vertices for $n = 3, 4, \ldots, E$.

Then we evaluate these diagrams using the exact propagator $\widetilde{\Delta}(k^2)$ for internal lines, and the exact 1PI vertices $\mathbf{V}_n$; external lines are assigned a factor of one.

This is because, in the LSZ formula, each Klein-Gordon wave operator becomes (in momentum space) a factor of $k^2_i + m^2$ that multiplies each external propagator, leaving behind only the residue of the pole in that propagator at $k^2_i = -m^2$; by construction, this residue is one.

We sum these tree diagrams to get the scattering amplitude; loop corrections have all been accounted for already in $\widetilde{\Delta}(k^2)$ and $\mathbf{V}_n$.

Order by order in g, this procedure is equivalent to computing the scattering amplitude by summing the usual set of contributing diagrams.

Thus we now know how to compute an arbitrary scattering amplitude to arbitrarily high order.

The procedure is the same in any quantum field theory; only the form of the propagators and vertices change, depending on the spins of the fields.

The tree-level diagrams of the final step can be thought of as the Feynman diagrams of a *quantum action* (or *effective action*, or *quantum effective action*) $\Gamma(\varphi)$.

There is a simple and interesting relationship between the effective action $\Gamma(\varphi)$ and the sum of connected diagrams with sources iW(J).

We derive it in section 21.

20 Two-Particle Elastic Scattering at One Loop

Prerequisite: 19

We now illustrate the general rules of section 19 by computing the two-particle elastic scattering amplitude, including all one-loop corrections, in φ^3 theory in six dimensions.

Elastic means that the number of outgoing particles (of each species, in more general contexts) is the same as the number of incoming particles (of each species).

We computed the amplitude for this process at tree level in section 10, with the result

$$i\mathcal{T}_{\text{tree}} = \tfrac{1}{i}(ig)^2\left[\tilde{\Delta}(-s) + \tilde{\Delta}(-t) + \tilde{\Delta}(-u)\right], \tag{20.1}$$

where $\tilde{\Delta}(-s) = 1/(-s + m^2 - i\epsilon)$ is the free-field propagator, and s, t, and u are the Mandelstam variables.

Later we will need to remember that s is positive, that t and u are negative, and that $s + t + u = 4m^2$.

The exact scattering amplitude is given by the diagrams of fig. (20.1), with all propagators and vertices interpreted as exact propagators and vertices.

Recall, however, that each external propagator contributes only the residue of the pole at $k^2 = -m^2$, and that this residue is one; thus the factor associated with each external line is simply one.

We get the one-loop approximation to the exact amplitude by using the one-loop expressions for the internal propagators and vertices.

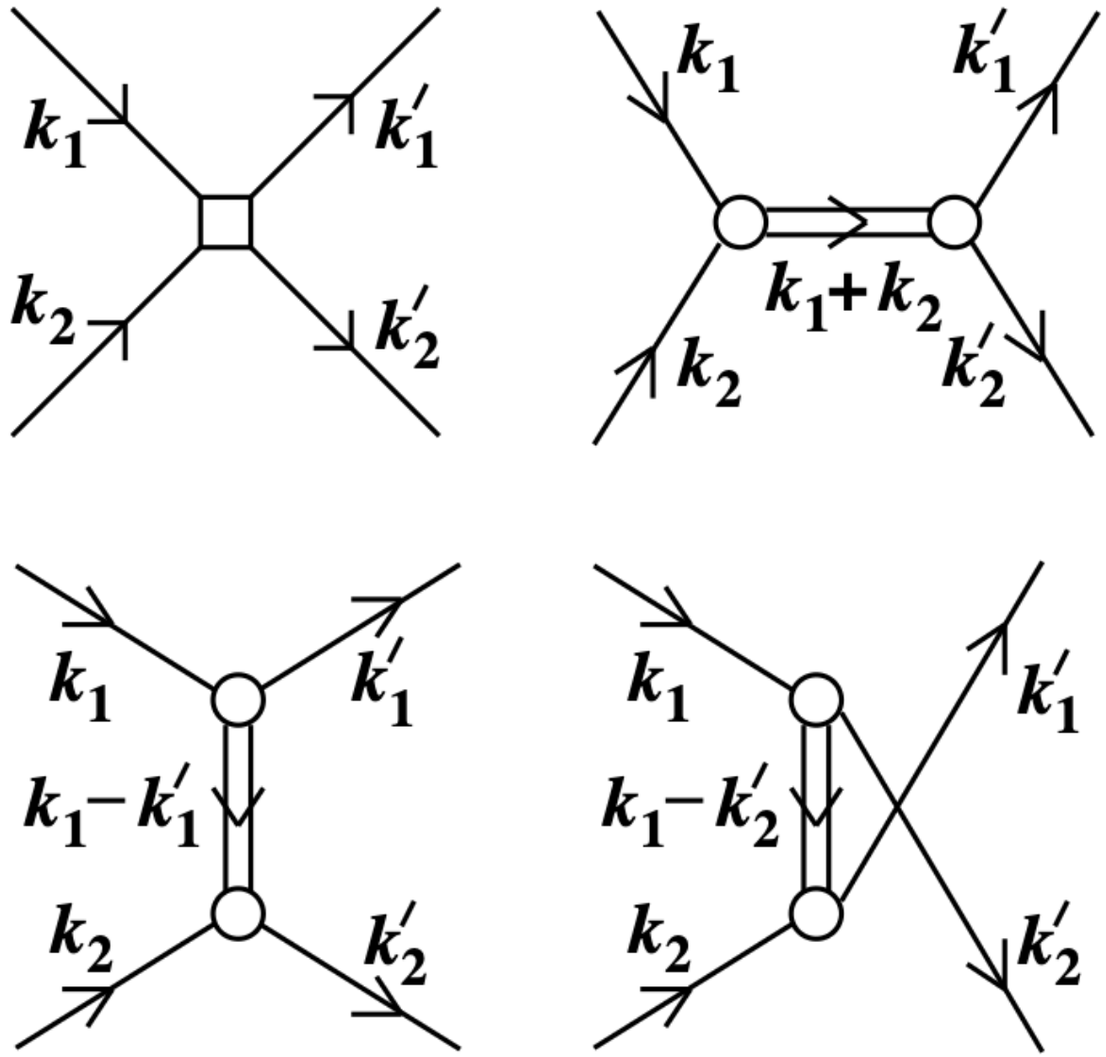


Figure 20.1: The Feynman diagrams contributing to the two-particle elastic scattering amplitude; a double line stands for the exact propagator $\frac{1}{i}\,\widetilde{\Delta}(k)$, a circle for the exact three-point vertex $\mathbf{V}_3(k_1,k_2,k_3)$, and a square for the exact four-point vertex $\mathbf{V}_4(k_1,k_2,k_3,k_4)$. An external line stands for the unit residue of the pole at $k^2 = -m^2$.

We thus have

$$i\mathcal{T}_{1-\text{loop}} = \frac{1}{i}\Big([i\mathbf{V}_3(s)]^2\tilde{\mathbf{\Delta}}(-s) + [i\mathbf{V}_3(t)]^2\tilde{\mathbf{\Delta}}(-t) + [i\mathbf{V}_3(u)]^2\tilde{\mathbf{\Delta}}(-u)\Big) \qquad (20.2)$$
$$+ i\mathbf{V}_4(s,t,u)\ ,$$

where, suppressing the $i\epsilon$'s,

$$\tilde{\mathbf{\Delta}}(-s) = \frac{1}{-s + m^2 - \Pi(-s)} \,, \tag{20.3}$$

$$\Pi(-s) = \tfrac{1}{2}\alpha \int_0^1 dx\, D_2(s) \ln\Big(D_2(s)/D_0\Big) - \tfrac{1}{12}\alpha(-s+m^2) \,, \tag{20.4}$$

$$\mathbf{V}_3(s)/g = 1 - \tfrac{1}{2}\alpha \int dF_3 \, \ln\Big(D_3(s)/m^2\Big) \,, \tag{20.5}$$

$$\mathbf{V}_4(s,t,u) = \tfrac{1}{6}g^2\alpha \int dF_4 \left[\frac{1}{D_4(s,t)} + \frac{1}{D_4(t,u)} + \frac{1}{D_4(u,s)}\right] . \tag{20.6}$$

Here $\alpha = g^2/(4\pi)^3$, the Feynman integration measure is

$$\begin{aligned} \int dF_n \, f(x) &= (n-1)! \int_0^1 dx_1 \ldots dx_n \, \delta(x_1 + \ldots + x_n - 1) f(x) \\ &= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ldots \int_0^{1-x_1-\ldots-x_{n-2}} dx_{n-1} \\ &\qquad \times f(x)\Big|_{x_n = 1-x_1-\ldots-x_{n-1}} \,, \end{aligned} \tag{20.7}$$

and we have defined

$$D_2(s) = -x(1-x)s + m^2 \ , \tag{20.8}$$

$$D_0 = +[1-x(1-x)]m^2 \ , \tag{20.9}$$

$$D_3(s) = -x_1x_2s + [1-(x_1+x_2)x_3]m^2 \ , \tag{20.10}$$

$$D_4(s,t) = -x_1x_2s - x_3x_4t + [1-(x_1+x_2)(x_3+x_4)]m^2 \ . \tag{20.11}$$

We obtain $\mathbf{V}_3(s)$ from the general three-point function $\mathbf{V}_3(k_1,k_2,k_3)$ by setting two of the three k^2_i to $-m^2$, and the third to $-s$.

We obtain $\mathbf{V}_4(s,t,u)$ from the general four-point function $\mathbf{V}_4(k_1,k_2,k_3,k_4)$ by setting all four k^2_i to $-m^2$, $(k_1 + k_2)^2$ to $-s$, $(k_1 + k_3)^2$ to $-t$, and $(k_1 + k_4)^2$ to $-u$.

Recall that the vertex functions are defined with all momenta treated as incoming; here we have identified $-k_3$ and $-k_4$ as the outgoing momenta.

Eqs. (20.2–20.11) are formidable expressions.

To gain some intuition about them, let us consider the limit of high-energy, fixed angle scattering, where we take s, |t|, and |u| all much larger than m^2.

Equivalently, we are considering the amplitude in the limit of zero particle mass.

We can then set $m^2 = 0$ in $D_2(s)$, $D_3(s)$, and $D_4(s,t)$.

For the self-energy, we get

$$\Pi(-s) = -\tfrac{1}{2}\alpha\, s \int_0^1 dx\, x(1-x) \left[\ln\left(\frac{-s}{m^2}\right) + \ln\left(\frac{x(1-x)}{1-x(1-x)}\right)\right] + \tfrac{1}{12}\alpha\, s$$
$$= -\tfrac{1}{12}\alpha\, s\Big[\ln(-s/m^2) + 3 - \pi\sqrt{3}\Big] . \tag{20.12}$$

Thus,

$$\tilde{\mathbf{\Delta}}(-s) = \frac{1}{-s - \Pi(-s)}$$
$$= -\frac{1}{s}\Big(1 + \tfrac{1}{12}\alpha\Big[\ln(-s/m^2) + 3 - \pi\sqrt{3}\Big]\Big) + O(\alpha^2) . \tag{20.13}$$

The appropriate branch of the logarithm is found by replacing s by s + iϵ.

For s real and positive, −s lies just below the negative real axis, and so

$$\ln(-s) = \ln s - i\pi . \tag{20.14}$$

For t (or u), which is negative, we have instead

$$\ln(-t) = \ln|t| ,$$
$$\ln t = \ln|t| + i\pi . \tag{20.15}$$

For the three-point vertex, we get

$$\begin{aligned}\mathbf{V}_3(s)/g &= 1 - \tfrac{1}{2}\alpha \int dF_3 \Big[\ln(-s/m^2) + \ln(x_1 x_2) \Big] \,, \\ &= 1 - \tfrac{1}{2}\alpha \Big[\ln(-s/m^2) - 3 \Big] \,,\end{aligned} \tag{20.16}$$

where the same comments about the appropriate branch apply.

For the four-point vertex, the integral over the Feynman parameters can be done in closed form, with the result

$$\begin{aligned}\int \frac{dF_4}{D_4(s,t)} &= -\frac{3}{s+t}\left(\pi^2 + \Big[\ln(s/t)\Big]^2\right) \\ &= +\frac{3}{u}\left(\pi^2 + \Big[\ln(s/t)\Big]^2\right) \,,\end{aligned} \tag{20.17}$$

where the second line follows from s + t + u = 0.

Putting all of this together, we have

$$\mathcal{T}_{1-\text{loop}} = g^2 \Big[F(s,t,u) + F(t,u,s) + F(u,s,t) \Big] \,, \tag{20.18}$$

where

$$F(s,t,u) \equiv -\frac{1}{s}\left(1 - \tfrac{11}{12}\alpha \Big[\ln(-s/m^2) + c \Big] - \tfrac{1}{2}\alpha \Big[\ln(t/u)\Big]^2\right) \,, \tag{20.19}$$

and $c = (6\pi^2 + \pi\sqrt{3} - 39)/11 = 2.33$.

This is a typical result of a loop calculation: the original tree-level amplitude is corrected by powers of logarithms of kinematic variables.

21 The Quantum Action

Prerequisite: 19

In section 19, we saw how to compute (in φ^3 theory in d = 6 dimensions) the 1PI vertex functions $\mathbf{V}_n(k_1,\ldots,k_4)$ for $n \geq 4$ via the skeleton expansion: draw all Feynman diagrams with n external lines that are one-, two-, and three-particle irreducible, and compute them using the exact propagator $\widetilde{\Delta}(k2)$ and three-point vertex function $\mathbf{V}_3(k_1,k_2,k_3)$.

We now define the *quantum action* (or *effective action*, or *quantum effective action*)

$$\Gamma(\varphi) \equiv \frac{1}{2}\int \frac{d^dk}{(2\pi)^dx}\,\widetilde{\varphi}(-k)\Big(k^2+m^2-\Pi(k^2)\Big)\widetilde{\varphi}(k)$$
$$+\sum_{n=3}^{\infty}\frac{1}{n!}\int \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_n}{(2\pi)^d}\,(2\pi)^d\delta^d(k_1+\ldots+k_n) \qquad (21.1)$$
$$\times\mathbf{V}_n(k_1,\ldots,k_n)\,\widetilde{\varphi}(k_1)\ldots\widetilde{\varphi}(k_n)\ ,$$

where $\tilde{\varphi}(k) = \int d^dx e^{-ikx}\varphi(x)$.

The quantum action has the property that the tree-level Feynman diagrams it generates give the complete scattering amplitude of the original theory.

In this section, we will determine the relationship between $\Gamma(\varphi)$ and the sum of connected diagrams with sources, iW(J), introduced in section 9.

Recall that W(J) is related to the path integral

$$Z(J) = \int \mathcal{D}\varphi \, \exp\left[iS(\varphi) + i \int d^d x \, J\varphi \right] , \tag{21.2}$$

where S = $\int d^d x \, \mathcal{L}$ is the action, via

$$Z(J) = \exp[iW(J)] \, . \tag{21.3}$$

Consider now the path integral

$$Z_\Gamma(J) \equiv \int \mathcal{D}\varphi \, \exp\left[i\Gamma(\varphi) + i \int d^d x \, J\varphi \right] \tag{21.4}$$

$$= \exp[iW_\Gamma(J)] \, . \tag{21.5}$$

$W_\Gamma(J)$ is given by the sum of connected diagrams (with sources) in which each line represents the exact propagator, and each n-point vertex represents the exact 1PI vertex $\mathbf{V}_n$.

$W_\Gamma(J)$ would be equal to W(J) if we included only tree diagrams in $W_\Gamma(J)$.

We can isolate the tree-level contribution to a path integral by means of the following trick.

Introduce a dimensionless parameter that we will call $\hbar$, and the path integral

$$Z_{\Gamma,\hbar}(J) \equiv \int \mathcal{D}\varphi \, \exp\left[\frac{i}{\hbar} \left(\Gamma(\varphi) + \int d^d x \, J\varphi \right) \right] \tag{21.6}$$

$$= \exp[iW_{\Gamma,\hbar}(J)] \, . \tag{21.7}$$

In a given connected diagram with sources, every propagator (including those that connect to sources) is multiplied by $\hbar$, every source by $1/\hbar$, and every vertex by $1/\hbar$.

The overall factor of $\hbar$ is then $\hbar^{\text{P–E–V}}$, where V is the number of vertices, E is the number of sources (equivalently, the number of external lines after we remove the sources), and P is the number of propagators (external and internal).

We next note that P–E–V is equal to L–1, where L is the number of closed loops.

This can be seen by counting the number of internal momenta and the constraints among them.

Specifically, assign an unfixed momentum to each internal line; there are P–E of these momenta.

Then the V vertices provide V constraints.

One linear combination of these constraints gives overall momentum conservation, and so does not constrain the internal momenta.

Therefore, the number of internal momenta left unfixed by the vertex constraints is (P–E)–(V–1), and the number of unfixed momenta is the same as the number of loops L.

So, $W_{\Gamma,\hbar}(J)$ can be expressed as a power series in $\hbar$ of the form

$$W_{\Gamma,\hbar}(J) = \sum_{L=0}^{\infty} \hbar^{L-1} W_{\Gamma,L}(J) \; . \tag{21.8}$$

If we take the formal limit of $\hbar \to 0$, the dominant term is the one with L = 0, which is given by the sum of tree diagrams only.

This is just what we want.

We conclude that

$$W(J) = W_{\Gamma, L=0}(J) \ . \tag{21.9}$$

Next we perform the path integral in eq. (21.6) by the method of stationary phase.

We find the point (actually, the field configuration) at which the exponent is stationary; this is given by the solution of the *quantum equation of motion*

$$\frac{\delta}{\delta\varphi(x)}\Gamma(\varphi) = -J(x) \ . \tag{21.10}$$

Let φ_J(x) denote the solution of eq. (21.10) with a specified source function J(x).

Then the stationary-phase approximation to $Z_{\Gamma,\hbar}$(J) is

$$Z_{\Gamma,\hbar}(J) = \exp\left[\frac{i}{\hbar}\left(\Gamma(\varphi_J) + \int d^d x \, J\varphi_J\right) + O(\hbar^0)\right] \ . \tag{21.11}$$

Combining the results of eqs. (21.7), (21.8), (21.9), and (21.11), we find

$$W(J) = \Gamma(\varphi_J) + \int d^d x \, J\varphi_J \ . \tag{21.12}$$

This is the main result of this section.

Let us explore it further.

Recall from section 9 that the vacuum expectation value of the field operator φ(x) is given by

$$\langle 0|\varphi(x)|0\rangle = \frac{\delta}{\delta J(x)} W(J)\Big|_{J=0} \ . \tag{21.13}$$

Now consider what we get if we do not set J = 0 after taking the derivative:

$$\langle 0|\varphi(x)|0\rangle_J \equiv \frac{\delta}{\delta J(x)} W(J) \ . \tag{21.14}$$

This is the vacuum expectation value of φ(x) in the presence of a nonzero source function J(x).

We can get some more information about it by using eq. (21.12) for W(J).

Making use of the product rule for derivatives, we have

$$\langle 0|\varphi(x)|0\rangle_J = \frac{\delta}{\delta J(x)} \Gamma(\varphi_J) + \varphi_J(x) + \int d^6 y \, J(y) \frac{\delta \varphi_J(y)}{\delta J(x)} \ . \tag{21.15}$$

We can evaluate the first term on the right-hand side by using the chain rule,

$$\frac{\delta}{\delta J(x)}\Gamma(\varphi_J) = \int d^6y \, \frac{\delta\Gamma(\varphi_J)}{\delta\varphi_J(y)} \, \frac{\delta\varphi_J(y)}{\delta J(x)} \ . \tag{21.16}$$

Then we can combine the first and third terms on the right-hand side of eq. (21.15) to get

$$\langle 0|\varphi(x)|0\rangle_J = \int d^6y \left[\frac{\delta\Gamma(\varphi_J)}{\delta\varphi_J(y)} + J(y)\right] \frac{\delta\varphi_J(y)}{\delta J(x)} + \varphi_J(x) \ . \tag{21.17}$$

Now we note from eq. (21.10) that the factor in large brackets on the right-hand side of eq. (21.17) vanishes, and so

$$\langle 0|\varphi(x)|0\rangle_J = \varphi_J(x) \ . \tag{21.18}$$

That is, the vacuum expectation value of the field operator φ(x) in the presence of a nonzero source function is also the solution to the quantum equation of motion, eq. (21.10).

We can also write the quantum action in terms of a *derivative expansion*,

$$\Gamma(\varphi) = \int d^dx \left[-\mathcal{U}(\varphi) - \tfrac{1}{2}\mathcal{Z}(\varphi)\partial^\mu\varphi\partial_\mu\varphi + \ldots \right], \tag{21.19}$$

where the ellipses stand for an infinite number of terms with more and more derivatives, and $\mathcal{U}(\varphi)$ and $\mathcal{Z}(\varphi)$ are ordinary functions (not functionals) of φ(x).

U(ϕ) is called the *quantum potential* (or *effective potential*, or *quantum effective potential*), and it plays an important conceptual role in theories with spontaneous symmetry breaking; see section 31.

However, it is rarely necessary to compute it explicitly, except in those cases where we are unable to do so.

22 Continuous Symmetries and Conserved Currents

Prerequisite: 8

Suppose we have a set of scalar fields ϕa(x), and a lagrangian density $\mathcal{L}(x) = \mathcal{L}(\varphi_a(x),\partial_\mu\varphi_a(x))$.

Consider what happens to $\mathcal{L}(x)$ if we make an infinitesimal change $\varphi_a(x)\rightarrow \varphi_a(x) + \delta\varphi_a(x)$ in each field.

We have $\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \delta\mathcal{L}(x)$, where $\delta\mathcal{L}(x)$ is given by the chain rule,

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\varphi_a(x)}\,\delta\varphi_a(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a(x))}\,\partial_\mu\delta\varphi_a(x)\ . \tag{22.1}$$

Next consider the classical equations of motion (also known as the Euler-Lagrange equations, or the *field equations*), given by the action principle

$$\frac{\delta S}{\delta\varphi_a(x)} = 0\ , \tag{22.2}$$

where S = $\int d^4y\mathcal{L}(y)$ is the action, and $\delta/\delta\varphi_a(x)$ is a functional derivative.

For definiteness, we work in four spacetime dimensions, though our results will apply in any number.

We have (with repeated indices implicitly summed)

$$\begin{aligned}
\frac{\delta S}{\delta\varphi_a(x)} &= \int d^4y\,\frac{\delta\mathcal{L}(y)}{\delta\varphi_a(x)} \\
&= \int d^4y\left[\frac{\partial\mathcal{L}(y)}{\partial\varphi_b(y)}\,\frac{\delta\varphi_b(y)}{\delta\varphi_a(x)} + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\varphi_b(y))}\,\frac{\delta(\partial_\mu\varphi_b(y))}{\delta\varphi_a(x)}\right] \\
&= \int d^4y\left[\frac{\partial\mathcal{L}(y)}{\partial\varphi_b(y)}\,\delta_{ba}\delta^4(y-x) + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\varphi_b(y))}\,\delta_{ba}\partial_\mu\delta^4(y-x)\right] \qquad (22.3) \\
&= \frac{\partial\mathcal{L}(x)}{\partial\varphi_a(x)} - \partial_\mu\,\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\ .
\end{aligned}$$

We can use this result to make the replacement

$$\frac{\partial\mathcal{L}(x)}{\partial\varphi_a(x)} \to \partial_\mu\,\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))} + \frac{\delta S}{\delta\varphi_a(x)} \qquad (22.4)$$

in eq. (22.1).

Then, combining two of the terms, we get

$$\delta\mathcal{L}(x) = \partial_\mu \left(\frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} \, \delta\varphi_a(x) \right) + \frac{\delta S}{\delta \varphi_a(x)} \delta\varphi_a(x) \; . \tag{22.5}$$

Next we identify the object in large parentheses in eq. (22.5) as the Noether current

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} \, \delta\varphi_a(x) \; . \tag{22.6}$$

Eq. (22.5) then implies

$$\partial_\mu j^\mu(x) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta\varphi_a(x) \; . \tag{22.7}$$

If the classical field equations are satisfied, then the second term on the right-hand side of eq. (22.7) vanishes.

The Noether current plays a special role if we can find a set of infinitesimal field transformations that leaves the lagrangian unchanged, or invariant.

In this case, we have $\delta\mathcal{L} = 0$, and we say that the lagrangian has a *continuous symmetry*.

From eq. (22.7), we then have $\partial_\mu j^\mu = 0$ whenever the field equations are satisfied, and we say that the Noether current is conserved.

In terms of its space and time components, this means that

$$\frac{\partial}{\partial t} j^0(x) + \nabla \cdot \mathbf{j}(x) = 0 \ . \tag{22.8}$$

If we interpret $j^0(x)$ as a charge density, and $\mathbf{j}(x)$ as the corresponding current density, then eq. (22.8) expresses the local conservation of this charge.

Let us see an example of this.

Consider a theory of a *complex* scalar field with lagrangian

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda (\varphi^\dagger \varphi)^2 \ . \tag{22.9}$$

We can also rewrite $\mathcal{L}$ in terms of two real scalar fields by setting $\varphi = (\varphi_1 + i\, \varphi_2)/\sqrt{2}$ to get

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_1 \partial_\mu \varphi_1 - \tfrac{1}{2}\partial^\mu \varphi_2 \partial_\mu \varphi_2 - \tfrac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \tfrac{1}{16}\lambda (\varphi_1^2 + \varphi_2^2)^2 \ . \tag{22.10}$$

In the form of eq. (22.9), it is obvious that $\mathcal{L}$ is left invariant by the transformation

$$\varphi(x) \to e^{-i\alpha} \varphi(x) \ , \tag{22.11}$$

where α is a real number.

This is called a $U(1)$ transformation, a transformation by a unitary 1×1 matrix.

In terms of φ_1 and φ_2, this transformation read

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} . \tag{21.12}$$

If we think of (φ_1 , φ_2) as a two-component vector, then eq. (22.12) is just a rotation of this vector in the plane by angle α.

Eq. (22.12) is called an *SO(2) transformation*, a transformation by an orthogonal 2×2 matrix with a special value of the determinant (namely +1, as opposed to−1, the only other possibility for an orthogonal matrix).

We have learned that a U(1) transformation can be mapped into an SO(2) transformation.

The infinitesimal form of eq. (22.11) is

$$\begin{aligned} \varphi(x) &\to \varphi(x) - i\alpha\varphi(x) , \\ \varphi^\dagger(x) &\to \varphi^\dagger(x) + i\alpha\varphi^\dagger(x) , \end{aligned} \tag{21.13}$$

where α is now infinitesimal.

In eq. (22.6), we should treat φ_1 and φ_2 as independent fields.

It is also conventional to scale the infinitesimal parameter out of the current, so that we have

$$
\begin{aligned}
\alpha\, j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\dagger)}\,\delta\varphi^\dagger \\
&= (-\partial^\mu \varphi^\dagger)(-i\alpha\varphi) + (-\partial^\mu \varphi)(+i\alpha\varphi^\dagger) \\
&= \alpha\, \mathrm{Im}(\varphi^\dagger \overleftrightarrow{\partial^\mu} \varphi)\ ,
\end{aligned}
\tag{22.14}
$$

where $A\overleftrightarrow{\partial}^\mu B \equiv A\partial^\mu B - (\partial^\mu A)B$.

Canceling out α, we find that the Noether current is

$$
j^\mu = \mathrm{Im}(\varphi^\dagger \overleftrightarrow{\partial^\mu} \varphi)\ . \tag{22.15}
$$

We can also repeat this exercise using the SO(2) form of the transformation.

For infinitesimal α, eq. (22.12) becomes $\delta\varphi_1 = +\alpha\varphi_2$ and $\delta\varphi_2 = -\alpha\varphi_1$.

Then the Noether current is given by

$$
\begin{aligned}
\alpha\, j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_1)}\,\delta\varphi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_2)}\,\delta\varphi_2 \\
&= (-\partial^\mu \varphi_1)(+\alpha\varphi_2) + (-\partial^\mu \varphi_2)(-\alpha\varphi_1) \\
&= \alpha\, (\varphi_1 \overleftrightarrow{\partial^\mu} \varphi_2)\ ,
\end{aligned}
\tag{22.16}
$$

which is (hearteningly) equivalent to eq. (22.14).

Let us define the Noether charge

$$Q \equiv \int d^3x\, j^0(x) = \int d^3x\ \mathrm{Im}(\varphi^\dagger \overleftrightarrow{\partial^0} \varphi)\ , \tag{22.17}$$

and investigate its properties.

If we integrate eq. (22.8) over d^3x, use Gauss's law to write the volume integral of $\nabla\cdot\mathbf{j}$ as a surface integral, and assume that the boundary conditions at infinity fix $\mathbf{j}(x) = 0$ on that surface, then we find that Q is constant in time.

To get a better idea of the physical implications of this, let us rewrite Q using the free-field expansions

$$\varphi(x) = \int \widetilde{dk} \left[a(\mathbf{k})e^{ikx} + b^*(\mathbf{k})e^{-ikx}\right]\ , \tag{22.18}$$

$$\varphi^\dagger(x) = \int \widetilde{dk} \left[b(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right]\ .$$

We have written $a^*(k)$ and $b^*(k)$ rather than $a^\dagger(k)$ and $b^\dagger(k)$ because so far our discussion has been about the classical field theory.

In a theory with interactions, these formulae (and their first time derivatives) are valid at any one particular time (say, $t = -\infty$).

Then, we can plug them into eq. (22.17), and find (after some manipulation similar to what we did for the hamiltonian in section 3)

$$Q = \int \widetilde{dk} \left[a^*(\mathbf{k})a(\mathbf{k}) - b(\mathbf{k})b^*(\mathbf{k})\right]\ . \tag{22.19}$$

In the quantum theory, this becomes an operator that counts the number of a particles minus the number of b particles.

This number is then time-independent, and so the scattering amplitude vanishes identically for any process that changes the value of Q.

This can be seen directly from the Feynman rules, which conserve Q at every vertex.

To better understand the implications of the Noether current in the quantum theory, we begin by considering the path integral,

$$Z(J) = \int \mathcal{D}\varphi \, e^{i[S + \int d^4y \, J_a \varphi_a]} \, . \tag{22.20}$$

The value of Z(J) is unchanged if we make the change of variable $\varphi_a(x) \rightarrow \varphi_a(x) + \delta\varphi_a(x)$, with $\delta\varphi_a(x)$ an arbitrary infinitesimal shift that (we assume) leaves the measure $\mathcal{D}\varphi$ invariant.

Thus we have

$$\begin{aligned} 0 &= \delta Z(J) \\ &= i \int \mathcal{D}\varphi \, e^{i[S + \int d^4y \, J_b \varphi_b]} \int d^4x \left(\frac{\delta S}{\delta \varphi_a(x)} + J_a(x) \right) \delta\varphi_a(x) \, . \end{aligned} \tag{22.21}$$

We can now take n functional derivatives with respect to $J_{a_j}(x_j)$, and then set J = 0, to get

$$0 = \int \mathcal{D}\varphi\, e^{iS} \int d^4x \Bigg[i\, \frac{\delta S}{\delta \varphi_a(x)}\, \varphi_{a_1}(x_1) \ldots \varphi_{a_n}(x_n) + \sum_{j=1}^{n} \varphi_{a_1}(x_1) \ldots \delta_{a a_j} \delta^4(x - x_j) \ldots \varphi_{a_n}(x_n) \Bigg] \delta\varphi_a(x)\ . \tag{22.22}$$

Since $\delta\varphi_a(x)$ is arbitrary, we can drop it (and the integral over d^4x).

Then, since the path integral computes the vacuum expectation value of the time-ordered product, we have

$$0 = i\langle 0|\mathrm{T}\frac{\delta S}{\delta \varphi_a(x)}\, \varphi_{a_1}(x_1) \ldots \varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n} \langle 0|\mathrm{T}\varphi_{a_1}(x_1) \ldots \delta_{a a_j} \delta^4(x - x_j) \ldots \varphi_{a_n}(x_n)|0\rangle\ . \tag{22.23}$$

These are the *Schwinger-Dyson equation*s for the theory.

To get a feel for them, let us look at free-field theory for a single real scalar field, for which $\delta S/\delta\varphi(x) = (\partial_x^2 - m^2)\varphi(x)$.

For n = 1 we get

$$(-\partial_x^2 + m^2) i\langle 0|\mathrm{T}\varphi(x)\varphi(x_1)|0\rangle = \delta^4(x - x_1)\ . \tag{22.24}$$

That the Klein-Gordon wave operator should sit outside the time-ordered product (and hence act on the time-ordering step functions) is clear from the path integral form of eq. (22.22).

We see from eq. (22.24) that the free-field propagator, $\Delta(x-x_1) = i\langle 0|T\varphi(x)\varphi(x_1)|0\rangle$, is a Green's function for the Klein-Gordon wave operator, a fact we first learned in section 8.

More generally, we can write

$$\langle 0|\mathrm{T}\frac{\delta S}{\delta\varphi_a(x)}\,\varphi_{a_1}(x_1)\dots\varphi_{a_n}(x_n)|0\rangle = 0 \quad \text{for} \quad x \neq x_{1,\dots,n}\ . \tag{22.25}$$

We see that the classical equation of motion is satisfied by a quantum field inside a correlation function, as long as its spacetime argument differs from those of all the other fields.

When this is not the case, we get extra contact terms.

Let us now consider a theory that has a continuous symmetry and a corresponding Noether current.

Take eq. (22.22), and set $\delta\varphi_a(x)$ to be the infinitesimal change in $\varphi_a(x)$ that results in $\delta\mathcal{L}(x) = 0$.

Now sum over the index a, and use eq. (22.7).

The result is the *Ward* (or *Ward-Takahashi*) *identity*

$$\begin{aligned} 0 = {} & \partial_\mu\langle 0|\mathrm{T}j^\mu(x)\varphi_{a_1}(x_1)\dots\varphi_{a_n}(x_n)|0\rangle \\ & + i\sum_{j=1}^{n}\langle 0|\mathrm{T}\varphi_{a_1}(x_1)\dots\delta\varphi_{a_j}(x)\delta^4(x-x_j)\dots\varphi_{a_n}(x_n)|0\rangle\ . \end{aligned} \tag{22.26}$$

Thus, conservation of the Noether current holds in the quantum theory, with the current inside a correlation function, up to contact terms with a specific form that depends on the details of the infinitesimal transformation that leaves $\boldsymbol{\mathcal{L}}$ invariant.

The Noether current is also useful in a slightly more general context.

Suppose we have a transformation of the fields such that $\delta\boldsymbol{\mathcal{L}}(x)$ is not zero, but instead is a total divergence: $\delta\boldsymbol{\mathcal{L}}(x) = \partial_\mu K^\mu(x)$ for some $K^\mu(x)$.

Then there is still a conserved current, now given by

$$j^\mu(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,\delta\varphi_a(x) - K^\mu(x)\ . \tag{22.27}$$

An example of this is provided by the symmetry of spacetime translations.

We transform the fields via $\varphi_a(x) \longrightarrow \varphi_a(x - a)$, where a^μ is a constant four-vector.

The infinitesimal version of this is $\varphi_a(x) \longrightarrow \varphi_a(x) - a^\nu\partial_\nu\varphi_a(x)$, and so we have $\delta\varphi_a(x) = -a^\nu\partial_\nu\varphi_a(x)$.

Under this transformation, we obviously have $\boldsymbol{\mathcal{L}}(x) \rightarrow \boldsymbol{\mathcal{L}}(x - a)$, and so $\delta\boldsymbol{\mathcal{L}}(x) = -a^\nu\partial_\nu\boldsymbol{\mathcal{L}}(x) = -\partial_\nu(a^\nu\boldsymbol{\mathcal{L}}(x))$.

Thus in this case $K_\nu(x) = -a^\nu\boldsymbol{\mathcal{L}}(x)$, and the conserved current is

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))}(-a^\nu \partial_\nu \varphi_a(x)) - a^\mu \mathcal{L}(x)$$
$$= a_\nu T^{\mu\nu}(x) \ , \tag{22.28}$$

where we have defined the *stress-energy* or *energy-momentum tensor*

$$T^{\mu\nu}(x) \equiv -\frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))}\, \partial^\nu \varphi_a(x) + g^{\mu\nu}\mathcal{L}(x) \ . \tag{22.29}$$

For a renormalizable theory of a set of real scalar fields $\varphi_a(x)$, the lagrangian takes the form

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_a \partial_\mu \varphi_a - V(\varphi) \ , \tag{22.30}$$

where $V(\varphi)$ is a polynomial in the φ_a's.

In this case

$$T^{\mu\nu} = \partial^\mu \varphi_a \partial^\nu \varphi_a + g^{\mu\nu}\mathcal{L} \ . \tag{22.31}$$

In particular,

$$T^{00} = \tfrac{1}{2}\Pi_a^2 + \tfrac{1}{2}(\nabla \varphi_a)^2 + V(\varphi) \ , \tag{22.32}$$

where $\Pi_a = \partial_0 \varphi_a$ is the canonical momentum conjugate to the field φ_a.

We recognize T^{00} as the *hamiltonian density* $\mathscr{H}$ that corresponds to the lagrangian density of eq. (22.30).

Then, by Lorentz symmetry, T^{0j} must be the corresponding momentum density.

We have

$$T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a \ . \tag{22.33}$$

To check that this is a sensible result, we use the free-field expansion for a set of real scalar fields [the same as eq. (22.18) but with b(**k**) = a(**k**) for each field]; then we find that the momentum operator is given by

$$P^j = \int d^3x \, T^{0j}(x) = \int \widetilde{dk} \ k^j \ a_a^\dagger(\mathbf{k}) a_a(\mathbf{k}) \ , \tag{22.34}$$

which is just what we would expect.

We therefore identify the *energy-momentum four-vector* as

$$P^\mu = \int d^3x \, T^{0\mu}(x) \ . \tag{22.35}$$

Recall that in section 2 we defined the *spacetime translation operator* as

$$T(a) \equiv \exp(-iP^\mu a_\mu) \ , \tag{22.36}$$

and announced that it had the property that

$$T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x - a) \ . \tag{22.37}$$

Now that we have an explicit formula for P^μ, we can check this.

This is easiest to do for infinitesimal a^μ; then eq. (22.37) becomes

$$[\varphi_a(x), P^\mu] = \tfrac{1}{i}\partial^\mu \varphi_a(x) \ . \tag{22.38}$$

This can indeed be verified by using the canonical commutation relations for $\phi_a(x)$ and $\Pi_a(x)$.

One more symmetry we can investigate is Lorentz symmetry.

If we make an infinitesimal Lorentz transformation, we have $\varphi_a(x) \rightarrow \varphi_a(x + \delta\omega \cdot x)$, where $\delta\omega \cdot x$ is shorthand for $\delta\omega^\nu{}_\varrho x^\varrho$.

This case is very similar to that of spacetime translations; the only difference is that the translation parameter a^ν is now x dependent, $a^\nu \rightarrow -\delta\omega^\nu{}_\varrho x^\varrho$.

The resulting conserved current is

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\nu T^{\mu\rho}(x) - x^\rho T^{\mu\nu}(x) \ , \tag{22.39}$$

and it obeys $\partial_\mu M^{\mu\nu\varrho} = 0$, with the derivative contracted with the first index.

$M^{\mu\nu\varrho}$ is antisymmetric on its second two indices; this comes about because $\delta\omega^{\nu\varrho}$ is antisymmetric.

The conserved charges associated with this current are

$$M^{\nu\rho} = \int d^3x\, \mathcal{M}^{0\nu\rho}(x)\ , \tag{22.40}$$

and these are the generators of the Lorentz group that were introduced in section 2.

Again, we can use the canonical commutation relations for the fields to check that the Lorentz generators have the right commutation relations, both with the fields and with each other.

23 Discrete Symmetries: *P* , *T* , *C*, and *Z*

In section 2, we studied the proper orthochronous Lorentz transformations, which are continuously connected to the identity.

In this section, we will consider the effects of parity,

$$\mathcal{P}^{\mu}{}_{\nu} = (\mathcal{P}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{23.1}$$

and time reversal,

$$\mathcal{T}^{\mu}{}_{\nu} = (\mathcal{T}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{23.2}$$

We will also consider certain other discrete transformations such as *charge conjugation*.

Recall from section 2 that for every proper orthochronous Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ there is an associated unitary operator U(Λ) with the property that

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \ . \tag{23.3}$$

Thus for parity and time-reversal, we expect that there are corresponding unitary operators

$$P \equiv U(\mathcal{P}) \ , \tag{23.4}$$
$$T \equiv U(\mathcal{T}) \ , \tag{23.5}$$

such that

$$P^{-1}\varphi(x)P = \varphi(\mathcal{P}x) \ , \tag{23.6}$$
$$T^{-1}\varphi(x)T = \varphi(\mathcal{T}x) \ . \tag{23.7}$$

There is, however, an extra possible complication.

Since the $\mathcal{P}$ and $\mathcal{T}$ matrices are their own inverses, a second parity or time-reversal transformation should transform all observables back into themselves.

Using eqs. (23.6) and (23.7), along with $\mathcal{P}^2 = 1$ and $\mathcal{T}^2 = 1$, we see that

$$P^{-2}\varphi(x)P^2 = \varphi(x) \ , \tag{23.8}$$
$$T^{-2}\varphi(x)T^2 = \varphi(x) \ . \tag{23.9}$$

Since φ(x) is a hermitian operator, it is in principle an observable, and so eqs. (23.8) and (23.9) are just what we expect.

However, another possibility for the parity transformation of the field, different from eqs. (23.6) and (23.7) but nevertheless consistent with eqs. (23.8) and (23.9), is

$$P^{-1}\varphi(x)P = -\varphi(\mathcal{P}x)\ , \tag{23.10}$$

$$T^{-1}\varphi(x)T = -\varphi(\mathcal{T}x)\ . \tag{23.11}$$

This possible extra minus sign cannot arise for proper orthochronous Lorentz transformations, because they are continuously connected to the identity, and for the identity transformation (that is, no transformation at all), we must obviously have the plus sign.

If the minus sign appears on the right-hand side, we say that the field is odd under parity (or time reversal).

If a scalar field is *odd under parity*, we sometimes say that it is a *pseudoscalar*.

So, how do we know which is right, eqs. (23.6) and (23.7), or eqs. (23.10) and (23.11)?

The general answer is that we get to choose, but there is a key principle to guide our choice: if at all possible, we want to define P and T so that the lagrangian density is even,

$$P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x)\ , \tag{23.12}$$

$$T^{-1}\mathcal{L}(x)T = +\mathcal{L}(\mathcal{T}x)\ . \tag{23.13}$$

Then, after we integrate over d^4x to get the action S, the action will be invariant.

This means that parity and time-reversal are conserved.

For theories with spin-zero fields only, it is clear that the choice of eqs. (23.6) and (23.7) always leads to eqs. (23.12) and (23.13), and so there is no reason to flirt with eqs. (23.10) and (23.11).

For theories that also include spin-one-half fields, certain scalar bilinears in these fields are necessarily odd under parity and time reversal, as we will see in section 40.

If a scalar field couples to such a bilinear, then eqs. (23.12) and (23.13) will hold if and only if we choose eqs. (23.10) and (23.11) for that scalar, and so that is what we must do.

There is one more interesting fact about the time-reversal operator T: it is *antiunitary*, rather than unitary.

Antiunitary means that $T^{-1}iT = -i$.

To see why this must be the case, consider a Lorentz transformation of the energy-momentum four-vector,

$$U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \ . \tag{23.14}$$

For parity and time-reversal, we therefore expect

$$P^{-1} P^\mu P = \mathcal{P}^\mu{}_\nu P^\nu \ , \tag{23.15}$$

$$T^{-1} P^\mu T = \mathcal{T}^\mu{}_\nu P^\nu \ . \tag{23.16}$$

In particular, for $\mu = 0$, we expect $P^{-1}HP = +H$ and $T^{-1}HT = -H$.

The first of these is fine; it says the hamiltonian is invariant under parity, which is what we want.

However, eq. (23.16) is a disaster: it says that the hamiltonian is invariant under time-reversal if and only if $H = -H$, which is possible only if $H = 0$.

Can we just put an extra minus sign on the right-hand side of eq. (23.16), as we did for eq. (23.11)?

The answer is no.

We constructed P^μ explicitly in terms of the fields in section 22, and it is easy to check that choosing eq. (23.11) for the fields does not yield an extra minus sign in eq. (23.16) for the energy-momentum four-vector.

Let us reconsider the origin of eq. (23.14).

We first recall that the spacetime translation operator

$$T(a) = \exp(-iP\cdot a) \ . \tag{23.17}$$

(which should not be confused with the time-reversal operator T) transforms a scalar field according to

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a) \ . \tag{23.18}$$

The spacetime translation operator is a scalar with a spacetime coordinate as a label; by analogy with eq. (23.3), we should have

$$U(\Lambda)^{-1}T(a)U(\Lambda) = T(\Lambda^{-1}a) \ . \tag{23.19}$$

Now, treat a^μ as infinitesimal in eq. (23.19) to get

$$\begin{aligned} U(\Lambda)^{-1}(I - ia_\mu P^\mu)U(\Lambda) &= I - i(\Lambda^{-1})_\nu{}^\mu a_\mu P^\nu \\ &= I - i\Lambda^\mu{}_\nu a_\mu P^\nu \ . \end{aligned} \tag{23.20}$$

For time-reversal, this becomes

$$T^{-1}(I - ia_\mu P^\mu)T = I - i\mathcal{T}^\mu{}_\nu a_\mu P^\nu \ . \tag{23.21}$$

If we now identify the coefficients of $-ia_\mu$ on each side, we get eq. (23.16), which is bad.

In order to get the extra minus sign that we need, we must impose the antiunitary condition

$$T^{-1} i T = -i \ . \tag{23.22}$$

We then find

$$T^{-1} P^\mu T = -\mathcal{T}^\mu{}_\nu P^\nu \tag{23.23}$$

instead of eq. (23.16).

This yields $T^{-1}HT = +H$, which is the correct expression of time-reversal invariance.

We turn now to other unitary operators that change the signs of scalar fields, but do nothing to their spacetime arguments.

Suppose we have a theory with real scalar fields $\varphi_a(x)$, and a unitary operator Z that obeys

$$Z^{-1}\varphi_a(x)Z = \eta_a\varphi_a(x)\ , \tag{23.24}$$

where η_a is either +1 or−1 for each field.

We will call Z a Z_2 *operator*, because Z_2 is the additive group of the integers modulo 2, which is equivalent to the multiplicative group of +1 and−1.

This also implies that $Z^2 = 1$, and so $Z^{-1} = Z$.

For theories with spin-zero fields only, the same is also true of P and T, but things are more subtle for higher spin, as we will see in Part II.)

Consider the theory of a complex scalar field $\varphi = (\varphi_1 + i\,\varphi_2)/\sqrt{2}$ that was introduced in section 22, with lagrangian

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 \tag{23.25}$$

$$= -\tfrac{1}{2}\partial^\mu\varphi_1\partial_\mu\varphi_1 - \tfrac{1}{2}\partial^\mu\varphi_2\partial_\mu\varphi_2 - \tfrac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - \tfrac{1}{16}\lambda(\varphi_1^2+\varphi_2^2)^2. \tag{23.26}$$

In the form of eq. (23.25), $\mathcal{L}$ is obviously invariant under the U(1) transformation

$$\varphi(x) \to e^{-i\alpha}\varphi(x)\ . \tag{23.27}$$

In the form of eq. (23.26), $\mathcal{L}$ is obviously invariant under the equivalent SO(2) transformation,

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} . \tag{23.28}$$

However, it is also obvious that $\mathcal{L}$ has an additional discrete symmetry,

$$\varphi(x) \leftrightarrow \varphi^\dagger(x) \tag{23.29}$$

in the form of eq. (23.25), or equivalently

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \to \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} . \tag{23.30}$$

in the form of eq. (2

This discrete symmetry is called *charge conjugation*.

It always occurs as a companion to a continuous U(1) symmetry.

In terms of the two real fields, it enlarges the group from SO(2) (the group of 2×2 orthogonal matrices with determinant +1) to O(2) (the group of 2×2 orthogonal matrices).

We can implement charge conjugation by means of a particular Z_2 operator C that obeys

$$C^{-1}\varphi(x)C = \varphi^\dagger(x) \ , \tag{23.31}$$

or equivalently

$$C^{-1}\varphi_1(x)C = +\varphi_1(x)\ , \tag{23.32}$$

$$C^{-1}\varphi_2(x)C = -\varphi_2(x)\ . \tag{23.33}$$

We then have

$$C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)\ , \tag{23.34}$$

and so charge conjugation is a symmetry of the theory.

Physically, it implies that the scattering amplitudes are unchanged if we exchange all the a-type particles (which have charge +1) with all the b-type particles (which have charge−1).

This means, in particular, that the a and b particles must have exactly the same mass.

We say that b is a's *antiparticle*.

More generally, we can also have Z_2 symmetries that are not related to antiparticles.

Consider, for example, φ^4 theory, where φ is a real scalar field with lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{24}\lambda\varphi^4\ . \tag{23.35}$$

If we define the Z_2 operator Z via

$$Z^{-1}\varphi(x)Z = -\varphi(x)\ , \tag{23.36}$$

then $\mathcal{L}$ is obviously invariant.

We therefore have $Z^{-1}HZ = H$, or equivalently $[Z,H] = 0$, where H is the hamiltonian.

If we assume that (as usual) the ground state is unique, then, since Z commutes with H, the ground state must also be an eigenstate of Z.

We can fix the phase of Z [which is undetermined by eq. (23.36)] via

$$Z|0\rangle = Z^{-1}|0\rangle = +|0\rangle \ . \tag{23.37}$$

then, using eqs. (23.36) and (23.37), we have

$$\begin{aligned} \langle 0|\varphi(x)|0\rangle &= \langle 0|ZZ^{-1}\varphi(x)ZZ^{-1}|0\rangle \\ &= -\langle 0|\varphi(x)|0\rangle \ . \end{aligned} \tag{23.38}$$

Since $\langle 0|\phi(x)|0\rangle$ is equal to minus itself, it must be zero.

Thus, as long as the ground state is unique, the Z_2 symmetry of φ^4 theory guarantees that the field has zero vacuum expectation value.

We therefore do not need to enforce this condition with a counterterm $Y\varphi$, as we did in φ^3 theory.

The assumption of a unique ground state does not necessarily hold, however, as we will see in section 30.

24 Nonabelian Symmetries

Consider the theory (introduced in section 22) of a two real scalar fields φ_1 and φ_2 with

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_1\partial_\mu\varphi_1 - \tfrac{1}{2}\partial^\mu\varphi_2\partial_\mu\varphi_2 - \tfrac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - \tfrac{1}{16}\lambda(\varphi_1^2+\varphi_2^2)^2 \ . \tag{24.1}$$

We can generalize this to the case of N real scalar fields φ_i with

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}m^2\varphi_i\varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2 \ , \tag{24.2}$$

where a repeated index is summed.

This lagrangian is clearly invariant under the SO(*N*) transformation

$$\varphi_i(x) \to R_{ij}\varphi_j(x) \ , \tag{24.3}$$

where R is an orthogonal matrix with a positive determinant: $R^{\mathrm{T}} = R^{-1}$, det $R = +1$.

This largrangian is also clearly invariant under the Z2 transformation $\varphi_i(x) \to -\varphi_i(x)$, which enlarges SO(N) to O(N); see section 23.

However, in this section we will be concerned only with the continuous SO(N) part of the symmetry.

Next we will need some results from group theory.

Consider an infinitesimal SO(N) transformation,

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2) \ . \tag{24.4}$$

Orthogonality of R_{ij} implies that θ_{ij} is real and antisymmetric.

It is convenient to express θ_{ij} in terms of a basis set of hermitian matrices $(T^a)_{ij}$.

The index a runs from 1 to 1/2 N(N−1), the number of linearly independent, hermitian, antisymmetric, N × N matrices.

We can, for example, choose each T^a to have a single nonzero entry −i above the main diagonal, and a corresponding +i below the main diagonal.

These matrices obey the normalization condition

$$\mathrm{Tr}(T^a T^b) = 2\delta^{ab} \ . \tag{24.5}$$

In terms of them, we can write

$$\theta_{jk} = -i\theta^a (T^a)_{jk} \ , \tag{24.6}$$

where θ^a is a set of 1/2 N(N−1) real, infinitesimal parameters.

The T^a's are the *generator matrices* of SO(N).

The product of any two SO(N) transformations is another SO(N) transformation; this implies that the commutator of any two generator matrices must be a linear combination of generator matrices,

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{24.7}$$

The numerical factors f^{abc} are the structure coefficients of the group, and eq. (24.7) specifies its Lie algebra.

If $f^{abc} = 0$, the group is abelian.

Otherwise, it is nonabelian.

Thus, U(1) and SO(2) are abelian groups (since they each have only one generator that obviously must commute with itself), and SO(N) for $N \geq 3$ is nonabelian.

If we multiply eq. (24.7) on the right by T^d, take the trace, and use eq. (24.5), we find

$$f^{abd} = -\tfrac{1}{2} i \,\mathrm{Tr}\Big([T^a, T^b] T^d \Big) \ . \tag{24.8}$$

Using the cyclic property of the trace, we find that f^{abc} must be completely antisymmetric.

Taking the complex conjugate of eq. (24.8) (and remembering that the T^a's are hermitian matrices), we find that f^{abc} must be real.

The simplest nonabelian group is SO(3).

In this case, we can choose $(T^a)_{ij} = -i\varepsilon^{aij}$, where ε^{ijk} is the completely antisymmetric Levi-Civita symbol, with $\varepsilon^{123} = +1$.

The commutation relations become

$$[T^a, T^b] = i\varepsilon^{abc} T^c \ . \tag{24.9}$$

That is, the structure coefficients of SO(3) are given by $f^{abc} = \varepsilon^{abc}$.

Consider now a theory with N complex scalar fields φ_i, and a lagrangian

$$\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \tfrac{1}{4}\lambda(\varphi_i^\dagger \varphi_i)^2 \ , \tag{24.10}$$

where a repeated index is summed.

This lagrangian is clearly invariant under the U(N) transformation

$$\varphi_i(x) \to U_{ij}\varphi_j(x) \ , \tag{24.11}$$

where U is a unitary matrix: $U^\dagger = U^{-1}$.

We can write $U_{ij} = e^{-i\theta} U_{ij}$, where θ is a real parameter and det $U_{ij} = +1$; U_{ij} is called a special unitary matrix.

Clearly the product of two special unitary matrices is another special unitary matrix; the N× N special unitary matrices form the group SU(N).

The group U(N) is the direct product of the group U(1) and the group SU(N); we write $U(N) = U(1) \times SU(N)$.

Consider an infinitesimal SU(N) transformation,

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + O(\theta^2) \ , \tag{24.12}$$

where θ^a is a set of real, infinitesimal parameters.

Unitarity of U implies that the generator matrices T^a are hermitian, and det U = +1 implies that each T^a is traceless.

This follows from the general matrix formula ln det A= Tr ln A.

The index a runs from 1 to N^2-1, the number of linearly independent, hermitian, traceless, N × N matrices.

We can choose these matrices to obey the normalization condition of eq. (24.5).

For SU(2), the generators can be chosen to be the Pauli matrices; the structure coefficients of SU(2) then turn out to be $f^{abc} = 2\varepsilon^{abc}$, the same as those of SO(3), up to an irrelevant overall factor, which could be removed by changing the numerical factor on right-hand side of eq. (24.5) from 2 to 1/2.

For SU(N), we can choose the T^a's in the following way.

First, there are the SO(N) generators, with one −i above the main diagonal a corresponding +i below; there are 1/2 N(N−1) of these.

Next, we get another set by putting one +1 above the main diagonal and a corresponding +1 below; there are 1/2 N(N−1) of these.

Finally, there are diagonal matrices with n 1's along the main diagonal, followed a single entry −n, followed by zeros [times an overall normalization constant to enforce eq. (24.5)]; there are N−1 of these. The total is $N^2 - 1$, as required.

However, if we examine the lagrangian of eq. (24.10) more closely, we find that it is actually invariant under a larger symmetry group, namely SO(2N).

To see this, write each complex scalar field in terms of two real scalar fields, $\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}$.

Then

$$\varphi_j^\dagger \varphi_j = \tfrac{1}{2}(\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2) \; . \tag{24.13}$$

Thus, we have 2N real scalar fields that enter $\mathcal{L}$ symmetrically, and so the actual symmetry group of eq. (24.10) is SO(2N), rather than just the obvious subgroup U(N).

We will, however, meet the SU(N) groups again in Parts II and III, where they will play a more important role.

25 Unstable Particles and Resonances

Consider a theory of two real scalar fields, φ and χ, with lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m_\varphi^2\varphi^2 - \tfrac{1}{2}\partial^\mu\chi\partial_\mu\chi - \tfrac{1}{2}m_\chi^2\chi^2 + \tfrac{1}{2}g\varphi\chi^2 + \tfrac{1}{6}h\varphi^3 \tag{25.1}$$

This theory is renormalizable in six dimensions, where g and h are dimensionless coupling constants.

Let us assume that $m_\varphi > 2m_\chi$.

Then it is kinematically possible for the φ particle to decay into two χ particles.

The amplitude for this process is given at tree level by the Feynman diagram of fig. (25.1), and is simply

$$\mathcal{T} = g$$

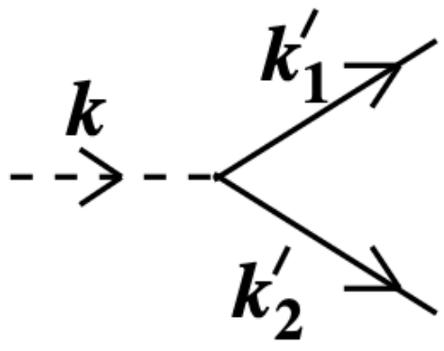


Figure 25.1: The tree-level Feynman diagram for the decay of a φ particle (dashed line) into two χ particles (solid lines).

We can also choose to define g as the value of the exact $\varphi\,\chi^2$ vertex function $\mathbf{V_3}(k,k'_1,k'_2)$ when all three particles are on shell: $k^2 = -m^2_\varphi$, $k'^2_1 = k'^2_2 = -m^2_\chi$.

This implies that

$$\mathcal{T} = g \tag{25.2}$$

exactly.

According to the formulae of section 11, the differential decay rate (in the rest frame of the initial φ particle) is

$$d\Gamma = \frac{1}{2m_\varphi}\, d\mathrm{LIPS}_2\, |\mathcal{T}|^2\ , \tag{25.3}$$

where dLIPS$_2$ is the Lorentz invariant phase space differential for two outgoing particles, introduced in section 11.

We must make a slight adaptation for six dimensions:

$$d\mathrm{LIPS}_2 \equiv (2\pi)^6 \delta^6(k_1'+k_2'-k)\, \widetilde{dk}_1'\, \widetilde{dk}_2' \ . \tag{25.4}$$

Here k= (m_φ,0) is the energy-momentum of the decaying particle, and

$$\widetilde{dk} = \frac{d^5k}{(2\pi)^5 2\omega} \tag{25.5}$$

is the Lorentz-invariant phase-space differential for one particle.

Recall that we can also write it as

$$\widetilde{dk} = \frac{d^6k}{(2\pi)^6}\, 2\pi\delta(k^2+m_\chi^2)\, \theta(k^0) \ , \tag{25.6}$$

where θ(x) is the unit step function.

Performing the integral over k^0 turns eq. (25.6) into eq. (25.5).

Repeating for six dimensions what we did in section 11 for four dimensions, we find

$$d\mathrm{LIPS}_2 = \frac{|\mathbf{k}'_1|^3}{4(2\pi)^4 m_\varphi}\, d\Omega \ , \tag{25.7}$$

where $|\mathbf{k}'_1|=1/2\ (m^2_\varphi - 4m^2_\chi)^{1/2}$ is the magnitude of the spatial momentum of one of the outgoing particles.

We can now plug this into eq. (25.3), and use $d\Omega = \Omega_5 = 2\pi^{5/2}\ /\Gamma(5/2) = 8/3\ \pi^2$.

We also need to divide by a symmetry factor of two, due to the presence of two identical particles in the final state.

The result is

$$\Gamma = \frac{1}{2}\cdot\frac{1}{2m_\varphi}\int d\mathrm{LIPS}_2\,|\mathcal{T}|^2 \tag{25.8}$$

$$= \tfrac{1}{12}\pi\alpha(1-4m^2_\chi/m^2_\varphi)^{3/2}\,m_\varphi \ , \tag{25.9}$$

where $\alpha = g^2/(4\pi)^3$.

However, as we discussed in section 11, we have a conceptual problem.

According to our development of the LSZ formula in section 5, each incoming and outgoing particle should correspond to a single-particle state that is an exact eigenstate of the exact hamiltonian.

This is clearly not the case for a particle that can decay.

Let us, then, compute something else instead: the correction to the φ propagator from a loop of χ particles, as shown in fig. (25.2).

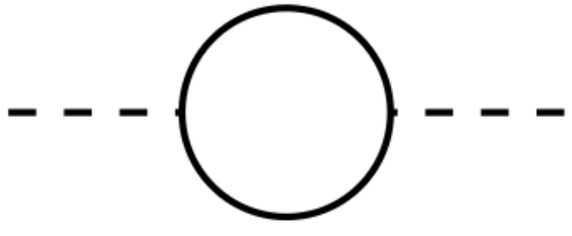


Figure 25.2: A loop of χ particles correcting the φ propagator.

The diagram is the same as the one we already analyzed in section 14, except that the internal propagators contain m_χ instead of m_φ.

There is also a contribution from a loop of φ particles, but we can ignore it if we assume that $h \ll g$.)

We have

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln D - A' k^2 - B' m_\varphi^2 \,, \tag{25.10}$$

where

$$D = x(1-x)k^2 + m_\chi^2 - i\epsilon \,, \tag{25.11}$$

and A and B are the finite counterterm coefficients that remain after the infinities have been absorbed.

We now try to fix A and B by imposing the usual on-shell conditions $\Pi(-m^2_\varphi) = 0$ and $\Pi'(-m^2_\varphi) = 0$.

But, we have a problem - for $k^2 = -\, m^2_\varphi$ and $m_\varphi > 2m_\chi$, D is negative for part of the range of x.

Therefore *l*n *D* has an imaginary part.

This imaginary part cannot be canceled by A′ and B′, since A′ and B′ must be real: they are coefficients of hermitian operators in the lagrangian.

The best we can do is Re $\Pi(-m^2_\varphi) = 0$ and Re $\Pi'(-m^2_\varphi) = 0$.

Imposing these gives

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/|D_0|) - \tfrac{1}{12}\alpha(k^2 + m_\varphi^2)\ , \tag{25.12}$$

where

$$D_0 = -x(1-x)m_\varphi^2 + m_\chi^2\ . \tag{25.13}$$

Now let us compute the imaginary part of $\Pi(k^2)$.

This arises from the integration range $x_- < x < x_+$, where $x_\pm = 1/2 \pm 1/2\,(1 + m^2_\chi/k^2)^{1/2}$ are the roots of $D = 0$ when $k^2 < -4m^2_\chi$.

In this range, Im ln $D = -i\pi$; the minus sign arises because, according to eq. (25.11), D has a small negative imaginary part.

Now we have

$$\begin{aligned} \mathrm{Im}\,\Pi(k^2) &= -\tfrac{1}{2}\pi\alpha \int_{x_-}^{x_+} dx\, D \\ &= -\tfrac{1}{12}\pi\alpha(1 + 4m_\chi^2/k^2)^{3/2}\, k^2 \end{aligned} \tag{25.14}$$

when $k^2 < -4m^2_\chi$.

Evaluating eq. (25.14) at $k^2 = -m^2_\varphi$, we get

$$\mathrm{Im}\,\Pi(-m^2_\varphi) = \tfrac{1}{12}\pi\alpha(1 - 4m^2_\chi/m^2_\varphi)^{3/2}\, m^2_\varphi \ . \tag{25.15}$$

From this and eq. (25.9), we see that

$$\mathrm{Im}\,\Pi(-m^2_\varphi) = m_\varphi\Gamma \ . \tag{25.16}$$

This is not an accident.

Instead, it is a general rule.

We will argue this in two ways: first, from the mathematics of Feynman diagrams, and second, from the physics of resonant scattering in quantum mechanics.

We begin with the mathematics of Feynman diagrams.

Return to the diagrammatic expression for $\Pi(k^2)$, before we evaluated any of the integrals:

$$\begin{aligned}\Pi(k^2) = -\tfrac{1}{2}ig^2 \int \frac{d^6\ell_1}{(2\pi)^6}\,\frac{d^6\ell_2}{(2\pi)^6}\,(2\pi)^6\delta^6(\ell_1+\ell_2-k) \\ \times \frac{1}{\ell_1^2 + m_\chi^2 - i\epsilon}\,\frac{1}{\ell_2^2 + m_\chi^2 - i\epsilon} \\ - (Ak^2 + Bm^2_\varphi) \ .\end{aligned} \tag{25.17}$$

Here, for later convenience, we have assigned the internal lines momenta ℓ_1 and ℓ_2, and explicitly included the momentum-conserving delta function that fixes one of them.

We can take the imaginary part of $\Pi(k^2)$ by using the identity

$$\frac{1}{x - i\epsilon} = P\frac{1}{x} + i\pi\delta(x) \ , \tag{25.18}$$

where P means the principal part.

We then get, in a shorthand notation,

$$\mathrm{Im}\,\Pi(k^2) = -\tfrac{1}{2}g^2 \int \left(P_1 P_2 - \pi^2 \delta_1 \delta_2\right) \ . \tag{25.19}$$

Next, we notice that the integral in eq. (25.17) is the Fourier transform of $[\Delta(x-y)]^2$, where

$$\Delta(x-y) = \int \frac{d^6k}{(2\pi)^6} \, \frac{e^{ik(x-y)}}{k^2 + m_\chi^2 - i\epsilon} \tag{25.20}$$

is the Feynman propagator.

Recall that we can get the retarded or advanced propagator (rather than the Feynman propagator) by replacing the ϵ in eq. (25.20) with, respectively, $-s\epsilon$ or $+s\epsilon$, where $s \equiv \mathrm{sign}(k^0)$.

Therefore, in eq. (25.19), replacing δ_1 with $-s_1\delta_1$ and δ_2 with $+s_2\delta_2$ yields an integral that is the real part of the Fourier transform of $\Delta_{\mathrm{ret}}(x-y)\Delta_{\mathrm{adv}}(x-y)$.

But this product is zero, because the first factor vanishes when $x_0 \geq y_0$, and the second when $x_0 \leq y_0$.

So we can subtract the modified integrand from the original without changing the value of the integral.

Thus we have

$$\mathrm{Im}\,\Pi(k^2) = \tfrac{1}{2} g^2 \pi^2 \int (1 + s_1 s_2)\delta_1 \delta_2 \ . \tag{25.21}$$

The factor of $1+ s_1 s_2$ vanishes if ℓ^0_1 and ℓ^0_2 have opposite signs, and equals 2 if they have the same sign.

Because the delta function in eq. (25.17) enforces $\ell^0_1 + \ell^0_2 = k^0$, and $k^0 = m_\varphi$ is positive, both ℓ^0_1 and ℓ^0_2 must be positive.

So we can replace the factor of $1+ s_1 s_2$ in eq. (25.21) with $2\theta(\ell^0_1)\theta(\ell^0_2)$.

Rearranging the numerical factors, we have

$$\begin{aligned}\mathrm{Im}\,\Pi(k^2) = \tfrac{1}{4} g^2 \int \frac{d^6\ell_1}{(2\pi)^6}\,\frac{d^6\ell_2}{(2\pi)^6}\,(2\pi)^6\delta^6(\ell_1+\ell_2-k) \\ \times\, 2\pi\delta(\ell_1^2 + m_\chi^2)\theta(\ell_1^0)\, 2\pi\delta(\ell_2^2 + m_\chi^2)\theta(\ell_2^0)\ .\end{aligned} \tag{25.22}$$

If we now set $k^2 = -m^2\varphi$, use eqs. (25.4) and (25.6), and recall that $\mathcal{T} = g$ the decay amplitude, we can rewrite eq. (25.22) as

$$\mathrm{Im}\,\Pi(-m_\varphi^2) = \tfrac{1}{4} \int d\mathrm{LIPS}_2\, |\mathcal{T}|^2\ . \tag{25.23}$$

Comparing eqs. (25.8) and (25.23), we see that we indeed have

$$\mathrm{Im}\,\Pi(-m_\varphi^2) = m_\varphi \Gamma \ . \tag{25.24}$$

This relation persists at higher orders in perturbation theory.

Our analysis can be generalized to give the Cutkosky rules for computing the imaginary part of any Feynman diagram, but this is beyond the scope of our current interest.

To get a more physical understanding of this result, recall that in non-relativistic quantum mechanics, a metastable state with energy E_0 and angular momentum quantum number ℓ shows up as a resonance in the partial-wave scattering amplitude,

$$f_\ell(E) \sim \frac{1}{E - E_0 + i\Gamma/2} \ . \tag{25.25}$$

If we imagine convolving this amplitude with a wave packet; $\tilde{\psi}(E)e^{-iEt}$, we will find a time dependence

$$\begin{aligned} \psi(t) &\sim \int dE \, \frac{1}{E - E_0 + i\Gamma/2} \, \tilde{\psi}(E) e^{-iEt} \\ &\sim e^{-iE_0 t - \Gamma t/2} \ . \end{aligned} \tag{25.26}$$

Therefore $|\psi(t)|^2 \sim e^{-\Gamma t}$, and we identify Γ as the inverse lifetime of the metastable state.

In the relativistic case, consider the scattering process $\chi\chi \to \chi\chi$.

The contributing diagrams from the effective action are those of fig. (20.1), where the exact internal propagator can be either φ or χ.

Suppose that the center-of-mass energy squared s is close to m^2_φ.

Since the φ propagator has a pole near $s = m^2_\varphi$, s-channel φ exchange, shown in fig. (25.3), makes the dominant contribution to the $\chi\chi$ scattering amplitude.

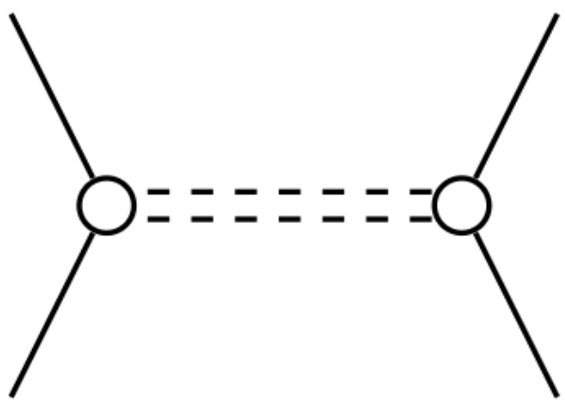


Figure 25.3: For s near m^2_φ, the dominant contribution to $\chi\chi$ scattering is s-channel φ exchange.

We then have

$$\mathcal{T} \simeq \frac{g^2}{-s + m_\varphi^2 - \Pi(-s)} \ . \tag{25.27}$$

Here we have used the fact that the exact $\varphi\chi\chi$ vertex has the value g when all three particles are on-shell.

Now let us write

$$s = (m_\varphi + \varepsilon)^2 \simeq m_\varphi^2 + 2m_\varphi\varepsilon \ , \tag{25.28}$$

where $\varepsilon \ll m_\varphi$ is the amount of energy by which our incoming particles are off resonance.

We find

$$\mathcal{T} \simeq \frac{-g^2/2m_\varphi}{\varepsilon + \Pi(-m_\varphi^2)/2m_\varphi} \,. \tag{25.29}$$

Recalling that Re $\Pi(-m^2_\varphi) = 0$, and comparing with eq. (25.25), we see that we should make the identification of eq. (25.24).

26 Infrared Divergences

In section 20, we computed the $\varphi\varphi \to \varphi\varphi$ scattering amplitude in φ^4 theory in six dimensions in the high-energy limit (s, |t|, and |u| all much larger than m^2).

We found that

$$\mathcal{T} = \mathcal{T}_0 \left[1 - \tfrac{11}{12}\alpha\Big(\ln(s/m^2) + O(m^0)\Big) + O(\alpha^2)\right], \tag{26.1}$$

where $\mathcal{T}_0 = -g^2(s^{-1} + t^{-1} + u^{-1})$ is the tree-level result, and the $O(m^0)$ term includes everything without a *large logarithm* that blows up in the limit $m \to 0$.

Suppose we are interested in the limit of massless particles.

The large log is then problematic, since it blows up in this limit.

What does this mean?

It means we have made a mistake - actually, two mistakes - in this section, we will remedy one of them.

Throughout the physical sciences, it is necessary to make various idealizations in order to make progress.

Recall the “massless springs” and “frictionless planes” of freshman mechanics.

Sometimes these idealizations can lead us into trouble, and that is one of the things that has gone wrong here.

We have assumed that we can isolate individual particles.

The reasoning behind this was explained in section 5, and it depends on the existence of an energy gap between the one-particle states and the multiparticle continuum.

However, this gap vanishes if the theory includes massless particles.

In this case, it is possible that the scattering process involved the creation of some extra very low energy (or soft) particles that escaped detection.

Or, there may have been some extra soft particles hiding in the initial state that discreetly participated in the scattering process.

Or, what was seen as a single high-energy particle may actually have been two or more particles that were moving colinearly and sharing the energy.

Let us, then, correct our idealization of a perfect detector and account for these possibilities.

We will work with φ^3 theory, initially in d spacetime dimensions.

Let $\mathcal{T}$ be the amplitude for some scattering process in φ^3 theory.

Now consider the possibility that one of the outgoing particles in this process splits into two, as shown in fig. (26.1).

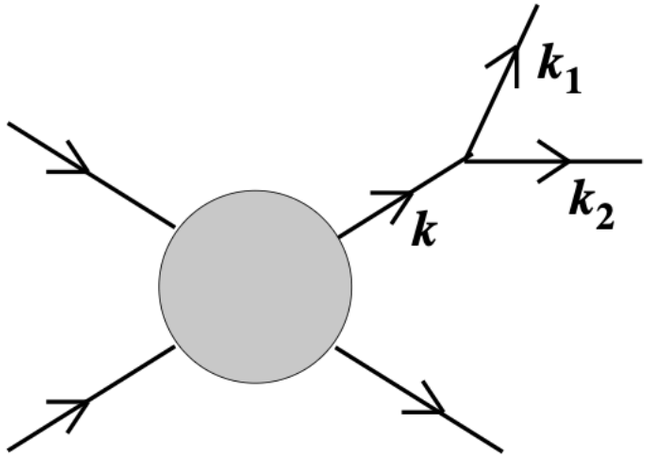


Figure 26.1: An outgoing particle splits into two. The gray circle stands for the sum of all diagrams contributing to the original amplitude i $\mathcal{T}$.

The amplitude for this new process is given in terms of $\mathcal{T}$ by

$$\mathcal{T}_{\text{split}} = ig\,\frac{-i}{k^2+m^2}\,\mathcal{T}\ ,$$

where $k = k_1 + k_2$, and k_1 and k_2 are the on-shell four-momenta of the two particles produced by the split.

For notational convenience, we drop our usual primes on the outgoing momenta.

The key point is this: in the massless limit, it is possible for $1/(k^2 + m^2)$ to diverge.

To understand the physical consequences of this possibility, we should compute an appropriate cross-section.

To get the cross section for the original process (without the split), we multiply $|\mathcal{T}|^2$ by $\widetilde{dk}$ as well as by similar differentials for other outgoing particles, and by an overall energy-momentum delta function.

For the process with the split, we multiply $|\mathcal{T}_{\text{split}}|^2$ by $1/2\ \widetilde{dk}_1\widetilde{dk}_2$ instead of $\widetilde{dk}$.

The factor of one-half is for counting of identical particles.)

If we assume that (due to some imperfection) our detector cannot tell whether or not the one particle actually split into two, then we should (according to the usual rules of quantum mechanics) add the probabilities for the two events, which are distinguishable in principle.

We can therefore define an effectively observable squared-amplitude via

$$|\mathcal{T}|^2_{\text{obs}}\,\widetilde{dk} = |\mathcal{T}|^2\,\widetilde{dk} + |\mathcal{T}_{\text{split}}|^2\,\tfrac{1}{2}\widetilde{dk}_1\widetilde{dk}_2 + \ldots\ . \tag{26.3}$$

Here the ellipses stand for all other similar processes involving emission of one or more extra particles in the final state, or absorption of one or more extra particles in the initial state.

We can simplify eq. (26.3) by including a factor of

$$1 = (2\pi)^{d-1}\,2\omega\,\delta^{d-1}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\widetilde{dk} \tag{26.4}$$

in the second term.

Now all terms in eq. (26.3) include a factor of $\widetilde{dk}$, so we can drop it.

Then, using eq. (26.2), we get

$$|\mathcal{T}|^2_{\rm obs} \equiv |\mathcal{T}|^2 \left[1 + \frac{g^2}{(k^2+m^2)^2}\,(2\pi)^{d-1}\,2\omega\,\delta^{d-1}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,\tfrac{1}{2}\widetilde{dk}_1\widetilde{dk}_2 + \ldots\right] . \qquad (26.5)$$

Now we come to the point: in the massless limit, the phase space integral in the second term in eq. (26.5) can diverge.

This is because, for m= 0,

$$k^2 = (k_1+k_2)^2 = -4\omega_1\omega_2\sin^2(\theta/2)\ , \qquad (26.6)$$

where θ is the angle between the spatial momenta k_1 and k_2, and $\omega_{1,2} = |\mathbf{k}_{1,2}|$.

Also, for m = 0,

$$\widetilde{dk}_1\widetilde{dk}_2 \sim (\omega_1^{d-3}\,d\omega_1)\,(\omega_2^{d-3}\,d\omega_2)\,(\sin^{d-3}\theta\,d\theta)\ . \qquad (26.7)$$

Therefore, for small θ

$$\frac{\widetilde{dk}_1\widetilde{dk}_2}{(k^2)^2} \sim \frac{d\omega_1}{\omega_1^{5-d}}\,\frac{d\omega_2}{\omega_2^{5-d}}\,\frac{d\theta}{\theta^{7-d}}\ . \qquad (26.8)$$

Thus the integral over each ω diverges at the low end for d ≤ 4, and the integral over θ diverges at the low end for d ≤ 6.

These divergent integrals would be cut off (and rendered finite) if we kept the mass m nonzero, as we will see below.

Our discussion leads us to expect that the m $\rightarrow$ 0 divergence in the second term of eq. (26.5) should cancel the m $\rightarrow$ 0 divergence in the loop correction to $|\mathcal{T}|^2$.

We will now see how this works (or fails to work) in detail for the familiar case of two-particle scattering in six spacetime dimensions, where $\mathcal{T}$ is given by eq. (26.1).

For d = 6, there is no problem with soft particles (corresponding to the small-ω divergence), but there is a problem with collinear particles (corresponding to the small-θ divergence).

Let us assume that our imperfect detector cannot tell one particle from two nearly collinear particles if the angle θ between their spatial momenta is less than some small angle δ.

Since we ultimately want to take the m$\rightarrow$ 0 limit, we will evaluate eq. (26.5) with $m^2/\mathbf{k}^2 \ll \delta^2 \ll 1$.

We can immediately integrate over d^5k_2 using the delta function, which results in setting $\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1$ everywhere.

Let β then be the angle between $\mathbf{k}_1$ (which is still to be integrated over) and $\mathbf{k}$ (which is fixed).

For two-particle scattering, $|\mathbf{k}|=1/2\ \sqrt{s}$ in the limit m$\rightarrow$ 0.

We then have

$$(2\pi)^5\, 2\omega\, \delta^5(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\, \tfrac{1}{2}\widetilde{dk}_1\widetilde{dk}_2 \rightarrow \frac{\Omega_4}{4(2\pi)^5}\,\frac{\omega}{\omega_1\omega_2}\,|\mathbf{k}_1|^4\, d|\mathbf{k}_1|\, \sin^3\beta\, d\beta\ , \qquad (26.9)$$

where $\Omega_4 = 2\pi^2$ is the area of the unit four-sphere.

Now let γ be the angle between $\mathbf{k}_2$ and $\mathbf{k}$.

The geometry of this trio of vectors implies $\theta = \beta + \gamma$, $|\mathbf{k}_1| = (\sin\gamma/\sin\theta)|\mathbf{k}|$, and $|\mathbf{k}_2| = (\sin\beta/\sin\theta)|\mathbf{k}|$.

All three of the angles are small and positive, and it then is useful to write $\beta = x\theta$ and $\gamma = (1-x)\theta$, with $0 \le x \le 1$ and $\theta \le \delta \ll 1$.

In the low mass limit, we can safely set m= 0 everywhere in eq. (26.5) except in the propagator, $1/(k^2 + m^2)$.

Then, expanding to leading order in both θ and m, we find (after some algebra)

$$k^2 + m^2 \simeq -x(1-x)\mathbf{k}^2 \left[\theta^2 + (m^2/\mathbf{k}^2) f(x)\right], \tag{26.10}$$

where $f(x) = (1-x+x^2)/(x-x^2)^2$.

Everywhere else in eq. (26.5), we can safely set $\omega_1 = |\mathbf{k}_1| = (1-x)\,|\mathbf{k}|$ and $\omega_2 = |\mathbf{k}_2| = x|\mathbf{k}|$.

Then, changing the integration variables in eq. (26.9) from $|\mathbf{k}_1|$ and β to x and θ, we get

$$|\mathcal{T}|^2_{\rm obs} = |\mathcal{T}|^2 \left[1 + \frac{g^2\Omega_4}{4(2\pi)^5} \int_0^1 x(1-x)dx \int_0^\delta \frac{\theta^3\, d\theta}{[\theta^2 + (m^2/\mathbf{k}^2) f(x)]^2} + \ldots\right]. \tag{26.11}$$

Performing the integral over θ yields

$$\tfrac{1}{2}\ln(\delta^2\mathbf{k}^2/m^2) - \tfrac{1}{2}\ln f(x) - \tfrac{1}{2}\ . \tag{26.12}$$

Then, performing the integral over x and using $\Omega_4 = 2\pi^2$ and $\alpha = g^2/(4\pi)^3$,we get

$$|\mathcal{T}|^2_{\rm obs} = |\mathcal{T}|^2\left[1 + \tfrac{1}{12}\alpha\Big(\ln(\delta^2\mathbf{k}^2/m^2) + c\Big) + \ldots\right], \tag{26.13}$$

where $c = (4 - 3\sqrt{3}\pi)/3 = -4.11$.

The displayed correction term accounts for the possible splitting of one of the two outgoing particles.

Obviously, there is an identical correction for the other outgoing particle.

Less obviously (but still true), there is an identical correction for each of the two incoming particles.

A glib explanation is that we are computing an effective amplitude-squared, and this is the same for the reverse process, with in and outgoing particles switched.

So in and out particles should be treated symmetrically.

Then, since we have a total of four in and out particles (before accounting for any splitting,

$$|\mathcal{T}|^2_{\rm obs} = |\mathcal{T}|^2\left[1 + \tfrac{4}{12}\alpha\Big(\ln(\delta^2\mathbf{k}^2/m^2) + c\Big) + O(\alpha^2)\right]. \tag{26.14}$$

We have now accounted for the O(α) corrections due to the failure of our detector to separate two particles whose spatial momenta are nearly parallel.

Combining this with eq. (26.1), and recalling that $\mathbf{k}^2 = 1/4\ s$, we get

$$
\begin{aligned}
|\mathcal{T}|^2_{\rm obs} &= |\mathcal{T}_0|^2 \left[1 - \tfrac{11}{6}\alpha\Big(\ln(s/m^2) + O(m^0)\Big) + O(\alpha^2)\right] \\
&\qquad \times \left[1 + \tfrac{1}{3}\alpha\Big(\ln(\delta^2 s/m^2) + O(m^0)\Big) + O(\alpha^2)\right] \\
&= |\mathcal{T}_0|^2 \Big[1 - \alpha\Big(\tfrac{3}{2}\ln(s/m^2) + \tfrac{1}{3}\ln(1/\delta^2) + O(m^0)\Big) \\
&\qquad\qquad + O(\alpha^2)\Big].
\end{aligned}
\tag{26.15}
$$

We now have two kinds of large logs.

One is $\ln(1/\delta^2)$; this factor depends on the properties of our detector.

If we build a very good detector, one for which $\alpha\ln(1/\delta^2)$ is not small, then we will have to do more work, and calculate higher-order corrections to eq. (26.15).

The other large log is our original nemesis $\ln(s/m^2)$.

This factor blows up in the massless limit.

This means that there is still a mistake hidden somewhere in our analysis.

27 Other Renormalization Schemes

To find the remaining mistake in eq. (26.15), we must review our renormalization procedure.

Recall our result from section 14 for the one-loop correction to the propagator,

$$\begin{aligned}\Pi(k^2) = &-\left[A + \tfrac{1}{6}\alpha\left(\tfrac{1}{\varepsilon} + \tfrac{1}{2}\right)\right]k^2 - \left[B + \alpha\left(\tfrac{1}{\varepsilon} + \tfrac{1}{2}\right)\right]m^2 \\ &+ \tfrac{1}{2}\alpha\int_0^1 dx\, D\ln(D/\mu^2) + O(\alpha^2)\ ,\end{aligned} \tag{27.1}$$

where $\alpha = g^2/(4\pi)^3$ and $D = x(1-x)k^2+m^2$.

The derivative of $\Pi(k^2)$ with respect to k^2 is

$$\begin{aligned}\Pi'(k^2) = &-\left[A + \tfrac{1}{6}\alpha\left(\tfrac{1}{\varepsilon} + \tfrac{1}{2}\right)\right] \\ &+ \tfrac{1}{2}\alpha\int_0^1 dx\, x(1-x)\left[\ln(D/\mu^2) + 1\right] + O(\alpha^2)\ .\end{aligned} \tag{27.2}$$

We previously determined A and B via the requirements $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$.

The first condition ensures that the exact propagator $\widetilde{\Delta}(k^2)$ has a pole at $k^2 = -m^2$, and the second ensures that the residue of this pole is one.

Recall that the field must be normalized in this way for the validity of the LSZ formula.

We now consider the massless limit.

We have $D = x(1-x)k^2$, and we should apparently try to impose $\Pi(0) = \Pi'(0) = 0$.

However, $\Pi(0)$ is now automatically zero for any values of A and B, while $\Pi'(0)$ is ill defined.

Physically, the problem is that the one-particle states are no longer separated from the multiparticle continuum by a finite gap in energy.

Mathematically, the pole in $\widetilde{\Delta}(k^2)$ at $k^2 = -m^2$ merges with the branch point at $k^2 = -4m^2$, and is no longer a simple pole.

The only way out of this difficulty is to change the *renormalization scheme*.

Let us first see what this means in the case $m \neq 0$, where we know what we are doing.

Let us try making a different choice of A and B.

Specifically, let

$$A = -\tfrac{1}{6}\alpha\tfrac{1}{\varepsilon} + O(\alpha^2) \,,$$
$$B = -\alpha\tfrac{1}{\varepsilon} + O(\alpha^2) \,. \tag{27.3}$$

Here we have chosen A and B to cancel the infinities, and nothing more; we say that A and B have no finite parts.

This choice represents a different *renormalization scheme*.

Our original choice (which, up until now, we have pretended was inescapable!) is called the on-shell or OS scheme.

The choice of eq. (27.3) is called the *modified minimal-subtraction* or MS (pronounced "emm-ess-bar") scheme. ["Modified" because we introduced μ via $g \to g\,\tilde{\mu}^{-\epsilon/2}$, with $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$; had we set $\mu = \tilde{\mu}$ instead, the scheme would be just plain minimal subtraction or MS.]

Now we have

$$\Pi_{\overline{\rm MS}}(k^2) = -\tfrac{1}{12}\alpha(k^2+6m^2) + \tfrac{1}{2}\alpha\int_0^1 dx\, D\ln(D/\mu^2) + O(\alpha^2)\ , \qquad (27.4)$$

as compared to our old result in the on-shell scheme,

$$\Pi_{\rm OS}(k^2) = -\tfrac{1}{12}\alpha(k^2+m^2) + \tfrac{1}{2}\alpha\int_0^1 dx\, D\ln(D/D_0) + O(\alpha^2)\ , \qquad (27.5)$$

where again $D = x(1-x)k^2 + m^2$, and $D_0 = [-x(1-x)+1]m^2$.

Notice that $\Pi_{\overline{MS}}(k^2)$ has a well-defined $m \to 0$ limit, whereas $\Pi_{\rm OS}(k^2)$ does not.

On the other hand, $\Pi_{\overline{MS}}(k^2)$ depends explicitly on the fake parameter μ, whereas $\Pi_{\rm OS}(k^2)$ does not.

What does this all mean?

First, in the $\overline{MS}$ scheme, the propagator $\Delta_{\overline{MS}}(k^2)$ will no longer have a pole at $k^2 = -m^2$.

The pole will be somewhere else.

However, by definition, the actual physical mass mph of the particle is determined by the location of this pole: $k^2 = -m^2_{ph}$.

Thus, the lagrangian parameter m is no longer the same as m_{ph}.

Furthermore, the residue of this pole is no longer one.

Let us call the residue R.

The LSZ formula must now be corrected by multiplying its right-hand side by a factor of $R^{-1/2}$ for each external particle (incoming or outgoing).

This is because it is the field $R^{-1/2}\ \varphi(x)$ that now has unit amplitude to create a one-particle state.

Note also that, in the LSZ formula, each Klein-Gordon wave operator should be $-\partial^2 + m^2_{ph}$, and not $-\partial^2 + m^2$; also, each external four-momentum should square to $-m^2_{ph}$, and not $-m^2$.

A review of the derivation of the LSZ formula clearly shows that each of these mass parameters must be the actual particle mass, and not the parameter in the lagrangian.

Finally, in the LSZ formula, each external line will contribute a factor of R when the associated Klein-Gordon wave operator hits the external propagator and cancels its momentum-space pole, leaving behind the residue R.

Combined with the correction factor of $R^{-1/2}$ for each field, we get a net factor of $R^{1/2}$ for each external line when using the $\overline{MS}$ scheme.

Internal lines each contribute a factor of $(-i)/(k^2 + m^2)$, where m is the lagrangian-parameter mass, and each vertex contributes a factor of $iZ_g g$, where g is the lagrangian-parameter coupling.

Let us now compute the relation between m and m_{ph}, and then compute R.

We have

$$\Delta_{\overline{\rm MS}}(k^2)^{-1} = k^2 + m^2 - \Pi_{\overline{\rm MS}}(k^2) \ , \tag{27.6}$$

and, by definition,

$$\Delta_{\overline{\rm MS}}(-m^2_{\rm ph})^{-1} = 0 \ . \tag{27.7}$$

Setting $k^2 = -m^2_{ph}$ in eq. (27.6), using eq. (27.7), and rearranging, we find

$$m^2_{\rm ph} = m^2 - \Pi_{\overline{\rm MS}}(-m^2_{\rm ph}) \ . \tag{27.8}$$

Since $\Pi_{\overline{MS}}(k^2)$ is $O(\alpha)$, we see that the difference between m^2_{ph} and m^2 is $O(\alpha)$.

Therefore, on the right-hand side, we can replace m^2_{ph} with m^2, and only make an error of $O(\alpha^2)$.

Thus

$$m^2_{\rm ph} = m^2 - \Pi_{\overline{\rm MS}}(-m^2) + O(\alpha^2) \ . \tag{27.9}$$

Working this out, we get

$$m^2_{\rm ph} = m^2 - \tfrac{1}{2}\alpha\left[\tfrac{1}{6}m^2 - m^2 + \int_0^1 dx\, D_0 \ln(D_0/\mu^2)\right] + O(\alpha^2) \ , \tag{27.10}$$

where $D_0 = [1-x(1-x)]m^2$.

Doing the integrals yields

$$m_{\rm ph}^2 = m^2\left[1+\tfrac{5}{12}\alpha\Big(\ln(\mu^2/m^2)+c'\Big)+O(\alpha^2)\right]. \tag{27.11}$$

where $c' = (34-3\pi\sqrt{3})/15 = 1.18$.

Now, physics should be independent of the fake parameter μ.

However, the right-hand side of eq. (27.11) depends explicitly on μ.

It must, be, then, that m and α take on different numerical values as μ is varied, in just the right way to leave physical quantities (like m_{ph}) unchanged.

We can use this information to find differential equations that tell us how m and α change with μ.

For example, take the logarithm of eq. (27.11) and divide by two to get

$$\ln m_{\rm ph} = \ln m + \tfrac{5}{12}\alpha\Big(\ln(\mu/m)+\tfrac{1}{2}c'\Big)+O(\alpha^2)\,. \tag{27.12}$$

Now differentiate with respect to $\ln\mu$ and require m_{ph} to remain fixed:

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}\ln m_{\rm ph} \\ &= \frac{1}{m}\frac{dm}{d\ln\mu} + \tfrac{5}{12}\alpha + O(\alpha^2)\,. \end{aligned} \tag{27.13}$$

To get the second line, we had to assume that $d\alpha/d\ln\mu = O(\alpha^2)$, which we will verify shortly.

Then, rearranging eq. (27.13) gives

$$\frac{dm}{d\ln\mu} = \left(-\tfrac{5}{12}\alpha + O(\alpha^2)\right) m \ . \tag{27.14}$$

The factor in large parentheses on the right is called the *anomalous dimension* of the mass parameter, and it is often given the name $\gamma_m(\alpha)$.

Turning now to the residue R, we have

$$R^{-1} = \frac{d}{dk^2}\left[\mathbf{\Delta}_{\overline{\rm MS}}(k^2)^{-1}\right]\Big|_{k^2=-m^2_{\rm ph}} \ . \tag{27.15}$$

Using eq. (27.6), we get

$$\begin{aligned} R^{-1} &= 1 - \Pi'_{\overline{\rm MS}}(-m^2_{\rm ph}) \\ &= 1 - \Pi'_{\overline{\rm MS}}(-m^2) + O(\alpha^2) \\ &= 1 + \tfrac{1}{12}\alpha\Big(\ln(\mu^2/m^2) + c''\Big) + O(\alpha^2) \ , \end{aligned} \tag{27.16}$$

where $c'' = (17 - 3\pi\sqrt{3})/3 = 0.23$.

We can also use $\overline{MS}$ to define the vertex function. We take

$$C = -\alpha\tfrac{1}{\bar\varepsilon} + O(\alpha^2) \ , \tag{27.17}$$

and so

$$\mathbf{V}_{3,\overline{\text{MS}}}(k_1,k_2,k_3) = g\left[1 - \tfrac{1}{2}\alpha\int dF_3\ \ln(D/\mu^2) + O(\alpha^2)\right] \tag{27.18}$$

where $D = xyk^2_1 + yzk^2_2 + zxk^2_3 + m^2$.

Let us now compute the $\phi\phi \to \phi\phi$ scattering amplitude in our fancy new renormalization scheme.

In the low-mass limit, repeating the steps that led to eq. (26.1), and including the LSZ correction factor $(R^{1/2})^4$, we get

$$\mathcal{T} = R^2\,\mathcal{T}_0\left[1 - \tfrac{11}{12}\alpha\Big(\ln(s/\mu^2) + O(m^0)\Big) + O(\alpha^2)\right], \tag{27.19}$$

where $\mathcal{T}_0 = -g^2(s^{-1}+t^{-1}+u^{-1})$ is the tree-level result.

Now using R from eq. (27.16), we find

$$\mathcal{T} = \mathcal{T}_0\left[1 - \alpha\Big(\tfrac{11}{12}\ln(s/\mu^2) + \tfrac{1}{6}\ln(\mu^2/m^2) + O(m^0)\Big) + O(\alpha^2)\right]. \tag{27.20}$$

To get an observable amplitude-squared with an imperfect detector, we must square eq. (27.20) and multiply it by the correction factor we derived in section 26,

$$|\mathcal{T}|^2_{\text{obs}} = |\mathcal{T}|^2\left[1 + \tfrac{1}{3}\alpha\Big(\ln(\delta^2 s/m^2) + O(m^0)\Big) + O(\alpha^2)\right], \tag{27.21}$$

where δ is the angular resolution of the detector.

Combining this with eq. (27.20), we get

$$|\mathcal{T}|^2_{\rm obs} = |\mathcal{T}_0|^2 \left[1 - \alpha\Big(\tfrac{3}{2}\ln(s/\mu^2) + \tfrac{1}{3}\ln(1/\delta^2) + O(m^0)\Big) + O(\alpha^2)\right]. \tag{27.22}$$

All factors of ln m^2 have disappeared!

Finally, we have obtained an expression that has a well-defined m$\to$ 0 limit.

Of course, μ is still a fake parameter, and so $|\mathcal{T}|^2_{\rm obs}$ cannot depend on it.

It must be, then, that the explicit dependence on μ in eq. (27.22) is canceled by the implicit μ dependence of α.

We can use this information to figure out how αmust vary with μ.

Noting that $|\mathcal{T}_0|^2 = O(g^4) = O(\alpha^2)$, we have

$$\ln|\mathcal{T}|^2_{\rm obs} = C_1 + 2\ln\alpha + 3\alpha\Big(\ln\mu + C_2\Big) + O(\alpha^2)\ , \tag{27.23}$$

where C_1 and C_2 are independent of μ and α (but depend on the Mandelstam variables).

Differentiating with respect to ln μ then gives

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}\ln|\mathcal{T}|^2_{\rm obs} \\ &= \frac{2}{\alpha}\frac{d\alpha}{d\ln\mu} + 3\alpha + O(\alpha^2)\ , \end{aligned} \tag{27.24}$$

or, after rearranging,

$$\frac{d\alpha}{d\ln\mu} = -\tfrac{3}{2}\alpha^2 + O(\alpha^3)\ . \tag{27.25}$$

The right-hand side of this equation is called the *beta function*.

Returning to eq. (27.22), we are free to choose any convenient value of μ that we might like.

To avoid introducing unnecessary large logs, we should choose $\mu^2 \sim$ s.

To compare the results at different values of s, we need to solve eq. (27.25).

Keeping only the leading term in the beta function, the solution is

$$\alpha(\mu_2) = \frac{\alpha(\mu_1)}{1 + \frac{3}{2}\alpha(\mu_1)\ln(\mu_2/\mu_1)}\ . \tag{27.26}$$

Thus, as μ increases, $\alpha(\mu)$ decreases.

A theory with this property is said to be *asymptotically free*.

In this case, the tree-level approximation (in the $\overline{MS}$ scheme with $\mu^2 \sim$ s) becomes better and better at higher and higher energies.

Of course, the opposite is true as well: as μ decreases, $\alpha(\mu)$ increases.

As we go to lower and lower energies, the theory becomes more and more strongly coupled.

If the particle mass is nonzero, this process stops at $\mu \sim$ m.

This is because the minimum value of s is $4m^2$, and so the factor of $\ln(s/\mu^2)$ becomes an unwanted large log for $\mu \ll$ m.

We should therefore not use values of μ below m.

Perturbation theory is still good at these low energies if $\alpha(m) \ll 1$.

If the particle mass is zero, $\alpha(\mu)$ continues to increase at lower and lower energies, and eventually perturbation theory breaks down.

This is a signal that the low-energy physics may be quite different from what we expect on the basis of a perturbative analysis.

In the case of ϕ^3 theory, we know what the correct low-energy physics is: the perturbative ground state is unstable against tunneling through the potential barrier, and there is no true ground state.

Asymptotic freedom is, in this case, a signal of this impending disaster.

Much more interesting is asymptotic freedom in a theory that does have a true ground state, such as quantum chromodynamics.

In this example, the particle excitations are colorless hadrons, rather than the quarks and gluons we would expect from examining the lagrangian.

If the sign of the beta function is positive, then the theory is infrared free.

The coupling increases as μ increases, and, at sufficiently high energy, perturbation theory breaks down.

On the other hand, the coupling decreases as we go to lower energies.

Once again, though, we should stop this process at $\mu \sim$ m if the particles have nonzero mass.

Quantum electrodynamics with massive electrons (but, of course, massless photons) is in this category.

Still more complicated behaviors are possible if the beta function has a zero at a nonzero value of α.

We briefly consider this case in the next section.

28 The Renormalization Group

In section 27 we introduced the $\overline{MS}$ renormalization scheme, and used the fact that physical observables must be independent of the fake parameter μ to figure out how the lagrangian parameters m and g must change with μ.

In this section we re-derive these results from a much more formal (but calculationally simpler) point of view, and see how they extend to all orders of perturbation theory.

Equations that tells us how the lagrangian parameters (and other objects that are not directly measurable, like correlation functions) vary with μ are collectively called the equations of the renormalization group.

Let us recall the lagrangian of our theory, and write it in two different ways.

In d = 6– ε dimensions, we have

$$\mathcal{L} = -\tfrac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} Z_m m^2 \varphi^2 + \tfrac{1}{6} Z_g g \tilde{\mu}^{\varepsilon/2} \varphi^3 + Y\varphi \tag{28.1}$$

and

$$\mathcal{L} = -\tfrac{1}{2} \partial^\mu \varphi_0 \partial_\mu \varphi_0 - \tfrac{1}{2} m_0^2 \varphi_0^2 + \tfrac{1}{6} g_0 \varphi_0^3 + Y_0 \varphi_0 \ . \tag{28.2}$$

The fields and parameters in eq. (28.1) are the renormalized fields and parameters.

And in particular, they are renormalized using the $\overline{MS}$ scheme, with $\mu^2 = 4\pi e^{-\gamma}\, \tilde{\mu}^2$.

The fields and parameters in eq. (28.2) are the bare fields and parameters.

Comparing eqs. (28.1) and (28.2) gives us the relationships between them:

$$\varphi_0(x) = Z_\varphi^{1/2} \varphi(x) \ , \tag{28.3}$$

$$m_0 = Z_\varphi^{-1/2} Z_m^{1/2} m \ , \tag{28.4}$$

$$g_0 = Z_\varphi^{-3/2} Z_g g \tilde{\mu}^{\varepsilon/2} \ , \tag{28.5}$$

$$Y_0 = Z_\varphi^{-1/2} Y \ . \tag{28.6}$$

Recall that, after using dimensional regularization, the infinities coming from loop integrals take the form of inverse powers of ε = 6 – d.

In the $\overline{MS}$ renormalization scheme, we choose the Z's to cancel off these powers of 1/ε, and nothing more.

Therefore the Z's can be written as

$$Z_\varphi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\varepsilon^n} \,, \tag{28.7}$$

$$Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{\varepsilon^n} \,, \tag{28.8}$$

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\varepsilon^n} \,, \tag{28.9}$$

where $\alpha = g^2/(4\pi)^3$.

Computing $\Pi(k^2)$ and $\mathbf{V}_3(k_1,k_2,k_3)$ in perturbation theory in the $\overline{MS}$ scheme gives us Taylor series in α for $a_n(\alpha)$, $b_n(\alpha)$, and $c_n(\alpha)$.

So far we have found

$$a_1(\alpha) = -\tfrac{1}{6}\alpha + O(\alpha^2) \,, \tag{28.10}$$

$$b_1(\alpha) = -\alpha + O(\alpha^2) \,, \tag{28.11}$$

$$c_1(\alpha) = -\alpha + O(\alpha^2) \,, \tag{28.12}$$

and that $a_n(\alpha)$, $b_n(\alpha)$, and $c_n(\alpha)$ are all at least $O(\alpha^2)$ for $n \geq 2$.

Next we turn to the trick that we will employ to compute the beta function for α, the anomalous dimension of m, and other useful things.

This is the trick: bare fields and parameters must be independent of μ.

Why is this so?

Recall that we introduced μ when we found that we had to regularize the theory to avoid infinities in the loop integrals of Feynman diagrams.

We argued at the time (and ever since) that physical quantities had to be independent of μ.

Thus μ is not really a parameter of the theory, but just a crutch that we had to introduce at an intermediate stage of the calculation.

In principle, the theory is completely specified by the values of the bare parameters, and, if we were smart enough, we would be able to compute the exact scattering amplitudes in terms of them, without ever introducing μ.

The point is this: since the exact scattering amplitudes are independent of μ, the bare parameters must be as well.

Let us start with g_0.

It is convenient to define

$$\alpha_0 \equiv g_0^2/(4\pi)^3 = Z_g^2 Z_\varphi^{-3} \tilde{\mu}^\varepsilon \alpha \ , \tag{28.13}$$

and also

$$G(\alpha,\varepsilon) \equiv \ln(Z_g^2 Z_\varphi^{-3}) \ . \tag{28.14}$$

From the general structure of eqs. (28.7) and (28.9), we have

$$G(\alpha,\varepsilon) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} \ , \tag{28.15}$$

where, in particular,

$$\begin{aligned} G_1(\alpha) &= 2c_1(\alpha) - 3a_1(\alpha) \\ &= -\tfrac{3}{2}\alpha + O(\alpha^2) \ . \end{aligned} \tag{28.16}$$

The logarithm of eq. (28.13) can now be written as

$$\ln \alpha_0 = G(\alpha,\varepsilon) + \ln \alpha + \varepsilon \ln \tilde{\mu} \ . \tag{28.17}$$

Next, differentiate eq. (28.17) with respect to $\ln \mu$, and require α_0 to be independent of it:

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu} \ln \alpha_0 \\ &= \frac{\partial G(\alpha,\varepsilon)}{\partial \alpha} \frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha} \frac{d\alpha}{d\ln\mu} + \varepsilon \ . \end{aligned} \tag{28.18}$$

Now regroup the terms, multiply by α, and use eq. (28.15) to get

$$0 = \left(1 + \frac{\alpha G_1'(\alpha)}{\varepsilon} + \frac{\alpha G_2'(\alpha)}{\varepsilon^2} + \ldots\right) \frac{d\alpha}{d\ln\mu} + \varepsilon\alpha \ . \tag{28.19}$$

Next we use some physical reasoning: dα/dln μ is the rate at which α must change to compensate for a small change in ln μ.

If compensation is possible at all, this rate should be finite in the ε→ 0 limit.

Therefore, in a renormalizable theory, we should have

$$\frac{d\alpha}{d\ln\mu} = -\varepsilon\alpha + \beta(\alpha) \ . \tag{28.20}$$

The first term, −εα, is fixed by matching the O(ε) terms in eq. (28.19).

The second term, the beta function β(α), is similarly determined by matching the O(ε^0) terms; the result is

$$\beta(\alpha) = \alpha^2 G_1'(\alpha) \ . \tag{28.21}$$

Terms that are higher-order in 1/ε must also cancel, and this determines all the other $G'_n(\alpha)$'s in terms of $G'_1(\alpha)$.

Thus, for example, cancellation of the O(ε^{-1}) terms fixes $G'_2(\alpha) = \alpha G'_1(\alpha)^2$.

These relations among the $G'_n(\alpha)$'s can of course be checked order by order in perturbation theory.

From eq. (28.21) and eq. (28.16), we find that the beta function is

$$\beta(\alpha) = -\tfrac{3}{2}\alpha^2 + O(\alpha^3) \ . \tag{28.22}$$

Hearteningly, this is the same result we found in section 27 by requiring the observed scattering cross section $|\mathcal{T}|^2_{\text{obs}}$ to be independent of μ.

However, simply as a matter of practical calculation, it is much easier to compute $G_1(\alpha)$ than it is to compute $|\mathcal{T}|^2_{\text{obs}}$.

Next consider the invariance of m_0.

We begin by defining

$$\begin{aligned} M(\alpha,\varepsilon) &\equiv \ln(Z_m^{1/2} Z_\varphi^{-1/2}) \\ &= \sum_{n=1}^{\infty} \frac{M_n(\alpha)}{\varepsilon^n} \ . \end{aligned} \tag{28.23}$$

From eqs. (28.10) and (28.12) we have

$$\begin{aligned} M_1(\alpha) &= \tfrac{1}{2} b_1(\alpha) - \tfrac{1}{2} a_1(\alpha) \\ &= -\tfrac{5}{12}\alpha + O(\alpha^2) \ . \end{aligned} \tag{28.24}$$

Then, from eq. (28.4), we have

$$\ln m_0 = M(\alpha,\varepsilon) + \ln m \ . \tag{28.25}$$

Take the derivative with respect to ln μ and require m_0 to be unchanged:

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}\ln m_0 \\ &= \frac{\partial M(\alpha,\varepsilon)}{\partial\alpha}\,\frac{d\alpha}{d\ln\mu} + \frac{1}{m}\,\frac{dm}{d\ln\mu}\,. \\ &= \frac{\partial M(\alpha,\varepsilon)}{\partial\alpha}\,\Big(-\varepsilon\alpha+\beta(\alpha)\Big) + \frac{1}{m}\,\frac{dm}{d\ln\mu}\,. \end{aligned} \tag{28.26}$$

Rearranging, we find

$$\begin{aligned} \frac{1}{m}\,\frac{dm}{d\ln\mu} &= \Big(\varepsilon\alpha-\beta(\alpha)\Big)\sum_{n=1}^{\infty}\frac{M_n'(\alpha)}{\varepsilon^n} \\ &= \alpha M_1'(\alpha)+\ldots\,, \end{aligned} \tag{28.27}$$

where the ellipses stand for terms with powers of 1/ε.

In a renormalizable theory, dm/dln μ should be finite in the $\varepsilon\to 0$ limit, and so these terms must actually all be zero.

Therefore, the anomalous dimension of the mass, defined via

$$\gamma_m(\alpha) \equiv \frac{1}{m}\,\frac{dm}{d\ln\mu}\,, \tag{28.28}$$

is given by

$$\begin{aligned} \gamma_m(\alpha) &= \alpha M_1'(\alpha) \\ &= -\tfrac{5}{12}\alpha + O(\alpha^2)\,. \end{aligned} \tag{28.29}$$

Comfortingly, this is just what we found in section 27.

Let us now consider the propagator in the $\overline{MS}$ renormalization scheme,

$$\tilde{\mathbf{\Delta}}(k^2) = i \int d^6x \, e^{ikx} \langle 0|\mathrm{T}\varphi(x)\varphi(0)|0\rangle \ . \tag{28.30}$$

The bare propagator,

$$\tilde{\mathbf{\Delta}}_0(k^2) = i \int d^6x \, e^{ikx} \langle 0|\mathrm{T}\varphi_0(x)\varphi_0(0)|0\rangle \ , \tag{28.31}$$

should be (by the now-familiar argument) independent of μ.

The bare and renormalized propagators are related by

$$\tilde{\mathbf{\Delta}}_0(k^2) = Z_\varphi \tilde{\mathbf{\Delta}}(k^2) \ . \tag{28.32}$$

Taking the logarithm and differentiating with respect to $\ln \mu$, we get

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu} \ln \tilde{\mathbf{\Delta}}_0(k^2) \\ &= \frac{d\ln Z_\varphi}{d\ln\mu} + \frac{d}{d\ln\mu} \ln \tilde{\mathbf{\Delta}}(k^2) \\ &= \frac{d\ln Z_\varphi}{d\ln\mu} + \frac{1}{\tilde{\mathbf{\Delta}}(k^2)} \left(\frac{\partial}{\partial \ln\mu} + \frac{d\alpha}{d\ln\mu}\frac{\partial}{\partial\alpha} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m} \right) \tilde{\mathbf{\Delta}}(k^2). \end{aligned} \tag{28.33}$$

We can write

$$\ln Z_\varphi = \frac{a_1(\alpha)}{\varepsilon} + \frac{a_2(\alpha) - \frac{1}{2}a_1^2(\alpha)}{\varepsilon^2} + \dots \ . \tag{28.34}$$

Then we have

$$\begin{aligned} \frac{d\ln Z_\varphi}{d\ln\mu} &= \frac{\partial \ln Z_\varphi}{\partial\alpha}\,\frac{d\alpha}{d\ln\mu} \\ &= \left(\frac{a_1'(\alpha)}{\varepsilon} + \dots\right)\Big(-\varepsilon\alpha + \beta(\alpha)\Big) \\ &= -\alpha a_1'(\alpha) + \dots \ , \end{aligned} \tag{28.35}$$

where the ellipses in the last line stand for terms with powers of 1/ε.

Since $\widetilde{\Delta}(k^2)$ should vary smoothly with μ in the $\varepsilon \to 0$ limit, these must all be zero.

We then define the *anomalous dimension* of the field

$$\gamma_\varphi(\alpha) \equiv \frac{1}{2}\,\frac{d\ln Z_\varphi}{d\ln\mu} \ . \tag{28.36}$$

From eq. (28.35) we find

$$\begin{aligned} \gamma_\varphi(\alpha) &= -\tfrac{1}{2}\alpha a_1'(\alpha) \\ &= +\tfrac{1}{12}\alpha + O(\alpha^2) \ . \end{aligned} \tag{28.37}$$

Eq. (28.33) can now be written as

$$\left(\frac{\partial}{\partial \ln \mu}+\beta(\alpha)\frac{\partial}{\partial \alpha}+\gamma_m(\alpha)m\frac{\partial}{\partial m}+2\gamma_\varphi(\alpha)\right)\tilde{\mathbf{\Delta}}(k^2)=0 \qquad (28.38)$$

in the $\varepsilon \to 0$ limit.

This is the Callan-Symanzik equation for the propagator.

The Callan-Symanzik equation is most interesting in the massless limit, and for a theory with a zero of the beta function at a nonzero value of α.

So, let us suppose that $\beta(\alpha_*) = 0$ for some $\alpha_* \neq 0$.

Then, for $\alpha = \alpha_*$ and $m = 0$, the Callan-Symanzik equation becomes

$$\left(\frac{\partial}{\partial \ln \mu}+2\gamma_\varphi(\alpha_*)\right)\tilde{\mathbf{\Delta}}(k^2)=0\,. \qquad (28.39)$$

The solution is

$$\tilde{\mathbf{\Delta}}(k^2)=\frac{C(\alpha_*)}{k^2}\left(\frac{\mu^2}{k^2}\right)^{-\gamma_\varphi(\alpha_*)}, \qquad (28.40)$$

where $C(\alpha_*)$ is an integration constant.

We used the fact that $\widetilde{\Delta}(k^2)$ has mass dimension -2 to get the k^2 dependence in addition to the $\tilde{\mu}$ dependence.

Thus the naive scaling law $\widetilde{\Delta}(k^2) \sim k^{-2}$ is changed to $\widetilde{\Delta}(k^2) \sim k^{-2[1-\gamma\varphi(\alpha_*)]}$.

This has applications in the theory of critical phenomena, which is beyond the scope of this book.

29 Effective Field Theory

So far we have been discussing only renormalizable theories.

In this section, we investigate what meaning can be assigned to nonrenormalizable theories, following an approach pioneered by Ken Wilson(who taught me QFT).

We will begin by analyzing a renormalizable theory from a new point of view.

Consider, as an example, φ^4 theory in four spacetime dimensions:

$$\mathcal{L} = -\tfrac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} Z_m m_{\rm ph}^2 \varphi^2 - \tfrac{1}{24} Z_\lambda \lambda_{\rm ph} \varphi^4 \ . \tag{29.1}$$

This example is actually problematic, because this theory is trivial, a technical term that we will explain later.

For now we proceed with a perturbative analysis.

We take the renormalizing Z factors to be defined in an on-shell scheme, and have emphasized this by writing the particle mass as m_{ph} and the coupling constant as λ_{ph}.

We define λ_{ph} as the value of the exact 1PI four-point vertex with zero external four-momenta:

$$\lambda_{\rm ph} \equiv \mathbf{V}_4(0,0,0,0) \ . \tag{29.2}$$

The path integral is given by

$$Z(J) = \int \mathcal{D}\varphi \, e^{iS + i\int J\varphi} \, , \tag{29.3}$$

where $S = \int d^4x \, \mathcal{L}$ and $\int J\varphi$ is short for $\int d^4x \, J\varphi$.

Our first step in analyzing this theory will be to perform the Wick rotation (applied to loop integrals in section 14) directly on the action.

We define a euclidean time $\tau \equiv it$.

Then we have

$$Z(J) = \int \mathcal{D}\varphi \, e^{-S_{\rm E} - \int J\varphi} \, , \tag{29.4}$$

where $S_{\rm E} = \int d^4x \, \mathcal{L}_{\rm E}$, $d^4x = d^3x \, d\tau$,

$$\mathcal{L}_{\rm E} = \tfrac{1}{2} Z_\varphi \partial_\mu\varphi\partial_\mu\varphi + \tfrac{1}{2} Z_m m_{\rm ph}^2 \varphi^2 + \tfrac{1}{24} Z_\lambda \lambda_{\rm ph} \varphi^4 \, , \tag{29.5}$$

and

$$\partial_\mu\varphi\partial_\mu\varphi = (\partial\varphi/\partial\tau)^2 + (\nabla\varphi)^2 \, . \tag{29.6}$$

Note that each term in S_E is always positive (or zero) for any field configuration φ(x).

This is the advantage of working in euclidean space: eq. (29.4), the *euclidean path integral*, is strongly damped (rather than rapidly oscillating) at large values of the field and/or its derivatives, and this makes its convergence properties more obvious.

Next, we Fourier transform to (euclidean) momentum space via

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \widetilde{\varphi}(k)\ . \tag{29.7}$$

The euclidean action becomes

$$\begin{aligned} S_{\rm E} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \widetilde{\varphi}(-k)\Big(Z_\varphi k^2 + Z_m m_{\rm ph}^2\Big)\widetilde{\varphi}(k) \\ + \frac{1}{24} Z_\lambda \lambda_{\rm ph} \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\delta^4(k_1{+}k_2{+}k_3{+}k_4) \\ \times\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\widetilde{\varphi}(k_3)\widetilde{\varphi}(k_4)\ . \end{aligned} \tag{29.8}$$

Note that $k^2 = \mathbf{k}^2 + k^2_\tau \geq 0$.

We now introduce an ultraviolet cutoff Λ.

It should be much larger than the particle mass m_{ph}, or any other energy scale of practical interest.

Then we perform the path integral over all $\phi(k)$ with $|\mathbf{k}| > \Lambda$.

We also take $\tilde{J} = 0$ for $|\mathbf{k}| > \Lambda$.

Then we find

$$Z(J) = \int \mathcal{D}\varphi_{|k|<\Lambda}\, e^{-S_{\rm eff}(\varphi;\Lambda) - \int J\varphi}\ , \tag{29.9}$$

where

$$e^{-S_{\rm eff}(\varphi;\Lambda)} = \int \mathcal{D}\varphi_{|k|>\Lambda}\, e^{-S_{\rm E}(\varphi)} \ . \tag{29.10}$$

$S_{\rm eff}(\varphi; \Lambda)$ is called the *Wilsonian effective action*.

We can write the corresponding lagrangian density as

$$\begin{aligned}\mathcal{L}_{\rm eff}(\varphi;\Lambda) = {} & \tfrac{1}{2}Z(\Lambda)\partial_\mu\varphi\partial_\mu\varphi + \tfrac{1}{2}m^2(\Lambda)\varphi^2 + \tfrac{1}{24}\lambda(\Lambda)\varphi^4 \\ & + \sum_{d\geq 6}\sum_i c_{d,i}(\Lambda)\mathcal{O}_{d,i} \ ,\end{aligned} \tag{29.11}$$

where the Fourier components of φ(x) are now cut off at $|\mathbf{k}| > \Lambda$:

$$\varphi(x) = \int_0^\Lambda \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \widetilde{\varphi}(k) \ . \tag{29.12}$$

The operators $O_{d,i}$ in eq. (29.11) consist of all terms that have mass dimension $d \geq 6$ and that are even under $\varphi \leftrightarrow -\varphi$; i is an index that distinguishes operators of the same dimension that are inequivalent after integrations by parts of any derivatives that act on the fields.

The operators must be even under $\varphi \leftrightarrow -\varphi$ in order to respect the $\varphi \leftrightarrow -\varphi$ symmetry of the original lagrangian.

The coefficients $Z(\Lambda)$, $m^2(\Lambda)$, $\lambda(\Lambda)$, and $c_{d,i}(\Lambda)$ in eq. (29.11) are all finite functions of Λ.

This is established by the following argument.

We can differentiate eq. (29.9) with respect to $J(x)$ to compute correlation functions of the renormalized field $\varphi(x)$, and correlation functions of renormalized fields are finite.

Using eq. (29.9), we can compute these correlation functions as a series of Feynman diagrams, with Feynman rules based on L_{eff} .

These rules include an ultraviolet cutoff Λ on the loop momenta, since the fields with higher momenta have already been integrated out.

Thus all of the loop integrals in these diagrams are finite.

Therefore the other parameters that enter the diagrams— $Z(\Lambda)$, $m^2(\Lambda)$, $\lambda(\Lambda)$, and $c_{d,i}(\Lambda)$—must be finite as well, in order to end up with finite correlation functions.

To compute these parameters, we can think of eq. (29.8) as the action for two kinds of fields, those with $|\mathbf{k}| < \Lambda$ and those with $|\mathbf{k}| > \Lambda$.

Then we draw all 1PI diagrams with external lines for $|\mathbf{k}| < \Lambda$ fields only.

For $\lambda_{ph} \ll 1$, the dominant contribution to $c_{d,i}(\Lambda)$ for an operator $O_{d,i}$ with 2n fields and d– 2n derivatives is then given by a one-loop diagram with 2n external lines (representing $|\mathbf{k}| < \Lambda$ fields), n vertices, and a $|\mathbf{k}| > \Lambda$ field circulating in the loop; see fig. (29.1).

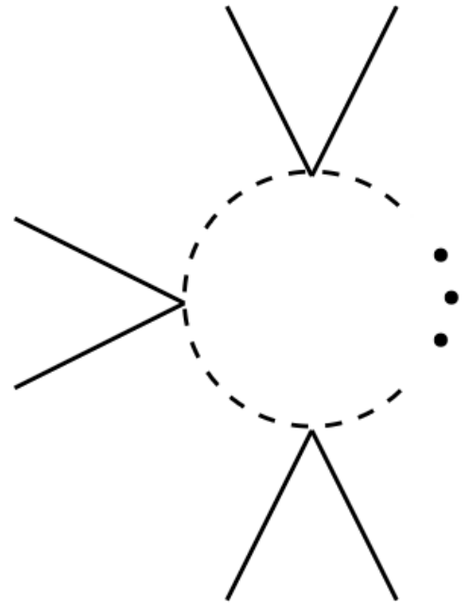


Figure 29.1: A one-loop 1PI diagram with 2n external lines. Each external line represents a field with $|\mathbf{k}| < \Lambda$. The internal (dashed) line represents a field with $|\mathbf{k}| > \Lambda$.

The simplest case to consider is $O_{2n,1} \equiv \varphi^{2n}$.

With 2n external lines, there are (2n)! ways of assigning the external momenta to the lines, but $2^n \times n \times 2$ of these give the same diagram: 2^n for exchanging the two external lines that meet at any one vertex; n for rotations of the diagram; and 2 for reflection of the diagram.

Since there are no derivatives on the external fields, we can set all of the external momenta to zero; then all (2n)!/(2^n2n) diagrams have the same value.

With a euclidean action, each internal line contributes a factor of $1/(k^2 + m^2_{ph})$, and each vertex contributes a factor of $-Z_\lambda \lambda_{ph} = -\lambda_{ph} + O(\lambda^2_{ph})$.

The vertex factor associated with the term

$c_{2n,1}(\Lambda)\varphi^{2n}$ in $\mathcal{L}_{\text{eff}}$ is $-(2n)!\, c_{2n,1}(\Lambda)$. Thus we have

$$-(2n)!\, c_{2n,1}(\Lambda) = \frac{(-\lambda_{\text{ph}})^n (2n)!}{2^n 2n} \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2_{\text{ph}}} \right)^n + O(\lambda^{n+1}_{\text{ph}}) \,. \tag{29.13}$$

For 2n ≥ 6, the integral converges, and we find

$$c_{2n,1}(\Lambda) = -\frac{(-\lambda_{\rm ph}/2)^n}{32\pi^2 n(n-2)}\,\frac{1}{\Lambda^{2n-4}} + O(\lambda_{\rm ph}^{n+1})\ . \tag{29.14}$$

We have taken $\Lambda \gg m_{\rm ph}$, and dropped terms down by powers of $m_{\rm ph}/\Lambda$.

For 2n = 4, we have to include the tree-level vertex; in this case, we have

$$-\lambda(\Lambda) = -Z_\lambda \lambda_{\rm ph} + \frac{3}{2}(-\lambda_{\rm ph})^2 \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m_{\rm ph}^2}\right)^2 + O(\lambda_{\rm ph}^3)\ . \tag{29.15}$$

This integral diverges.

To evaluate it, we note that the one-loop contribution to the exact four-point vertex is given by the same diagram, but with fields of all momenta circulating in the loop.

Thus we have

$$-\mathbf{V}_4(0,0,0,0) = -Z_\lambda \lambda_{\rm ph} + \frac{3}{2}(-\lambda_{\rm ph})^2 \int_0^\infty \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m_{\rm ph}^2}\right)^2 + O(\lambda_{\rm ph}^3)\ . \tag{29.16}$$

Then, using $\mathbf{V}_4(0,0,0,0) = \lambda_{\rm ph}$ and subtracting eq. (29.15) from eq. (29.16), we get

$$-\lambda_{\rm ph} + \lambda(\Lambda) = \frac{3}{2}(-\lambda_{\rm ph})^2 \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m_{\rm ph}^2} \right)^2 + O(\lambda_{\rm ph}^3) \ . \tag{29.17}$$

Evaluating the (now finite!) integral and rearranging, we have

$$\lambda(\Lambda) = \lambda_{\rm ph} + \frac{3}{16\pi^2} \lambda_{\rm ph}^2 \left[\ln(\Lambda/m_{\rm ph}) - \tfrac{1}{2} \right] + O(\lambda_{\rm ph}^3) \ . \tag{29.18}$$

Note that this result has the problem of a large log; the second term is smaller than the first only if $\lambda_{\rm ph} \ln(\Lambda/m_{\rm ph}) \ll 1$.

To cure this problem, we must change the renormalization scheme.

We will take up this issue shortly, but first let us examine the case of two external lines while continuing to use the on-shell scheme.

For the case of two external lines, the one-loop diagram has just one vertex, and by momentum conservation, the loop integral is completely indepedent of the external momentum.

This implies that the one-loop contribution to $Z(\Lambda)$ vanishes, and so we have

$$Z(\Lambda) = 1 + O(\lambda_{\rm ph}^2) \ . \tag{29.19}$$

The one-loop diagram does, however, give a nonzero contribution to $m^2(\Lambda)$; after including the tree-level term, we find

$$-m^2(\Lambda) = -Z_m m_{\rm ph}^2 + \frac{1}{2}(-\lambda_{\rm ph})\int_\Lambda^\infty \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m_{\rm ph}^2} + O(\lambda_{\rm ph}^2)\ . \tag{24.20}$$

This integral diverges.

To evaluate it, recall that the one-loop contribution to the exact particle mass-squared is given by the *same* diagram, but with fields of all momenta circulating in the loop.

Thus we have

$$-m_{\rm ph}^2 = -Z_m m_{\rm ph}^2 + \frac{1}{2}(-\lambda_{\rm ph})\int_0^\infty \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m_{\rm ph}^2} + O(\lambda_{\rm ph}^2)\ . \tag{24.21}$$

Then, subtracting eq. (29.20) from eq. (29.21), we get

$$-m_{\rm ph}^2 + m^2(\Lambda) = \frac{1}{2}(-\lambda_{\rm ph})\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m_{\rm ph}^2} + O(\lambda_{\rm ph}^2)\ . \tag{24.22}$$

Evaluating the (now finite!) integral and rearranging, we have

$$m^2(\Lambda) = m_{\rm ph}^2 - \frac{\lambda_{\rm ph}}{16\pi^2}\left[\Lambda^2 - m_{\rm ph}^2\ln(\Lambda^2/m_{\rm ph}^2)\right] + O(\lambda_{\rm ph}^2)\ . \tag{24.23}$$

We see that we now have an even worse situation than we did with the large log in $\lambda(\Lambda)$: the correction term is quadratically divergent.

As already noted, to fix these problems we must change the renormalization scheme.

In the context of an effective action with a specific value of the cutoff Λ_0, there is a simple way to do so: we simply treat this effective action as the fundamental starting point, with $Z(\Lambda_0)$, $m^2(\Lambda_0)$, $\lambda(\Lambda_0)$, and $c_{d,i}(\Lambda_0)$ as input parameters.

We then see what physics emerges at energy scales well below Λ_0.

We can set $Z(\Lambda_0) = 1$, with the understanding that the field no longer has the LSZ normalization and that we will have to correct the LSZ formula to account for this.

We will also assume that the parameters $\lambda(\Lambda_0)$, $m^2(\Lambda_0)$, and $c_{d,i}(\Lambda_0)$ are all small when measured in units of the cutoff:

$$\lambda(\Lambda_0) \ll 1 \,, \tag{29.24}$$

$$|m^2(\Lambda_0)| \ll \Lambda_0^2 \,, \tag{29.25}$$

$$c_{d,i}(\Lambda_0) \ll \Lambda_0^{-(d-4)} \,. \tag{29.26}$$

The proposal to treat the effective action as the fundamental starting point may not seem very appealing.

For one thing, we now have an infinite number of parameters to specify, rather than two!

Also, we now have an explicit cutoff in place, rather than trying to have a theory that works at all energy scales.

On the other hand, it may well be that quantum field theory does not work at arbitrarily high energies.

For example, quantum fluctuations in spacetime itself should become important above the Planck scale, which is given by the inverse square root of Newton's constant, and has a numerical value of $\sim 10^{19}$ GeV (compared to, say, the proton mass, which is $\sim$ 1 GeV).

So, let us leave the cutoff Λ_0 in place for now.

We will then make a two-pronged analysis.

First, we will see what happens at much lower energies.

Then, we will see what happens if we try to take the limit $\Lambda_0 \to \infty$.

We begin by examining lower energies.

To make things more tractable, we will set

$$c_{d,i}(\Lambda_0) = 0 \; ; \tag{29.27}$$

later we will examine the effects of a more general choice.

A nice way to see what happens at lower energies is to integrate out some more high-energy degrees of freedom.

Let us, then, perform the functional integral over Fourier modes $\varphi(k)$ with $\Lambda < |\mathbf{k}| < \Lambda_0$; we have

$$e^{-S_{\rm eff}(\varphi;\Lambda)} = \int \mathcal{D}\varphi_{\Lambda<|k|<\Lambda_0} \, e^{-S_{\rm eff}(\varphi;\Lambda_0)} \; . \tag{29.28}$$

We can do this calculation in perturbation theory, mimicking the procedure that we used earlier.

We find

$$m^2(\Lambda) = m^2(\Lambda_0) + \frac{1}{2}\lambda(\Lambda_0)\int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m^2(\Lambda_0)} + \dots\,, \tag{29.29}$$

$$\lambda(\Lambda) = \lambda(\Lambda_0) - \frac{3}{2}\lambda^2(\Lambda_0)\int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2(\Lambda_0)}\right)^2 + \dots\,, \tag{29.30}$$

$$c_{2n,1}(\Lambda) = -\frac{(-1)^n}{2^n 2n}\lambda^n(\Lambda_0)\int_\Lambda^{\Lambda_0} \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2(\Lambda_0)}\right)^n + \dots\,, \tag{29.31}$$

where the ellipses stand for higher-order corrections.

For Λ not too much less than Λ_0 (and, in particular, for $|m^2(\Lambda_0)| \ll \Lambda^2$), we find

$$m^2(\Lambda) = m^2(\Lambda_0) + \frac{1}{16\pi^2}\lambda(\Lambda_0)\left(\Lambda_0^2 - \Lambda^2\right) + \dots\,, \tag{29.32}$$

$$\lambda(\Lambda) = \lambda(\Lambda_0) - \frac{3}{16\pi^2}\lambda^2(\Lambda_0)\ln\frac{\Lambda_0}{\Lambda} + \dots\,, \tag{29.33}$$

$$c_{2n,1}(\Lambda) = -\frac{(-1)^n}{32\pi^2 2^n n(n-2)}\lambda^n(\Lambda_0)\left(\frac{1}{\Lambda^{2n-4}} - \frac{1}{\Lambda_0^{2n-4}}\right) + \dots\,. \tag{29.34}$$

We see from this that the corrections to $m^2(\Lambda)$, which is the only coefficient with positive mass dimension, are dominated by contributions from the high end of the integral.

On the other hand, the corrections to $c_{d,i}(\Lambda)$, coefficients with negative mass dimension, are dominated by contributions from the low end of the integral.

And the corrections to $\lambda(\Lambda)$, which is dimensionless, come equally from all portions of the range of integration.

For the $c_{d,i}(\Lambda)$, this means that their starting values at Λ_0 were not very important, as long as eq. (29.26) is obeyed.

Nonzero starting values would contribute another term of order $1/\Lambda_0{}^{2n-4}$ to $c_{2n,1}(\Lambda)$, but all such terms are less important than the one of order $1/\Lambda_0{}^{2n-4}$ that comes from doing the integral down to $|k| = \Lambda$.

Similarly, nonzero values of $c_{d,i}(\Lambda 0)$ would make subdominant contributions to $\lambda(\Lambda)$.

As an example, consider the contribution of the diagram in fig. (29.2).

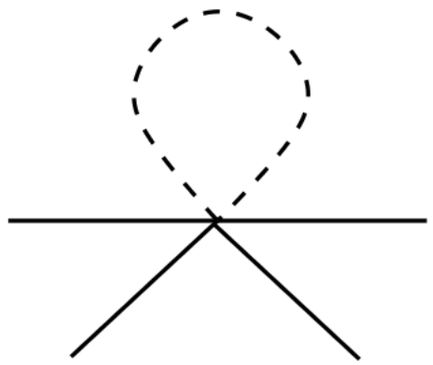


Figure 29.2: A one-loop contribution to the ϕ^4 vertex for fields with $|\mathbf{k}| < \Lambda$. The internal (dashed) line represents a field with $|\mathbf{k}| > \Lambda$.

Ignoring numerical factors, the vertex factor is $c_{6,1}(\Lambda_0)$, and the loop integral is the same as the one that enters into $m^2(\Lambda)$; it yields a factor of $\Lambda^2{}_0 - \Lambda^2 \sim \Lambda^2{}_0$.

Thus the contribution of this diagram to $\lambda(\Lambda)$ is of order $c_{6,1}(\Lambda_0)\Lambda^2{}_0$.

This is a pure number that, according to eq. (29.26), is small.

This contribution is missing the logarithmic enhancement factor $\ln(\Lambda_0/\Lambda)$ that we see in eq. (29.33).

On the other hand, for $m^2(\Lambda)$, there are infinitely many contributions of order Λ^2_0 when $c_{d,i}(\Lambda_0) \neq 0$.

These must add up to give $m^2(\Lambda)$ a value that is much smaller.

Indeed, we want to continue the process down to lower and lower values of Λ, with $m^2(\Lambda)$ dropping until it becomes of order $m2_{ph}$ at $\Lambda \sim m_{ph}$.

For this to happen, there must be very precise cancellations among all the terms of order Λ^2_0 that contribute to $m^2(\Lambda)$.

In some sense, it is more "natural" to have $m^2ph \sim \lambda(\Lambda_0)\ \Lambda^2_0$, rather than to arrange for these very precise cancellations.

This philosophical issue is called the *fine-tuning problem*, and it generically arises in theories with spin-zero fields.

In theories with higher-spin fields only, the action typically has more symmetry when these fields are massless, and this typically prevents divergences that are worse than logarithmic.

These theories are said to be technically natural, while theories with spin-zero fields (with physical masses well below the cutoff) generally are not.

(The only exceptions are theories where supersymmetry relates spin-zero and spin-one-half fields; the spin-zero fields then inherent the technical naturalness of their spin-one-half partners.)

For now, in ϕ^4 theory, we will simply accept the necessity of fine-tuning in order to have $m_{ph} \ll \Lambda$.

Returning to eqs. (29.32–29.34), we can recast them as differential equations that tell us how these parameters change with the value of the cutoff Λ. I

In particular, let us do this for $\lambda(\Lambda)$.

We take the derivative of eq. (29.33) with respect to Λ, multiply by Λ, and then set $\Lambda_0 = \Lambda$ to get

$$\frac{d}{d\ln\Lambda}\lambda(\Lambda) = \frac{3}{16\pi^2}\lambda^2(\Lambda) + \dots \ . \tag{29.35}$$

Notice that the right-hand side of eq. (29.33) is apparently the same as the beta function $\beta(\lambda) \equiv d\lambda/d\ln\mu$ that can be calculated, where it represents the rate of change in the $\overline{MS}$ parameter λ that was needed to compensate for a change in the $\overline{MS}$ renormalization scale μ.

Eq. (29.33) gives us a new physical interpretation of the beta function: it is the rate of change in the coefficient of the φ^4 term in the effective action as we vary the ultraviolet cutoff in that action.

Actually, though, there is a technical detail: it is really $Z(\Lambda)^{-2}\lambda(\Lambda)$ that is most closely analogous to the $\overline{MS}$ parameter λ.

This is because, if we rescale φ so that it has a canonical kinetic term of $1/2\ \partial_\mu\varphi\partial_\mu\varphi$, then the coefficient of the φ^4 term is $Z(\Lambda)^{-2}\lambda(\Lambda)$.

Since $Z(\Lambda) = 1 + O(\lambda^2(\Lambda))$, this has no effect at the one-loop level, but it does matter at two loops.

We can account for the effect of this *wave function renormalization* (in all the couplings) by writing, instead of eq. (29.11),

$$\begin{aligned}\mathcal{L}_{\text{eff}}(\varphi;\Lambda) = {} & \tfrac{1}{2}Z(\Lambda)\partial_\mu\varphi\partial_\mu\varphi + \tfrac{1}{2}Z(\Lambda)m^2(\Lambda)\varphi^2 + \tfrac{1}{24}Z^2(\Lambda)\lambda(\Lambda)\varphi^4 \\ & + \sum_{d\geq 6}\sum_i Z^{n_{d,i}/2}(\Lambda)c_{d,i}(\Lambda)\mathcal{O}_{d,i}\ ,\end{aligned} \tag{29.36}$$

where $n_{d,i}$ is the number of fields in the operator $O_{d,i}$.

Now the beta function for λ is universal up through two loops.

At three and higher loops, differences with the $\overline{MS}$ beta function can arise, due to the different underlying definitions of the coupling λ in the cutoff scheme and the $\overline{MS}$ scheme.

We now have the overall picture of Wilson's approach to quantum field theory.

First, define a quantum field theory via an action with an explicit momentum cutoff in place.[1]

Then, lower the cutoff by integrating out higher-momentum degrees of freedom.

As a result, the coefficients in the effective action will change.

If the field theory is weakly coupled—which in practice means eqs. (29.24–29.26) are obeyed—then the coefficients of the operators with negative mass dimension will start to take on the values we would have computed for them in perturbation theory, regardless of their precise initial values.

If we continuously rescale the fields to have canonical kinetic terms, then the dimensionless coupling constant(s) will change according to their beta functions.

The final results, at an energy scale E well below the initial cutoff Λ_0, are the same as we would predict via renormalized perturbation theory, up to small corrections by powers of E/Λ_0.

The advantage of the Wilson scheme is that it gives a nonperturbative definition of the theory which is applicable even if the theory is not weakly coupled.

With a spacetime lattice providing the cutoff, other techniques (typically requiring large-scale computer calculations) can be brought to bear on strongly-coupled theories.

The Wilson scheme also allows us to give physical meaning to nonrenormalizable theories.

Given an action for a nonrenormalizable theory, we can regard it as an effective action.

We should then impose a momentum cutoff Λ_0, where Λ_0 can be defined by saying that the coefficient of every operator O_i with mass dimension $D_i > 4$ is given by c_i/Λ_0^{Di-4} with $c_i \leq 1$.

Then we can use this theory for physics at energies below Λ_0.

At energies E far below Λ_0, the effective theory will look like a renormalizable one, up to corrections by powers of E/Λ_0.

This renormalizable theory might simply be a free-field theory with no interactions, or no theory at all if there are no particles with physical masses well below Λ_0.

We now turn to the final issue: can we remove the cutoff completely?

Returning to the example of φ^4 theory, let us suppose that we are somehow able to compute the exact beta function.

Then we can integrate the renormalization-group equation $d\lambda/d\ln\Lambda = \beta(\lambda)$ from $\Lambda = m_{\rm ph}$ to $\Lambda = \Lambda_0$ to get

$$\int_{\lambda(m_{\rm ph})}^{\lambda(\Lambda_0)} \frac{d\lambda}{\beta(\lambda)} = \ln\frac{\Lambda_0}{m_{\rm ph}} \ . \tag{29.37}$$

We would like to take the limit $\Lambda_0 \to \infty$.

Obviously, the right-hand side of eq. (29.37) becomes infinite in this limit, and so the left-hand side must as well.

However, it may not. Recall that, for small λ, $\beta(\lambda)$ is positive, and it increases faster than λ.

If this is true for all λ, then the left-hand side of eq. (29.37) will approach a fixed, finite value as we take the upper limit of integration to infinity.

This yields a maximum possible value for the initial cutoff, given by

$$\ln\frac{\Lambda_{\rm max}}{m_{\rm ph}} \equiv \int_{\lambda(m_{\rm ph})}^{\infty} \frac{d\lambda}{\beta(\lambda)} \ . \tag{29.38}$$

If we approximate the exact beta function with its leading term, $3\lambda^2/16\pi^2$,and use the leading term in eq. (29.18) to get $\lambda(m_{ph}) = \lambda_{ph}$, then we find

$$\Lambda_{\rm max} \simeq m_{\rm ph}\, e^{16\pi^2/3\lambda_{\rm ph}} \ . \tag{29.39}$$

The existence of a maximum possible value for the cutoff means that we cannot take the limit as the cutoff goes to infinity; we must use an effective action with a cutoff as our starting point.

If we insist on taking the cutoff to infinity, then the only possible value of λ_{ph} is $\lambda_{ph} = 0$.

Thus, φ^4 theory is trivial in the limit of infinite cutoff: there are no interactions.

There is much evidence for this, but as yet no rigorous proof.

The same is true of quantum electrodynamics, as was first conjectured by Landau; in this case, Λ_{max} is known as the location of the *Landau pole*.

However, the cutoff can be removed if the beta function grows no faster than λ at large λ; then the left-hand side of eq. (29.37) would diverge as we take the upper limit of integration to infinity.

Or, $\beta(\lambda)$ could drop to zero (and then become negative) at some finite value λ_*.

Then, if $\lambda_{ph} < \lambda_*$, the left-hand side of eq. (29.37) would diverge as the upper limit of integration approaches λ_*.

In this case, the effective coupling at higher and higher energies would remain fixed at λ_*, and $\lambda = \lambda_*$ is called an *ultravioldet* fixed point of the renormalization group.

If the beta function is negative for $\lambda = \lambda(m_{ph})$, the theory is said to be asymptotically free, and $\lambda(\Lambda)$ decreases as the cutoff is increased. In this case, there is no barrier to taking the limit $\Lambda \to \infty$.

In four space-time dimensions, the only asymptotically free theories are nonabelian gauge theories; see section 69.

30 Spontaneous Symmetry Breaking

Consider φ^4 theory, where φ is a real scalar field with lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{24}\lambda\varphi^4 \ . \tag{30.1}$$

As we discussed in section 23, this theory has a Z_2 symmetry: $\mathcal{L}$ is invariant under $\varphi(x) \to -\varphi(x)$, and we can define a unitary operator Z that implements this:

$$Z^{-1}\varphi(x)Z = -\varphi(x) \ . \tag{30.2}$$

We also have $Z^2 = 1$, and so $Z^{-1} = Z$. Since unitarity implies $Z^{-1} = Z^\dagger$, this makes Z hermitian as well as unitary.

Now suppose that the parameter m^2 is, in spite of its name, negative rather than positive.

We can write $\mathcal{L}$ in the form

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi) \ , \tag{30.3}$$

where the potential is

$$
\begin{aligned}
V(\varphi) &= \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{24}\lambda\varphi^4 \\
&= \tfrac{1}{24}\lambda(\varphi^2 - v^2)^2 - \tfrac{1}{24}\lambda v^4 \ .
\end{aligned}
\tag{30.4}
$$

In the second line, we have defined

$$
v \equiv +(6|m^2|/\lambda)^{1/2} \ . \tag{30.5}
$$

We can (and will) drop the last, constant, term in eq. (30.4).

From eq. (30.4) it is clear that there are two classical field configurations that minimize the energy: $\varphi(x) = +v$ and $\varphi(x) = -v$.

This is in contrast to the usual case of positive m^2, for which the minimum-energy classical field configuration is $\varphi(x) = 0$.

We can expect that the quantum theory will follow suit.

For $m^2 < 0$, there will be two ground states, $|0+\rangle$ and $|0-\rangle$, with the property that

$$
\begin{aligned}
\langle 0+|\varphi(x)|0+\rangle &= +v \ , \\
\langle 0-|\varphi(x)|0-\rangle &= -v \ ,
\end{aligned}
\tag{30.6}
$$

up to quantum corrections from loop diagrams that we will treat in detail in section 30.

These two ground states are exchanged by the operator Z,

$$Z|0+\rangle = |0-\rangle \ , \tag{30.7}$$

and they are orthogonal: $\langle 0+|0-\rangle = 0$.

This last claim requires some comment.

Consider a similar problem in quantum mechanics,

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{24}\lambda(x^2 - v^2)^2 \ . \tag{30.8}$$

There are two approximate ground states in this case, specified by the approximate wave functions

$$\psi_\pm(x) = \langle x|0\pm\rangle \sim \exp[-\omega(x \mp v)^2/2] \ , \tag{30.9}$$

where $\omega = (\lambda v^2/3)^{1/2}$ is the frequency of small oscillations about the minimum.

However, the true ground state is a symmetric linear combination of these.

The antisymmetric linear combination has a slightly higher energy, due to the effects of quantum tunneling.

We can regard a field theory as an infinite set of oscillators, one for each point in space, each with a hamiltonian like eq. (30.8), and coupled together by the $(\nabla\varphi)^2$ term in the field-theory hamiltonian.

There is a tunneling amplitude for each oscillator, but to turn the field-theoretic state $|0+\rangle$ into $|0-\rangle$, all the oscillators have to tunnel, and so the tunneling amplitude gets raised to the power of the number of oscillators, that is, to the power of infinity (more precisely, to a power that scales like the volume of space).

Therefore, in the limit of infinite volume, $\langle 0+|0-\rangle$ vanishes.

Thus we can pick either $|0+\rangle$ or $|0-\rangle$ to use as the ground state.

Let us choose $|0+\rangle$.

Then we can define a shifted field,

$$\rho(x) = \varphi(x) - v \ , \tag{30.10}$$

which obeys $\langle 0+|\rho(x)|0+\rangle = 0$.

We must still worry about loop corrections, which we will do at the end of this section.

The potential becomes

$$\begin{aligned} V(\varphi) &= \tfrac{1}{24}\lambda[(\rho+v)^2 - v^2]^2 \\ &= \tfrac{1}{6}\lambda v^2\rho^2 + \tfrac{1}{6}\lambda v\rho^3 + \tfrac{1}{24}\lambda\rho^4 \ , \end{aligned} \tag{30.11}$$

and so the lagrangian is now

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{6}\lambda v^2\rho^2 - \tfrac{1}{6}\lambda v\rho^3 - \tfrac{1}{24}\lambda\rho^4 \ . \tag{30.12}$$

We see that the coefficient of the ρ^2 term is $1/6\ \lambda v^2 = |m^2|$.

This coefficient should be identified as $1/2\ m^2_\rho$, where m_ρ is the mass of the corresponding ρ particle.

Also, we see that the shifted field now has a cubic as well as a quartic interaction.

Eq. (30.12) specifies a perfectly sensible, renormalizable quantum field theory, but it no longer has an obvious Z_2 symmetry.

We say that the Z_2 symmetry is *spontaneously broken*.

This leads to a question about renormalization.

If we include renormalizing Z factors in the original lagrangian, we get

$$\mathcal{L} = -\tfrac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} Z_m m^2 \varphi^2 - \tfrac{1}{24} Z_\lambda \lambda \varphi^4 \ . \qquad (30.13)$$

For positive m^2, these three Z factors are sufficient to absorb infinities for $d \leq 4$, where the mass dimension of λ is positive or zero.

On the other hand, looking at the lagrangian for negative m^2 after the shift, eq. (30.12), we would seem to need an extra Z factor for the ρ^3 term.

Also, once we have a ρ^3 term, we would expect to need to add a ρ term to cancel tadpoles.

So, the question is, are the original three Z factors sufficient to absorb all the divergences in the Feynman diagrams derived from eq. (30.13)?

The answer is yes.

To see why, consider the quantum action (introduced in section 21)

$$\begin{aligned}\Gamma(\varphi) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\widetilde{\varphi}(-k)\Big(k^2+m^2-\Pi(k^2)\Big)\widetilde{\varphi}(k) \\ +\sum_{n=3}^{\infty}\frac{1}{n!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\,(2\pi)^4\delta^4(k_1+\ldots+k_n) \\ \times \mathbf{V}_n(k_1,\ldots,k_n)\,\widetilde{\varphi}(k_1)\ldots\widetilde{\varphi}(k_n)\;,\end{aligned} \tag{30.14}$$

computed with $m^2 > 0$.

The ingredients of $\Gamma(\varphi)$—the self-energy $\Pi(k^2)$ and the exact 1PI vertices $\mathbf{V}_n$—are all made finite and well-defined (in, say, the $\overline{MS}$ renormalization scheme) by adjusting the three Z factors in eq. (30.13).

Furthermore, for $m^2 > 0$, the quantum action inherits the Z_2 symmetry of the classical action.

To see this directly, we note that $\mathbf{V}_n$ must zero for odd n, simply because there is no way to draw a 1PI diagram with an odd number of external lines using only a four-point vertex.

Thus $\Gamma(\varphi)$ also has the Z_2 symmetry.

This is a simple example of a more general result that we proved in problem 21.2: the quantum action inherits any linear symmetry of the classical action, provided that it is also a symmetry of the integration measure $\mathcal{D}\varphi$.

Linear means that the transformed fields are linear functions of the original ones.

The integration measure is almost always invariant; when it is not, the symmetry is said to be anomalous.

We will meet an anomalous symmetry in section 75.

Once we have computed the quantum action for $m^2 > 0$, we can go ahead and consider the case of $m^2 < 0$.

Recall from section 21 that the quantum equation of motion in the presence of a source is $\delta\Gamma/\delta\varphi(x) = -J(x)$, and that the solution of this equation is also the vacuum expectation value of $\varphi(x)$.

Now set $J(x) = 0$, and look for a translationally invariant (that is, constant) solution $\varphi(x) = v$.

If there is more than one such solution, we want the one(s) with the lowest energy.

This is equivalent to minimizing the quantum potential $U(\varphi)$, where

$$\Gamma(\varphi) = \int d^4x \left[-\mathcal{U}(\varphi) - \tfrac{1}{2}\mathcal{Z}(\varphi)\partial^\mu\varphi\partial_\mu\varphi + \dots \right] , \tag{30.15}$$

and the ellipses stand for terms with more derivatives.

In a weakly coupled theory, we can expect the loop-corrected potential $U(\varphi)$ to be qualitatively similar to the classical potential $V(\varphi)$.

Therefore, for $m^2 < 0$, we expect that there are two minima of $U(\varphi)$ with equal energy, located at $\varphi(x) = \pm v$, where $v = \langle 0|\varphi(x)|0\rangle$ is the exact vacuum expectation value of the field.

Thus we have a description of spontaneous symmetry breaking in the quantum theory based on the quantum action, and the quantum action is made finite by adjusting only the three Z factors that appear in the original, symmetric form of the lagrangian.

In the next section, we will see how this works in explicit calculations.

31 Broken Symmetry and Loop Corrections

Consider φ^4 theory, where φ is a real scalar field with lagrangian

$$\mathcal{L} = -\tfrac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} Z_m m^2 \varphi^2 - \tfrac{1}{24} Z_\lambda \lambda \varphi^4 \ . \tag{31.1}$$

In d= 4 spacetime dimensions, the coupling λ is dimensionless.

We begin by considering the case $m^2 > 0$, where the Z_2 symmetry of $\mathcal{L}$ under $\varphi \to -\varphi$ is manifest.

We wish to compute the three renormalizing Z factors.

We work in d = 4 − ε dimensions, and take $\lambda \to \lambda\, \tilde{\mu}^{\varepsilon}$ (where $\tilde{\mu}$ has dimensions of mass) so that λ remains dimensionless.

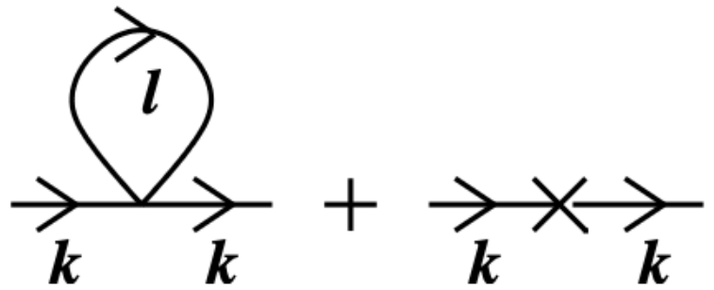
The propagator correction Π(k2) is given by the diagrams of fig. (31.1), which yield

$$i\Pi(k^2) = \tfrac{1}{2}(-i\lambda\tilde{\mu}^{\varepsilon})\tfrac{1}{i}\tilde{\Delta}(0) - i(Ak^2 + Bm^2)\ , \tag{31.2}$$

where A = Z_φ − 1 and B = Z_m − 1, and

$$\tilde{\Delta}(0) = \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{\ell^2 + m^2}\ . \tag{31.3}$$

Figure 31.1: O(λ) corrections to Π(k^2).



Using the usual bag of tricks from section 14, we find

$$\tilde{\mu}^{\varepsilon}\tilde{\Delta}(0) = \frac{-i}{(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + \ln(\mu^2/m^2)\right]m^2\ , \tag{31.4}$$

where $\mu^2 = 4\pi e^{-\gamma}\ \tilde{\mu}^{\,2}$.

Thus

$$\Pi(k^2) = \frac{\lambda}{2(4\pi)^2}\left[\frac{2}{\varepsilon} + 1 + \ln(\mu^2/m^2)\right]m^2 - Ak^2 - Bm^2 \ . \tag{31.5}$$

From eq. (31.5) we see that we must have

$$A = O(\lambda^2) \ , \tag{31.6}$$

$$B = \frac{\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \kappa_B\right) + O(\lambda^2) \ , \tag{31.7}$$

where κ_B is a finite constant (that may depend on μ).

In the $\overline{MS}$ renormalization scheme, we take $\kappa_B = 0$, but we will leave κ_B arbitrary for now.

Next we turn to the vertex correction, given by the diagram of fig. (31.2), plus two others with $k_2 \leftrightarrow k_3$ and $k_2 \leftrightarrow k_4$; all momenta are treated as incoming.

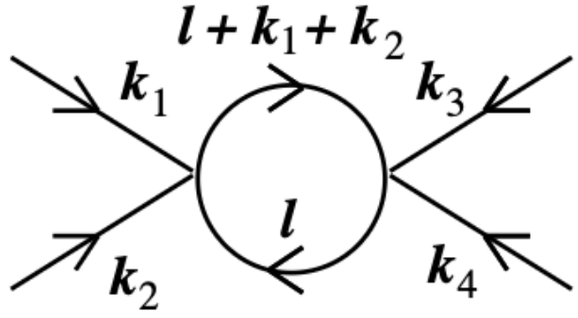


Figure 31.2: The $O(\lambda^2)$ correction to $\mathbf{V}_4(k_1, k_2, k_3, k_4)$. Two other diagrams, obtained from this one via $k_2 \leftrightarrow k_3$ and $k_2 \leftrightarrow k_4$, also contribute.

$\widetilde{dk}$ $\overline{MS}$

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\boldsymbol{\mathcal{L}}$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ the the the the the the $\mathbf{V}_4(k_1,k_2,k_3,k_4)$

We have

$$i\mathbf{V}_4(k_1,k_2,k_3,k_4) = -iZ_\lambda\lambda + \tfrac{1}{2}(-i\lambda)^2\left(\tfrac{1}{i}\right)^2\left[iF(-s)+iF(-t)+iF(-u)\right] + O(\lambda^3)\ . \tag{31.8}$$

Here we have defined $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_3)^2$, $u = -(k_1 + k_4)^2$, and

$$\begin{aligned} iF(k^2) &\equiv \tilde{\mu}^\varepsilon \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{((\ell+k)^2+m^2)(\ell^2+m^2)} \\ &= \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} + \int_0^1 dx\,\ln(\mu^2/D)\right]\ , \end{aligned} \tag{31.9}$$

where $D= x(1-x)k^2 + m^2$.

Setting $Z_\lambda = 1 + C$ in eq. (31.8), we see that we need

$$C = \frac{3\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \kappa_C\right) + O(\lambda^2)\ , \tag{31.10}$$

where κ_C is a finite constant.

We may as well pause to compute the beta function, $\beta(\lambda) = d\lambda/d\ln\mu$, where the derivative is taken with the bare coupling λ_0 held fixed, and the finite parts of the counterterms set to zero, in accord with the $\overline{MS}$ prescription.

We have

$$\lambda_0 = Z_\lambda Z_\varphi^{-2} \lambda \tilde{\mu}^\varepsilon \ , \tag{31.11}$$

with

$$\ln\left(Z_\lambda Z_\varphi^{-2}\right) = \frac{3\lambda}{16\pi^2}\,\frac{1}{\varepsilon} + O(\lambda^2) \ . \tag{31.12}$$

Let $L_1(\lambda)$ be the coefficient of $1/\varepsilon$ in eq. (31.12).

Our analysis in section 28 shows that the beta function is then given by $\beta(\lambda) = \lambda 2L'_1(\lambda)$.

Thus we find

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \ . \tag{31.13}$$

The beta function is positive, which means that the theory becomes more and more strongly coupled at higher and higher energies.

Now we consider the more interesting case of $m^2 < 0$, which results in the spontaneous breakdown of the Z_2 symmetry.

Following the procedure of section 30, we set $\varphi(x) = \rho(x) + v$, where $v = (6|m^2|/\lambda)^{1/2}$ minimizes the potential (without Z factors).

Then the lagrangian becomes (with Z factors)

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi \partial^\mu \rho \partial_\mu \rho - \tfrac{1}{2}(\tfrac{3}{4}Z_\lambda - \tfrac{1}{4}Z_m) m_\rho^2 \rho^2$$
$$+ \tfrac{1}{2}(Z_m - Z_\lambda)(3/\lambda\tilde{\mu}^\varepsilon)^{1/2} m_\rho^3 \rho \qquad (31.14)$$
$$- \tfrac{1}{6}Z_\lambda (3\lambda\tilde{\mu}^\varepsilon)^{1/2} m_\rho \rho^3 - \tfrac{1}{24}Z_\lambda \lambda \tilde{\mu}^\varepsilon \rho^4 \ ,$$

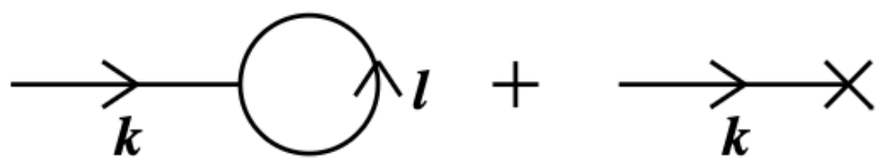
where $m^2_\rho = 2|m^2|$.

Now we can compute various one-loop corrections.

We begin with the vacuum expectation value of ρ.

The O(λ) correction is given by the diagrams of fig. (31.3).

Figure 31.3: The O(λ) correction to the vacuum expectation value of the ρ field.



The three-point vertex factor is $-iZ_\lambda g_3$, where g_3 can be read off of eq. (31.14):

$$g_3 = (3\lambda\tilde{\mu}^\varepsilon)^{1/2} m_\rho \ . \qquad (31.15)$$

The one-point vertex factor is iY, where Y can also be read off of eq. (31.14):

$$Y = \tfrac{1}{2}(Z_m - Z_\lambda)(3/\lambda\tilde{\mu}^\varepsilon)^{1/2} m_\rho^3 \ . \qquad (31.16)$$

Following the discussion of section 9, we then find that

$$\langle 0|\rho(x)|0\rangle = \left(iY + \tfrac{1}{2}(-iZ_\lambda g_3)\tfrac{1}{i}\tilde{\Delta}(0)\right)\int d^4y\, \tfrac{1}{i}\Delta(x-y)\ , \tag{31.17}$$

plus higher-order corrections.

Using eqs. (31.15) and (31.16), and eq. (31.4) with $m^2 \to m^2_\rho$, the factor in large parentheses in eq. (31.17) becomes

$$\frac{i}{2}(3/\lambda)^{1/2}m_\rho^3\left(Z_m - Z_\lambda + \frac{\lambda}{16\pi^2}\left[\frac{2}{\varepsilon} + 1 + \ln(\mu^2/m_\rho^2)\right] + O(\lambda^2)\right)\ . \tag{31.18}$$

Using $Z_m = 1 + B$ and $Z_\lambda = 1 + C$, with B and C from eqs. (31.7) and (31.10), the factor in large parentheses in eq. (31.18) becomes

$$\frac{\lambda}{16\pi^2}\left[\kappa_B - \kappa_C + 1 + \ln(\mu^2/m_\rho^2)\right]\ . \tag{31.19}$$

All the 1/ε's have canceled.

The remaining finite vacuum expectation value for ρ(x) can now be removed by choosing

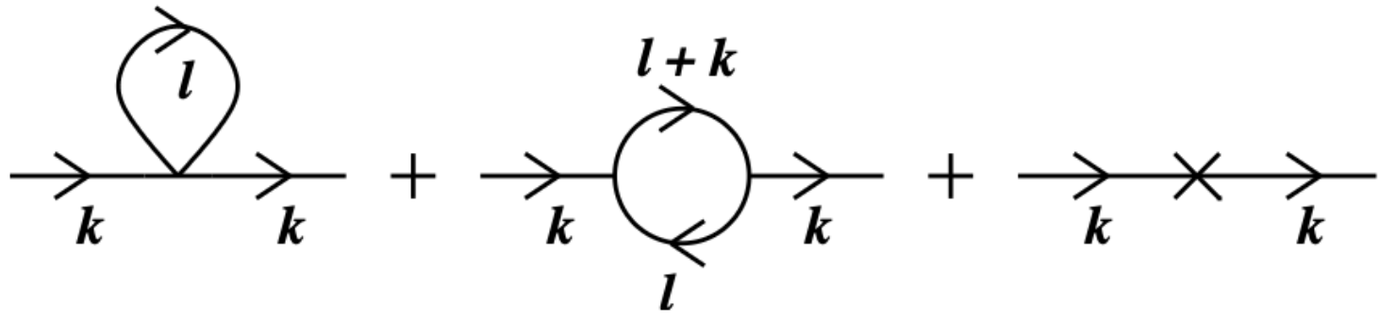
$$\kappa_B - \kappa_C = -1 - \ln(\mu^2/m_\rho^2)\ . \tag{31.20}$$

This will also cancel all diagrams with one-loop tadpoles.

Next we consider the ρ propagator.

The diagrams contributing to the O(λ) correction are shown in fig. (31.4).

Figure 31.4: O(λ) corrections to the ρ propagator.



The counterterm insertion is −iX, where, again reading off of eq. (31.14),

$$X = Ak^2 + (\tfrac{3}{4}C - \tfrac{1}{4}B)m_\rho^2 \ . \tag{31.21}$$

Putting together the results of eq. (31.2) for the first diagram (with $m^2 \to m^2_\rho$), eq. (31.9) for the second (ditto), and eq. (31.21) for the third, we get

$$\begin{aligned}
\Pi(k^2) &= -\tfrac{1}{2}(\lambda\tilde{\mu}^\varepsilon)\tfrac{1}{i}\tilde{\Delta}(0) + \tfrac{1}{2}g_3^2 F(k^2) - X + O(\lambda^2) \\
&= \frac{\lambda}{32\pi^2}\, m_\rho^2 \left[\frac{2}{\varepsilon} + 1 + \ln(\mu^2/m_\rho^2)\right] \\
&\quad + \frac{\lambda}{32\pi^2}\, m_\rho^2 \left[\frac{2}{\varepsilon} + \int_0^1 dx\, \ln(\mu^2/D)\right] \\
&\quad - Ak^2 - (\tfrac{3}{4}C - \tfrac{1}{4}B)m_\rho^2 + O(\lambda^2).
\end{aligned} \tag{31.22}$$

Again using eqs. (31.7) and (31.10) for B and C, we see that all the 1/ε's cancel, and we're left with

$$\Pi(k^2) = \frac{\lambda}{32\pi^2} m_\rho^2 \left[1 + \ln(\mu^2/m_\rho^2) + \int_0^1 dx \, \ln(\mu^2/D) + \tfrac{1}{2}(9\kappa_C - \kappa_B) \right] + O(\lambda^2) \, . \tag{31.23}$$

We can now choose to work in an OS scheme, where we require $\Pi(-m^2_\rho) = 0$ and $\Pi'(-m^2_\rho) = 0$.

We see that, to this order in λ, $\Pi(k^2)$ is independent of k^2.

Thus, we automatically have $\Pi'(-m^2_\rho) = 0$, and we can choose $9\kappa_C - \kappa_B$ to fix $\Pi(-m^2_\rho) = 0$.

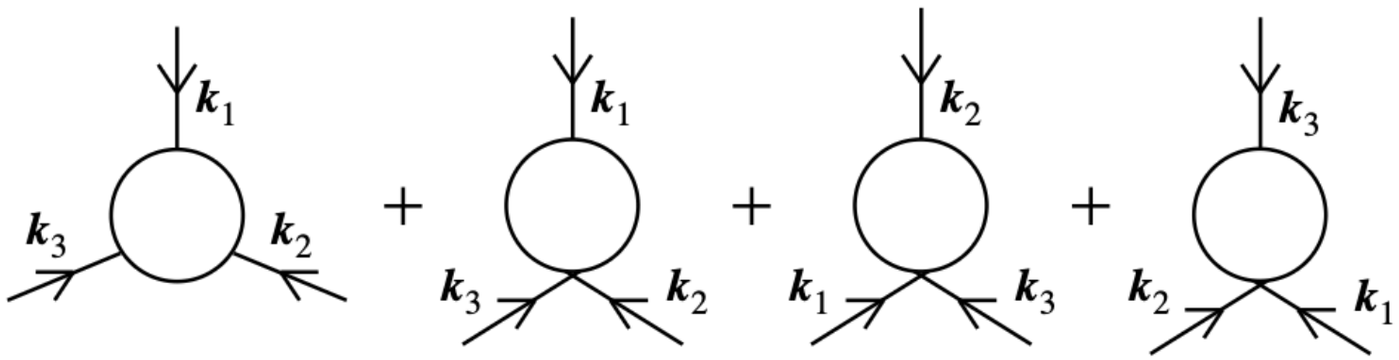
Together with eq. (31.20), this completely determines κ_B and κ_C to this order in λ.

Next we consider the one-loop correction to the three-point vertex, given by the diagrams of fig. (31.5).

We wish to show that the infinities are canceled by the value of $Z_\lambda = 1 + C$ that we have already determined.

The first diagram in fig. (31.5) is finite, and so for our purposes we can ignore it.

Figure 31.5: O(λ) corrections to the vertex for three ϱ fields.



The remaining three, plus the original vertex, sum up to give

$$i\mathbf{V}_3(k_1,k_2,k_3)_{\rm div} = -iZ_\lambda g_3 + \tfrac{1}{2}(-i\lambda)(-ig_3)\left(\tfrac{1}{i}\right)^2 \times\left[iF(k_1^2)+iF(k_2^2)+iF(k_3^2)\right] + O(\lambda^{5/2})\,, \tag{31.24}$$

where the subscript div means that we are keeping only the divergent part.

Using eq. (31.9), we have

$$\mathbf{V}_3(k_1,k_2,k_3)_{\rm div} = -g_3\left(1+C-\frac{3\lambda}{16\pi^2}\,\frac{1}{\varepsilon}+O(\lambda^2)\right)\,. \tag{31.25}$$

From eq. (31.10), we see that the divergent terms do indeed cancel to this order in λ.

Finally, we have the correction to the four-point vertex.

In this case, the divergent diagrams are just those of fig. (31.1), and so the calculation of the divergent part of $\mathbf{V}_4$ is exactly the same as it is when $m^2 > 0$ (but with m_ρ in place of m).

Since we have already done that calculation (it was how we determined C in the first place), we need not repeat it.

We have thus seen how we can compute the divergent parts of the counterterms in the simpler case of $m^2 > 0$, where the Z_2 symmetry is unbroken, and that these counterterms will also serve to cancel the divergences in the more complicated case of $m^2 < 0$, where the Z_2 symmetry is spontaneously broken.

This a general rule for renormalizable theories with spontaneous symmetry breaking, regardless of the nature of the symmetry group.

32 Spontaneous Breaking of Continuous Symmetries

Consider the theory (introduced in section 22) of a complex scalar field φ with

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2 \ . \tag{32.1}$$

This lagrangian is obviously invariant under the U(1) transformation

$$\varphi(x) \to e^{-i\alpha}\varphi(x) \ , \tag{32.2}$$

where α is a real number.

Now suppose that m^2 is negative.

The minimum of the potential of eq. (32.1) is achieved for

$$\varphi(x) = \tfrac{1}{\sqrt{2}} v e^{-i\theta} \ , \tag{32.3}$$

where

$$v = (4|m^2|/\lambda)^{1/2} \ , \tag{32.4}$$

and the phase θ is arbitrary; the factor of root-two in eq. (32.3) is conventional.

Thus we have a continuous family of minima of the potential, parameterized by θ.

Under the U(1) transformation of eq. (32.2), θ changes to θ + α; thus the different minimum-energy field configurations are all related to each other by the symmetry.

In the quantum theory, we therefore expect to find a continuous family of ground states, labeled by θ, with the property that

$$\langle\theta|\varphi(x)|\theta\rangle = \tfrac{1}{\sqrt{2}} v e^{-i\theta} \ . \tag{32.5}$$

Also, according to the discussion in section 30, we expect $\langle\theta'|\theta\rangle = 0$ for $\theta' \neq \theta$.

Returning to classical language, there is a flat direction in field space that we can move along without changing the energy.

The physical consequence of this is the existence of a massless particle called a Goldstone boson.

Let us see how this works in more detail.

We first choose the phase $\theta = 0$, and then write

$$\varphi(x) = \frac{1}{\sqrt{2}}[v + a(x) + ib(x)] \ , \tag{32.6}$$

where a and b are real scalar fields.

Substituting eq. (32.6) into eq. (32.1), we find

$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{2}\partial^\mu a \partial_\mu a - \tfrac{1}{2}\partial^\mu b \partial_\mu b \\ & - |m^2| a^2 - \tfrac{1}{2}\lambda^{1/2}|m| a(a^2 + b^2) - \tfrac{1}{16}\lambda(a^2 + b^2)^2 \ . \end{aligned} \tag{32.7}$$

We see from this that the a field has a mass given by $1/2\ m^2_a = |m^2|$.

The b field, on the other hand, is massless, and we identify it as the Goldstone boson.

A different parameterization brings out the role of the massless field more clearly.

We write

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \ , \tag{32.8}$$

where ρ and χ are real scalar fields.

Substituting eq. (32.8) into eq. (32.1), we get

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}\left(1+\frac{\rho}{v}\right)^2\partial^\mu\chi\partial_\mu\chi \\ - |m^2|\rho^2 - \tfrac{1}{2}\lambda^{1/2}|m|\rho^3 - \tfrac{1}{16}\lambda\rho^4 \ . \tag{32.9}$$

We see from this that the ϱfield has a mass given by $1/2\ m^2_\rho = |m^2|$, and that the χ field is massless.

These are the same particle masses we found using the parameterization of eq. (32.6).

This is not an accident: the particle masses and scattering amplitudes are independent of field redefinitions.

Note that the χ field does not appear in the potential at all.

Thus it parameterizes the flat direction.

In terms of the ρ and χ fields, the U(1) transformation takes the simple form $\chi(x) \rightarrow \chi(x) + \alpha$.

Does the masslessness of the χ field survive loop corrections?

It does.

To see this, we note that if the χ field remains massless, its exact propagator $\widetilde{\Delta}_\chi(k^2)$ should have a pole at $k^2 = 0$; equivalently, the self-energy $\Pi_\chi(k^2)$, related to the propagator by $\widetilde{\Delta}_\chi(k^2) = 1/[k^2 - \Pi\chi(k^2)]$, should satisfy $\Pi(0) = 0$.

We can evaluate $\Pi_\chi(0)$ by summing all 1PI diagrams with two external χ lines, each with fourmomentum $k = 0$.

We note from eq. (32.9) that the derivatives acting on the χ fields imply that the vertex factors for the $\rho\chi\chi$ and $\rho\rho\chi\chi$ vertices are each proportional to $k_1 \cdot k_2$, where k_1 and k_2 are the momenta of the two χ lines that meet at that vertex.

Since the external lines have zero momentum, the attached vertices vanish; hence, $\Pi_\chi(0) = 0$, and the χ particle remains massless.

The same conclusion can be reached by considering the quantum action $\Gamma(\varphi)$, which includes all loop corrections.

According to our discussion in section 29, the quantum action has the same symmetries as the classical action.

Therefore, in the case at hand,

$$\Gamma(\varphi) = \Gamma(e^{-i\alpha}\varphi) \ . \tag{32.10}$$

Spontaneous symmetry breaking occurs if the minimum of $\Gamma(\varphi)$ is at a constant, nonzero value of φ.

Because of eq. (32.10), the phase of this constant is arbitrary.

Therefore, there must be a flat direction in field space, corresponding to the phase of φ(x).

The physical consequence of this flat direction is a massless particle, the Goldstone boson.

All of this has a straightforward extension to the nonabelian case.

Consider

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}m^2\varphi_i\varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2 \ , \tag{32.11}$$

where a repeated index is summed.

This lagrangian is invariant under the infinitesimal SO(N) transformation

$$\delta\varphi_i = -i\theta^a(T^a)_{ij}\varphi_j \ , \tag{32.12}$$

where i runs from 1 to 1/2 N(N−1), θ^a is a set of 1/2 N(N−1) real, infinitesimal parameters, and each antisymmetric generator matrix T^a has a single nonzero entry −i above the main diagonal, and a corresponding +i below the main diagonal.

Now let us take $m^2 < 0$ in eq. (32.11).

The minimum of the potential is achieved for $\varphi_i(x) = v_i$, where $v^2 = v_i v_i = 4|m^2|/\lambda$, and the direction in which the N-component vector $\vec{v}$ points is arbitrary.

In the quantum theory, we interpret v_i as the vacuum expectation value (VEV for short) of the quantum field $\varphi_i(x)$.

We can choose our coordinate system so that $v_i = v\delta_{iN}$; that is, the VEV lies entirely in the last component.

Now consider making an infinitesimal SO(N) transformation.

This changes the VEV; we have

$$\begin{aligned} v_i &\to v_i - i\theta^a (T^a)_{ij} v_j \\ &= v\delta_{iN} - i\theta^a (T^a)_{iN} v \ . \end{aligned} \tag{32.13}$$

For some choices of θ^a, the second term on the right-hand side of eq. (32.13) vanishes.

This happens if the corresponding T^a has no nonzero entry in the last column.

There are N−1 T^a's with a nonzero entry in the last column: those with the −i in the first row and last column, in the second row and last column, etc, down to the $N-1^{th}$ row and last column.

These T^a's are said to be broken generators.

A generator is broken if $(T^a)_{ij}v_j \neq 0$, and unbroken if $(T^a)_{ij}v_j = 0$.

An infinitesimal SO(N) transformation that involves a broken generator changes the VEV of the field, but not the energy.

Thus, each broken generator corresponds to a *flat direction* in field space.

Each flat direction implies the existence of a corresponding massless particle.

This is Goldstone's theorem: there is one massless Goldstone boson for each broken generator.

The unbroken generators, on the other hand, do not change the VEV of the field.

Therefore, after rewriting the lagrangian in terms of shifted fields (each with zero VEV), there should still be a manifest symmetry corresponding to the set of unbroken generators.

In the present case, the number of unbroken generators is 1/2 N(N−1) − (N−1) = 1/2 (N−1)(N−2).

This is the number of generators of SO(N−1).

Therefore, we expect SO(N−1) to be an obvious symmetry of the lagrangian after it is written in terms of shifted fields.

Let us see how this works in the present case.

We can rewrite eq. (32.11) as

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - V(\varphi) \ , \tag{32.14}$$

with

$$V(\varphi) = \tfrac{1}{16}\lambda(\varphi_i\varphi_i - v^2)^2 \ , \tag{32.15}$$

where $v = (4|m^2|/\lambda)^{1/2}$, and the repeated index i is implicitly summed from 1 to N.

Now let $\varphi_N(x) = v + \rho(x)$, and plug this into eq. (32.14).

With the repeated index i now implicitly summed from 1 to N−1, we have

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - V(\rho,\varphi)\ , \tag{32.16}$$

where

$$\begin{aligned} V(\rho,\varphi) &= \tfrac{1}{16}\lambda[(v+\rho)^2 + \varphi_i\varphi_i - v^2]^2 \\ &= \tfrac{1}{16}\lambda(2v\rho + \rho^2 + \varphi_i\varphi_i)^2 \\ &= \tfrac{1}{4}\lambda v^2\rho^2 + \tfrac{1}{4}\lambda v\rho(\rho^2 + \varphi_i\varphi_i) + \tfrac{1}{16}\lambda(\rho^2 + \varphi_i\varphi_i)^2\ . \end{aligned} \tag{32.17}$$

There is indeed a manifest SO(N−1) symmetry in eqs. (32.16) and (32.17).

Also, the N−1 φ_i fields are massless: they are the expected N−1 Goldstone bosons.

Part II

Spin One Half

33 Representations of the Lorentz Group

In section 2, we saw that we could define a unitary operator $U(\Lambda)$ that implemented a Lorentz transformation on a scalar field φ(x) via

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x) \ . \tag{33.1}$$

As shown in section 2, this implies that the derivative of the field transforms as

$$U(\Lambda)^{-1}\partial^{\mu}\varphi(x)U(\Lambda) = \Lambda^{\mu}{}_{\rho}\bar{\partial}^{\rho}\varphi(\Lambda^{-1}x) \ , \tag{33.2}$$

where the bar on the derivative means that it is with respect to the argument $\bar{x} = \Lambda^{-1}x$.

Eq. (33.2) suggests that we could define a *vector field* A^{μ}(x) that would transform as

$$U(\Lambda)^{-1}A^{\rho}(x)U(\Lambda) = \Lambda^{\mu}{}_{\rho}A^{\rho}(\Lambda^{-1}x) \ , \tag{33.3}$$

or a *tensor field* $B^{\mu\nu}$(x) that would transform as

$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}B^{\rho\sigma}(\Lambda^{-1}x) \ . \tag{33.4}$$

Note that if $B^{\mu\nu}$ is either symmetric, $B^{\mu\nu}(x) = B^{\nu\mu}(x)$, or antisymmetric, $B^{\mu\nu}(x) = -B^{\nu\mu}(x)$, then the symmetry is preserved by the Lorentz transformation.

Also, if we take the trace to get $T(x) \equiv g_{\mu\nu}B^{\mu\nu}(x)$, then, using $g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$, we find that T(x) transforms like a scalar field,

$$U(\Lambda)^{-1}T(x)U(\Lambda) = T(\Lambda^{-1}x) \ . \tag{33.5}$$

Thus, given a tensor field $B^{\mu\nu}(x)$ with no particular symmetry, we can write

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4}g^{\mu\nu}T(x) \ , \tag{33.6}$$

where $A^{\mu\nu}$ is antisymmetric ($A^{\mu\nu} = -A^{\nu\mu}$) and $S^{\mu\nu}$ is symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and traceless ($g_{\mu\nu}S^{\mu\nu} = 0$).

The key point is that the fields $A^{\mu\nu}$, $S^{\mu\nu}$, and T do not mix with each other under Lorentz transformations.

Is it possible to further break apart these fields into still smaller sets that do not mix under Lorentz transformations?

How do we make this decomposition into *irreducible representations* of the Lorentz group for a field carrying n vector indices?

Are there any other kinds of indices we could consistently assign to a field?

If so, how do these behave under a Lorentz transformation?

The answers to these questions are to be found in the theory of **group representations**.

Let us see how this works for the Lorentz group (in four spacetime dimensions).

Consider a field (not necessarily hermitian) that carries a generic Lorentz index, $\varphi_A(x)$.

Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\varphi_A(x)U(\Lambda) = L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x) \ , \tag{33.7}$$

where $L_A{}^B(\Lambda)$ is a matrix that depends on Λ.

These finite-dimensional matrices must obey the group composition rule

$$L_A{}^B(\Lambda')L_B{}^C(\Lambda) = L_A{}^C(\Lambda'\Lambda) \ . \tag{33.8}$$

We say that the matrices $L_A{}^B(\Lambda)$ form a representation of the Lorentz group.

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$U(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \ , \tag{33.9}$$

where the operators $M^{\mu\nu}$ are the generators of the Lorentz group.

As shown in section 2, the generators obey the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu\leftrightarrow\nu)\Big) - (\rho\leftrightarrow\sigma) \ , \tag{33.10}$$

which specify the *Lie algebra* of the Lorentz group.

We can identify the components of the angular momentum operator $\vec{J}$ as $J_i \equiv 1/2\ \varepsilon_{ijk} M^{jk}$ and the components of the boost operator $\vec{K}$ as $K_i \equiv M^{i0}$.

We then find from eq. (33.10) that

$$[J_i, J_j] = +i\varepsilon_{ijk} J_k \ , \tag{33.11}$$

$$[J_i, K_j] = +i\varepsilon_{ijk} K_k \ , \tag{33.12}$$

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k \ . \tag{33.13}$$

For an infinitesimal transformation, we also have

$$L_A{}^B(1+\delta\omega) = \delta_A{}^B + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A{}^B \ , \tag{33.14}$$

Eq. (33.7) then becomes

$$[\varphi_A(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi_A(x) + (S^{\mu\nu})_A{}^B \varphi_B(x) \ , \tag{33.15}$$

where $\mathcal{L}^{\mu\nu} \equiv 1/i(x^\mu \partial^\nu - x^\nu \partial^\mu)$.

Both the differential operators $\mathcal{L}^{\mu\nu}$ and the representation matrices $(S^{\mu\nu})_A{}^B$ must separately obey the same commutation relations as the generators themselves.

Our problem now is to find all possible sets of finite-dimensional matrices that obey eq. (33.10), or equivalently eqs. (33.11–33.13).

Although the operators $M^{\mu\nu}$ must be hermitian, the matrices $(S^{\mu\nu})_A{}^B$ need not be.

If we restrict our attention to eq. (33.11) alone, we know (from standard results in the quantum mechanics of angular momentum) that we can find three (2j+1) × (2j+1) hermitian matrices $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ that obey eq. (33.11), and that the eigenvalues of (say) $\mathcal{T}_3$ are −j, −j+1,…, +j, where j has the possible values 0, 1/2 , 1,....

We further know that these matrices constitute all of the inequivalent, irreducible representations of the Lie algebra of SO(3), the rotation group in three dimensions.

Inequivalent means not related by a unitary transformation; irreducible means cannot be made block-diagonal by a unitary transformation.

(The standard derivation assumes that the matrices are hermitian, but allowing nonhermitian matrices does not enlarge the set of solutions.)

Also, when j is a half integer, a rotation by 2π results in an overall minus sign; these representations of the Lie algebra of SO(3) are therefore actually not representations of the group SO(3), since a 2π rotation should be equivalent to no rotation.

As we saw in section 24, the Lie algebra of SO(3) is the same as the Lie algebra of SU(2); the half-integer representations of this Lie algebra do qualify as representations of the group SU(2).

We would like to extend these conclusions to encompass the full set of eqs. (33.11–33.13).

In order to do so, it is helpful to define some nonhermitian operators whose physical significance is obscure, but which simplify the commutation relations.

These are

$$N_i \equiv \tfrac{1}{2}(J_i - iK_i) \ , \tag{33.16}$$

$$N_i^\dagger \equiv \tfrac{1}{2}(J_i + iK_i) \ . \tag{33.17}$$

In terms of N_i and $N^\dagger_i$, eqs. (33.11–33.13) become

$$[N_i, N_j] = i\varepsilon_{ijk} N_k \ , \tag{33.18}$$

$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger \ , \tag{33.19}$$

$$[N_i, N_j^\dagger] = 0 \ . \tag{33.20}$$

We see that we have two different SU(2) Lie algebras that are exchanged by hermitian conjugation.

As we just discussed, a representation of the SU(2) Lie algebra is specified by an integer or half integer; we therefore conclude that a representation of the Lie algebra of the Lorentz group in four spacetime dimensions is specified by two integers or half-integers n and n′.

We will label these representations as (2n+1, 2n′+1); the number of components of a representation is then (2n+1)(2n′+1).

Different components within a representation can also be labeled by their angular momentum representations.

To do this, we first note that, from eqs. (33.16) and (33.17), we have $J_i = N_i + N^\dagger_i$.

Thus, deducing the allowed values of j given n and n′ becomes a standard problem in the addition of angular momenta.

The general result is that the allowed values of j are |n−n′|, |n−n′|+1,…, n+n′, and each of these values appears exactly once.

The four simplest and most often encountered representations are (1,1), (2,1), (1,2), and (2,2).

These are given special names:

$$\begin{aligned}(1,1) &= \textit{Scalar} \text{ or } \textit{singlet} \\ (2,1) &= \textit{Left-handed spinor} \\ (1,2) &= \textit{Right-handed spinor} \\ (2,2) &= \textit{Vector}\end{aligned} \tag{33.21}$$

It may seem a little surprising that (2,2) is to be identified as the vector representation.

To see that this must be the case, we first note that the vector representation is irreducible: all the components of a four-vector mix with each other under a general Lorentz transformation.

Secondly, the vector representation has four components.

The only candidate irreducible representations are (4,1), (1,4), and (2,2).

The first two of these contain angular momenta j = 3/2 only, whereas (2,2) contains j = 0 and j = 1.

This is just right for a four-vector, whose time component is a scalar under spatial rotations, and whose space components are a three-vector.

In order to gain a better understanding of what it means for (2,2) to be the vector representation, we must first investigate the spinor representations (1,2) and (2,1), which contain angular momenta j =1/2 only.

34 Left- and Right-Handed Spinor Fields

Consider a *left-handed spinor field* $\psi_a(x)$, also known as a *left-handed Weyl field,* which is in the (2,1) representation of the Lie algebra of the Lorentz group.

Here the index a is a *left-handed spinor index* that takes on two possible values.

Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x) \ , \tag{34.1}$$

where $L_a{}^b(\Lambda)$ is a matrix in the (2,1) representation.

These matrices satisfy the group composition rule

$$L_a{}^b(\Lambda')L_b{}^c(\Lambda) = L_a{}^c(\Lambda'\Lambda) \ . \tag{34.2}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^b \ , \tag{34.3}$$

where $(S_L^{\mu\nu})_a{}^b = -(S_L^{\nu\mu})_a{}^b$ is a set of 2×2 matrices that obey the same commutation relations as the generators $M^{\mu\nu}$, namely

$$[S_{\rm L}^{\mu\nu}, S_{\rm L}^{\rho\sigma}] = i\Big(g^{\mu\rho} S_{\rm L}^{\nu\sigma} - (\mu \leftrightarrow \nu)\Big) - (\rho \leftrightarrow \sigma) \ . \tag{34.4}$$

Using

$$U(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu} M^{\mu\nu} \ , \tag{34.5}$$

eq. (34.1) becomes

$$[\psi_a(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_{\rm L}^{\mu\nu})_a{}^b \psi_b(x) \ , \tag{34.6}$$

where $\mathcal{L}^{\mu\nu} = 1/i(x^\mu \partial^\nu - x^\nu \partial^\mu)$.

The $\mathcal{L}^{\mu\nu}$ term in eq. (34.6) would also be present for a scalar field, and is not the focus of our current interest; we will suppress it by evaluating the fields at the spacetime origin, $x^\mu = 0$.

Recalling that $M^{ij} = \varepsilon^{ijk} J_k$, where J_k is the angular momentum operator, we have

$$\varepsilon^{ijk}[\psi_a(0), J_k] = (S_{\rm L}^{ij})_a{}^b \psi_b(0) \ . \tag{34.7}$$

Recall that the (2,1) representation of the Lorentz group includes angular momentum j = 1/2 only.

For a spin-one-half operator, the standard convention is that the matrix on the right-hand side of eq. (34.7) is $1/2\ \varepsilon^{ijk}\sigma_k$, where we have suppressed the row index a and the column index b, and where σ_k is a Pauli matrix:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{34.8}$$

We therefore conclude that

$$(S_{\rm L}^{ij})_a{}^b = \tfrac{1}{2}\varepsilon^{ijk}\sigma_k \ ,$$

Thus, for example, setting i =1 and j =2 yields $(S_L{}^{12})_a{}^b = 12\ \varepsilon^{12k}\sigma_k = 1/2\ \sigma_3$, and so $(S_L{}^{12})_1{}^1 = +\ 1/2$, $(S_L{}^{12})_2{}^2 = -1/2$, and $(S_L{}^{12})_1{}^2 = (S_L{}^{12})_2{}^1 = 0$.

Once we have the (2,1) representation matrices for the angular momentum operator Ji, we can easily get them for the boost operator $K_k = M^{k0}$.

This is because $J_k = N_k + N^\dagger{}_k$ and $K_k = i(N_k - N^\dagger{}_k)$, and, acting on a field in the (2,1) representation, $N\dagger k$ is zero.

Therefore, the representation matrices for K_k are simply i times those for J_k, and so

$$(S_{\rm L}^{k0})_a{}^b = \tfrac{1}{2} i\sigma_k \ . \tag{34.10}$$

Now consider taking the hermitian conjugate of the left-handed spinor field $\psi_a(x)$.

Recall that hermitian conjugation swaps the two SU(2) Lie algebras that comprise the Lie algebra of the Lorentz group.

Therefore, the hermitian conjugate of a field in the (2,1) representation should be a field in the (1,2) representation; such a field is called a *right-handed spinor field* or a *right-handed Weyl field*.

We will distinguish the indices of the (1,2) representation from those of the (2,1) representation by putting dots over them.

Thus, we write

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot{a}}(x) \ . \tag{34.11}$$

Under a Lorentz transformation, we have

$$U(\Lambda)^{-1}\psi^\dagger_{\dot{a}}(x)U(\Lambda) = R_{\dot{a}}{}^{\dot{b}}(\Lambda)\psi^\dagger_{\dot{b}}(\Lambda^{-1}x) \ , \tag{34.12}$$

where $R_{\dot{a}}{}^{\dot{b}}(\Lambda)$ is a matrix in the (1,2) representation.

These matrices satisfy the group composition rule

$$R_{\dot{a}}{}^{\dot{b}}(\Lambda')R_{\dot{b}}{}^{\dot{c}}(\Lambda) = R_{\dot{a}}{}^{\dot{c}}(\Lambda'\Lambda) \ . \tag{34.13}$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, we can write

$$R_{\dot{a}}{}^{\dot{b}}(1+\delta\omega) = \delta_{\dot{a}}{}^{\dot{b}} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot{a}}{}^{\dot{b}} \ , \tag{34.14}$$

where $(S^{\mu\nu}_R)^{\dot{b}}_{\dot{a}} = -(S^{\nu\mu}_R)^{\dot{b}}_{\dot{a}}$ is a set of 2×2 matrices that obey the same commutation relations as the generators $M^{\mu\nu}$.

We then have

$$[\psi^\dagger_{\dot{a}}(0), M^{\mu\nu}] = (S^{\mu\nu}_{\rm R})_{\dot{a}}{}^{\dot{b}}\psi^\dagger_{\dot{b}}(0) \ . \tag{34.15}$$

Taking the hermitian conjugate of this equation, we get

$$[M^{\mu\nu}, \psi_a(0)] = [(S^{\mu\nu}_{\rm R})_{\dot{a}}{}^{\dot{b}}]^* \psi_b(0) \ . \tag{34.16}$$

Comparing this with eq. (34.6), we see that

$$(S^{\mu\nu}_{\rm R})_{\dot{a}}{}^{\dot{b}} = -[(S^{\mu\nu}_{\rm L})_a{}^b]^* \ . \tag{34.17}$$

In the previous section, we examined the Lorentz-transformation properties of a field carrying two vector indices.

To help us get better acquainted with the properties of spinor indices, let us now do the same for a field that carries two (2,1) indices.

Call this field $C_{ab}(x)$. Under a Lorentz transformation, we have

$$U(\Lambda)^{-1} C_{ab}(x) U(\Lambda) = L_a{}^c(\Lambda) L_b{}^d(\Lambda) C_{cd}(\Lambda^{-1}x) \ . \tag{34.18}$$

The question we wish to address is whether or not the four components of C_{ab} can be grouped into smaller sets that do not mix with each other under Lorentz transformations.

To answer this question, recall from quantum mechanics that two spin-one-half particles can be in a state of total spin zero, or total spin one.

Furthermore, the single spin-zero state is the unique antisymmetric combination of the two spin-one-half states, and the three spin-one states are the three symmetric combinations of the two spin-one-half states.

We can write this schematically as $2 \otimes 2 = 1_A \oplus 3_S$, where we label the representation of SU(2) by the number of its components, and the subscripts S and A indicate whether that representation appears in the symmetric or anti-symmetric combination of the two 2's.

For the Lorentz group, the relevant relation is $(2,1) \otimes (2,1) = (1,1)_A \oplus (3,1)_S$.

This implies that we should be able to write

$$C_{ab}(x) = \varepsilon_{ab} D(x) + G_{ab}(x) \ , \tag{34.19}$$

where D(x) is a scalar field, $\varepsilon_{ab} = -\varepsilon_{ba}$ is an antisymmetric set of constants, and $G_{ab}(x) = G_{ba}(x)$.

The symbol ε_{ab} is uniquely determined by its symmetry properties up to an overall constant; we will choose $\varepsilon_{21} = -\varepsilon_{12} = +1$.

Since D(x) is a Lorentz scalar, eq. (34.19) is consistent with eq. (34.18) only if

$$L_a{}^c(\Lambda) L_b{}^d(\Lambda) \varepsilon_{cd} = \varepsilon_{ab} \ . \tag{34.20}$$

This means that ε_{ab} is an invariant symbol of the Lorentz group: it does not change under a Lorentz transformation that acts on all of its indices.

In this way, ε_{ab} is analogous to the metric $g_{\mu\nu}$, which is also an invariant symbol, since

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu} \ . \tag{34.21}$$

We use $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ to raise and lower vector indices, and we can use ε_{ab} and and its inverse ε^{ab} to raise and lower left-handed spinor indices.

Here we define ε^{ab} via

$$\varepsilon^{12} = \varepsilon_{21} = +1 \ , \qquad \varepsilon^{21} = \varepsilon_{12} = -1 \ . \tag{34.22}$$

With this definition, we have

$$\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c \ , \qquad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c \ . \tag{34.23}$$

We can then define

$$\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x) \ . \tag{34.24}$$

We also have (suppressing the spacetime argument of the field)

$$\psi_a = \varepsilon_{ab}\psi^b = \varepsilon_{ab}\varepsilon^{bc}\psi_c = \delta_a{}^c\psi_c \ , \tag{34.25}$$

as we would expect.

However, the antisymmetry of ε^{ab} means that we must be careful with minus signs; for example, eq. (34.24) can be written in various ways, such as

$$\psi^a = \varepsilon^{ab}\psi_b = -\varepsilon^{ba}\psi_b = -\psi_b\varepsilon^{ba} = \psi_b\varepsilon^{ab} \ . \tag{34.26}$$

We must also be careful about signs when we contract indices, since

$$\psi^a \chi_a = \varepsilon^{ab} \psi_b \chi_a = -\varepsilon^{ba} \psi_b \chi_a = -\psi_b \chi^b \ . \tag{34.27}$$

In section 35, we will (mercifully) develop an index-free notation that automatically keeps track of these essential (but annoying) minus signs.

An exactly analogous discussion applies to the second SU(2) factor; from the group-theoretic relation (1,2) ⊗ (1,2) = $(1,1)_A \oplus (1,3)_S$, we can deduce the existence of an invariant symbol $\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$.

We will normalize $\varepsilon^{\dot{a}\dot{b}}$ according to eq. (34.22).

Then eqs. (34.23–34.27) hold if all the undotted indices are replaced by dotted indices.

Now consider a field carrying one undotted and one dotted index, $A_{a\dot{a}}$ (x).

Such a field is in the (2,2) representation, and in section 33 we concluded that the (2,2) representation was the vector representation.

We would more naturally write a field in the vector representation as Aμ(x).

There must, then, be a dictionary that gives us the components of $A_{a\dot{a}}$(x) in terms of the components of A_μ(x); we can write this as

$$A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x) \ , \tag{34.28}$$

where $\sigma^{\mu}_{a\dot{a}}$ is another invariant symbol.

That such a symbol must exist can be deduced from the group-theoretic relation

$$(2,1) \otimes (1,2) \otimes (2,2) = (1,1) \oplus \dots \ . \tag{34.29}$$

As we will see in section 35, it turns out to be consistent with our already established conventions for $S_L{}^{\mu\nu}$ and $S_R{}^{\mu\nu}$ to choose

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma}) \ . \tag{34.30}$$

Thus, for example, $\sigma^3_{1\dot{1}} = +1, \sigma^3_{2\dot{2}} = -1, \sigma^3_{1\dot{2}} = \sigma^3_{2\dot{1}} = 0$.

In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol.

For example, we can deduce the existence of $g_{\mu\nu} = g_{\nu\mu}$ from

$$(2,2) \otimes (2,2) = (1,1)_{\rm S} \oplus (1,3)_{\rm A} \oplus (3,1)_{\rm A} \oplus (3,3)_{\rm S} \ . \tag{34.31}$$

Another invariant symbol, the *Levi-Civita symbol*, follows from

$$(2,2) \otimes (2,2) \otimes (2,2) \otimes (2,2) = (1,1)_{\rm A} \oplus \dots \ , \tag{34.32}$$

where the subscript A denotes the completely antisymmetric part.

The Levi-Civita symbol is $\varepsilon^{\mu\nu\rho\sigma}$, which is antisymmetric on exchange of any pair of its indices, and is normalized via $\varepsilon^{0123} = +1$.

To see that $\varepsilon^{\mu\nu\rho\sigma}$ is invariant, we note that $\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}$ is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to $\varepsilon^{\mu\nu\rho\sigma}$.

The constant of proportionality works out to be $\det \Lambda$, which is +1 for a proper Lorentz transformation.

We are finally ready to answer a question we posed at the beginning of section 33.

There we considered a field $B^{\mu\nu}(x)$ carrying two vector indices, and we decomposed it as

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4} g^{\mu\nu} T(x) \ , \tag{34.33}$$

where $A^{\mu\nu}$ is antisymmetric ($A^{\mu\nu} = -A^{\nu\mu}$) and $S^{\mu\nu}$ is symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and traceless ($g_{\mu\nu}S^{\mu\nu} = 0$).

We asked whether further decomposition into still smaller irreducible representations was possible.

The answer to this question can be found in eq. (34.31).

Obviously, T(x) corresponds to (1,1), and $S^{\mu\nu}(x)$ to (3,3).

But, according to eq. (34.31), the antisymmetric field $A^{\mu\nu}(x)$ should correspond to $(3,1) \oplus (1,3)$.

A field in the (3,1) representation carries a symmetric pair of left-handed (undotted) spinor indices; its hermitian conjugate is a field in the (1,3) representation that carries a symmetric pair of right-handed (dotted) spinor indices.

We should, then, be able to find a mapping, analogous to eq. (34.28), that gives $A^{\mu\nu}(x)$ in terms of a field $G_{a\dot{b}}(x)$ and its hermitian conjugate $G^{\dagger}_{a\dot{b}}(x)$.

This mapping is provided by the generator matrices $S_L{}^{\mu\nu}$ and $S_R{}^{\mu\nu}$.

We first note that the Pauli matrices are traceless, and so eqs. (34.9) and (34.10) imply that $(S_L{}^{\mu\nu})_a{}^a = 0$.

Using eq. (34.24), we can rewrite this as $\varepsilon^{ab}(S_L{}^{\mu\nu})_{ab} = 0$.

Since ε^{ab} is antisymmetric, $(S_L{}^{\mu\nu})_{ab}$ must be symmetric on exchange of its two spinor indices.

An identical argument shows that $(S_R{}^{\mu\nu})_{ab}$ must be symmetric on exchange of its two spinor indices.

Furthermore, according to eqs. (34.9) and (34.10), we have

$$(S_{\rm L}^{10})_a{}^b = -i(S_{\rm L}^{23})_a{}^b \ . \tag{34.34}$$

This can be written covariantly with the Levi-Civita symbol as

$$(S_{\rm L}^{\mu\nu})_a{}^b = -\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{{\rm L}\,\rho\sigma})_a{}^b \ . \tag{34.35}$$

Similarly,

$$(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b} = +\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{{\rm R}\,\rho\sigma})_{\dot a}{}^{\dot b}\ . \tag{34.36}$$

Eq. (34.36) follows from taking the complex conjugate of eq. (34.35) and using eq. (34.17).

Now, given a field $G_{ab}(x)$ in the (3,1) representation, we can map it into a *self-dual antisymmetric tensor* $G^{\mu\nu}(x)$ via

$$G^{\mu\nu}(x) \equiv (S_{\rm L}^{\mu\nu})^{ab} G_{ab}(x)\ . \tag{34.37}$$

By *self-dual*, we mean that $G^{\mu\nu}(x)$ obeys

$$G^{\mu\nu}(x) = -\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}(x)\ . \tag{34.38}$$

Taking the hermitian conjugate of eq. (34.37), and using eq. (34.17), we get

$$G^{\dagger\mu\nu}(x) = -(S_{\rm R}^{\mu\nu})^{\dot a\dot b} G^\dagger_{\dot a\dot b}(x)\ , \tag{34.39}$$

which is *anti-self-dual*,

$$G^{\dagger\mu\nu}(x) = +\tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma} G^\dagger_{\rho\sigma}(x)\ . \tag{34.40}$$

Given a hermitian antisymmetric tensor field $A^{\mu\nu}(x)$, we can extract its self-dual and anti-self-dual parts via

$$G^{\mu\nu}(x) = \tfrac{1}{2}A^{\mu\nu}(x) - \tfrac{i}{4}\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}(x)\ , \tag{34.41}$$

$$G^{\dagger\mu\nu}(x) = \tfrac{1}{2}A^{\mu\nu}(x) + \tfrac{i}{4}\varepsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}(x)\ . \tag{34.42}$$

Then we have

$$A^{\mu\nu}(x) = G^{\mu\nu}(x) + G^{\dagger\mu\nu}(x) \ . \tag{34.43}$$

The field $G^{\mu\nu}(x)$ is in the (3,1) representation, and the field $G^{\dagger\mu\nu}(x)$ is in the (1,3) representation; these do not mix under Lorentz transformations.

35 Manipulating Spinor Indices

In section 34 we introduced the invariant symbols ε^{ab}, ε_{ab}, $\varepsilon_{\dot{a}\dot{b}}$, and $\varepsilon^{\dot{a}\dot{b}}$, where

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = +1 \ , \qquad \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1 \ . \tag{35.1}$$

We use the ε symbols to raise and lower spinor indices, contracting the second index on the ε.

If we contract the first index instead, then there is an extra minus sign.

Another invariant symbol is

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma}) \ , \tag{35.2}$$

where I is the 2×2 identity matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{35.3}$$

are the Pauli matrices.

Now let us consider some combinations of invariant symbols with some indices contracted, such as $g_{\mu\nu}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}$.

This object must also be invariant.

Then, since it carries two undotted and two dotted spinor indices, it must be proportional to $\varepsilon_{ab}\,\varepsilon_{\dot{a}\dot{b}}$.

Using eqs. (35.1) and (35.2), we can laboriously check this; it turns out to be correct.

The proportionality constant works out to be minus two:

$$\sigma^{\mu}_{a\dot{a}}\sigma_{\mu b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} \ . \tag{35.4}$$

Similarly, $\varepsilon^{ab}\,\varepsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}$ must be proportional to $g_{\mu\nu}$, and the proportionality constant is again minus two:

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}} = -2g^{\mu\nu} \ . \tag{35.5}$$

Next, let's see what we can learn about the generator matrices $(S^{\mu\nu}_L)^b_a$ and $(S^{\mu\nu}_R)^{\dot{b}}_{\dot{a}}$ from the fact that ε^{ab}, $\varepsilon^{\dot{a}\dot{b}}$, and $\sigma^{\mu}_{a\dot{a}}$ are all invariant symbols.

Begin with

$$\varepsilon_{ab} = L(\Lambda)_a{}^c L(\Lambda)_b{}^d \varepsilon_{cd} \ , \tag{35.6}$$

which expresses the Lorentz invariance of ε_{ab}.

For an infinitesimal transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, we have

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b \ , \tag{35.7}$$

and eq. (35.6) becomes

$$\begin{aligned} \varepsilon_{ab} &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[\,(S_{\rm L}^{\mu\nu})_a{}^c\varepsilon_{cb} + (S_{\rm L}^{\mu\nu})_b{}^d\varepsilon_{ad}\,\Big] + O(\delta\omega^2) \\ &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[\,-(S_{\rm L}^{\mu\nu})_{ab} + (S_{\rm L}^{\mu\nu})_{ba}\,\Big] + O(\delta\omega^2)\ . \end{aligned} \tag{35.8}$$

Since eq. (35.8) holds for any choice of $\delta\omega_{\mu\nu}$, it must be that the factor in square brackets vanishes.

Thus we conclude that $(S_L{}^{\mu\nu})_{ab} = (S_L{}^{\mu\nu})_{ba}$, which we had already deduced in section 34 by a different method.

Similarly, starting from the Lorentz invariance of $\varepsilon^{\dot a\dot b}$, we can show that $(S_R^{\mu\nu})_{\dot a\dot b} = (S_R^{\mu\nu})_{\dot b\dot a}$.

Next, start from

$$\sigma^\rho_{a\dot a} = \Lambda^\rho{}_\tau L(\Lambda)_a{}^b R(\Lambda)_{\dot a}{}^{\dot b}\sigma^\tau_{b\dot b}\ ,$$

which expresses the Lorentz invariance of $\sigma^\rho_{a\dot a}$.

For an infinitesimal transformation, we have

$$\Lambda^{\rho}{}_{\tau} = \delta^{\rho}{}_{\tau} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm V}^{\mu\nu})^{\rho}{}_{\tau} \ , \tag{35.10}$$

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b \ , \tag{35.11}$$

$$R_{\dot a}{}^{\dot b}(1+\delta\omega) = \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b} \ , \tag{35.12}$$

where

$$(S_{\rm V}^{\mu\nu})^{\rho}{}_{\tau} \equiv \tfrac{1}{i}(g^{\mu\rho}\delta^{\nu}{}_{\tau} - g^{\nu\rho}\delta^{\mu}{}_{\tau}) \ . \tag{35.13}$$

Substituting eqs. (35.10–35.13) into eq. (35.9) and isolating the coefficient of $\delta\omega_{\mu\nu}$ yields

$$(g^{\mu\rho}\delta^{\nu}{}_{\tau} - g^{\nu\rho}\delta^{\mu}{}_{\tau})\sigma^{\tau}_{a\dot a} + i(S_{\rm L}^{\mu\nu})_a{}^b\sigma^{\rho}_{b\dot a} + i(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b}\sigma^{\rho}_{a\dot b} = 0 \ . \tag{35.14}$$

Now multiply by $\sigma_{\rho a\dot a}$ to get

$$\sigma^{\mu}_{c\dot c}\sigma^{\nu}_{a\dot a} - \sigma^{\nu}_{c\dot c}\sigma^{\mu}_{a\dot a} + i(S_{\rm L}^{\mu\nu})_a{}^b\sigma^{\rho}_{b\dot a}\sigma_{\rho c\dot c} + i(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b}\sigma^{\rho}_{a\dot b}\sigma_{\rho c\dot c} = 0 \ . \tag{35.15}$$

Next use eq. (35.4) in each of the last two terms to get

$$\sigma^{\mu}_{c\dot c}\sigma^{\nu}_{a\dot a} - \sigma^{\nu}_{c\dot c}\sigma^{\mu}_{a\dot a} + 2i(S_{\rm L}^{\mu\nu})_{ac}\varepsilon_{\dot a\dot c} + 2i(S_{\rm R}^{\mu\nu})_{\dot a\dot c}\varepsilon_{ac} = 0 \ . \tag{35.16}$$

If we multiply eq. (35.16) by $\varepsilon^{\dot a\dot c}$, and remember that $\varepsilon^{\dot a\dot c}(S_R^{\mu\nu})_{\dot a\dot c} = 0$ and that $\varepsilon^{\dot a\dot c}\varepsilon_{\dot a\dot c} = -2$, we get a formula for $(S_L{}^{\mu\nu})_{ac}$, namely

$$(S_{\rm L}^{\mu\nu})_{ac} = \tfrac{i}{4}\varepsilon^{\dot a\dot c}(\sigma^{\mu}_{a\dot a}\sigma^{\nu}_{c\dot c} - \sigma^{\nu}_{a\dot a}\sigma^{\mu}_{c\dot c}) \ . \tag{35.17}$$

Similarly, if we multiply eq. (35.16) by ε^{ac}, we get

$$(S_{\rm R}^{\mu\nu})_{\dot a\dot c} = \tfrac{i}{4}\varepsilon^{ac}(\sigma^{\mu}_{a\dot a}\sigma^{\nu}_{c\dot c} - \sigma^{\nu}_{a\dot a}\sigma^{\mu}_{c\dot c}) \ . \tag{35.18}$$

These formulae can be made to look a little nicer if we define

$$\bar\sigma^{\mu\dot a a} \equiv \varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^{\mu}_{b\dot b} \ . \tag{35.19}$$

Numerically, it turns out that

$$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma) \ . \tag{35.20}$$

Using $\tilde\sigma^{\mu}$, we can write eqs. (35.17) and (35.18) as

$$(S_{\rm L}^{\mu\nu})_a{}^b = +\tfrac{i}{4}(\sigma^{\mu}\bar\sigma^{\nu} - \sigma^{\nu}\bar\sigma^{\mu})_a{}^b \ , \tag{35.21}$$

$$(S_{\rm R}^{\mu\nu})^{\dot a}{}_{\dot b} = -\tfrac{i}{4}(\bar\sigma^{\mu}\sigma^{\nu} - \bar\sigma^{\nu}\sigma^{\mu})^{\dot a}{}_{\dot b} \ . \tag{35.22}$$

In eq. (35.22), we have suppressed a contracted pair of undotted indices arranged as $^{c}{}_{c}$, and in eq. (35.21), we have suppressed a contracted pair of dotted indices arranged as $_{\dot c}{}^{\dot c}$.

We will adopt this as a general convention: a missing pair of contracted, undotted indices is understood to be written as $^{c}{}_{c}$, and a missing pair of contracted, dotted indices is understood to be written as $_{\dot c}{}^{\dot c}$.

Thus, if χ and ψ are two left-handed Weyl fields, we have

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot{a}}\psi^{\dagger\dot{a}} \ . \tag{25.23}$$

We expect Weyl fields to describe spin-one-half particles, and (by the spin-statistics theorem) these particles must be fermions.

Therefore the corresponding fields must *anticommute*, rather than commute.

That is, we should have

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x) \ . \tag{25.24}$$

Thus we can rewrite eq. (35.23) as

$$\chi\psi = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \psi\chi \ . \tag{25.25}$$

The second equality follows from anticommutation of the fields, and the third from switching ${}^{a}_{a}$ to ${}^{a}_{a}$ (which introduces an extra minus sign).

Eq. (35.25) tells us that $\chi\psi = \psi\chi$, which is a nice feature of this notation.

Furthermore, if we take the hermitian conjugate of $\chi\psi$, we get

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = (\psi_a)^\dagger(\chi^a)^\dagger = \psi^\dagger_{\dot{a}}\chi^{\dagger\dot{a}} = \psi^\dagger\chi^\dagger \ . \tag{25.26}$$

That $(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger$ is just what we would expect if we ignored the indices completely.

Of course, by analogy with eq. (35.25), we also have $\psi^\dagger\chi^\dagger = \chi^\dagger\psi^\dagger$.

In order to tell whether a spinor field is left-handed or right-handed when its spinor index is suppressed, we will adopt the convention that a right-handed field is always written as the hermitian conjugate of a left-handed field.

Thus, a right-handed field is always written with a dagger, and a left-handed field is always written without a dagger.

Let's try computing the hermitian conjugate of something a little more complicated:

$$\psi^\dagger \bar{\sigma}^\mu \chi = \psi^\dagger_{\dot{a}} \bar{\sigma}^{\mu \dot{a} c} \chi_c \ . \tag{35.27}$$

This behaves like a vector field under Lorentz transformations,

$$U(\Lambda)^{-1} [\psi^\dagger \bar{\sigma}^\mu \chi] U(\Lambda) = \Lambda^\mu{}_\nu [\psi^\dagger \bar{\sigma}^\nu \chi] \ . \tag{35.28}$$

To avoid clutter, we suppressed the spacetime argument of the fields; as usual, it is x on the left-hand side and Λ^{-1}x on the right.

The hermitian conjugate of eq. (35.27) is

$$\begin{aligned} [\psi^\dagger \bar{\sigma}^\mu \chi]^\dagger &= [\psi^\dagger_{\dot{a}} \bar{\sigma}^{\mu \dot{a} c} \chi_c]^\dagger \\ &= \chi^\dagger_{\dot{c}} (\bar{\sigma}^{\mu a \dot{c}})^* \psi_a \\ &= \chi^\dagger_{\dot{c}} \bar{\sigma}^{\mu \dot{c} a} \psi_a \\ &= \chi^\dagger \bar{\sigma}^\mu \psi \ . \end{aligned} \tag{35.29}$$

In the third line, we used the hermiticity of the matrices $\bar{\sigma}^\mu = (I, -\bar{\sigma})$.

We will get considerably more practice with this notation in the following sections.

36 Lagrangians for Spinor Fields

Suppose we have a left-handed spinor field ψ_a.

We would like to find a suitable lagrangian for it.

This lagrangian must be Lorentz invariant, and it must be hermitian.

We would also like it to be quadratic in ψ_a and its hermitian conjugate $\psi_a{}^\dagger$, because this will lead to a linear equation of motion, with plane-wave solutions.

We want plane-wave solutions because these describe free particles, the starting point for a theory of interacting particles.

Let us begin with terms with no derivatives.

The only possibility is $\psi\psi = \psi^a\psi_a = \varepsilon^{ab}\psi_b\psi_a$, plus its hermitian conjugate.

Because of anticommutation of the fields ($\psi_b\psi_a = -\psi_a\psi_b$), this expression does not vanish (as it would if the fields commuted), and so we can use it as a term in $\boldsymbol{\mathcal{L}}$.

Next we need a term with derivatives.

The obvious choice is $\partial^\mu\psi\partial_\mu\psi$, plus its hermitian conjugate.

This, however, yields a hamiltonian that is unbounded below, which is unacceptable.

To get a bounded hamiltonian, the kinetic term must involve both ψ and $\psi^\dagger$.

A candidate is $i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi$.

This not hermitian, but

$$
\begin{aligned}
(i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi)^\dagger &= (i\psi_{\dot{a}}^\dagger \, \bar{\sigma}^{\mu \dot{a} c} \partial_\mu \psi_c)^\dagger \\
&= -i\partial_\mu \psi_{\dot{c}}^\dagger \, (\bar{\sigma}^{\mu a \dot{c}})^* \psi_a \\
&= -i\partial_\mu \psi_{\dot{c}}^\dagger \, \bar{\sigma}^{\mu \dot{c} a} \psi_a \\
&= i\psi_{\dot{c}}^\dagger \, \bar{\sigma}^{\mu \dot{c} a} \partial_\mu \psi_a - i\partial_\mu (\psi_{\dot{c}}^\dagger \, \bar{\sigma}^{\mu \dot{c} a} \psi_a). \\
&= i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - i\partial_\mu (\psi^\dagger \bar{\sigma}^\mu \psi) \ .
\end{aligned}
\tag{36.1}
$$

In the third line, we used the hermiticity of the matrices $\bar{\sigma}_\mu = (\mathrm{I}, -\vec{\sigma})$.

In the fourth line, we used $-(\partial \mathrm{A})\mathrm{B} = \mathrm{A}\partial\mathrm{B} - \partial(\mathrm{AB})$.

In the last line, the second term is a total divergence, and vanishes (with suitable boundary conditions on the fields at infinity) when we integrate it over $\mathrm{d}^4\mathrm{x}$ to get the action S.

Thus $i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi$ has the hermiticity properties necessary for a term in $\mathcal{L}$.

Our complete lagrangian for ψ is then

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m \psi\psi - \tfrac{1}{2} m^* \psi^\dagger \psi^\dagger \ , \tag{36.2}$$

where m is a complex parameter with dimensions of mass.

The phase of m is actually irrelevant: if m = $|m|e^{i\alpha}$, we can set ψ = $e^{-i\alpha/2}\,\tilde{\psi}$ in eq. (36.2); then we get a lagrangian for $\tilde{\psi}$ that is identical to eq. (36.2), but with m replaced by |m|.

So we can, without loss of generality, take m to be real and positive in the first place, and that is what we will do, setting m^* = m in eq. (36.2)

The equation of motion for ψ is then

$$0 = -\frac{\delta S}{\delta \psi^\dagger} = -i\bar{\sigma}^\mu \partial_\mu \psi + m\psi^\dagger \ , \tag{36.3}$$

Restoring the spinor indices, this reads

$$0 = -i\bar{\sigma}^{\mu \dot{a} c} \partial_\mu \psi_c + m \psi^{\dagger \dot{a}} \ . \tag{36.4}$$

Taking the hermitian conjugate (or, equivalently, computing −δS/δψ), we get

$$\begin{aligned} 0 &= +i(\bar{\sigma}^{\mu a \dot{c}})^* \, \partial_\mu \psi^\dagger_{\dot{c}} + m \psi^a \\ &= +i\bar{\sigma}^{\mu \dot{c} a} \partial_\mu \psi^\dagger_{\dot{c}} + m \psi^a \\ &= -i\sigma^\mu_{a\dot{c}} \, \partial_\mu \psi^{\dagger \dot{c}} + m \psi_a \ . \end{aligned} \tag{36.5}$$

In the second line, we used the hermiticity of the matrices $\bar{\sigma}_\mu = (\mathrm{I}, -\vec{\sigma})$.

In the third, we lowered the undotted index, and switched ${}^{\dot{c}}{}_{\dot{c}}$ to ${}_{\dot{c}}{}^{\dot{c}}$, which gives an extra minus sign.

Eqs. (36.5) and (36.4) can be combined to read

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\,\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} = 0 \,. \tag{36.6}$$

We can write this more compactly by introducing the 4×4 *gamma matrices*

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix} . \tag{36.7}$$

Using the sigma-matrix relations,

$$\begin{aligned} (\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_a{}^c &= -2g^{\mu\nu}\delta_a{}^c \,, \\ (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{c}} &= -2g^{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}} \,, \end{aligned} \tag{36.8}$$

which are most easily derived from the numerical formulae $\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma})$ and $\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$, we see that the gamma matrices obey

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \,, \tag{36.9}$$

where {A,B} ≡ AB+ BA denotes the anticommutator, and there is an understood 4 × 4 identity matrix on the right-hand side.

We also introduce a four-component *Majorana field*

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger \dot{c}} \end{pmatrix} \tag{36.10}$$

Then eq. (36.6) becomes

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0 \ . \tag{36.11}$$

This is the *Dirac equation*.

We first encountered it in section 1, where the gamma matrices were given different names ($\beta = \gamma_0$ and $\alpha^k = \gamma^0\gamma^k$).

Also, in section 1 we were trying (and failing) to interpret Ψ as a wave function, rather than as a quantum field.

Now consider a theory of two left-handed spinor fields with an SO(2) symmetry,

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2} m \psi_i \psi_i - \tfrac{1}{2} m \psi_i^\dagger \psi_i^\dagger \ , \tag{36.12}$$

where the spinor indices are suppressed and i = 1, 2 is implicitly summed.

As in the analogous case of two scalar fields discussed in sections 22 and 23, this lagrangian is invariant under the SO(2) transformation

We can write the lagrangian so that the SO(2) symmetry appears as a U(1) symmetry instead; let

$$\chi = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \,, \tag{36.14}$$

$$\xi = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \,. \tag{36.15}$$

In terms of these fields, we have

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger \chi^\dagger \,. \tag{36.16}$$

Eq. (36.16) is invariant under the U(1) version of eq. (36.13),

$$\begin{aligned} \chi &\to e^{-i\alpha}\chi \,, \\ \xi &\to e^{+i\alpha}\xi \,. \end{aligned} \tag{36.17}$$

Next, let us derive the equations of motion that we get from eq. (36.16), following the same procedure that ultimately led to eq. (36.6).

The result is

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\,\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0 \,. \tag{36.18}$$

We can now define a four-component *Dirac field*

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} \,, \tag{36.19}$$

which obeys the Dirac equation, eq. (36.11).

We have annoyingly used the same symbol Ψ to denote both a Majorana field and a Dirac field; these are different objects, and so we must always announce which is meant when we write Ψ.

We can also write the lagrangian, eq. (36.16), in terms of the Dirac field Ψ, eq. (36.19).

First we take the hermitian conjugate of Ψ to get

$$\Psi^\dagger = (\chi_{\dot{a}}^\dagger\,,\ \xi^a)\ . \tag{36.20}$$

Introduce the matrix

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}\ . \tag{36.21}$$

Numerically, $\beta = \gamma^0$.

However, the spinor index structure of β and γ^0 is different, and so we will distinguish them.

Given β, we define

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi_{\dot{a}}^\dagger)\ . \tag{36.22}$$

Then we have

$$\overline{\Psi}\Psi = \xi^a\chi_a + \chi_{\dot{a}}^\dagger\xi^{\dagger\dot{a}}\ . \tag{36.23}$$

Also,

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi = \xi^a\sigma^\mu_{a\dot{c}}\,\partial_\mu\xi^{\dagger\dot{c}} + \chi^\dagger_{\dot{a}}\,\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu\chi_c\ . \tag{36.24}$$

Using A∂B = −(∂A)B + ∂(AB), the first term on the right-hand side of eq. (36.24) can be rewritten as

$$\xi^a\sigma^\mu_{a\dot{c}}\,\partial_\mu\xi^{\dagger\dot{c}} = -(\partial_\mu\xi^a)\sigma^\mu_{a\dot{c}}\,\xi^{\dagger\dot{c}} + \partial_\mu(\xi^a\sigma^\mu_{a\dot{c}}\,\xi^{\dagger\dot{c}})\ . \tag{36.25}$$

Then the first term on the right-hand side of eq. (36.25) can be rewritten as

$$-(\partial_\mu\xi^a)\sigma^\mu_{a\dot{c}}\,\xi^{\dagger\dot{c}} = +\xi^{\dagger\dot{c}}\sigma^\mu_{a\dot{c}}\,\partial_\mu\xi^a = +\xi^\dagger_{\dot{c}}\,\bar{\sigma}^{\mu\dot{c}a}\partial_\mu\xi_a\ . \tag{36.26}$$

Here we used anticommutation of the fields to get the first equality, and switched ${}^{\dot{c}}{}_{\dot{c}}$ to ${}_{\dot{c}}{}^{\dot{c}}$ and ${}_a{}^a$ to ${}^a{}_a$ (thus generating two minus signs) to get the second.

Combining eqs. (36.24–36.26), we get

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + \partial_\mu(\xi\sigma^\mu\xi^\dagger)\ . \tag{36.27}$$

Therefore, up to an irrelevant total divergence, we have

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi\ . \tag{36.28}$$

This form of the lagrangian is invariant under the U(1) transformation

$$\begin{aligned}\Psi &\to e^{-i\alpha}\,\Psi\ ,\\ \overline{\Psi} &\to e^{+i\alpha}\,\overline{\Psi}\ ,\end{aligned} \tag{36.29}$$

which, given eq. (36.19), is the same as eq. (36.17).

The Noether current associated with this symmetry is

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi \ . \tag{36.30}$$

In quantum electrodynamics, the electromagnetic current is $e\overline{\Psi}\gamma^\mu\Psi,$ where e is the charge of the electron.

As in the case of a complex scalar field with a U(1) symmetry, there is an additional discrete symmetry, called *charge conjugation,* that enlarges SO(2) to O(2).

Charge conjugation simply exchanges χ and ξ.

We can define a unitary *charge conjugation operator C* that implements this,

$$\begin{aligned} C^{-1}\chi_a(x)C &= \xi_a(x) \ , \\ C^{-1}\xi_a(x)C &= \chi_a(x) \ , \end{aligned} \tag{36.31}$$

where, for the sake of precision, we have restored the spinor index and spacetime argument.

We then have $C^{-1}\boldsymbol{\mathcal{L}}(\mathrm{x})C = \boldsymbol{\mathcal{L}}(\mathrm{x})$.

To express eq. (36.31) in terms of the Dirac field, eq. (36.19), we first introduce the charge conjugation matrix

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix} . \tag{36.32}$$

Next we notice that, if we take the transpose of $\overline{\Psi}$, eq. (36.22), we get

$$\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi^a \\ \chi^\dagger_{\dot{a}} \end{pmatrix} . \tag{36.33}$$

Then, if we multiply by C, we get a field that we will call Ψ^{C}, the *charge conjugate* of Ψ,

$$\Psi^{\mathrm{C}} \equiv \mathcal{C}\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix} . \tag{36.34}$$

We see that Ψ^{C} is the same as the original field Ψ, eq. (36.19), except that the roles of χ and ξ have been switched.

We therefore have

$$C^{-1}\Psi(x)C = \Psi^{\mathrm{C}}(x) \tag{36.35}$$

for a Dirac field.

The charge conjugation matrix has a number of useful properties.

As a numerical matrix, it obeys

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} , \tag{36.36}$$

and we can also write it as

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix}. \tag{36.37}$$

A result that we will need later is

$$\begin{aligned} \mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} &= \begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} 0 & \sigma^{\mu}_{b\dot{c}} \\ \bar{\sigma}^{\mu\dot{b}c} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{ce} & 0 \\ 0 & \varepsilon^{\dot{c}\dot{e}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varepsilon^{ab}\sigma^{\mu}_{b\dot{c}}\varepsilon^{\dot{c}\dot{e}} \\ \varepsilon_{\dot{a}\dot{b}}\bar{\sigma}^{\mu\dot{b}c}\varepsilon_{ce} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\bar{\sigma}^{\mu a e} \\ -\sigma^{\mu}_{\dot{a}e} & 0 \end{pmatrix}. \end{aligned} \tag{36.38}$$

The minus signs in the last line come from raising or lowering an index by contracting with the first (rather than the second) index of an ε symbol.

Comparing with

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu}_{e\dot{a}} \\ \bar{\sigma}^{\mu\dot{e}a} & 0 \end{pmatrix}, \tag{36.39}$$

we see that

$$\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{\mathrm{T}}. \tag{36.40}$$

Now let us return to the Majorana field, eq. (36.10).

It is obvious that a Majorana field is its own charge conjugate, that is, $\Psi^C = \Psi$.

This condition is analogous to the condition $\phi^\dagger = \phi$ that is satisfied by a real scalar field.

A Dirac field, with its U(1) symmetry, is analogous to a complex scalar field, while a Majorana field, which has no U(1) symmetry, is analogous to a real scalar field.

We can write our original lagrangian for a single left-handed spinor field, eq. (36.2), in terms of a Majorana field, eq. (36.10), by retracing eqs. (36.20–36.28) with $\chi \to \psi$ and $\xi \to \psi$.

The result is

$$\mathcal{L} = \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi \ . \tag{36.41}$$

However, we cannot yet derive the equation of motion from eq. (36.41) because it does not yet incorporate the Majorana condition $\Psi^C = \Psi$.

To remedy this, we use eq. (36.36) to write the Majorana condition $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$ as $\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$.

Then we can replace Ψ in eq. (36.41) by ΨTC to get

$$\mathcal{L} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \ . \tag{36.42}$$

The equation of motion that follows from this lagrangian is once again the Dirac equation.

We can also recover the Weyl components of a Dirac or Majorana field by means of a suitable projection matrix.

Define

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}, \tag{36.43}$$

where the subscript 5 is simply part of the traditional name of this matrix, rather than the value of some index.

Then we can define left and right projection matrices

$$\begin{aligned} P_{\mathrm{L}} &\equiv \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}, \\ P_{\mathrm{R}} &\equiv \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}. \end{aligned} \tag{36.44}$$

Thus we have, for a Dirac field,

$$\begin{aligned} P_{\mathrm{L}}\Psi &= \begin{pmatrix} \chi_c \\ 0 \end{pmatrix}, \\ P_{\mathrm{R}}\Psi &= \begin{pmatrix} 0 \\ \xi^{\dagger\dot{c}} \end{pmatrix}. \end{aligned} \tag{36.45}$$

The matrix γ_5 can also be expressed as

$$\begin{aligned}\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \ ,\end{aligned} \tag{36.46}$$

where $\varepsilon_{0123}=-1$.

Finally, let us consider the behavior of a Dirac or Majorana field under a Lorentz transformation.

Recall that left- and right-handed spinor fields transform according to

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L(\Lambda)_a{}^c\,\psi_c(\Lambda^{-1}x) \ , \tag{36.47}$$

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R(\Lambda)_{\dot a}{}^{\dot c}\,\psi^\dagger_{\dot c}(\Lambda^{-1}x) \ , \tag{36.48}$$

where, for an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$,

$$L(1+\delta\omega)_a{}^c = \delta_a{}^c + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^c \ , \tag{36.49}$$

$$R(1+\delta\omega)_{\dot a}{}^{\dot c} = \delta_{\dot a}{}^{\dot c} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot c} \ , \tag{36.50}$$

and where

$$(S^{\mu\nu}_{\rm L})_a{}^c = +\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^c \ , \tag{36.51}$$

$$(S^{\mu\nu}_{\rm R})^{\dot a}{}_{\dot c} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot c} \ . \tag{36.52}$$

From these formulae, and the definition of γ^μ, eq. (36.7), we can see that

$$\tfrac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} +(S_{\rm L}^{\mu\nu})_a{}^c & 0 \\ 0 & -(S_{\rm R}^{\mu\nu})^{\dot a}{}_{\dot c} \end{pmatrix} \equiv S^{\mu\nu} \ . \tag{36.53}$$

Then, for either a Dirac or Majorana field Ψ, we can write

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x) \ , \tag{36.54}$$

where, for an infinitesimal transformation, the 4×4 matrix $D(\Lambda)$ is

$$D(1+\delta\omega) = 1 + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu} \ , \tag{36.55}$$

with $S^{\mu\nu}$ given by eq. (36.53).

The minus sign in front of $S_R{}^{\mu\nu}$ in eq. (36.53)is compensated by the switch from a $^{\dot c}{}_{\dot c}$ contraction in eq. (36.50) to a $_{\dot c}{}^{\dot c}$ contraction in eq. (36.54).

37 Canonical Quantization of Spinor Fields I

Consider a left-handed Weyl field ψ with lagrangian

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \ . \tag{37.1}$$

The canonically conjugate momentum to the field $\psi_a(x)$ is the

$$\begin{aligned}\pi^a(x) &\equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a(x))} \\ &= i\psi^\dagger_{\dot{a}}(x)\bar{\sigma}^{0\dot{a}a} \ .\end{aligned} \tag{37.2}$$

The hamiltonian is

$$\begin{aligned}\mathcal{H} &= \pi^a \partial_0 \psi_a - \mathcal{L} \\ &= i\psi^\dagger_{\dot{a}}\bar{\sigma}^{0\dot{a}a}\psi_a - \mathcal{L} \\ &= -i\psi^\dagger\bar{\sigma}^i\partial_i\psi + \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \ .\end{aligned} \tag{37.3}$$

The appropriate canonical *anticommutation* relations are

$$\{\psi_a(\mathbf{x},t), \psi_c(\mathbf{y},t)\} = 0 \ , \tag{37.4}$$

$$\{\psi_a(\mathbf{x},t), \pi^c(\mathbf{y},t)\} = i\delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{37.5}$$

Substituting in eq. (37.2) for π^c, we get

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\}\bar{\sigma}^{0\dot{c}c} = \delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{37.6}$$

Then, using $\bar{\sigma}_0 = \sigma^0 = \mathrm{I}$, we have

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{37.7}$$

or, equivalently,

$$\{\psi^a(\mathbf{x},t),\psi^{\dagger\dot{c}}(\mathbf{y},t)\}=\bar{\sigma}^{0\dot{c}a}\,\delta^3(\mathbf{x}-\mathbf{y})\ . \tag{37.8}$$

We can also translate this into four-component notation for either a Dirac or a Majorana field.

A Dirac field is defined in terms of two left-handed Weyl fields χ and ξ via

$$\Psi\equiv\begin{pmatrix}\chi_c\\ \xi^{\dagger\dot{c}}\end{pmatrix}\ . \tag{37.9}$$

We also define

$$\overline{\Psi}\equiv\Psi^\dagger\beta=(\xi^a,\,\chi^\dagger_{\dot{a}})\ , \tag{37.10}$$

where

$$\beta\equiv\begin{pmatrix}0 & \delta^{\dot{a}}{}_{\dot{c}}\\ \delta_a{}^c & 0\end{pmatrix}\ . \tag{37.11}$$

The lagrangian is

$$\begin{aligned}\mathcal{L}&=i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi+i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi-m(\chi\xi+\xi^\dagger\chi^\dagger)\\ &=i\overline{\Psi}\gamma^\mu\partial_\mu\Psi-m\overline{\Psi}\Psi\ .\end{aligned} \tag{37.12}$$

The fields χ and ξ each obey the canonical anticommutation relations of eq. (37.5).

This translates into

$$\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\}=0\ , \tag{37.13}$$

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\}=(\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{37.14}$$

where α and β are four-component spinor indices, and

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix} . \tag{37.15}$$

Eqs. (37.13) and (37.14) can also be derived directly from the four-component form of the lagrangian, eq. (37.12), by noting that the canonically conjugate momentum to the field Ψ is $\partial\mathcal{L}/\partial(\partial_0\Psi) = i\Psi\gamma^0$, and that $(\gamma^0)^2 = 1$.

A Majorana field is defined in terms of a single left-handed Weyl field ψ via

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} . \tag{37.16}$$

We also define

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\psi^a, \psi^\dagger_{\dot{a}}) . \tag{37.17}$$

A Majorana field obeys the Majorana condition

$$\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C} , \tag{37.18}$$

where

$$\mathcal{C} \equiv \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix} \tag{37.19}$$

is the charge conjugation matrix.

The lagrangian is

$$\begin{aligned}\mathcal{L} &= i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \tfrac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger) \\ &= \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi \\ &= \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi\ .\end{aligned} \tag{37.20}$$

The field ψ obeys the canonical anticommutation relations of eq. (37.5).

This translates into

$$\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{37.21}$$

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{37.22}$$

where α and β are four-component spinor indices.

To derive eqs. (37.21) and (37.22) directly from the four-component form of the lagrangian, eq. (37.20), requires new formalism for the quantization of constrained systems.

This is because the canonically conjugate momentum to the field Ψ is $\partial\mathcal{L}/\partial(\partial_0\Psi) = 1/2\Psi^{\mathrm{TC}}\gamma^0$, and this is linearly related to Ψ itself; this relation constitutes a constraint that must be solved before imposition of the anticommutation relations.

In this case, solving the constraint simply returns us to the Weyl formalism with which we began.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ the the the the $\mathbf{V}_4(k_1,k_2,k_3,k_4)$

The equation of motion that follows from either eq. (37.12) or eq. (37.20) is the Dirac equation,

$$(-i\not{\partial} + m)\Psi = 0 \ . \tag{37.23}$$

Here we have introduced the *Feynman slash*: given any four-vector a^μ, we define

$$\not{a} \equiv a_\mu \gamma^\mu \ . \tag{37.24}$$

To solve the Dirac equation, we first note that if we act on it with $-i\not{\partial} + m$, we get

$$\begin{aligned} 0 &= (i\not{\partial} + m)(-i\not{\partial} + m)\Psi \\ &= (\not{\partial}\not{\partial} + m^2)\Psi \\ &= (-\partial^2 + m^2)\Psi \ . \end{aligned} \tag{37.25}$$

Here we have used

$$\begin{aligned} \not{a}\not{a} &= a_\mu a_\nu \gamma^\mu \gamma^\nu \\ &= a_\mu a_\nu \left(\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \right) \\ &= a_\mu a_\nu \left(-g^{\mu\nu} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \right) \\ &= -a_\mu a_\nu g^{\mu\nu} + 0 \\ &= -a^2 \ . \end{aligned} \tag{37.26}$$

From eq. (37.25), we see that Ψ obeys the Klein-Gordon equation.

Therefore, the Dirac equation has plane-wave solutions.

Let us consider a specific solution of the form

$$\Psi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx} \ . \tag{37.27}$$

where $p^0 = \omega \equiv (\mathbf{p}^2 + m^2)^{1/2}$, and $u(\mathbf{p})$ and $v(\mathbf{p})$ are four-component constant spinors.

Plugging eq. (37.27) into the eq. (37.23), we get

$$(\not{p} + m)u(\mathbf{p})e^{ipx} + (-\not{p} + m)v(\mathbf{p})e^{-ipx} = 0 \ . \tag{37.28}$$

Thus we require

$$\begin{aligned} (\not{p} + m)u(\mathbf{p}) &= 0 \ , \\ (-\not{p} + m)v(\mathbf{p}) &= 0 \ . \end{aligned} \tag{37.29}$$

Each of these equations has two linearly independent solutions that we will call $u_{\pm}(\mathbf{p})$ and $v_{\pm}(\mathbf{p})$; their detailed properties will be worked out in the next section.

The general solution of the Dirac equation can then be written as

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^{\dagger}(\mathbf{p})v_s(\mathbf{p})e^{-ipx} \right] \ , \tag{37.30}$$

where the integration measure is as usual

$$\widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3 2\omega} \ . \tag{37.31}$$

The four-component spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ obey the equations

$$\begin{aligned}(\not{p} + m)u_s(\mathbf{p}) &= 0 \,, \\ (-\not{p} + m)v_s(\mathbf{p}) &= 0 \,.\end{aligned} \tag{38.1}$$

Each of these equations has two solutions, which we label via s = + and s = −.

For $m \neq 0$, we can go to the rest frame, $\mathbf{p} = \mathbf{0}$.

We will then distinguish the two solutions by the eigenvalue of the spin matrix

$$S_z = \tfrac{i}{4}[\gamma^1, \gamma^2] = \tfrac{i}{2}\gamma^1\gamma^2 = \begin{pmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & \frac{1}{2}\sigma_3 \end{pmatrix}. \tag{38.2}$$

Specifically, we will require

$$\begin{aligned}S_z u_\pm(\mathbf{0}) &= \pm\tfrac{1}{2}u_\pm(\mathbf{0}) \,, \\ S_z v_\pm(\mathbf{0}) &= \mp\tfrac{1}{2}v_\pm(\mathbf{0}) \,.\end{aligned} \tag{38.3}$$

The reason for the opposite sign for the v spinor is that this choice results in

$$\begin{aligned}[J_z, b^\dagger_\pm(\mathbf{0})] &= \pm\tfrac{1}{2}b^\dagger_\pm(\mathbf{0}) \,, \\ [J_z, d^\dagger_\pm(\mathbf{0})] &= \pm\tfrac{1}{2}d^\dagger_\pm(\mathbf{0}) \,,\end{aligned} \tag{38.4}$$

where J_z is the z component of the angular momentum operator.

Eq. (38.4) implies that $b^\dagger_+(\mathbf{0})$ and $d^\dagger_+(\mathbf{0})$ each creates a particle with spin up along the z axis.

We will verify eq. (38.4) in problem 39.2.

For $\mathbf{p} = \mathbf{0}$, we have $\not{p} = -m\gamma^0$, where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{38.5}$$

Eqs. (38.1) and (38.3) are then easy to solve.

Choosing (for later convenience) a specific normalization and phase for each of $u_\pm(\mathbf{0})$ and $v_\pm(\mathbf{0})$, we get

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$
$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{38.6}$$

For later use we also compute the barred spinors

$$\begin{aligned} \overline{u}_s(\mathbf{p}) &\equiv u_s^\dagger(\mathbf{p})\beta\,, \\ \overline{v}_s(\mathbf{p}) &\equiv v_s^\dagger(\mathbf{p})\beta\,, \end{aligned} \tag{38.7}$$

where

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{38.8}$$

satisfies

$$\beta^{\mathrm{T}} = \beta^{\dagger} = \beta^{-1} = \beta \, . \tag{38.9}$$

We get

$$\begin{aligned} \overline{u}_{+}(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,1,\,0)\,, \\ \overline{u}_{-}(\mathbf{0}) &= \sqrt{m}\,(0,\,1,\,0,\,1)\,, \\ \overline{v}_{+}(\mathbf{0}) &= \sqrt{m}\,(0,\,-1,\,0,\,1)\,, \\ \overline{v}_{-}(\mathbf{0}) &= \sqrt{m}\,(1,\,0,\,-1,\,0)\,. \end{aligned} \tag{38.10}$$

We can now find the spinors corresponding to an arbitrary three-momentum **p** by applying to $u_s(\mathbf{0})$ and $v_s(\mathbf{0})$ the matrix $D(\Lambda)$ that corresponds to an appropriate boost.

This is given by

$$D(\Lambda) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})\,, \tag{38.11}$$

where $\hat{\mathbf{p}}$ is a unit vector in the **p** direction, $K^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{2}\gamma^j\gamma^0$ is the boost matrix, and $\eta \equiv \sinh^{-1}(|\mathbf{p}|/m)$ is the rapidity.

Thus we have

$$\begin{aligned} u_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})\,, \\ v_s(\mathbf{p}) &= \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})\,. \end{aligned} \tag{38.12}$$

We also have

$$\begin{aligned}\overline{u}_s(\mathbf{p}) &= \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})\ ,\\ \overline{v}_s(\mathbf{p}) &= \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})\ .\end{aligned} \tag{38.13}$$

This follows from $\overline{K^j} = K^j$, where for any general combination of gamma matrices,

$$\overline{A} \equiv \beta A^\dagger \beta\ . \tag{38.14}$$

In particular, it turns out that

$$\begin{aligned}\overline{\gamma^\mu} &= \gamma^\mu\ ,\\ \overline{S^{\mu\nu}} &= S^{\mu\nu}\ ,\\ \overline{i\gamma_5} &= i\gamma_5\ ,\\ \overline{\gamma^\mu\gamma_5} &= \gamma^\mu\gamma_5\ ,\\ \overline{i\gamma_5 S^{\mu\nu}} &= i\gamma_5 S^{\mu\nu}\ .\end{aligned} \tag{38.15}$$

The barred spinors satisfy the equations

$$\begin{aligned}\overline{u}_s(\mathbf{p})(\not{p}+m) &= 0\ ,\\ \overline{v}_s(\mathbf{p})(-\not{p}+m) &= 0\ .\end{aligned} \tag{38.16}$$

It is not very hard to work out $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ from eq. (38.12), but it is even easier to use various tricks that will sidestep any need for the explicit formulae.

Consider, for example, $\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p})$; from eqs. (38.12) and (38.13), we see that $\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = \overline{u}_{s'}(\mathbf{0})u_s(\mathbf{0})$, and this is easy to compute from eqs. (38.6) and (38.10).

We find

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= +2m\,\delta_{s's}\ , \\
\overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= -2m\,\delta_{s's}\ , \\
\overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= 0\ , \\
\overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= 0\ .
\end{aligned} \tag{38.17}$$

Also useful are the *Gordon identities*,

$$\begin{aligned}
2m\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) &= \overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]u_s(\mathbf{p})\ , \\
-2m\,\overline{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) &= \overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]v_s(\mathbf{p})\ .
\end{aligned} \tag{38.18}$$

To derive them, start with

$$\gamma^\mu \not{p} = \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}\,] = -p^\mu - 2iS^{\mu\nu}p_\nu\ , \tag{38.19}$$

$$\not{p}'\gamma^\mu = \tfrac{1}{2}\{\gamma^\mu, \not{p}'\} - \tfrac{1}{2}[\gamma^\mu, \not{p}'\,] = -p'^\mu + 2iS^{\mu\nu}p'_\nu\ . \tag{38.20}$$

Add eqs. (38.19) and (38.20), sandwich them between u′ and u spinors (or v′ and v spinors), and use eqs. (38.1) and (38.16).

An important special case is p′ = p; then, using eq. (38.17), we find

$$\begin{aligned}
\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) &= 2p^\mu\delta_{s's}\ , \\
\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) &= 2p^\mu\delta_{s's}\ .
\end{aligned} \tag{38.21}$$

With a little more effort, we can also show

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0\,,$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0\,. \tag{38.22}$$

We will need eqs. (38.21) and (38.22) in the next section.

Consider now the spin sums $\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p})$ and $\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p})$, each of which is a 4×4 matrix.

The sum over eigenstates of S_z should remove any memory of the spin-quantization axis, and so the result should be expressible in terms of the four-vector p^μ and various gamma matrices, with all vector indices contracted.

In the rest frame, $\not{p} = -m\gamma^0$, and it is easy to check that $\sum_{s=\pm} u_s(\mathbf{0})\overline{u}_s(\mathbf{0}) = m\gamma^0 + m$ and $\sum_{s=\pm} v_s(\mathbf{0})\overline{v}_s(\mathbf{0}) = m\gamma^0 - m$.

We therefore conclude that

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m\,,$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m\,. \tag{38.23}$$

We will make extensive use of eq. (38.23) when we calculate scattering cross sections for spin-one-half particles.

From eq. (38.23), we can get $u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$, etc, by applying appropriate spin projection matrices.

In the rest frame, we have

$$\begin{aligned}
\tfrac{1}{2}(1+2sS_z)u_{s'}(\mathbf{0}) &= \delta_{ss'}\,u_{s'}(\mathbf{0})\ , \\
\tfrac{1}{2}(1-2sS_z)v_{s'}(\mathbf{0}) &= \delta_{ss'}\,v_{s'}(\mathbf{0})\ .
\end{aligned} \tag{38.24}$$

In order to boost these projection matrices to a more general frame, we first recall that

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{38.25}$$

This allows us to write $S_z = i/2\ \gamma^1\gamma^2$ as $S_z = -1/2\ \gamma_5\gamma^3\gamma^0$.

In the rest frame, we can write γ^0 as $-\not{p}/m$, and γ^3 as $\not{z}$, where $z^\mu = (0, \hat{\mathbf{z}})$; thus we have

$$S_z = \tfrac{1}{2m}\gamma_5\not{z}\not{p}\ . \tag{38.26}$$

Now we can boost S_z to any other frame simply by replacing $\not{z}$ and $\not{p}$ with their values in that frame.

Note that, in any frame, z^μ satisfies $z^2 = 1$ and $\mathbf{z}\cdot\mathbf{p} = 0$.

Boosting eq. (38.24) then yields

$$\begin{aligned}
\tfrac{1}{2}(1-s\gamma_5\not{z})u_{s'}(\mathbf{p}) &= \delta_{ss'}\,u_{s'}(\mathbf{p})\ , \\
\tfrac{1}{2}(1-s\gamma_5\not{z})v_{s'}(\mathbf{p}) &= \delta_{ss'}\,v_{s'}(\mathbf{p})\ ,
\end{aligned} \tag{38.27}$$

where we have used eq. (38.1) to eliminate $\not{p}$.

Combining eqs. (38.23) and (38.27) we get

$$
\begin{aligned}
u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m) \ , \\
v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} - m) \ .
\end{aligned}
\tag{38.28}
$$

It is interesting to consider the extreme relativistic limit of this formula.

Let us take the three-momentum to be in the z direction, so that it is parallel to the spin-quantization axis.

The component of the spin in the direction of the three-momentum is called the *helicity*.

A fermion with helicity +1/2 is said to be *right-handed*, and a fermion with helicity −1/2 is said to be *left-handed*.

For rapidity η, we have

$$
\begin{aligned}
\tfrac{1}{m}p^\mu &= (\cosh\eta, 0, 0, \sinh\eta) \ , \\
z^\mu &= (\sinh\eta, 0, 0, \cosh\eta) \ .
\end{aligned}
\tag{38.29}
$$

The first equation is simply the definition of η, and the second follows from $z^2 = 1$ and $p \cdot z = 0$ (along with the knowledge that a boost of a four-vector in the z direction does not change its x and y components).

In the limit of large η, we see that

$$
z^\mu = \tfrac{1}{m}p^\mu + O(e^{-\eta}) \ .
\tag{38.30}
$$

Hence, in eq. (38.28), we can replace $\not{z}$ with $\not{p}/m$, and then use the matrix relation $(\not{p}/m)(-\not{p} \pm m) = \mp(-\not{p} \pm m)$, which holds for $p^2 = -m^2$.

For consistency, we should then also drop the m relative to $\not{p}$, since it is down by a factor of $O(e^{-\eta})$.

We get

$$
\begin{aligned}
u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &\to \tfrac{1}{2}(1 + s\gamma_5)(-\not{p}) \ , \\
v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &\to \tfrac{1}{2}(1 - s\gamma_5)(-\not{p}) \ .
\end{aligned}
\tag{38.31}
$$

The spinor corresponding to a right-handed fermion (helicity +1/2) is $u_+(\mathbf{p})$ for a b-type particle and $v_-(\mathbf{p})$ for a d-type particle.

According to eq. (38.31), either of these is projected by 1/2 $(1 + \gamma_5)$ = diag(0,0,1,1) onto the lower two components only.

In terms of the Dirac field $\Psi(x)$, this is the part that corresponds to the right-handed Weyl field.

Similarly, left-handed fermions are projected (in the extreme relativistic limit) onto the upper two spinor components only, corresponding to the left-handed Weyl field.

The case of a massless particle follows from the extreme relativistic limit of a massive particle.

In particular, eqs. (38.1), (38.16), (38.17), (38.21), (38.22), and (38.23) are all valid with m = 0, and eq. (38.31) becomes exact.

Finally, for our discussion of parity, time reversal, and charge conjugation in section 40, we will need a number of relationships among the u and v spinors.

First, note that $\beta u_s(\mathbf{0}) = +u_s(\mathbf{0})$ and $\beta v_s(\mathbf{0}) = -v_s(\mathbf{0})$.

Also, $\beta K^j = -K^j\beta$.

We then have

$$\begin{aligned} u_s(-\mathbf{p}) &= +\beta u_s(\mathbf{p}) \,, \\ v_s(-\mathbf{p}) &= -\beta v_s(\mathbf{p}) \,. \end{aligned} \tag{38.32}$$

Next, we need the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{38.33}$$

which obeys

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} \,, \tag{38.34}$$

$$\beta\mathcal{C} = -\mathcal{C}\beta \,, \tag{38.35}$$

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}} \,. \tag{38.36}$$

Using eqs. (38.6), (38.10), and (38.33), we can show that $Cu_s(\mathbf{0})^T = v\mathbf{s(0)}$ and $Cv\mathbf{s(0)T} = u\mathbf{s(0)}$.

Also, eq. (38.36) implies $C^{-1}K^jC = -(K^j)^T$.

From this we can conclude that

$$\begin{aligned} \mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} &= v_s(\mathbf{p}) \ , \\ \mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} &= u_s(\mathbf{p}) \ . \end{aligned} \tag{38.37}$$

Taking the complex conjugate of eq. (38.37), and using $\overline{u}^{\mathrm{T}*} = \overline{u}^{\dagger} = \beta u$, we get

$$\begin{aligned} u_s^*(\mathbf{p}) &= \mathcal{C}\beta v_s(\mathbf{p}) \ , \\ v_s^*(\mathbf{p}) &= \mathcal{C}\beta u_s(\mathbf{p}) \ . \end{aligned} \tag{38.38}$$

Next, note that $\gamma_5 u_s(\mathbf{0}) = +s v_{-s}(\mathbf{0})$ and $\gamma_5 v_s(\mathbf{0}) = -s u_{-s}(\mathbf{0})$, and that $\gamma_5 K^j = K^j \gamma_5$.

Therefore

$$\begin{aligned} \gamma_5 u_s(\mathbf{p}) &= +s\, v_{-s}(\mathbf{p}) \ , \\ \gamma_5 v_s(\mathbf{p}) &= -s\, u_{-s}(\mathbf{p}) \ . \end{aligned} \tag{38.39}$$

Combining eqs. (38.32), (38.38), and (38.39) results in

$$\begin{aligned} u_{-s}^*(-\mathbf{p}) &= -s\, \mathcal{C}\gamma_5 u_s(\mathbf{p}) \ , \\ v_{-s}^*(-\mathbf{p}) &= -s\, \mathcal{C}\gamma_5 v_s(\mathbf{p}) \ . \end{aligned} \tag{38.40}$$

We will need eq. (38.32) in our discussion of parity, eq. (38.37) in our discussion of charge conjugation, and eq. (38.40) in our discussion of time reversal.

39 Canonical Quantization of Spinor Fields II

A Dirac field Ψ with lagrangian

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi \tag{39.1}$$

obeys the canonical anticommutation relations

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0\ , \tag{39.2}$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{39.3}$$

and has the Dirac equation

$$(-i\partial\!\!\!/ + m)\Psi = 0 \tag{39.4}$$

as its equation of motion.

The general solution is

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]\ , \tag{39.5}$$

where $b_s(\mathbf{p})$ and $d^\dagger_s(\mathbf{p})$ are operators; the properties of the four-component spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ were belabored in the previous section.

Let us express $b_s(\mathbf{p})$ and $d^\dagger_s(\mathbf{p})$ in terms of $\Psi(x)$ and $\overline{\Psi}(x)$.

We begin with

$$\int d^3x\, e^{-ipx}\Psi(x) = \sum_{s'=\pm}\left[\tfrac{1}{2\omega}b_{s'}(\mathbf{p})u_{s'}(\mathbf{p}) + \tfrac{1}{2\omega}e^{2i\omega t}d^\dagger_{s'}(-\mathbf{p})v_{s'}(-\mathbf{p})\right] \qquad (39.6)$$

Next, multiply on the left by $\overline{u}_s(\mathbf{p})\gamma^0$, and use $\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$ and $\overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) = 0$ from section 38.

The result is

$$b_s(\mathbf{p}) = \int d^3x\, e^{-ipx}\,\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\ . \qquad (39.7)$$

Note that $b_s(\mathbf{p})$ is time independent.

To get $b^\dagger_s(\mathbf{p})$, take the hermitian conjugate of eq. (39.7), using

$$\begin{aligned}\left[\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\right]^\dagger &= \overline{\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)} \\ &= \overline{\Psi}(x)\overline{\gamma^0}u_s(\mathbf{p}) \\ &= \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})\ ,\end{aligned} \qquad (39.8)$$

where, for any general combination of gamma matrices A,

$$\overline{A} \equiv \beta A^\dagger \beta\ . \qquad (39.9)$$

Thus we find

$$b^\dagger_s(\mathbf{p}) = \int d^3x\, e^{ipx}\,\overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})\ . \qquad (39.10)$$

To extract $d^\dagger_s(\mathbf{p})$ from $\Psi(x)$, we start with

$$\int d^3x\, e^{ipx}\Psi(x) = \sum_{s'=\pm}\left[\tfrac{1}{2\omega}e^{-2i\omega t}b_{s'}(-\mathbf{p})u_{s'}(-\mathbf{p}) + \tfrac{1}{2\omega}d^\dagger_{s'}(\mathbf{p})v_{s'}(\mathbf{p})\right]. \tag{39.11}$$

Next, multiply on the left by $\overline{v}_s(\mathbf{p})\gamma^0$, and use $\overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$ and $\overline{v}_s(\mathbf{p})\gamma^0 u_{s'}(-\mathbf{p}) = 0$ from section 38.

The result is

$$d^\dagger_s(\mathbf{p}) = \int d^3x\, e^{ipx}\,\overline{v}_s(\mathbf{p})\gamma^0\Psi(x)\,. \tag{39.12}$$

To get $d_s(\mathbf{p})$, take the hermitian conjugate of eq. (39.12), which yields

$$d_s(\mathbf{p}) = \int d^3x\, e^{-ipx}\,\overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})\,. \tag{39.13}$$

Next, let us work out the anticommutation relations of the b and d operators (and their hermitian conjugates).

From eq. (39.2), it is immediately clear that

$$\begin{aligned}\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} &= 0\,,\\ \{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= 0\,,\\ \{b_s(\mathbf{p}), d^\dagger_{s'}(\mathbf{p}')\} &= 0\,,\end{aligned} \tag{39.14}$$

because these involve only the anticommutator of Ψ with itself, and this vanishes.

Of course, hermitian conjugation also yields

$$\begin{aligned}
\{b_s^\dagger(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= 0\,, \\
\{d_s^\dagger(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} &= 0\,, \\
\{b_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= 0\,.
\end{aligned} \tag{39.15}$$

Now consider

$$\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{-ipx+ip'y}\, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x), \overline{\Psi}(y)\}\gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{-i(p-p')x}\, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 u_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, \overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'}\,.
\end{aligned} \tag{39.16}$$

In the first line, we are free to set $x^0 = y^0$ because $b_s(\mathbf{p})$ and $b^\dagger_{s'}(\mathbf{p}')$ are actually time independent.

In the third, we used $(\gamma^0)^2 = 1$, and in the fourth, $\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$.

Similarly,

$$\begin{aligned}
\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{ipx-ip'y}\, \overline{v}_s(\mathbf{p})\gamma^0\{\Psi(x), \overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{i(p-p')x}\, \overline{v}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, \overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'}\,.
\end{aligned} \tag{39.17}$$

And finally,

$$\begin{aligned}\{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{-ipx-ip'y}\, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x), \overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\ &= \int d^3x\, e^{-i(p+p')x}\, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\ &= (2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')\, \overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) \\ &= 0\,. \end{aligned} \tag{39.18}$$

According to the discussion in section 3, eqs. (39.14–39.18) are exactly what we need to describe the creation and annihilation of fermions.

In this case, we have two different kinds: b-type and d-type, each with two possible spin states, s = + and s = −.

Next, let us evaluate the hamiltonian

$$H = \int d^3x\, \overline{\Psi}(-i\gamma^i\partial_i + m)\Psi \tag{39.19}$$

in terms of the b and d operators.

We have

$$
\begin{aligned}
(-i\gamma^i\partial_i + m)\Psi &= \sum_{s=\pm}\int \widetilde{dp}\,\Big(-i\gamma^i\partial_i + m\Big)\Big(\, b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\Big) \\
&= \sum_{s=\pm}\int \widetilde{dp}\,\Big[\, b_s(\mathbf{p})(+\gamma^i p_i + m)u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^i p_i + m)v_s(\mathbf{p})e^{-ipx}\,\Big] \\
&= \sum_{s=\pm}\int \widetilde{dp}\,\Big[\, b_s(\mathbf{p})(\gamma^0\omega)u_s(\mathbf{p})e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^0\omega)v_s(\mathbf{p})e^{-ipx}\,\Big].
\end{aligned}
\tag{39.20}
$$

Therefore

$$
\begin{aligned}
H &= \sum_{s,s'}\int \widetilde{dp}\;\widetilde{dp}'\;d^3x\,\Big(b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'x} + d_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'x}\Big) \\
&\qquad\qquad \times\omega\,\Big(b_s(\mathbf{p})\gamma^0 u_s(\mathbf{p})e^{ipx} - d_s^\dagger(\mathbf{p})\gamma^0 v_s(\mathbf{p})e^{-ipx}\Big) \\
&= \sum_{s,s'}\int \widetilde{dp}\;\widetilde{dp}'\;d^3x\;\omega\,\Big[\; b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{-i(p'-p)x} \\
&\qquad\qquad - b_{s'}^\dagger(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{u}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{-i(p'+p)x} \\
&\qquad\qquad + d_{s'}(\mathbf{p}')b_s(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\;e^{+i(p'+p)x} \\
&\qquad\qquad - d_{s'}(\mathbf{p}')d_s^\dagger(\mathbf{p})\;\overline{v}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\;e^{+i(p'-p)x}\,\Big]
\end{aligned}
$$

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\boldsymbol{\mathcal{L}}$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ the the the the the $\mathbf{V}_4(k_1,k_2,k_3,k_4)$

$$
\begin{aligned}
&= \sum_{s,s'} \int \widetilde{dp}\ \tfrac{1}{2}\Big[\, b^\dagger_{s'}(\mathbf{p}) b_s(\mathbf{p})\ \overline{u}_{s'}(\mathbf{p}) \gamma^0 u_s(\mathbf{p}) \\
&\qquad\qquad - b^\dagger_{s'}(-\mathbf{p}) d^\dagger_s(\mathbf{p})\ \overline{u}_{s'}(-\mathbf{p}) \gamma^0 v_s(\mathbf{p})\ e^{+2i\omega t} \\
&\qquad\qquad + d_{s'}(-\mathbf{p}) b_s(\mathbf{p})\ \overline{v}_{s'}(-\mathbf{p}) \gamma^0 u_s(\mathbf{p})\ e^{-2i\omega t} \\
&\qquad\qquad - d_{s'}(\mathbf{p}) d^\dagger_s(\mathbf{p})\ \overline{v}_{s'}(\mathbf{p}) \gamma^0 v_s(\mathbf{p}) \,\Big] \\
&= \sum_{s} \int \widetilde{dp}\ \omega \Big[\, b^\dagger_s(\mathbf{p}) b_s(\mathbf{p}) - d_s(\mathbf{p}) d^\dagger_s(\mathbf{p}) \,\Big].
\end{aligned}
\tag{39.21}
$$

Using eq. (39.17), we can rewrite this as

$$
H = \sum_{s=\pm} \int \widetilde{dp}\ \omega \Big[\, b^\dagger_s(\mathbf{p}) b_s(\mathbf{p}) + d^\dagger_s(\mathbf{p}) d_s(\mathbf{p}) \,\Big] - 4\mathcal{E}_0 V\ , \tag{39.22}
$$

where $\mathcal{E}_0 = 1/2\ (2\pi)^{-3} \int d^3k\ \omega$ is the zero-point energy per unit volume that we found for a real scalar field in section 3, and $V = (2\pi)^3 \delta^3(0) = \int d^3x$ is the volume of space.

That the zero-point energy is negative rather than positive is characteristic of fermions; that it is larger in magnitude by a factor of four is due to the four types of particles that are associated with a Dirac field.

We can cancel off this constant energy by including a constant term $\Omega_0 = -4\mathcal{E}_0$ in the original lagrangian density; from here on, we will assume that this has been done.

The ground state of the hamiltonian (39.22) is the *vacuum state* $|0\rangle$ that is annihilated by every $b_s(\mathbf{p})$ and $d_s(\mathbf{p})$,

$$b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0 \ . \tag{39.23}$$

Then, we can interpret the $b^\dagger_s(p)$ operator as creating a b-type particle with momentum p, energy $\omega = (p^2 + m^2)^{1/2}$, and spin S_z =1/2 s, and the $d^\dagger_s(p)$ operator as creating a d-type particle with the same properties.

The b-type and d-type particles are distinguished by the value of the charge $Q = \int d^3x\, j^0$, where $j^\mu = \Psi\gamma^\mu\Psi$ is the Noether current associated with the invariance of $\mathcal{L}$ under the U(1) transformation $\Psi \rightarrow e^{-i\alpha}\Psi$, $\quad \overline{\Psi} \rightarrow e^{+i\alpha}\ \overline{\Psi}$

Following the same procedure that we used for the hamiltonian, we can show that

$$\begin{aligned} Q &= \int d^3x\ \overline{\Psi}\gamma^0\Psi \\ &= \sum_{s=\pm} \int \widetilde{dp}\ \left[b^\dagger_s(\mathbf{p}) b_s(\mathbf{p}) + d_s(\mathbf{p}) d^\dagger_s(\mathbf{p}) \right] \\ &= \sum_{s=\pm} \int \widetilde{dp}\ \left[b^\dagger_s(\mathbf{p}) b_s(\mathbf{p}) - d^\dagger_s(\mathbf{p}) d_s(\mathbf{p}) \right] + \text{constant} \ . \end{aligned} \tag{39.24}$$

Thus the conserved charge Q counts the total number of b-type particles minus the total number of d-type particles. (We are free to shift the overall value of Q to remove the constant term, and so we shall.)

In quantum electrodynamics, we will identify the b-type particles as electrons and the d-type particles as positrons.

Now consider a Majorana field Ψ with lagrangian

$$\mathcal{L} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\partial\!\!\!/\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \ . \tag{39.25}$$

The equation of motion for Ψ is once again the Dirac equation, and so the general solution is once again given by eq. (39.5).

However, Ψ must also obey the Majorana condition $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$.

Starting from the barred form of eq. (39.5),

$$\overline{\Psi}(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s^\dagger(\mathbf{p})\overline{u}_s(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})\overline{v}_s(\mathbf{p})e^{ipx}\right], \tag{39.26}$$

we have

$$\mathcal{C}\overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s^\dagger(\mathbf{p})\,\mathcal{C}\overline{u}_s^{\mathrm{T}}(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})\,\mathcal{C}\overline{v}_s^{\mathrm{T}}(\mathbf{p})e^{ipx}\right]. \tag{39.27}$$

From section 38, we have

$$\begin{aligned}\mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} &= v_s(\mathbf{p})\ ,\\ \mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} &= u_s(\mathbf{p})\ ,\end{aligned} \tag{39.28}$$

and so

$$\mathcal{C}\overline{\Psi}^{\mathrm{T}}(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx} + d_s(\mathbf{p})u_s(\mathbf{p})e^{ipx}\right] \ . \tag{39.29}$$

Comparing eqs. (39.5) and (39.29), we see that we will have $\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$ if

$$d_s(\mathbf{p}) = b_s(\mathbf{p}) \ . \tag{39.30}$$

Thus a free Majorana field can be written as

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right] \ . \tag{39.31}$$

The anticommutation relations for a Majorana field,

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{39.32}$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{39.33}$$

can be used to show that

$$\begin{aligned} \{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} &= 0 \ , \\ \{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega \delta_{ss'} \ , \end{aligned} \tag{39.34}$$

as we would expect.

The hamiltonian for the Majorana field Ψ is

$$\begin{aligned} H &= \tfrac{1}{2} \int d^3x \ \Psi^{\mathrm{T}} \mathcal{C} (-i\gamma^i \partial_i + m) \Psi \\ &= \tfrac{1}{2} \int d^3x \ \overline{\Psi} (-i\gamma^i \partial_i + m) \Psi \ , \end{aligned} \tag{39.35}$$

and we can work through the same manipulations that led to eq. (39.21); the only differences are an extra overall factor of one-half, and $d_s(\mathbf{p}) = b_s(\mathbf{p})$.

Thus we get

$$H = \tfrac{1}{2} \sum_{s=\pm} \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - b_s(\mathbf{p}) b_s^\dagger(\mathbf{p}) \right]. \tag{39.36}$$

Note that this would reduce to a constant if we tried to use commutators rather than anticommutators in eq. (39.34), a reflection of the spin-statistics theorem.

Using eq. (39.34) as it is, we find

$$H = \sum_{s=\pm} \int \widetilde{dp}\ \omega\, b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - 2\mathcal{E}_0 V. \tag{39.37}$$

Again, we can (and will) cancel off the zero-point energy by including a term $\Omega_0 = -2\mathcal{E}_0$ in the original lagrangian density.

The Majorana lagrangian has no U(1) symmetry.

Thus there is no associated charge, and only one kind of particle (with two possible spin states).

40 Parity, Time Reversal, and Charge Conjugation

Recall that, under a Lorentz transformation Λ implemented by the unitary operator $U(\Lambda)$, a Dirac (or Majorana) field transforms as

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x) \ . \tag{40.1}$$

For an infinitesimal transformation . $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$, the matrix $D(\Lambda)$ is given by

$$D(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu} \ , \tag{40.2}$$

where the Lorentz generator matrices are

$$S^{\mu\nu} = \tfrac{i}{4}[\gamma^{\mu},\gamma^{\nu}] \ . \tag{40.3}$$

In this section, we will consider the two Lorentz transformations that cannot be reached via a sequence of infinitesimal transformations away from the identity: parity and time reversal.

We begin with parity.

Define the parity transformation

$$\mathcal{P}^{\mu}{}_{\nu} = (\mathcal{P}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{40.4}$$

and the corresponding unitary operator

$$P \equiv U(\mathcal{P}) \ . \tag{40.5}$$

Now we have

$$P^{-1}\Psi(x)P = D(\mathcal{P})\Psi(\mathcal{P}x) \ . \tag{40.6}$$

The question we wish to answer is, what is the matrix D(P)?

First of all, if we make a second parity transformation, we get

$$P^{-2}\Psi(x)P^2 = D(\mathcal{P})^2\Psi(x) \ , \tag{40.7}$$

and it is tempting to conclude that we should have $D(P)^2 = 1$, so that we return to the original field.

This is correct for scalar fields, since they are themselves observable.

With fermions, however, it takes an even number of fields to construct an observable.

Therefore we need only require the weaker condition $D(P)^2 = \pm 1$.

We will also require the particle creation and annihilation operators to transform in a simple way.

Because

$$P^{-1}\mathbf{P}P = -\mathbf{P} \ , \tag{40.8}$$

$$P^{-1}\mathbf{J}P = +\mathbf{J} \ , \tag{40.9}$$

where **P** is the total three-momentum operator and **J** is the total angular momentum operator, a parity transformation should reverse the three-momentum while leaving the spin direction unchanged.

We therefore require

$$P^{-1}b_s^\dagger(\mathbf{p})P = \eta\, b_s^\dagger(-\mathbf{p})\,,$$
$$P^{-1}d_s^\dagger(\mathbf{p})P = \eta\, d_s^\dagger(-\mathbf{p})\,, \tag{40.10}$$

where η is a possible phase factor that (by the previous argument about observables) should satisfy $\eta^2 = \pm 1$.

We could in principle assign different phase factors to the b and d operators, but we choose them to be the same so that the parity transformation is compatible with the Majorana condition $d_s(\mathbf{p}) = b_s(\mathbf{p})$.

Writing the mode expansion of the free field

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right], \tag{40.11}$$

the parity transformation reads

$$\begin{aligned}
&P^{-1}\Psi(x)P \\
&\quad= \sum_{s=\pm}\int \widetilde{dp}\left[\left(P^{-1}b_s(\mathbf{p})P\right)u_s(\mathbf{p})e^{ipx} + \left(P^{-1}d_s^\dagger(\mathbf{p})P\right)v_s(\mathbf{p})e^{-ipx}\right] \\
&\quad= \sum_{s=\pm}\int \widetilde{dp}\left[\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})e^{ipx} + \eta d_s^\dagger(-\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right] \\
&\quad= \sum_{s=\pm}\int \widetilde{dp}\left[\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})e^{ip\mathcal{P}x} + \eta d_s^\dagger(\mathbf{p})v_s(-\mathbf{p})e^{-ip\mathcal{P}x}\right].
\end{aligned} \tag{40.12}$$

In the last line, we have changed the integration variable from **p** to **−p**.

We now use a result from section 38, namely that

$$\begin{aligned} u_s(-\mathbf{p}) &= +\beta u_s(\mathbf{p}) \,, \\ v_s(-\mathbf{p}) &= -\beta v_s(\mathbf{p}) \,, \end{aligned} \tag{40.13}$$

where

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{40.14}$$

Then, if we choose η =−i, eq. (40.12) becomes

$$\begin{aligned} P^{-1}\Psi(x)P &= \sum_{s=\pm} \int \widetilde{dp} \left[ib_s(\mathbf{p})\beta u_s(\mathbf{p}) e^{ip\mathcal{P}x} + id_s^\dagger(\mathbf{p})\beta v_s(\mathbf{p}) e^{-ip\mathcal{P}x} \right] \\ &= i\beta\, \Psi(\mathcal{P}x) \,. \end{aligned} \tag{40.15}$$

Thus we see that D(P) = iβ. (We could also have chosen η = i, resulting in D(P) = −iβ; either choice is acceptable.)

The factor of i has an interesting physical consequence.

Consider a state of an electron and positron with zero center-of-mass momentum,

$$|\phi\rangle = \int \widetilde{dp}\, \phi(\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle \ ; \tag{40.16}$$

here $\varphi(\mathbf{p})$ is the momentum-space wave function.

Let us assume that the vacuum is parity invariant: $\mathrm{P}|0\rangle = \mathrm{P}^{-1}|0\rangle = |0\rangle$.

Let us also assume that the wave function has definite parity: $\varphi(-\mathbf{p}) = (-)^\ell \varphi(\mathbf{p})$.

Then, applying the inverse parity operator on $|\varphi\rangle$, we get

$$\begin{aligned} P^{-1}|\phi\rangle &= \int \widetilde{dp}\, \phi(\mathbf{p}) \Big(P^{-1} b_s^\dagger(\mathbf{p}) P\Big)\Big(P^{-1} d_{s'}^\dagger(-\mathbf{p}) P\Big) P^{-1}|0\rangle \ . \\ &= (-i)^2 \int \widetilde{dp}\, \phi(\mathbf{p}) b_s^\dagger(-\mathbf{p}) d_{s'}^\dagger(\mathbf{p})|0\rangle \\ &= (-i)^2 \int \widetilde{dp}\, \phi(-\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle \\ &= -(-)^\ell |\phi\rangle \ . \end{aligned} \tag{40.17}$$

Thus, the parity of this state is opposite to that of its wave function; an electron-positron pair has an intrinsic parity of -1.

This also applies to a pair of Majorana fermions.

This influences the selection rules for fermion pair annihilation in theories that conserve parity. (A pair of electrons also has negative intrinsic parity, but this is less interesting because the electrons are prevented from annihilating by charge conservation.)

Let us see what eq. (40.15) implies for the two Weyl fields that comprise the Dirac field.

Recalling that

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger \dot{a}} \end{pmatrix}, \tag{40.18}$$

we see from eqs. (40.14) and (40.15) that

$$\begin{aligned} P^{-1}\chi_a(x)P &= i\xi^{\dagger \dot{a}}(\mathcal{P}x) \ , \\ P^{-1}\xi^{\dagger \dot{a}}(x)P &= i\chi_a(\mathcal{P}x) \ . \end{aligned} \tag{40.19}$$

Thus a parity transformation exchanges a left-handed field for a right-handed one.

If we take the hermitian conjugate of eq. (40.19), then raise the index on one side while lowering it on the other (and remember that this introduces a relative minus sign!), we get

$$\begin{aligned} P^{-1}\chi^{\dagger \dot{a}}(x)P &= i\xi_a(\mathcal{P}x) \ , \\ P^{-1}\xi_a(x)P &= i\chi^{\dagger \dot{a}}(\mathcal{P}x) \ . \end{aligned} \tag{40.20}$$

Comparing eqs. (40.19) and (40.20), we see that they are compatible with the Majorana condition $\chi_a(x) = \xi_a(x)$.

Next we take up time reversal.

Define the time-reversal transformation

$$\mathcal{T}^{\mu}{}_{\nu} = (\mathcal{T}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \tag{40.21}$$

and the corresponding operator

$$T \equiv U(\mathcal{T}) \ . \tag{40.22}$$

Now we have

$$T^{-1}\Psi(x)T = D(\mathcal{T})\Psi(\mathcal{T}x) \ . \tag{40.23}$$

The question we wish to answer is, what is the matrix D(T)

As with parity, we can conclude that $D(T)^2 = \pm 1$, and we will require the particle creation and annihilation operators to transform in a simple way.

Because

$$T^{-1}\mathbf{P}T = -\mathbf{P} \ , \tag{40.24}$$

$$T^{-1}\mathbf{J}T = -\mathbf{J} \ , \tag{40.25}$$

where **P** is the total three-momentum operator and **J** is the total angular momentum operator, a time-reversal transformation should reverse the direction of both the three-momentum and the spin.

We therefore require

$$
\begin{aligned}
T^{-1} b_s^\dagger(\mathbf{p}) T &= \zeta_s \, b_{-s}^\dagger(-\mathbf{p}) \; , \\
T^{-1} d_s^\dagger(\mathbf{p}) T &= \zeta_s \, d_{-s}^\dagger(-\mathbf{p}) \; .
\end{aligned}
\tag{40.26}
$$

This time we allow for possible s-dependence of the phase factor.

Also, we recall from section 23 that T must be an antiunitary operator, so that $T^{-1} i T = -i$.

Then we have

$$
\begin{aligned}
& T^{-1} \Psi(x) T \\
& \quad = \sum_{s=\pm} \int \widetilde{dp} \left[\left(T^{-1} b_s(\mathbf{p}) T \right) u_s^*(\mathbf{p}) e^{-ipx} + \left(T^{-1} d_s^\dagger(\mathbf{p}) T \right) v_s^*(\mathbf{p}) e^{ipx} \right] \\
& \quad = \sum_{s=\pm} \int \widetilde{dp} \left[\zeta_s^* b_{-s}(-\mathbf{p}) u_s^*(\mathbf{p}) e^{-ipx} + \zeta_s d_{-s}^\dagger(-\mathbf{p}) v_s^*(\mathbf{p}) e^{ipx} \right] \\
& \quad = \sum_{s=\pm} \int \widetilde{dp} \left[\zeta_{-s}^* b_s(\mathbf{p}) u_{-s}^*(-\mathbf{p}) e^{ip\mathcal{T}x} + \zeta_{-s} d_s^\dagger(\mathbf{p}) v_{-s}^*(-\mathbf{p}) e^{-ip\mathcal{T}x} \right] .
\end{aligned}
\tag{40.27}
$$

In the last line, we have changed the integration variable from **p** to **−p**, and the summation variable from s to −s.

We now use a result from section 38, namely that

$$
\begin{aligned}
u_{-s}^*(-\mathbf{p}) &= -s \, \mathcal{C} \gamma_5 u_s(\mathbf{p}) \; , \\
v_{-s}^*(-\mathbf{p}) &= -s \, \mathcal{C} \gamma_5 v_s(\mathbf{p}) \; .
\end{aligned}
\tag{40.28}
$$

Then, if we choose $\zeta_s = s$, eq. (40.27) becomes

$$T^{-1}\Psi(x)T = \mathcal{C}\gamma_5\Psi(\mathcal{T}x) \ . \tag{40.29}$$

Thus we see that $D(T) = C\gamma_5$. (We could also have chosen $\zeta_s = -s$, resulting in $D(T) = -C\gamma_5$; either choice is acceptable.)

As with parity, we can consider the effect of time reversal on the Weyl fields.

Using eqs. (40.18), (40.29),

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ab} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix}, \tag{40.30}$$

and

$$\gamma_5 = \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}, \tag{40.31}$$

we see that

$$\begin{aligned} T^{-1}\chi_a(x)T &= +\chi^a(\mathcal{T}x) \ , \\ T^{-1}\xi^{\dagger\dot{a}}(x)T &= -\xi^\dagger_{\dot{a}}(\mathcal{T}x) \ . \end{aligned} \tag{40.32}$$

Thus left-handed Weyl fields transform into left-handed Weyl fields (and right-handed into right-handed) under time reversal.

If we take the hermitian conjugate of eq. (40.32), then raise the index on one side while lowering it on the other (and remember that this introduces a relative minus sign!), we get

$$T^{-1}\chi^{\dagger\dot{a}}(x)T = -\chi^{\dagger}_{\dot{a}}(\mathcal{T}x) \ ,$$
$$T^{-1}\xi_{a}(x)T = +\xi^{a}(\mathcal{T}x) \ . \qquad (40.33)$$

Comparing eqs. (40.32) and (40.33), we see that they are compatible with the Majorana condition $\chi_a(x) = \xi_a(x)$.

It is interesting and important to evaluate the transformation properties of fermion bilinears of the form $\overline{\Psi}A\Psi$, where A is some combination of gamma matrices.

We will consider A's that satisfy $\overline{A} = A$, where $\overline{A} \equiv \beta A^{\dagger}\beta$; in this case, $\overline{\Psi}A\Psi$ is hermitian.

Let us begin with parity transformations.

From $\overline{\Psi} = \Psi^{\dagger}\beta$ and eq. (40.15) we get

$$P^{-1}\overline{\Psi}(x)P = -i\overline{\Psi}(\mathcal{P}x)\beta \ , \qquad (40.34)$$

Combining eqs. (40.15) and (40.34) we find

$$P^{-1}\Big(\overline{\Psi}A\Psi\Big)P = \overline{\Psi}\Big(\beta A\beta\Big)\Psi \ , \qquad (40.35)$$

where we have suppressed the spacetime arguments (which transform in the obvious way).

For various particular choices of A we have

$$\begin{aligned}
\beta 1 \beta &= +1 \,, \\
\beta i\gamma_5 \beta &= -i\gamma_5 \,, \\
\beta \gamma^0 \beta &= +\gamma^0 \,, \\
\beta \gamma^i \beta &= -\gamma^i \,, \\
\beta \gamma^0 \gamma_5 \beta &= -\gamma^0 \gamma_5 \,, \\
\beta \gamma^i \gamma_5 \beta &= +\gamma^i \gamma_5 \,.
\end{aligned} \tag{40.36}$$

Therefore, the corresponding hermitian bilinears transform as

$$\begin{aligned}
P^{-1}\left(\overline{\Psi}\Psi\right)P &= +\,\overline{\Psi}\Psi \,, \\
P^{-1}\left(\overline{\Psi} i\gamma_5 \Psi\right)P &= -\,\overline{\Psi} i\gamma_5 \Psi \,, \\
P^{-1}\left(\overline{\Psi}\gamma^\mu \Psi\right)P &= +\,\mathcal{P}^\mu{}_\nu \overline{\Psi}\gamma^\nu \Psi \,, \\
P^{-1}\left(\overline{\Psi}\gamma^\mu \gamma_5 \Psi\right)P &= -\,\mathcal{P}^\mu{}_\nu \overline{\Psi}\gamma^\nu \gamma_5 \Psi \,,
\end{aligned} \tag{40.37}$$

Thus we see that $\overline{\Psi}\Psi$ and $\overline{\Psi}\gamma^\mu\Psi$ are even under a parity transformation, while $\overline{\Psi} i\gamma_5 \Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are odd.

We say that $\overline{\Psi}\Psi$ is a scalar, $\overline{\Psi}\gamma^\mu\Psi$ is a vector or *polar vector*, $\overline{\Psi} i\gamma_5 \Psi$ is a pseudoscalar, and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ is a *pseudovector* or *axial vector*.

Turning to time reversal, from eq. (40.29) we get

$$T^{-1}\overline{\Psi}(x)T = \overline{\Psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1} \ . \tag{40.38}$$

Combining eqs. (40.29) and (40.38), along with $T^{-1}AT = A^*$, we find

$$T^{-1}\Big(\overline{\Psi}A\Psi\Big)T = \overline{\Psi}\Big(\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\Big)\Psi \ , \tag{40.39}$$

where we have suppressed the spacetime arguments (which transform in the obvious way).

Recall that $C^{-1}\gamma^\mu C = -(\gamma^\mu)T$ and that $C^{-1}\gamma_5 C = \gamma_5$.

Also, γ^0 and γ_5 are real, hermitian, and square to one, while γ^i is antihermitian.

Finally, γ_5 anticommutes with γ^μ.

Using all of this info, we find

$$\begin{aligned}
\gamma_5\mathcal{C}^{-1}1^*\mathcal{C}\gamma_5 &= +1 \ , \\
\gamma_5\mathcal{C}^{-1}(i\gamma_5)^*\mathcal{C}\gamma_5 &= -i\gamma_5 \ , \\
\gamma_5\mathcal{C}^{-1}(\gamma^0)^*\mathcal{C}\gamma_5 &= +\gamma^0 \ , \\
\gamma_5\mathcal{C}^{-1}(\gamma^i)^*\mathcal{C}\gamma_5 &= -\gamma^i \ , \\
\gamma_5\mathcal{C}^{-1}(\gamma^0\gamma_5)^*\mathcal{C}\gamma_5 &= +\gamma^0\gamma_5 \ , \\
\gamma_5\mathcal{C}^{-1}(\gamma^i\gamma_5)^*\mathcal{C}\gamma_5 &= -\gamma^i\gamma_5 \ .
\end{aligned} \tag{40.40}$$

Therefore,

$$
\begin{aligned}
T^{-1}\Big(\overline{\Psi}\Psi\Big)T &= +\overline{\Psi}\Psi\ , \\
T^{-1}\Big(\overline{\Psi}i\gamma_5\Psi\Big)T &= -\overline{\Psi}i\gamma_5\Psi\ , \\
T^{-1}\Big(\overline{\Psi}\gamma^\mu\Psi\Big)T &= -\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi\ , \\
T^{-1}\Big(\overline{\Psi}\gamma^\mu\gamma_5\Psi\Big)T &= -\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi\ .
\end{aligned}
\tag{40.41}
$$

Thus we see that $\overline{\Psi}\Psi$ is even under time reversal, while $\overline{\Psi}i\gamma_5\Psi$ $\overline{\Psi}\gamma^\mu\Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ are odd.

For completeness we will also consider the transformation properties of bilinears under charge conjugation.

Recall that

$$
\begin{aligned}
C^{-1}\Psi(x)C &= \mathcal{C}\overline{\Psi}^{\mathrm{T}}(x)\ , \\
C^{-1}\overline{\Psi}(x)C &= \Psi^{\mathrm{T}}(x)\mathcal{C}\ .
\end{aligned}
\tag{40.42}
$$

The bilinear $\overline{\Psi}A\Psi$ therefore transforms as

$$
C^{-1}\Big(\overline{\Psi}A\Psi\Big)C = \Psi^{\mathrm{T}}\mathcal{C}A\mathcal{C}\overline{\Psi}^{\mathrm{T}}\ . \tag{40.43}
$$

Since all indices are contracted, we can rewrite the right-hand side as its transpose, with an extra minus sign for exchanging the order of the two fermion fields.

We get

$$C^{-1}\left(\overline{\Psi}A\Psi\right)C = -\overline{\Psi}\mathcal{C}^{\mathrm{T}}A^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\Psi \ . \tag{40.44}$$

Recalling that $C^{\mathrm{T}} = C^{-1} = -C$, we have

$$C^{-1}\left(\overline{\Psi}A\Psi\right)C = \overline{\Psi}\left(\mathcal{C}^{-1}A^{\mathrm{T}}\mathcal{C}\right)\Psi \ . \tag{40.45}$$

Once again we can go through the list:

$$\begin{aligned}
\mathcal{C}^{-1}1^{\mathrm{T}}\mathcal{C} &= +1 \ , \\
\mathcal{C}^{-1}(i\gamma_5)^{\mathrm{T}}\mathcal{C} &= +i\gamma_5 \ , \\
\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} &= -\gamma^\mu \ , \\
\mathcal{C}^{-1}(\gamma^\mu\gamma_5)^{\mathrm{T}}\mathcal{C} &= +\gamma^\mu\gamma_5 \ .
\end{aligned} \tag{40.46}$$

Therefore,

$$\begin{aligned}
C^{-1}\left(\overline{\Psi}\Psi\right)C &= +\overline{\Psi}\Psi \ , \\
C^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)C &= +\overline{\Psi}i\gamma_5\Psi \ , \\
C^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)C &= -\overline{\Psi}\gamma^\mu\Psi \ , \\
C^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)C &= +\overline{\Psi}\gamma^\mu\gamma_5\Psi \ .
\end{aligned} \tag{40.47}$$

Thus we see that $\overline{\Psi}\Psi$, $\overline{\Psi}i\gamma_5\Psi$, $\overline{\Psi}\gamma^\mu\Psi$ are even under charge conjugation, $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ is odd.

For a Majorana field, we have $C^{-1}\Psi C = \Psi$ and $C^{-1}\overline{\Psi}C = \overline{\Psi}$; this implies $C^{-1}(\overline{\Psi}A\Psi)C = \overline{\Psi}A\Psi$ for any combination of gamma matrices A.

Since eq. (40.47) tells that $C^{-1}(\overline{\Psi}\gamma^\mu\Psi)C = -\overline{\Psi}\gamma^\mu\Psi$ for either a Dirac or Majorana field, it must be that $\overline{\Psi}\gamma^\mu\Psi = 0$ for a Majorana field.

Let us consider the combined effects of the three transformations (C, P, and T) on the bilinears.

From eqs. (40.37), (40.41), and (40.47), we have

$$\begin{aligned}
(CPT)^{-1}\left(\overline{\Psi}\Psi\right)CPT &= +\overline{\Psi}\Psi\,, \\
(CPT)^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)CPT &= +\overline{\Psi}i\gamma_5\Psi\,, \\
(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)CPT &= -\overline{\Psi}\gamma^\mu\Psi\,, \\
(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)CPT &= -\overline{\Psi}\gamma^\mu\gamma_5\Psi\,,
\end{aligned} \tag{40.48}$$

where we have used $P^\mu{}_\nu T^\nu{}_\rho = -\delta^\mu{}_\rho$.

We see that $\overline{\Psi}\Psi$ and $\overline{\Psi}i\gamma_5\Psi$ are both even under CPT, while $\overline{\Psi}\gamma^\mu\Psi$ and $\overline{\Psi}\gamma^\mu\gamma_5\Psi$ both odd.

These are (it turns out) examples of a more general rule: a fermion bilinear with n vector indices (and no uncontracted spinor indices) is even (odd) under CPT if n is even (odd).

This also applies if we allow derivatives acting on the fields, since each component of ∂_μ is odd under the combination PT and even under C.

For scalar and vector fields, it is always possible to choose the phase factors in the C, P, and T transformations so that, overall, they obey the same rule: a hermitian combination of fields and derivatives is even or odd depending on the total number of uncontracted vector indices.

Putting this together with our result for fermion bilinears, we see that any hermitian combination of any set of fields (scalar, vector, Dirac, Majorana) and their derivatives that is a Lorentz scalar (and so carries no indices) is even under CPT.

Since the lagrangian must be formed out of such combinations, we have $\mathcal{L}(x) \to \mathcal{L}(-x)$ under CPT, and so the action $S = \int d^4x \mathcal{L}$ is invariant.

This is the CPT theorem.

41 LSZ Reduction for Spin-One-Half Particles

Let us now consider how to construct appropriate initial and final states for scattering experiments.

We will first consider the case of a Dirac field Ψ, and assume that its interactions respect the U(1) symmetry that gives rise to the conserved current $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ and its associated charge Q.

In the free theory, we can create a state of one particle by acting on the vacuum state with a creation operator:

$$|p, s, +\rangle = b_s^\dagger(\mathbf{p})|0\rangle \ , \tag{41.1}$$

$$|p, s, -\rangle = d_s^\dagger(\mathbf{p})|0\rangle \ , \tag{41.2}$$

where the label ± on the ket indicates the value of the U(1) charge Q, and

$$b_s^\dagger(\mathbf{p}) = \int d^3x\, e^{ipx}\, \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})\ , \tag{41.3}$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x\, e^{ipx}\, \overline{v}_s(\mathbf{p})\gamma^0 \Psi(x)\ . \tag{41.4}$$

Recall that $b^\dagger_s(\mathbf{p})$ and $d^\dagger_s(\mathbf{p})$ are time independent in the free theory.

The states $|p,s,\pm\rangle$ have the Lorentz-invariant normalization

$$\langle p,s,q|p',s',q'\rangle = (2\pi)^3\, 2\omega\, \delta^3(\mathbf{p}-\mathbf{p}')\, \delta_{ss'}\, \delta_{qq'}\ , \tag{41.5}$$

where $\omega = (p^2 + m^2)^{1/2}$.

Let us consider an operator that (in the free theory) creates a particle with definite spin and charge, localized in momentum space near $\mathbf{p}_1$, and localized in position space near the origin:

$$b_1^\dagger \equiv \int d^3p\, f_1(\mathbf{p}) b^\dagger_{s_1}(\mathbf{p})\ , \tag{41.6}$$

where

$$f_1(\mathbf{p}) \propto \exp[-(\mathbf{p}-\mathbf{p}_1)^2/4\sigma^2] \tag{41.7}$$

is an appropriate wave packet, and σ is its width in momentum space.

If we time evolve (in the Schrodinger picture) the state created by this time-independent operator, then the wave packet will propagate (and spread out).

The particle will thus be localized far from the origin as $t \to \pm\infty$.

If we consider instead an initial state of the form $|i\rangle = b^\dagger_1 b^\dagger_2 |0\rangle$, where $p_1 \neq p_2$, then we have two particles that are widely separated in the far past.

Let us guess that this still works in the interacting theory.

One complication is that $b^\dagger_s(\mathbf{p})$ will no longer be time independent, and so $b^\dagger_1$, eq. (41.6), becomes time dependent as well.

Our guess for a suitable initial state for a scattering experiment is then

$$|i\rangle = \lim_{t\to-\infty} b_1^\dagger(t) b_2^\dagger(t)|0\rangle \ . \tag{41.8}$$

By appropriately normalizing the wave packets, we can make $\langle i|i\rangle = 1$, and we will assume that this is the case.

Similarly, we can consider a final state

$$|f\rangle = \lim_{t\to+\infty} b_{1'}^\dagger(t) b_{2'}^\dagger(t)|0\rangle \ , \tag{41.9}$$

where $p'_1 \neq p'_2$, and $\langle f | f \rangle = 1$.

This describes two widely separated particles in the far future.

We could also consider acting with more creation operators, if we are interested in the production of some extra particles in the collision of two, or using $d^\dagger$ operators instead of $b^\dagger$ operators for some or all of the initial and final particles.

Now the scattering amplitude is simply given by $\langle f \,|\, i \rangle$.

We need to find a more useful expression for $\langle f \,|\, i \rangle$.

To this end, let us note that

$$
\begin{aligned}
& b_1^\dagger(-\infty) - b_1^\dagger(+\infty) \\
& \qquad = -\int_{-\infty}^{+\infty} dt\; \partial_0 b_1^\dagger(t) \\
& = -\int d^3p\; f_1(\mathbf{p}) \int d^4x\; \partial_0 \Big(e^{ipx}\, \overline{\Psi}(x) \gamma^0 u_{s_1}(\mathbf{p}) \Big) \;. \\
& = -\int d^3p\; f_1(\mathbf{p}) \int d^4x\; \overline{\Psi}(x) \Big(\gamma^0 \overleftarrow{\partial}_0 - i\gamma^0 p^0 \Big) u_{s_1}(\mathbf{p}) e^{ipx} \\
& = -\int d^3p\; f_1(\mathbf{p}) \int d^4x\; \overline{\Psi}(x) \Big(\gamma^0 \overleftarrow{\partial}_0 - i\gamma^i p_i - im \Big) u_{s_1}(\mathbf{p}) e^{ipx} \\
& = -\int d^3p\; f_1(\mathbf{p}) \int d^4x\; \overline{\Psi}(x) \Big(\gamma^0 \overleftarrow{\partial}_0 - \gamma^i \overrightarrow{\partial}_i - im \Big) u_{s_1}(\mathbf{p}) e^{ipx} \\
& = -\int d^3p\; f_1(\mathbf{p}) \int d^4x\; \overline{\Psi}(x) \Big(\gamma^0 \overleftarrow{\partial}_0 + \gamma^i \overleftarrow{\partial}_i - im \Big) u_{s_1}(\mathbf{p}) e^{ipx} \\
& = i \int d^3p\; f_1(\mathbf{p}) \int d^4x\; \overline{\Psi}(x) (+i \overleftarrow{\not\partial} + m) u_{s_1}(\mathbf{p}) e^{ipx} \;.
\end{aligned}
\tag{41.10}
$$

The first equality is just the fundamental theorem of calculus.

To get the second, we substituted the definition of $b^{\dagger}_1(t)$, and combined the d^3x from this definition with the dt to get d^4x.

The third comes from straightforward evaluation of the time derivatives.

The fourth uses $(\not{p} + m)u_s(\mathbf{p}) = 0$.

The fifth writes ip_i as ∂_i acting on e^{ipx}.

The sixth uses integration by parts to move the ∂_i onto the field $\overline{\Psi}(x)$; here the wave packet is needed to avoid a surface term.

The seventh simply identifies $\gamma^0\partial_0 + \gamma^i\partial_i$ as .

In free-field theory, the right-hand side of eq. (41.10) is zero, since $\Psi(x)$ obeys the Dirac equation, which, after barring it, reads

$$\overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m) = 0\ . \tag{41.11}$$

In an interacting theory, however, the right-hand side of eq. (41.10) will not be zero.

We will also need the hermitian conjugate of eq. (41.10), which (after some slight rearranging) reads

$$\begin{aligned} b_1(+\infty) - b_1(-\infty) \\ = i\int d^3p\ f_1(\mathbf{p}) \int d^4x\ e^{-ipx}\ \overline{u}_{s_1}(\mathbf{p})(-i\not\partial + m)\Psi(x)\ , \end{aligned} \tag{41.12}$$

and the analogous formulae for the d operators,

$$d_1^\dagger(-\infty) - d_1^\dagger(+\infty) = -i\int d^3p\, f_1(\mathbf{p}) \int d^4x\, e^{ipx}\, \overline{v}_{s_1}(\mathbf{p})(-i\not{\partial} + m)\Psi(x)\ , \tag{41.13}$$

$$d_1(+\infty) - d_1(-\infty) = -i\int d^3p\, f_1(\mathbf{p}) \int d^4x\, \overline{\Psi}(x)(+i\overleftarrow{\not{\partial}} + m)v_{s_1}(\mathbf{p})e^{-ipx}\ . \tag{41.14}$$

Let us now return to the scattering amplitude we were considering,

$$\langle f|i\rangle = \langle 0|b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle\ . \tag{41.15}$$

Note that the operators are in time order.

Thus, if we feel like it, we can put in a time-ordering symbol without changing anything:

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\, b_{2'}(+\infty)b_{1'}(+\infty)b_1^\dagger(-\infty)b_2^\dagger(-\infty)|0\rangle\ . \tag{41.16}$$

The symbol T means the product of operators to its right is to be ordered, not as written, but with operators at later times to the left of those at earlier times.

However, there is an extra minus sign if this rearrangement involves an odd number of exchanges of these anticommuting operators.

Now let us use eqs. (41.10) and (41.12) in eq. (41.16).

The time-ordering symbol automatically moves all $b_{i'}$ $(-\infty)$'s to the right, where they annihilate $|0\rangle$.

Similarly, all $b^\dagger_i(+\infty)$'s move to the left, where they annihilate $\langle 0|$.

The wave packets no longer play a key role, and we can take the $\sigma \to 0$ limit in eq. (41.7), so that $f_1(\mathbf{p}) = \delta^3(\mathbf{p} - \mathbf{p}_1)$.

The initial and final states now have a delta-function normalization, the multiparticle generalization of eq. (41.5).

We are left with the *Lehmann-Symanzik-Zimmermann reduction formula* for spin-one-half particles,

$$
\begin{aligned}
\langle f|i\rangle = i^4 \int d^4x_1\, d^4x_2\, d^4x_{1'}\, d^4x_{2'} \\
\times e^{-ip'_1 x'_1}\, [\overline{u}_{s_{1'}}(\mathbf{p}_{1'})(-i\not{\partial}_{1'} + m)]_{\alpha_{1'}} \\
\times e^{-ip'_2 x'_2}\, [\overline{u}_{s_{2'}}(\mathbf{p}_{2'})(-i\not{\partial}_{2'} + m)]_{\alpha_{2'}} \\
\times \langle 0|\, \mathrm{T}\, \Psi_{\alpha_{2'}}(x_{2'}) \Psi_{\alpha_{1'}}(x_{1'}) \overline{\Psi}_{\alpha_1}(x_1) \overline{\Psi}_{\alpha_2}(x_2) |0\rangle \\
\times [(+i\overleftarrow{\not{\partial}}_1 + m) u_{s_1}(\mathbf{p}_1)]_{\alpha_1}\, e^{ip_1 x_1} \\
\times [(+i\overleftarrow{\not{\partial}}_2 + m) u_{s_2}(\mathbf{p}_2)]_{\alpha_2}\, e^{ip_2 x_2}\ .
\end{aligned}
\tag{41.17}
$$

The generalization of the LSZ formula to other processes should be clear; insert a time-ordering symbol, and make the following replacements:

$$b_s^\dagger(\mathbf{p})_{\rm in} \to +i \int d^4x\, \overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m)u_s(\mathbf{p})\, e^{+ipx} \ , \tag{41.18}$$

$$b_s(\mathbf{p})_{\rm out} \to +i \int d^4x\, e^{-ipx}\, \overline{u}_s(\mathbf{p})(-i\not\partial + m)\Psi(x) \ , \tag{41.19}$$

$$d_s^\dagger(\mathbf{p})_{\rm in} \to -i \int d^4x\, e^{+ipx}\, \overline{v}_s(\mathbf{p})(-i\not\partial + m)\Psi(x) \ , \tag{41.20}$$

$$d_s(\mathbf{p})_{\rm out} \to -i \int d^4x\, \overline{\Psi}(x)(+i\overleftarrow{\not\partial} + m)v_s(\mathbf{p})\, e^{-ipx} \ , \tag{41.21}$$

where we have used the subscripts "in" and "out" to denote $t \to -\infty$ and $t \to +\infty$, respectively.

All of this holds for a Majorana field as well.

In that case, $d_s(\mathbf{p}) = b_s(\mathbf{p})$, and we can use either eq. (41.18) or eq. (41.20) for the incoming particles, and either eq. (41.19) or eq. (41.21) for the outgoing particles, whichever is more convenient.

The Majorana condition $\overline{\Psi} = \Psi^{\rm T}C$ guarantees that the results will be equivalent.

As in the case of a scalar field, we cheated a little in our derivation of the LSZ formula, because we assumed that the creation operators of free field theory would work comparably in the interacting theory.

After performing an analysis that is entirely analogous to what we did for the scalar in section 5, we come to the same conclusion: the LSZ formula holds provided the field is properly normalized.

For a Dirac field, we must require

$$\langle 0|\Psi(x)|0\rangle = 0 \,, \tag{41.22}$$

$$\langle p,s,+|\Psi(x)|0\rangle = 0 \,, \tag{41.23}$$

$$\langle p,s,-|\Psi(x)|0\rangle = v_s(p)e^{-ipx} \,, \tag{41.24}$$

$$\langle p,s,+|\overline{\Psi}(x)|0\rangle = \overline{u}_s(p)e^{-ipx} \,, \tag{41.25}$$

$$\langle p,s,-|\overline{\Psi}(x)|0\rangle = 0 \,, \tag{41.26}$$

where $\langle 0|0\rangle = 1$, and the one-particle states are normalized according to eq. (41.5).

The zeros on the right-hand sides of eqs. (41.23) and (41.26) are required by charge conservation.

To see this, start with $[\Psi(x),Q] = +\Psi(x)$, take the matrix elements indicated, and use $Q|0\rangle = 0$ and $Q|p,s,\pm\rangle = \pm|p,s,\pm\rangle$.

The zero on the right-hand side of eq. (41.22) is required by Lorentz invariance.

To see this, start with $[\Psi(0),M^{\mu\nu}] = S^{\mu\nu}\Psi(0)$, and take the expectation value in the vacuum state $|0\rangle$.

If $|0\rangle$ is Lorentz invariant (as we will assume), then it is annihilated by the Lorentz generators $M^{\mu\nu}$, which means that we must have $S^{\mu\nu}\langle 0|\Psi(0)|0\rangle = 0$; this is possible for all μ and ν only if $\langle 0|\Psi(0)|0\rangle = 0$, which (by translation invariance) is possible only if $\langle 0|\Psi(x)|0\rangle = 0$.

The right-hand sides of eqs. (41.24) and (41.25) are similarly fixed by Lorentz invariance: only the overall scale might be different in an interacting theory.

However, the LSZ formula is correctly normalized if and only if eqs. (41.24) and (41.25) hold as written.

We will enforce this by rescaling (or, one might say, renormalizing) $\Psi(x)$ by an overall constant.

This is just a change of the name of the operator of interest, and does not affect the physics.

However, the rescaled $\Psi(x)$ will obey eqs. (41.24) and (41.25). (These two equations are related by charge conjugation, and so actually constitute only one condition on Ψ.)

For a Majorana field, there is no conserved charge, and we have

$$\langle 0|\Psi(x)|0\rangle = 0 \,, \tag{41.27}$$

$$\langle p,s|\Psi(x)|0\rangle = v_s(p)e^{-ipx} \,, \tag{41.28}$$

$$\langle p,s|\overline{\Psi}(x)|0\rangle = \overline{u}_s(p)e^{-ipx} \,, \tag{41.29}$$

instead of eqs. (41.22–41.26).

The renormalization of Ψ necessitates including appropriate Z factors in the lagrangian.

Consider, for example,

$$\mathcal{L} = iZ\overline{\Psi}\partial\!\!\!/\Psi - Z_m m\overline{\Psi}\Psi - \tfrac{1}{4}Z_g g(\overline{\Psi}\Psi)^2 \,, \tag{41.30}$$

where Ψ is a Dirac field, and g is a coupling constant.

We choose the three constants Z, Z_m, and Z_g so that the following three conditions are satisfied: m is the mass of a single particle; g is fixed by some appropriate scattering cross section; and eq. (41.24) and is obeyed.

[Eq. (41.25) then follows by charge conjugation.]

Next, we must develop the tools needed to compute the correlation functions $\langle 0|\mathrm{T}\Psi_{\alpha_{1'}}(x_{1'})\ldots\overline{\Psi}_{\alpha_1}(x_1)\ldots|0\rangle$ in an interacting quantum field theory.

42 The Free Fermion Propagator

Consider a free Dirac field

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]\,, \tag{42.1}$$

$$\overline{\Psi}(y) = \sum_{s'=\pm}\int \widetilde{dp}'\,\left[b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'y} + d_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'y}\right]\,, \tag{42.2}$$

where

$$b_s(\mathbf{p})|0\rangle = d_s(\mathbf{p})|0\rangle = 0\,, \tag{42.3}$$

and

$$\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'}\,, \tag{42.4}$$

$$\{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'}\,, \tag{42.5}$$

and all the other possible anticommutators between b and d operators (and their hermitian conjugates) vanish.

We wish to compute the Feynman propagator

$$S(x-y)_{\alpha\beta} \equiv i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle \ , \tag{42.6}$$

where T denotes the time-ordered product,

$$\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y) \equiv \theta(x^0-y^0)\Psi_\alpha(x)\overline{\Psi}_\beta(y) - \theta(y^0-x^0)\overline{\Psi}_\beta(y)\Psi_\alpha(x) \ , \tag{42.7}$$

and $\theta(t)$ is the unit step function.

Note the minus sign in the second term; this is needed because $\Psi_\alpha(x)\Psi_\beta(y) = -\Psi_\beta(y)\Psi_\alpha(x)$ when $x_0 \neq y_0$.

We can now compute $\langle 0|\, \Psi_\alpha(x)\Psi_\beta(y)\, |0\rangle$ and $\langle 0|\Psi_\beta(y)\Psi_\alpha(x)|0\rangle$ by inserting eqs. (42.1) and (42.2), and then using eqs. (42.3–42.5). We get

$$\begin{aligned}
&\langle 0|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle \\
&\quad = \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp}'\, e^{ipx}\, e^{-ip'y}\, u_s(\mathbf{p})_\alpha \overline{u}_{s'}(\mathbf{p}')_\beta\, \langle 0|b_s(\mathbf{p})b^\dagger_{s'}(\mathbf{p}')|0\rangle \\
&\quad = \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp}'\, e^{ipx}\, e^{-ip'y}\, u_s(\mathbf{p})_\alpha \overline{u}_{s'}(\mathbf{p}')_\beta\, (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega\delta_{ss'} \\
&\quad = \sum_{s} \int \widetilde{dp}\, e^{ip(x-y)}\, u_s(\mathbf{p})_\alpha \overline{u}_s(\mathbf{p})_\beta \\
&\quad = \int \widetilde{dp}\, e^{ip(x-y)}\, (-\not{p}+m)_{\alpha\beta} \ .
\end{aligned} \tag{42.8}$$

To get the last line, we used a result from section 38.

Similarly,

$$\begin{aligned}
&\langle 0|\overline{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle \\
&\quad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}'\, e^{-ipx}\, e^{ip'y}\, v_s(\mathbf{p})_\alpha \overline{v}_{s'}(\mathbf{p}')_\beta \,\langle 0|d_{s'}(\mathbf{p}')d_s^\dagger(\mathbf{p})|0\rangle \\
&\quad = \sum_{s,s'} \int \widetilde{dp}\,\widetilde{dp}'\, e^{-ipx}\, e^{ip'y}\, v_s(\mathbf{p})_\alpha \overline{v}_{s'}(\mathbf{p}')_\beta \,(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\delta_{ss'} \\
&\quad = \sum_{s} \int \widetilde{dp}\, e^{-ip(x-y)}\, v_s(\mathbf{p})_\alpha \overline{v}_s(\mathbf{p})_\beta \\
&\quad = \int \widetilde{dp}\, e^{-ip(x-y)}\, (-\not{p} - m)_{\alpha\beta}\ . \qquad (42.9)
\end{aligned}$$

We can combine eqs. (42.8) and (42.9) into a compact formula for the time- ordered product by means of the identity

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}f(p)}{p^2+m^2-i\epsilon} &= i\theta(x^0-y^0)\int \widetilde{dp}\, e^{ip(x-y)}\, f(p) \\
&\quad + i\theta(y^0-x^0)\int \widetilde{dp}\, e^{-ip(x-y)}\, f(-p)\ , \qquad (42.10)
\end{aligned}$$

where *f*(p) is a polynomial in p; the derivation of eq. (42.10) was sketched in section 8.

We get

$$\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}\ , \tag{42.11}$$

and so

$$S(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}\ . \tag{42.12}$$

Note that S(x– y) is a Green's function for the Dirac wave operator:

$$\begin{aligned}(-i\not{\partial}_x+m)_{\alpha\beta}S(x-y)_{\beta\gamma} &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(\not{p}+m)_{\alpha\beta}(-\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(p^2+m^2)\delta_{\alpha\gamma}}{p^2+m^2-i\epsilon} \\ &= \delta^4(x-y)\delta_{\alpha\gamma}\ . \end{aligned} \tag{42.13}$$

Similarly,

$$\begin{aligned}S(x-y)_{\alpha\beta}(+i\overleftarrow{\not{\partial}}_y+m)_{\beta\gamma} &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not{p}+m)_{\alpha\beta}(\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(p^2+m^2)\delta_{\alpha\gamma}}{p^2+m^2-i\epsilon} \\ &= \delta^4(x-y)\delta_{\alpha\gamma}\ . \end{aligned} \tag{42.14}$$

We can also consider $\langle 0|\mathrm{T}\Psi\alpha(\mathrm{x})\Psi\beta(\mathrm{y})|0\rangle$ and $\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle$, but it is easy to see that now there is no way to pair up a b with a $\mathrm{b}^\dagger$ or a d with a $\mathrm{d}^\dagger$, and so

$$\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)|0\rangle = 0 \,, \tag{42.15}$$

$$\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = 0 \,. \tag{42.16}$$

Next, consider a Majorana field

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right] \,, \tag{42.17}$$

$$\overline{\Psi}(y) = \sum_{s'=\pm} \int \widetilde{dp}' \left[b_{s'}^\dagger(\mathbf{p}') \overline{u}_{s'}(\mathbf{p}') e^{-ip'y} + b_{s'}(\mathbf{p}') \overline{v}_{s'}(\mathbf{p}') e^{ip'y} \right] \,. \tag{42.18}$$

It is easy to see that $\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle$ is the same as it is in the Dirac case; the only difference in the calculation is that we would have b and $\mathrm{b}^\dagger$ in place of d and $\mathrm{d}^\dagger$ in the second line of eq. (42.9), and this does not change the final result.

Thus,

$$i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle = S(x-y)_{\alpha\beta} \,, \tag{42.19}$$

where S(x− y) is given by eq. (42.12).

However, eqs. (42.15) and (42.16) no longer hold for a Majorana field.

Instead, the Majorana condition $\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$, which can be rewritten as $\Psi^{\mathrm{T}} = \overline{\Psi}\mathcal{C}^{-1}$, implies

$$
\begin{aligned}
i\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)|0\rangle &= i\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\gamma(y)|0\rangle(\mathcal{C}^{-1})_{\gamma\beta} \\
&= [S(x-y)\mathcal{C}^{-1}]_{\alpha\beta}\ .
\end{aligned}
\tag{42.20}
$$

Similarly, using $C^T = C^{-1}$, we can write the Majorana condition as $\overline{\Psi}^T = C^{-1}\Psi$, and so

$$
\begin{aligned}
i\langle 0|\mathrm{T}\overline{\Psi}_\alpha(x)\overline{\Psi}_\beta(y)|0\rangle &= i(\mathcal{C}^{-1})_{\alpha\gamma}\langle 0|\mathrm{T}\Psi_\gamma(x)\overline{\Psi}_\beta(y)|0\rangle \\
&= [\mathcal{C}^{-1}S(x-y)]_{\alpha\beta}\ .
\end{aligned}
\tag{42.21}
$$

Of course, $C^{-1} = -C$, but it will prove more convenient to leave eqs. (42.20) and (42.21) as they are.

We can also consider the vacuum expectation value of a time-ordered product of more than two fields.

In the Dirac case, we must have an equal number of Ψ's and $\overline{\Psi}$'s to get a nonzero result; and then, the Ψ's and $\overline{\Psi}$'s must pair up to form propagators.

There is an extra minus sign if the ordering of the fields in their pairs is an odd permutation of the original ordering.

For example,

$$
\begin{aligned}
i^2\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)\Psi_\gamma(z)\overline{\Psi}_\delta(w)|0\rangle = &+ S(x-y)_{\alpha\beta}\,S(z-w)_{\gamma\delta} \\
&- S(x-w)_{\alpha\delta}\,S(z-y)_{\gamma\beta}\ .
\end{aligned}
\tag{42.22}
$$

In the Majorana case, we may as well let all the fields be Ψ's (since we can always replace a $\overline{\Psi}$ with $\Psi^{\mathrm{T}}\mathrm{C}$).

Then we must pair them up in all possible ways.

There is an extra minus sign if the ordering of the fields in their pairs is an odd permutation of the original ordering.

For example,

$$\begin{aligned} i^2\langle 0|\mathrm{T}\Psi_\alpha(x)\Psi_\beta(y)\Psi_\gamma(z)\Psi_\delta(w)|0\rangle \;=\; &+[S(x-y)\mathcal{C}^{-1}]_{\alpha\beta}\,[S(z-w)\mathcal{C}^{-1}]_{\gamma\delta} \\ &-[S(x-z)\mathcal{C}^{-1}]_{\alpha\gamma}\,[S(y-w)\mathcal{C}^{-1}]_{\beta\delta} \\ &+[S(x-w)\mathcal{C}^{-1}]_{\alpha\delta}\,[S(y-z)\mathcal{C}^{-1}]_{\beta\gamma}\;. \end{aligned} \tag{42.23}$$

Note that the ordering within a pair does not matter, since

$$[S(x-y)\mathcal{C}^{-1}]_{\alpha\beta} = -[S(y-x)\mathcal{C}^{-1}]_{\beta\alpha}\;. \tag{42.24}$$

This follows from anticommutation of the fields and eq. (42.20).

43 The Path Integral for Fermion Fields

We would like to write down a path integral formula for the vacuum-expectation value of a time-ordered product of free Dirac or Majorana fields.

Recall that for a real scalar field with

$$\begin{aligned}\mathcal{L}_0 &= -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 \\ &= -\tfrac{1}{2}\varphi(-\partial^2 + m^2)\varphi - \tfrac{1}{2}\partial_\mu(\varphi\partial^\mu\varphi)\ ,\end{aligned} \tag{43.1}$$

we have

$$\langle 0|\mathrm{T}\varphi(x_1)\dots|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta J(x_1)}\dots Z_0(J)\Big|_{J=0}\ , \tag{43.2}$$

where

$$Z_0(J) = \int \mathcal{D}\varphi\ \exp\left[i\int d^4x\,(\mathcal{L}_0 + J\varphi)\right]. \tag{43.3}$$

In this formula, we use the epsilon trick (see section 6) of replacing m^2 with $m^2 - i\epsilon$ to construct the vacuum as the initial and final state.

Then we get

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\,d^4y\,J(x)\Delta(x-y)J(y)\right], \tag{43.4}$$

where the Feynman propagator

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \tag{43.5}$$

is the inverse of the Klein-Gordon wave operator:

$$(-\partial_x^2 + m^2)\Delta(x-y) = \delta^4(x-y)\ . \tag{43.6}$$

For a complex scalar field with

$$\begin{aligned}\mathcal{L}_0 &= -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi \\ &= -\varphi^\dagger(-\partial^2 + m^2)\varphi - \partial_\mu(\varphi^\dagger\partial^\mu\varphi)\ ,\end{aligned} \tag{43.7}$$

we have instead

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J^\dagger(x_1)}\ldots\frac{1}{i}\frac{\delta}{\delta J(y_1)}\ldots Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0}\ , \tag{43.8}$$

where

$$\begin{aligned}Z_0(J^\dagger,J) &= \int\mathcal{D}\varphi^\dagger\mathcal{D}\varphi\ \exp\left[i\int d^4x\,(\mathcal{L}_0 + J^\dagger\varphi + \varphi^\dagger J)\right] \\ &= \exp\left[i\int d^4x\,d^4y\,J^\dagger(x)\Delta(x-y)J(y)\right]\ .\end{aligned} \tag{43.9}$$

We treat J and J†as independent variables when evaluating eq. (43.8).

In the case of a fermion field, we should have something similar, except that we need to account for the extra minus signs from anticommutation.

For this to work out, a functional derivative with respect to an anticommuting variable must itself be treated as anticommuting.

Thus if we define an anticommuting source η(x) for a Dirac field, we can write

$$\frac{\delta}{\delta\eta(x)}\int d^4y\left[\overline{\eta}(y)\Psi(y)+\overline{\Psi}(y)\eta(y)\right]=-\overline{\Psi}(x)\,, \tag{43.10}$$

$$\frac{\delta}{\delta\overline{\eta}(x)}\int d^4y\left[\overline{\eta}(y)\Psi(y)+\overline{\Psi}(y)\eta(y)\right]=+\Psi(x)\,. \tag{43.11}$$

The minus sign in eq. (43.10) arises because the $\delta/\delta\eta$ must pass through $\overline{\Psi}$ before reaching η.

Thus, consider a free Dirac field with

$$\begin{aligned}\mathcal{L}_0 &= i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi \\ &= -\overline{\Psi}(-i\partial\!\!\!/ + m)\Psi\,.\end{aligned} \tag{43.12}$$

A natural guess for the appropriate path-integral formula, based on analogy with eq. (43.9), is

$$\begin{aligned}&\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\ldots\overline{\Psi}_{\beta_1}(y_1)\ldots|0\rangle \\ &\qquad=\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}\;\ldots\;i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\;\ldots\;Z_0(\overline{\eta},\eta)\Big|_{\eta=\overline{\eta}=0}\,,\end{aligned} \tag{43.13}$$

where

$$\begin{aligned}Z_0(\overline{\eta},\eta) &= \int\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;\exp\left[i\int d^4x\,(\mathcal{L}_0+\overline{\eta}\Psi+\overline{\Psi}\eta)\right] \\ &= \exp\left[i\int d^4x\,d^4y\;\overline{\eta}(x)S(x-y)\eta(y)\right],\end{aligned} \tag{43.14}$$

and the Feynman propagator

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \, \frac{(-\not{p} + m)e^{ip(x-y)}}{p^2 + m^2 - i\epsilon} \tag{43.15}$$

is the inverse of the Dirac wave operator:

$$(-i\not{\partial}_x + m)S(x-y) = \delta^4(x-y) \; . \tag{43.16}$$

Note that each δ/δη in eq. (43.13) comes with a factor of i rather than the usual 1/i; this reflects the extra minus sign of eq. (43.10).

We treat η and $\overline{\eta}$ as independent variables when evaluating eq. (43.13).

It is straightforward to check (by working out a few examples) that eqs. (43.13–43.16) do indeed reproduce the result of section 42 for the vacuum expectation value of a time-ordered product of Dirac fields.

This is really all we need to know.

Recall that, for a complex scalar field with interactions specified by $\mathcal{L}_1(\varphi^\dagger,\varphi)$, we have

$$Z(J^\dagger, J) \;\propto\; \exp\left[i \int d^4x \, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}, \frac{1}{i}\frac{\delta}{\delta J^\dagger(x)} \right) \right] Z_0(J^\dagger, J) \; , \tag{43.17}$$

where the overall normalization is fixed by Z(0,0) = 1.

Thus, for a Dirac field with interactions specified by $\mathcal{L}_1(\overline{\Psi}, \Psi)$, we have

$$Z(\overline{\eta}, \eta) \;\propto\; \exp\left[i \int d^4x \; \mathcal{L}_1\left(i\frac{\delta}{\delta\eta(x)}, \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)} \right) \right] Z_0(\overline{\eta}, \eta) \;, \tag{43.18}$$

where again the overall normalization is fixed by Z(0,0) = 1.

Vacuum expectation values of time-ordered products of Dirac fields in an interacting theory will now be given by eq. (43.13), but with Z_0 $(\overline{\eta}, \eta)$ replaced by Z $(\overline{\eta}, \eta)$.

Then, just as for a scalar field, this will lead to a Feynman-diagram expansion for Z $(\overline{\eta}, \eta)$.

There are two extra complications: we must keep track of the spinor indices, and we must keep track of the extra minus signs from anticommutation.

Both tasks are straightforward; we will take them up in section 45.

Next, let us consider a Majorana field with

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\partial\!\!\!/\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi \\ &= -\tfrac{1}{2}\Psi^{\mathrm{T}}\mathcal{C}(-i\partial\!\!\!/ + m)\Psi \;. \end{aligned} \tag{43.19}$$

A natural guess for the appropriate path-integral formula, based on analogy with eq. (43.2), is

$$\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta\eta_{\alpha_1}(x_1)} \;\ldots\; Z_0(\eta)\Big|_{\eta=0} \;, \tag{43.20}$$

where

$$Z_0(\eta) = \int \mathcal{D}\Psi \, \exp\left[i \int d^4x \, (\mathcal{L}_0 + \eta^{\mathrm{T}} \Psi) \right]$$
$$= \exp\left[-\frac{i}{2} \int d^4x \, d^4y \, \eta^{\mathrm{T}}(x) S(x-y) \mathcal{C}^{-1} \eta(y) \right] . \tag{43.21}$$

The Feynman propagator $S(x-y)C^{-1}$ is the inverse of the Majorana wave operator $\mathcal{C}(-i\not{\partial} + m)$:

$$\mathcal{C}(-i\not{\partial}_x + m) S(x-y) \mathcal{C}^{-1} = \delta^4(x-y) \, . \tag{43.22}$$

The extra minus sign in eq. (43.21), as compared with eq. (43.14), arises because all functional derivatives in eq. (43.20) are accompanied by 1/i, rather than half by 1/i and half by i, as in eq. (43.13).

It is now straightforward to check (by working out a few examples) that eqs. (43.20–43.22) do indeed reproduce the result of section 42 for the vacuum expectation value of a time-ordered product of Majorana fields.

44 Formal Development of Fermionic Path Integrals

In Section 43, we formally defined the fermionic path integral for a free Dirac field Ψ via

$$Z_0(\overline{\eta}, \eta) = \int \mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \, \exp\left[i \int d^4x \, \overline{\Psi}(i\not{\partial} - m)\Psi + \overline{\eta}\Psi + \overline{\Psi}\eta \right]$$
$$= \exp\left[i \int d^4x \, d^4y \, \overline{\eta}(x) S(x-y) \eta(y) \right] , \tag{44.1}$$

where the Feynman propagator S(x− y) is the inverse of the Dirac wave operator:

$$(-i\partial\!\!\!/_x + m)S(x-y) = \delta^4(x-y)\ . \tag{44.2}$$

We would like to find a mathematical framework that allows us to derive this formula, rather than postulating it by analogy.

Consider a set of *anticommuting numbers* or *Grassmann variables* ψ_i that obey

$$\{\psi_i, \psi_j\} = 0\ , \tag{44.3}$$

where i = 1,...,n.

Let us begin with the very simplest case of n = 1, and thus a single anticommuting number ψ that obeys $\psi^2 = 0$.

We can define a function $f(\psi)$ of such an object via a Taylor expansion; because $\psi^2 = 0$, this expansion ends with the second term:

$$f(\psi) = a + \psi b\ . \tag{44.4}$$

The reason for writing the coefficient b to the right of the variable ψ will become clear in a moment.

Next we would like to define the derivative of $f(\psi)$ with respect to ψ.

Before we can do so, we must decide if $f(\psi)$ itself is to be commuting or anticommuting; generally we will be interested in functions that are themselves commuting.

In this case, a in eq. (44.4) should be treated as an ordinary commuting number, but b should be treated as an anticommuting number: $\{b, b\} = \{b, \psi\} = 0$.

In this case, $f(\psi) = a + \psi b = a - b\psi$.

Now we can define two kinds of derivatives.

The *left derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far left:

$$\partial_\psi f(\psi) = +b \, . \tag{44.5}$$

Similarly, the *right derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far right:

$$f(\psi)\overleftarrow{\partial}_\psi = -b \, . \tag{44.6}$$

Generally, when we write a derivative with respect to a Grassmann variable, we mean the left derivative.

However, in section 37, when we wrotethe canonical momentum for a fermionic field ψ as $\pi = \partial\mathcal{L}/\partial(\partial_0\psi)$, we actually meant the right derivative. (This is a standard, though rarely stated, convention.)

Correspondingly, we wrote the hamiltonian density as $\mathcal{H} = \pi\partial_0\psi - \mathcal{L}$, with $\partial\partial_0\psi$ to the right of π.

Finally, we would like to define a definite integral, analogous to integrating a real variable x from minus to plus infinity.

The key features of such an integral over x (when it converges) are linearity,

$$\int_{-\infty}^{+\infty} dx\, cf(x) = c\int_{-\infty}^{+\infty} dx\, f(x)\ , \tag{44.7}$$

and invariance under shifts of the dependent variable x by a constant:

$$\int_{-\infty}^{+\infty} dx\, f(x+a) = \int_{-\infty}^{+\infty} dx\, f(x)\ . \tag{44.8}$$

Up to an overall numerical factor that is the same for every $f(\psi)$, the only possible nontrivial definition of $\int d\psi\, f(\psi)$ that is both linear and shift invariant is

$$\int d\psi\, f(\psi) = b\ . \tag{44.9}$$

Now let us generalize this to n >1.

We have

$$f(\psi) = a + \psi_i b_i + \tfrac{1}{2}\psi_{i_1}\psi_{i_2}c_{i_1 i_2} + \ldots + \tfrac{1}{n!}\psi_{i_1}\ldots\psi_{i_n}d_{i_1\ldots i_n}\ , \tag{44.10}$$

where the indices are implicitly summed.

Here we have written the coefficients to the right of the variables to facilitate left-differentiation.

These coefficients are completely antisymmetric on exchange of any two indices.

The left derivative of $f(\psi)$ with respect to ψ_j is

$$\frac{\partial}{\partial\psi_j} f(\psi) = b_j + \psi_i c_{ji} + \ldots + \frac{1}{(n-1)!}\psi_{i_2}\ldots\psi_{i_n} d_{ji_2\ldots i_n} \ . \tag{44.11}$$

Next we would like to find a linear, shift-invariant definition of the integral of $f(\psi)$.

Note that the antisymmetry of the coefficients implies that

$$d_{i_1\ldots i_n} = d\,\varepsilon_{i_1\ldots i_n} \ . \tag{44.12}$$

where d is a just a number (ordinary if f is commuting and n is even, Grassmann if f is commuting and n is odd, etc.), and $\varepsilon_{i_1\ldots i_n}$ is the completely antisymmetric Levi-Civita symbol with $\varepsilon_{1\ldots n} = +1$.

This number d is a candidate (in fact, up to an overall numerical factor, the only candidate!) for the integral of $f(\psi)$

$$\int d^n\psi\, f(\psi) = d \ . \tag{44.13}$$

Although eq. (44.13) really tells us everything we need to know about $d^n\psi$, we can, if we like, write $d^n\psi = d\psi_n\ldots d\psi_1$ (note the backwards ordering), and treat the individual differentials as anticommuting: $\{d\psi_i, d\psi_j\} = 0$, $\{d\psi_i, \psi_j\} = 0$.

Then we take $\int d\psi_i = 0$ and $\int d\psi_i\psi_j = \delta_{ij}$ as our basic formulae, and use them to derive eq. (44.13).

Let us work out some consequences of eq. (44.13).

Consider what happens if we make a linear change of variable,

$$\psi_i = J_{ij}\psi'_j \ , \tag{44.14}$$

where J_{ji} is a matrix of commuting numbers (and therefore can be written on either the left or right of ψ'_j).

We now have

$$f(\psi) = a + \ldots + \tfrac{1}{n!}(J_{i_1 j_1}\psi'_{j_1}) \ldots (J_{i_n j_n}\psi'_{j_n})\varepsilon_{i_1 \ldots i_n} d \ . \tag{44.15}$$

Next we use

$$\varepsilon_{i_1 \ldots i_n} J_{i_1 j_1} \ldots J_{i_n j_n} = (\det J)\varepsilon_{j_1 \ldots j_n} \ , \tag{44.16}$$

which holds for any n × n matrix J, to get

$$f(\psi) = a + \ldots + \tfrac{1}{n!}\psi'_{i_1} \ldots \psi'_{i_n}\varepsilon_{i_1 \ldots i_n}(\det J) d \tag{44.17}$$

If we now integrate $f(\psi)$ over $d^n\psi'$, eq. (44.13) tells us that the result is (det J)d.

Thus

$$\int d^n\psi \, f(\psi) = (\det J)^{-1} \int d^n\psi' \, f(\psi) \ . \tag{44.18}$$

Recall that, for integrals over commuting real numbers x_i with $x_i = J_{ij}\, x'_j$, we have instead

$$\int d^n x\, f(x) = (\det J)^{+1} \int d^n x'\, f(x)\ . \tag{44.19}$$

Note the opposite sign on the power of the determinant.

Now consider a quadratic form $\psi^T M\psi = \psi_i M_{ij} \psi_j$, where M is an anti-symmetric matrix of commuting numbers (possibly complex).

Let's evaluate the gaussian integral $\int d^n\psi \exp(1/2\ \psi^T M\psi)$.

For example, for n = 2, we have

$$M = \begin{pmatrix} 0 & +m \\ -m & 0 \end{pmatrix}, \tag{44.20}$$

and $\psi^T M\psi = 2m\psi_1\psi_2$.

Thus $\exp(\ 1/2\ \psi^T M\psi) = 1 + m\psi_1\psi_2$, and so

$$\int d^n\psi\ \exp(\tfrac{1}{2}\psi^T M\psi) = m\ . \tag{44.21}$$

For larger n, we use the fact that a complex antisymmetric matrix can be brought to a block-diagonal form via

$$U^{\mathrm{T}} M U = \begin{pmatrix} 0 & +m_1 & & \\ -m_1 & 0 & & \\ & & & \ddots \end{pmatrix} , \tag{44.22}$$

where U is a unitary matrix, and each m_I is real and positive. (If n is odd there is a final row and column of all zeroes; from here on, we assume n is even.)

We can now let $\psi_i = U_{ij}\psi'_j$; then, we have

$$\int d^n\psi \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi) = (\det U)^{-1} \prod_{I=1}^{n/2} \int d^2\psi_I \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M_I \psi) \, , \tag{44.23}$$

where M_I represents one of the 2×2 blocks in eq. (44.22).

Each of these two-dimensional integrals can be evaluated using eq. (44.21), and so

$$\int d^n\psi \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi) = (\det U)^{-1} \prod_{I=1}^{n/2} m_I \, . \tag{44.24}$$

Taking the determinant of eq. (44.22), we get

$$(\det U)^2 (\det M) = \prod_{I=1}^{n/2} m_I^2 \, . \tag{44.25}$$

We can therefore rewrite the right-hand side of eq. (44.24) as

$$\int d^n\psi \, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi) = (\det M)^{1/2} \, . \tag{44.26}$$

In this form, there is a sign ambiguity associated with the square root; it is resolved by eq. (44.24).

However, the overall sign (more generally, any overall numerical factor) will never be of concern to us, so we can use eq. (44.26) without worrying about the correct branch of the square root.

It is instructive to compare eq. (44.26) with the corresponding gaussian integral for commuting real numbers,

$$\int d^n x \, \exp(-\tfrac{1}{2} x^{\mathrm{T}} M x) = (2\pi)^{n/2} (\det M)^{-1/2} \, . \tag{44.27}$$

Here M is a complex symmetric matrix.

Again, note the opposite sign on the power of the determinant.

Now let us introduce the notion of *complex* Grassmann variables via

$$\begin{aligned} \chi &\equiv \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \, , \\ \bar{\chi} &\equiv \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \, . \end{aligned} \tag{44.28}$$

We can invert this to get

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix} . \tag{44.29}$$

The determinant of this transformation matrix is –i, and so

$$d^2\psi = d\psi_2 d\psi_1 = (-i)^{-1} d\chi \, d\bar{\chi} \ . \tag{44.30}$$

Also, $\psi_1\psi_2 = -i\bar{\chi}\chi$.

Thus we have

$$\int d\chi \, d\bar{\chi} \, \bar{\chi}\chi = (-i)(-i)^{-1} \int d\psi_2 d\psi_1 \, \psi_1\psi_2 = 1 \ . \tag{44.31}$$

Thus, if we have a function

$$f(\chi, \bar{\chi}) = a + \chi b + \bar{\chi} c + \bar{\chi}\chi d \ , \tag{44.32}$$

its integral is

$$\int d\chi \, d\bar{\chi} \, f(\chi, \bar{\chi}) = d \ . \tag{44.33}$$

In particular,

$$\int d\chi \, d\bar{\chi} \, \exp(m\bar{\chi}\chi) = m \ . \tag{44.34}$$

Let us now consider n complex Grassmann variables χ_i and their complex conjugates, $\bar{\chi}_i$. .

We define

$$d^n\chi\, d^n\bar{\chi} \equiv d\chi_n d\bar{\chi}_n \dots d\chi_1 d\bar{\chi}_1 \ . \tag{44.35}$$

Then under a change of variable, $\chi_i = J_{ij}\chi'_j$ and $\bar{\chi}_i = K_{ij}\bar{\chi}'_j$, we have

$$d^n\chi\, d^n\bar{\chi} = (\det J)^{-1}(\det K)^{-1}\, d^n\chi'\, d^n\bar{\chi}' \ . \tag{44.36}$$

Note that we need not require $K_{ij} = J^*_{ij}$, because, as far as the integral is concerned, it is does not matter whether or not $\bar{\chi}_i$ is the complex conjugate of χ_i.

We now have enough information to evaluate $d^n\chi d^n\bar{\chi}\, \exp(\chi^\dagger M\chi)$, where M is a general complex matrix.

We make the change of variable $\chi = U\chi'$ and $\chi^\dagger = \chi'^\dagger V$, where U and V are unitary matrices with the property that VMU is diagonal with positive real entries m_i.

Then we get

$$\begin{aligned}\int d^n\chi\, d^n\bar{\chi}\, \exp(\chi^\dagger M\chi) &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n}\int d\chi_i d\bar{\chi}_i\, \exp(m_i\bar{\chi}_i\chi_i) \\ &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n} m_i \\ &= \det M \ .\end{aligned} \tag{44.37}$$

This can be compared to the analogous integral for commuting complex variables $z_i = (x_i + iy_i)/\sqrt{2}$ and $\bar{z} = (x_i - iy_i)/\sqrt{2}$, with $d^n z\, d^n \bar{z} = d^n x d^n y$, namely

$$\int d^n z\, d^n \bar{z}\, \exp(-z^\dagger M z) = (2\pi)^n (\det M)^{-1} \ . \tag{44.38}$$

We can now generalize eqs. (44.26) and (44.37) by shifting the integration variables, and using shift invariance of the integrals.

Thus, by making the replacement $\psi \rightarrow \psi - M^{-1}\eta$ in eq. (44.26), we get

$$\int d^n \psi\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}} M \psi + \eta^{\mathrm{T}} \psi) = (\det M)^{1/2} \exp(\tfrac{1}{2}\eta^{\mathrm{T}} M^{-1} \eta) \ . \tag{44.39}$$

(In verifying this, remember that M and its inverse are both antisymmetric.)

Similarly, by making the replacements $\chi \rightarrow \chi - M^{-1}\eta$ and $\chi^\dagger \rightarrow \chi^\dagger - \eta^\dagger M^{-1}$ in eq. (44.37), we get

$$\int d^n \chi\, d^n \bar{\chi}\, \exp(\chi^\dagger M \chi + \eta^\dagger \chi + \chi^\dagger \eta) = (\det M) \exp(-\eta^\dagger M^{-1} \eta) \ . \tag{44.40}$$

We can now see that eq. (44.1) is simply a particular case of eq. (44.40), with the index on the complex Grassmann variable generalized to include both the ordinary spin index α and the continuous spacetime argument x of the field $\Psi_\alpha(x)$.

Similarly, eq. (43.21) for the path integral for a free Majorana field is simply a particular case of eq. (44.39).

In both cases, the determinant factors are constants (that is, independent of the fields and sources) that we simply absorb into the overall normalization of the path integral.

We will meet determinants that cannot be so neatly absorbed in sections 53 and 71.

45 The Feynman Rules for Dirac Fields

In this section we will derive the Feynman rules for Yukawa theory, a theory with a Dirac field Ψ (with mass m) and a real scalar field ϕ (with mass M), interacting via

$$\mathcal{L}_1 = g\varphi\overline{\Psi}\Psi \ , \tag{45.1}$$

where g is a coupling constant.

In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

In four spacetime dimensions, φ has mass dimension $[\varphi] = 1$ and Ψ has mass dimension $[\Psi] = 3/2$; thus the coupling constant g is dimensionless: $[g] = 0$.

As discussed in section 12, this is generally the most interesting situation.

Note that $\mathcal{L}_1$ is invariant under the U(1) transformation $\Psi \rightarrow e^{-i\alpha}\Psi$, as is the free Dirac lagrangian.

Thus, the corresponding Noether current $\Psi\gamma^\mu\Psi$ is still conserved, and the associated charge Q (which counts the number of b-type particles minus the number of d-type particles) is constant in time.

We can think of Q as electric charge, and identify the b-type particle as the electron e^-, and the d-type particle as the positron e^+.

The scalar particle is electrically neutral (and could, for example, be thought of as the Higgs boson; see section 88).

We now use the general result of sections 9 and 43 to write

$$Z(\overline{\eta},\eta,J) \propto \exp\left[ig\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta \eta_\alpha(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta \overline{\eta}_\alpha(x)}\right)\right] Z_0(\overline{\eta},\eta,J)\ , \tag{45.2}$$

where

$$Z_0(\overline{\eta},\eta,J) = \exp\left[i\int d^4x\, d^4y\ \overline{\eta}(x)S(x-y)\eta(y)\right]$$

$$\times \exp\left[\frac{i}{2}\int d^4x\, d^4y\ J(x)\Delta(x-y)J(y)\right], \tag{45.3}$$

and

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\ \frac{(-\not{p}+m)e^{ip(x-y)}}{p^2+m^2-i\epsilon}\ , \tag{45.4}$$

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\ \frac{e^{ik(x-y)}}{k^2+M^2-i\epsilon} \tag{45.5}$$

are the appropriate Feynman propagators for the corresponding free fields.

We impose the normalization Z(0,0,0) = 1, and write

$$Z(\overline{\eta}, \eta, J) = \exp[iW(\overline{\eta}, \eta, J)] \ . \tag{45.6}$$

Then iW$(\overline{\eta}, \eta, J)$ can be expressed as a series of connected Feynman diagrams with sources.

We use a dashed line to stand for the scalar propagator 1/iΔ(x− y), and a solid line to stand for the fermion propagator 1/iS(x− y).

The only allowed vertex joins two solid lines and one dashed line; the associated vertex factor is ig.

The blob at the end of a dashed line stands for the φ source $\int$id^{4}xJ(x), and the blob at the end of a solid line for either the Ψ source $\int$ id^{4}x$\overline{\eta}(x)$, or the Ψ source $\int$ id^{4}xη(x).

To tell which is which, we adopt the "arrow rule" of problem 9.3: the blob stands for $\int$id^{4}xη(x) if the arrow on the attached line points away from the blob, and the blob stands for $\int$ id^{4}x $\overline{\eta}(x)$ if the arrow on the attached line points towards the blob.

Because L1 involves one Ψ and one Ψ, we also have the rule that, at each vertex, one arrow must point towards the vertex, and one away.

The first few tree diagrams that contribute to iW$(\overline{\eta}, \eta, J)$ are shown in fig. (45.1).

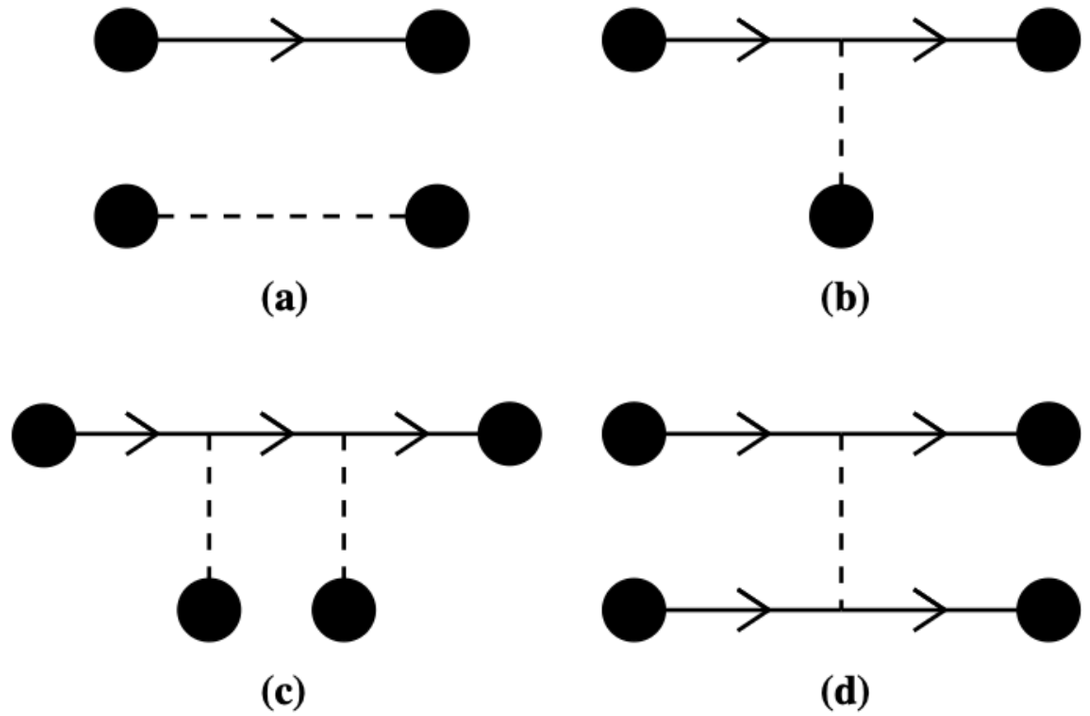


Figure 45.1: Tree contributions to $\mathrm{iW}(\overline{\eta}, \eta, J)$ with four or fewer sources.

We omit tadpole diagrams; as in φ^3 theory, these can be cancelled by shifting the φ field, or, equivalently, adding a term linear in φ to $\mathcal{L}$.

The LSZ formula is valid only after all tadpole diagrams have been cancelled in this way.

The spin indices on the fermionic sources and propagators are all contracted in the obvious way.

For example, the complete expression corresponding to fig. (45.1)(b) is

$$\begin{aligned} \text{Fig. (45.1)(b)} = i^3\left(\tfrac{1}{i}\right)^3 (ig) \int d^4x\, d^4y\, d^4z\, d^4w \\ \times \left[\overline{\eta}(x) S(x-y) S(y-z) \eta(z)\right] \\ \times \Delta(y-w) J(w) \ . \end{aligned} \tag{45.7}$$

Our main purpose in this section is to compute the tree-level amplitudes for various two-body elastic scattering processes, such as $e^-\varphi \to e^-\varphi$ and $e^+e^- \to \varphi\varphi$; for these, we will need to evaluate the tree-level contributions to connected correlation functions of the form $\langle 0|\mathrm{T}\Psi\overline{\Psi}\varphi\varphi|0\rangle_{\mathrm{C}}$.

Other processes of interest include $e^-e^- \to e^-e^-$ and $e^+e^- \to e^+e^-$; for these, we will need to evaluate the tree-level contributions to connected correlation functions of the form $\langle 0|\mathrm{T}\Psi\overline{\Psi}\Psi\overline{\Psi}|0\rangle_{\mathrm{C}}$.

For $\langle 0|\mathrm{T}\Psi\overline{\Psi}\varphi\varphi|0\rangle_{\mathrm{C}}$, the relevant tree-level contribution to $\mathrm{iW}^{\cdot}(\overline{\eta},\eta,J)$ is given by fig. (45.1)(c).

We have

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_\alpha(x)\overline{\Psi}_\beta(y)\varphi(z_1)\varphi(z_2)|0\rangle_{\mathrm{C}} \\
&\quad = \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\, i\frac{\delta}{\delta\eta_\beta(y)}\,\frac{1}{i}\frac{\delta}{\delta J(z_1)}\,\frac{1}{i}\frac{\delta}{\delta J(z_2)} iW(\overline{\eta},\eta,J)\Big|_{\overline{\eta}=\eta=J=0} \\
&\quad = \left(\tfrac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \\
&\qquad\qquad \times [S(x-w_1)S(w_1-w_2)S(w_2-y)]_{\alpha\beta} \\
&\qquad\qquad \times \Delta(z_1-w_1)\Delta(z_2-w_2) \\
&\quad + \Big(z_1 \leftrightarrow z_2\Big) + O(g^4)\ .
\end{aligned} \tag{45.8}$$

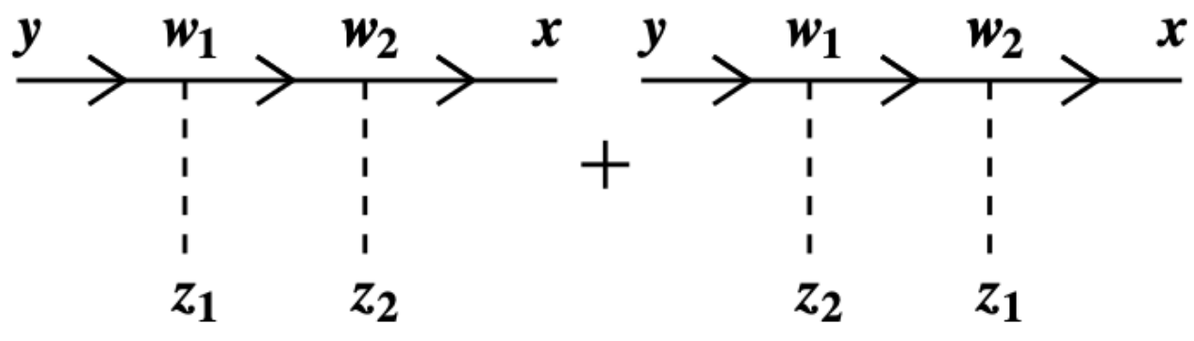


Figure 45.2: Diagrams corresponding to eq. (45.8).

For $\langle 0|\mathrm{T}\Psi\overline{\Psi}\Psi\overline{\Psi}|0\rangle_{\mathrm{C}}$, the relevant tree-level contribution to $\mathrm{iW}(\overline{\eta},\eta,J)$ is given by fig. (45.1)(d), which has a symmetry factor S = 2.

We have

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\overline{\Psi}_{\beta_1}(y_1)\Psi_{\alpha_2}(x_2)\overline{\Psi}_{\beta_2}(y_2)|0\rangle_{\mathrm{C}} \\
&\quad = \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_1}(x_1)}\, i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\,\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_{\alpha_2}(x_2)}\, i\frac{\delta}{\delta\eta_{\beta_2}(y_2)} iW(\overline{\eta},\eta,J)\Big|_{\overline{\eta}=\eta=J=0} \,. \qquad (45.9)
\end{aligned}$$

The two ηderivatives can act on the two η's in the diagram in two different ways; ditto for the two $\overline{\eta}$ derivatives.

This results in four different terms, but two of them are algebraic duplicates of the other two; this duplication cancels the symmetry factor (which is a general result for tree diagrams).

We get

$$\begin{aligned}
&\langle 0|\mathrm{T}\Psi_{\alpha_1}(x_1)\overline{\Psi}_{\beta_1}(y_1)\Psi_{\alpha_2}(x_2)\overline{\Psi}_{\beta_2}(y_2)|0\rangle_{\mathrm{C}} \\
&\quad = \left(\tfrac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \\
&\qquad\qquad \times [S(x_1-w_1)S(w_1-y_1)]_{\alpha_1\beta_1} \\
&\qquad\qquad \times \Delta(w_1-w_2) \\
&\qquad\qquad \times [S(x_2-w_2)S(w_2-y_2)]_{\alpha_2\beta_2} \\
&\qquad - \Big((y_1,\beta_1)\leftrightarrow(y_2,\beta_2)\Big) + O(g^4)\,. \qquad (45.10)
\end{aligned}$$

The corresponding diagrams, with sources removed, are shown in fig. (45.3).

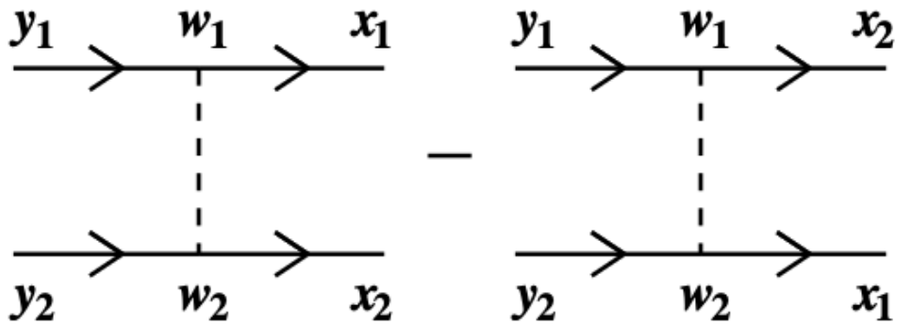


Figure 45.3: Diagrams corresponding to eq. (45.10).

Note that we now have a relative minus sign between the two diagrams, due to the anticommutation of the derivatives with respect to $\overline{\eta}$.

In general, the overall sign for a diagram can be determined by the following procedure.

First, draw each diagram with all the fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom).

Next, in each diagram, note the ordering (from top to bottom) of the labels on theright endpoints of the fermion lines.

If this ordering is an even permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive, and if it is an odd permutation, the sign is negative.

(This rule arises because endpoints with arrows pointing away from the vertex come from derivatives with respect to $\overline{\eta}$ that anticommute.

Of course, we could equally well put the right endpoints in a fixed order, and get the sign from the permutation of the left endpoints, which come from derivatives with
respect to η that anticommute.)

Also, in loop diagrams, a closed fermion loop yields an extra minus sign; we will discuss this rule in section 51.

Let us now consider a particular scattering process: $e^- \varphi \to e^- \varphi$.

The scattering amplitude is

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\, a(\mathbf{k}')_{\mathrm{out}} b_{s'}(\mathbf{p}')_{\mathrm{out}} b_s^\dagger(\mathbf{p})_{\mathrm{in}} a^\dagger(\mathbf{k})_{\mathrm{in}}\,|0\rangle\ . \tag{45.11}$$

Next we make the replacements

$$b_s^\dagger(\mathbf{p})_{\mathrm{in}} \to i\int d^4y\ \overline{\Psi}(y)(+i\overleftarrow{\not\partial} + m)u_s(\mathbf{p})\, e^{+ipy}\ , \tag{45.12}$$

$$b_{s'}(\mathbf{p}')_{\mathrm{out}} \to i\int d^4x\ e^{-ip'x}\ \overline{u}_{s'}(\mathbf{p}')(-i\not\partial + m)\Psi(x)\ , \tag{45.13}$$

$$a^\dagger(\mathbf{k})_{\mathrm{in}} \to i\int d^4z_1\ e^{+ikz_1}(-\partial^2 + m^2)\varphi(z_1)\ , \tag{45.14}$$

$$a(\mathbf{k}')_{\mathrm{out}} \to i\int d^4z_2\ e^{-ik'z_2}(-\partial^2 + m^2)\varphi(z_2) \tag{45.15}$$

in eq. (45.11), and then use eq. (45.8).

The wave operators (either Klein-Gordon or Dirac) act on the external propagators, and convert them to delta functions.

After using eqs. (45.4) and (45.5) for the internal propagators, all dependence on the various spacetime coordinates is in the form of plane-wave factors, as in section 10.

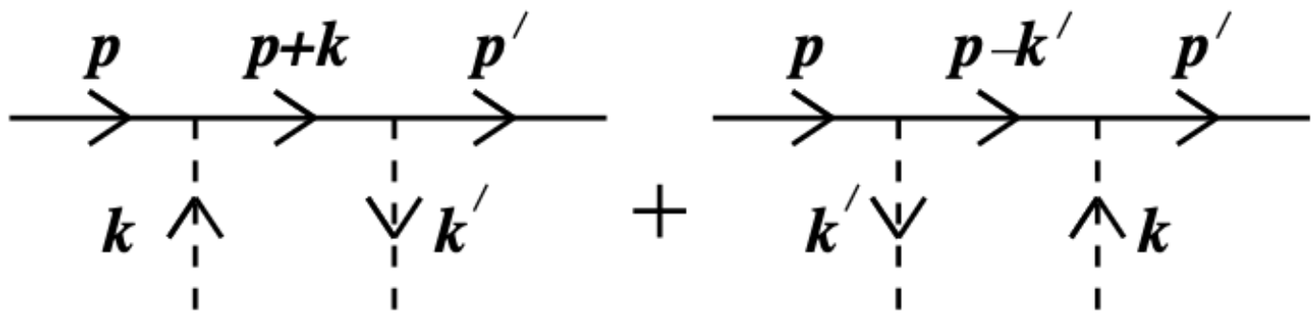
Integrating over the internal coordinates then generates delta functions that conserve four-momentum at each vertex.

The only new feature arises from the spinor factors $u_s(\mathbf{p})$ and $\overline{u}_{s'}(\mathbf{p}')$.

We find that $u_s(\mathbf{p})$ is associated with the external fermion line whose arrow points towards the vertex, and that $\overline{u}_{s'}(\mathbf{p}')$ is associated with the external fermion line whose arrow points away from the vertex.

We can therefore draw the momentum-space diagrams of fig. (45.4).

Figure 45.4: Diagrams for $e^-\varphi \to e^-\varphi$, corresponding to eq. (45.16).



Since there is only one fermion line in each diagram, the relative sign is positive.

The tree-level $e^-\varphi \to e^-\varphi$ scattering amplitude is then given by

$$i\mathcal{T}_{e^-\varphi\to e^-\varphi} = \tfrac{1}{i}(ig)^2\,\overline{u}_{s'}(\mathbf{p}')\left[\frac{-\not{p}-\not{k}+m}{-s+m^2}+\frac{-\not{p}+\not{k}'+m}{-u+m^2}\right]u_s(\mathbf{p})\ , \qquad (45.16)$$

where $s = -(p+k)^2$ and $u = -(p-k')^2$.

(We can safely ignore the $i\epsilon$'s in the propagators, because their denominators cannot vanish for any physically allowed values of s and u.)

Next consider the process $e^+\varphi \to e^+\varphi$.

We now have

$$\langle f|i\rangle = \langle 0|\,\mathrm{T}\, a(\mathbf{k}')_{\rm out} d_{s'}(\mathbf{p}')_{\rm out} d_s^\dagger(\mathbf{p})_{\rm in} a^\dagger(\mathbf{k})_{\rm in}\,|0\rangle\ . \tag{45.17}$$

The relevant replacements are

$$d_s^\dagger(\mathbf{p})_{\rm in} \to -i\int d^4x\; e^{+ipx}\;\overline{v}_s(\mathbf{p})(-i\not\partial + m)\Psi(x)\ , \tag{45.18}$$

$$d_{s'}(\mathbf{p}')_{\rm out} \to -i\int d^4y\;\overline{\Psi}(y)(+i\overleftarrow{\not\partial} + m)v_{s'}(\mathbf{p}')\,e^{-ipy}\ , \tag{45.19}$$

$$a^\dagger(\mathbf{k})_{\rm in} \to i\int d^4z_1\; e^{+ikz_1}(-\partial^2 + m^2)\varphi(z_1)\ , \tag{45.20}$$

$$a(\mathbf{k}')_{\rm out} \to i\int d^4z_2\; e^{-ikz_2}(-\partial^2 + m^2)\varphi(z_2)\ . \tag{45.21}$$

We substitute these into eq. (45.17), and then use eq. (45.8).

This ultimately leads to the momentum-space Feynman diagrams of fig. (45.5).

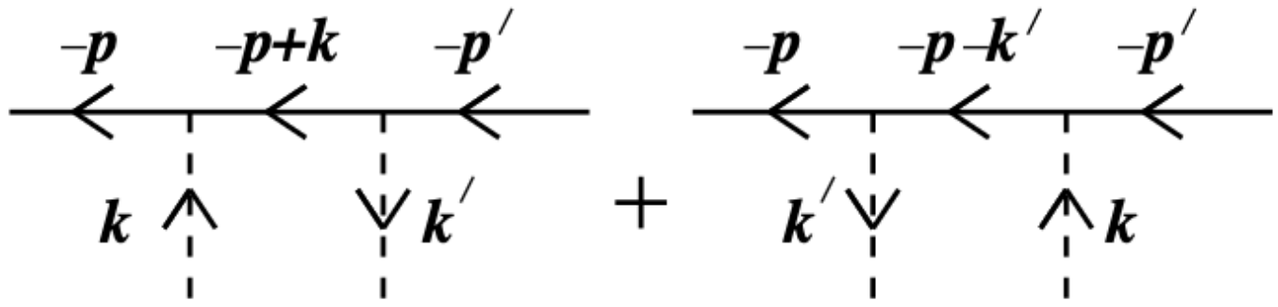


Figure 45.5: Diagrams for $e^+\varphi \to e^+\varphi$, corresponding to eq. (45.22).

Note that we must now label the external fermion lines with minus their four-momenta; this is characteristic of d-type particles.

(The same phenomenon occurs for a complex scalar field).

Regarding the spinor factors, we find that $-\overline{v}_s(p)$ is associated with the external fermion line whose arrow points away from the vertex, and $-v_{s'}(p')$ with the external fermion line whose arrow points towards the vertex.

The minus signs attached to each v and $\overline{v}$ can be consistently dropped, however, as they only affect the overall sign of the amplitude (and not the relative signs among contributing diagrams).

The tree-level expression for the $e^+\varphi \to e^+\varphi$ amplitude is then

$$i\mathcal{T}_{e^+\varphi\to e^+\varphi} = \tfrac{1}{i}(ig)^2\,\overline{v}_s(\mathbf{p})\left[\frac{\not{p}-\not{k}'+m}{-u+m^2}+\frac{\not{p}+\not{k}+m}{-s+m^2}\right]v_{s'}(\mathbf{p}')\ , \qquad (45.22)$$

where again $s = -(p+k)^2$ and $u = -(p-k')^2$

In a normal class, student would work out a few more of these before continuing……

We can abstract the following set of Feynman rules.

1. For each *incoming electron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with the electron's four-momentum, p_i.

2. For each *outgoing electron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with the electron's four-momentum, p'_i.

3. For each *incoming positron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with *minus* the positron's four-momentum, $-p_i$.

4. For each *outgoing positron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with *minus* the positron's four-momentum, $-p'_i$.

5. For each *incoming scalar*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with the scalar's four-momentum, k_i.

6. For each *outgoing scalar*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with the scalar's four-momentum, k'_i.

7. The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one dashed line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent.*

8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9. The value of a diagram consists of the following factors:

   for each incoming or outgoing scalar, 1;

   for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

   for each outgoing electron, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

   for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

   for each outgoing positron, $v_{s'_i}(\mathbf{p}'_i)$;

   for each vertex, ig;

   for each internal scalar, $-i/(k^2 + M^2 - i\epsilon)$;

   for each internal fermion, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$.

10. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any.

11. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative).

12. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of the contributing diagrams.

There are additional rules for counterterms and loops; in particular, each closed fermion loop contributes an extra minus sign.

We will postpone discussion of loop corrections to section 51.

Let us apply these rules to $e^+e^- \to \varphi\varphi$.

Let the initial electron and positron have four-momenta p_1 and p_2, respectively, and the two final scalars have four-momenta k'_1 and k'_2.

The relevant diagrams are shown in fig. (45.6); there is only one fermion line, and so the relative sign is positive.

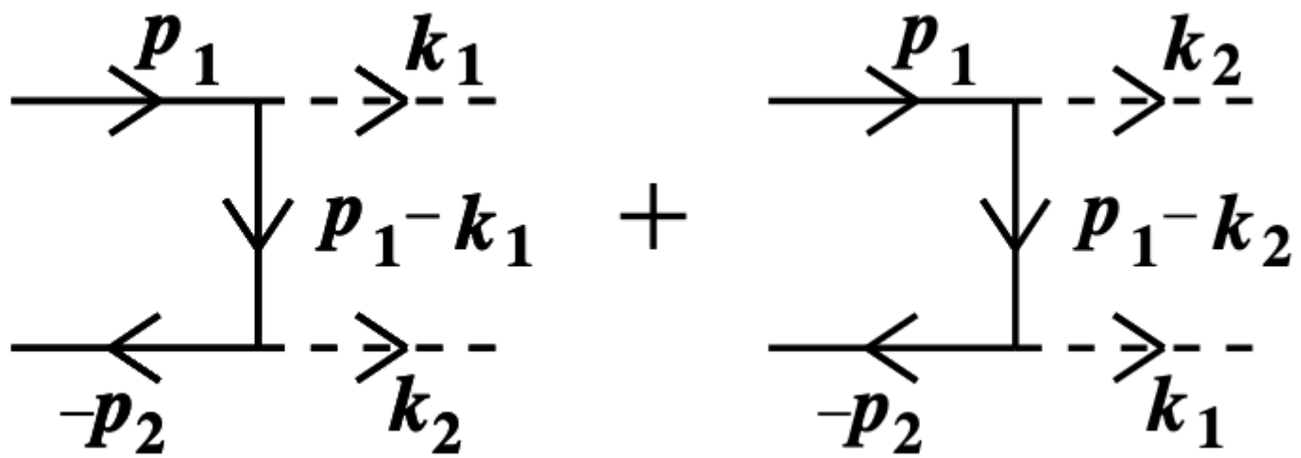


Figure 45.6: Diagrams for $e^+e^- \to \varphi\varphi$, corresponding to eq. (45.23).

The result is

$$i\mathcal{T}_{e^+e^-\to\varphi\varphi} = \tfrac{1}{i}(ig)^2\,\overline{v}_{s_2}(\mathbf{p}_2)\left[\frac{-\not{p}_1 + \not{k}'_1 + m}{-t+m^2} + \frac{-\not{p}_1 + \not{k}'_2 + m}{-u+m^2}\right] u_{s_1}(\mathbf{p}_1)\ , \qquad (45.23)$$

where $t = -(p_1 - k'_1)^2$ and $u = -(p_1 - k'_2)^2$.

Next, consider e−e−⟶ e−e−.

Let the initial electrons have four-momenta p_1 and p_2, and the final electrons have four-momenta p'_1 and p'_2.

The relevant diagrams are shown in fig. (45.7), and according to rule #11 the relative sign is negative.

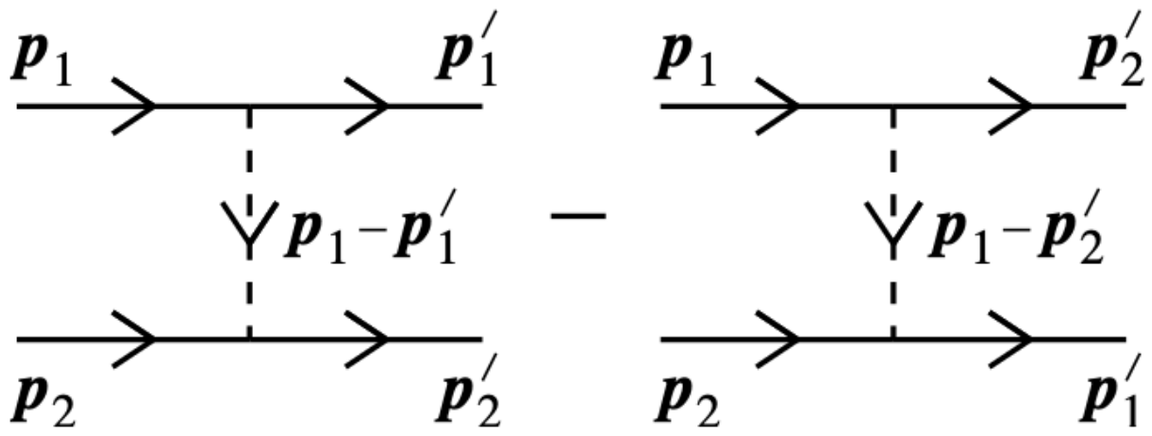


Figure 45.7: Diagrams for $e^-e^- \to e^-e^-$, corresponding to eq. (45.24).

Thus the result is

$$i\mathcal{T}_{e^-e^-\to e^-e^-} = \tfrac{1}{i}(ig)^2 \left[\frac{(\overline{u}_1' u_1)(\overline{u}_2' u_2)}{-t + M^2} - \frac{(\overline{u}_2' u_1)(\overline{u}_1' u_2)}{-u + M^2} \right], \tag{45.24}$$

where u_1 is short for $u_{s1}(\mathbf{p}_1)$, etc., and $t = -(p_1 - p'_1)^2$, $u = -(p_1 - p'_2)^2$.

One more: $e^+e^- \to e^+e^-$.

Let the initial electron and positron have four-momenta p_1 and p_2, respectively, and the final electron and positron have four-momenta p'_1 and p'_2, respectively.

The relevant diagrams are shown in fig. (45.8).

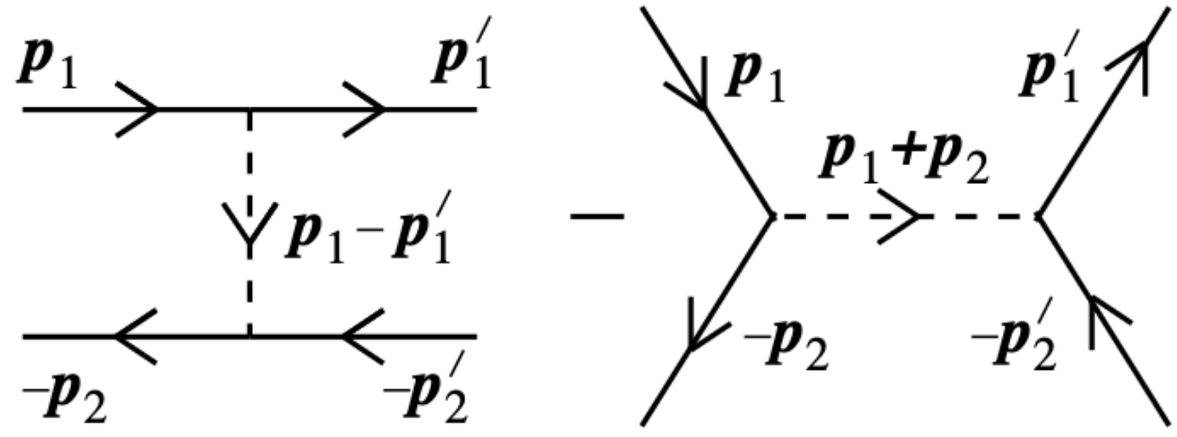


Figure 45.8: Diagrams for $e^+e^- \to e^+e^-$, corresponding to eq. (45.25).

If we redraw them in the the standard form of rule #11, as shown in fig. (45.9), we see that the relative sign is negative.

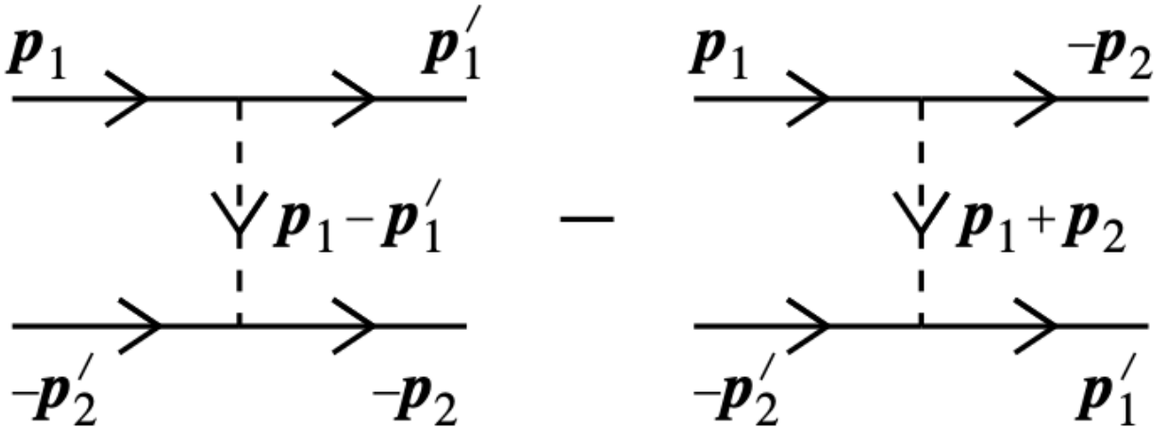


Figure 45.9: Same as fig. (45.8), but with the diagrams redrawn in the standard form given in rule #11.

Thus the result is

$$i\mathcal{T}_{e^+e^-\to e^+e^-} = \tfrac{1}{i}(ig)^2 \left[\frac{(\overline{u}_1' u_1)(\overline{v}_2 v_2')}{-t + M^2} - \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{-u + M^2} \right], \qquad (45.25)$$

where $s = -(p_1 + p_2)^2$ and $t = -(p_1 - p'_1)^2$.

46 Spin Sums

In the last section, we calculated various tree-level scattering amplitudes in Yukawa theory.

For example, for $e^-\varphi \to e^-\varphi$ we found

$$\mathcal{T} = g^2\, \overline{u}_{s'}(\mathbf{p}')\left[\frac{-\not{p} - \not{k} + m}{-s + m^2} + \frac{-\not{p} + \not{k}' + m}{-u + m^2}\right] u_s(\mathbf{p}) \ , \tag{46.1}$$

where $s = -(p+k)$ and $u = -(p-k')^2$.

In order to compute the corresponding cross section, we must evaluate $|\mathcal{T}|^2 = \mathcal{T}\ \mathcal{T}^*$.

We begin by simplifying eq. (46.1) a little; we use $(\not{p} + m)u_s(\mathbf{p}) = 0$ to replace the $-\not{p}$ in each numerator with m.

We then abbreviate eq. (46.1) as

$$\mathcal{T} = \overline{u}' A u \ , \tag{46.2}$$

where

$$A \equiv g^2 \left[\frac{-\not{k} + 2m}{m^2 - s} + \frac{\not{k}' + 2m}{m^2 - u}\right] . \tag{46.3}$$

Then we have

$$\mathcal{T}^* = \overline{\mathcal{T}} = \overline{\overline{u}' A u} = \overline{u}\overline{A}u' \ , \tag{46.4}$$

where in general $\overline{A} \equiv \beta A^\dagger \beta$, and, for the particular A of eq. (46.3), $\overline{A} = A$.

Thus we have

$$\begin{aligned} |\mathcal{T}|^2 &= (\overline{u}' A u)(\overline{u} A u') \\ &= \sum_{\alpha\beta\gamma\delta} \overline{u}'_\alpha A_{\alpha\beta} u_\beta \overline{u}_\gamma A_{\gamma\delta} u'_\delta \\ &= \sum_{\alpha\beta\gamma\delta} u'_\delta \overline{u}'_\alpha A_{\alpha\beta} u_\beta \overline{u}_\gamma A_{\gamma\delta} \\ &= \mathrm{Tr}\Big[(u'\overline{u}') A (u\overline{u}) A\Big] \ . \end{aligned} \tag{46.5}$$

Next, we use a result from section 38:

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m) \ , \tag{46.6}$$

where s = ± tells us whether the spin is up or down along the spin quantization axis z.

We then have

$$|\mathcal{T}|^2 = \tfrac{1}{4}\mathrm{Tr}\Big[(1 - s'\gamma_5 \not{z}')(-\not{p}' + m) A (1 - s\gamma_5 \not{z})(-\not{p} + m) A\Big] \ . \tag{46.7}$$

We now simply need to take traces of products of gamma matrices; we will work out the technology for this in the next section.

However, in practice, we are often not interested in (or are unable to easily measure or prepare) the spin states of the scattering particles.

Thus, if we know that an electron with momentum p′ landed in our detector, but know nothing about its spin, we should sum $|\mathcal{T}|^2$ over the two possible spin states of this outgoing electron.

Similarly, if the spin state of the initial electron is not specially prepared for each scattering event, then we should *average* $|\mathcal{T}|^2$ over the two possible spin states of this initial electron.

Then we can use

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m \tag{46.8}$$

in place of eq. (46.6).

Let us, then, take $|\mathcal{T}|^2$, sum over all final spins, and average over all initial spins, and call the result $\langle|\mathcal{T}|^2\rangle$.

In the present case, we have

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &\equiv \tfrac{1}{2}\sum_{s,s'}|\mathcal{T}|^2 \\ &= \tfrac{1}{2}\mathrm{Tr}\Big[(-\not{p}' + m)A(-\not{p} + m)A\Big]\ ,\end{aligned} \tag{46.9}$$

which is much less cumbersome than eq. (46.7).

Next let's try something a little harder, namely $e^+e^- \to e^+e^-$ found in section 45 that

$$\mathcal{T} = g^2 \left[\frac{(\overline{u}_1' u_1)(\overline{v}_2 v_2')}{M^2 - t} - \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{M^2 - s} \right]. \tag{46.10}$$

We then have

$$\overline{\mathcal{T}} = g^2 \left[\frac{(\overline{u}_1 u_1')(\overline{v}_2' v_2)}{M^2 - t} - \frac{(\overline{u}_1 v_2)(\overline{v}_2' u_1')}{M^2 - s} \right]. \tag{46.11}$$

When we multiply T by $\overline{\mathcal{T}}$, we will get four terms.

We want to arrange the factors in each of them so that every u and every v stands just to the left of the corresponding $\overline{u}$ and $\overline{v}$. In this way, we get

$$\begin{aligned} |\mathcal{T}|^2 = & + \frac{g^4}{(M^2-t)^2} \operatorname{Tr}\left[u_1 \overline{u}_1 u_1' \overline{u}_1'\right] \operatorname{Tr}\left[v_2' \overline{v}_2' v_2 \overline{v}_2\right] \\ & + \frac{g^4}{(M^2-s)^2} \operatorname{Tr}\left[u_1 \overline{u}_1 v_2 \overline{v}_2\right] \operatorname{Tr}\left[v_2' \overline{v}_2' u_1' \overline{u}_1'\right] \\ & - \frac{g^4}{(M^2-t)(M^2-s)} \operatorname{Tr}\left[u_1 \overline{u}_1 v_2 \overline{v}_2 v_2' \overline{v}_2' u_1' \overline{u}_1'\right] \\ & - \frac{g^4}{(M^2-s)(M^2-t)} \operatorname{Tr}\left[u_1 \overline{u}_1 u_1' \overline{u}_1' v_2' \overline{v}_2' v_2 \overline{v}_2\right] . \end{aligned} \tag{46.12}$$

Then we average over initial spins and sum over final spins, and use eq. (46.8) and

$$\sum_{s=\pm} v_s(\mathbf{p}) \overline{v}_s(\mathbf{p}) = -\not{p} - m \, . \tag{46.13}$$

We then must evaluate traces of products of up to four gamma matrices.

47 Gamma Matrix Technology

In this section, we will learn some tricks for handling gamma matrices.

We need the following information as a starting point:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \,, \tag{47.1}$$

$$\gamma_5^2 = 1 \,, \tag{47.2}$$

$$\{\gamma^\mu, \gamma_5\} = 0 \,, \tag{47.3}$$

$$\mathrm{Tr}\, 1 = 4 \,. \tag{47.4}$$

Now consider the trace of the product of n gamma matrices.

We have

$$\begin{aligned} \mathrm{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= \mathrm{Tr}[\gamma_5^2 \gamma^{\mu_1} \gamma_5^2 \dots \gamma_5^2 \gamma^{\mu_n}] \\ &= \mathrm{Tr}[(\gamma_5 \gamma^{\mu_1} \gamma_5) \dots (\gamma_5 \gamma^{\mu_n} \gamma_5)] \\ &= \mathrm{Tr}[(-\gamma_5^2 \gamma^{\mu_1}) \dots (-\gamma_5^2 \gamma^{\mu_n})] \\ &= (-1)^n \,\mathrm{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] \,. \end{aligned} \tag{47.5}$$

We used eq. (47.2) to get the first equality, the cyclic property of the trace for the second, eq. (47.3) for the third, and eq. (47.2) again for the fourth.

If n is odd, eq. (47.5) tells us that this trace is equal to minus itself, and must therefore be zero:

$$\mathrm{Tr}[\,\text{odd \# of } \gamma^\mu\text{'s}\,] = 0\ . \tag{47.6}$$

Similarly,

$$\mathrm{Tr}[\,\gamma_5\,(\,\text{odd \# of } \gamma^\mu\text{'s}\,)\,] = 0\ . \tag{47.7}$$

Next, consider $\mathrm{Tr}[\gamma^\mu\gamma^\nu]$.

We have

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^\nu\gamma^\mu] \\ &= \tfrac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \\ &= -g^{\mu\nu}\,\mathrm{Tr}\,1 \\ &= -4g^{\mu\nu}\ .\end{aligned} \tag{47.8}$$

The first equality follows from the cyclic property of the trace, the second averages the left- and right-hand sides of the first, the third uses eq. (47.1), and the fourth uses eq. (47.4).

A slightly nicer way of expressing eq. (47.8) is to introduce two arbitrary four-vectors a^μ and b^μ, and write

$$\mathrm{Tr}[\not{a}\not{b}] = -4(ab)\ , \tag{47.9}$$

where $\not{a} = a_\mu\gamma^\mu$, $\not{b} = b_\mu\gamma^\mu$, and $(ab) = a^\mu b_\mu$.

Next consider $\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}]$.

We evaluate this by moving $\not{a}$ to the right, using eq. (47.1), which is now more usefully written as

$$\not{a}\not{b} = -\not{b}\not{a} - 2(ab) \ . \tag{47.10}$$

Using this repeatedly, we have

$$\begin{aligned}
\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] &= -\mathrm{Tr}[\not{b}\not{a}\not{c}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}] \\
&= +\mathrm{Tr}[\not{b}\not{c}\not{a}\not{d}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}] \\
&= -\mathrm{Tr}[\not{b}\not{c}\not{d}\not{a}] - 2(ad)\mathrm{Tr}[\not{b}\not{c}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}]
\end{aligned} \tag{47.11}$$

Now we note that the first term on the right-hand side of the last line is, by the cyclic property of the trace, equal to minus the left-hand side.

We can then move this term to the left-hand side to get

$$2\,\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = \ -2(ad)\mathrm{Tr}[\not{b}\not{c}] + 2(ac)\mathrm{Tr}[\not{b}\not{d}] - 2(ab)\mathrm{Tr}[\not{c}\not{d}] \ . \tag{47.12}$$

Finally, we evaluate each $\mathrm{Tr}[\not{a}\not{b}]$ with eq. (47.9), and divide by two:

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4\Big[(ad)(bc) - (ac)(bd) + (ab)(cd)\Big] \ . \tag{47.13}$$

This is our final result for this trace.

Clearly, we can use the same technique to evaluate the trace of the product of any even number of gamma matrices.

Next, let's consider traces that involve γ_5's and γ^μ's. Since $\{\gamma_5,\gamma^\mu\} = 0$, we can always bring all the γ_5's together by moving them through the γ^μ's (generating minus signs as we go).

Then, since $\gamma_5^2 = 1$, we end up with either one γ_5 or none.

So we need only consider $\mathrm{Tr}[\gamma_5\gamma^{\mu 1} \ldots\gamma^{\mu n}]$.

And, according to eq. (47.7), we need only be concerned with even n.

Recall that an explicit formula for γ_5 is

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{47.14}$$

Eq. (47.13) then implies

$$\mathrm{Tr}\,\gamma_5 = 0 \ . \tag{47.15}$$

Similarly, we can show that

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0 \ . \tag{47.16}$$

Finally, consider $\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]$.

The only way to get a nonzero result is to have the four vector indices take on four different values.

If we consider the special case $\mathrm{Tr}[\gamma_5\gamma^3\gamma^2\gamma^1\gamma^0]$, plug in eq. (47.14), and then use $(\gamma^i)^2 = -1$ and $(\gamma^0)^2 = 1$, we get $i(-1)^3\ \mathrm{Tr}\ 1 = -4i$, or equivalently

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\varepsilon^{\mu\nu\rho\sigma}\ , \tag{47.17}$$

where $\varepsilon^{0123} = \varepsilon^{3210} = +1$.

Another category of gamma matrix combinations that we will eventually encounter is $\gamma^\mu \not{a} \ldots \gamma_\mu$.

The simplest of these is

$$\begin{aligned}\gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} \\ &= -g_{\mu\nu}g^{\mu\nu} \\ &= -d\ .\end{aligned} \tag{47.18}$$

To get the second equality, we used the fact that $g_{\mu\nu}$ is symmetric, and so only the symmetric part of $\gamma^\mu\gamma^\nu$ contributes.

In the last line, d is the number of spacetime dimensions.

Of course, our entire spinor formalism has been built around $d = 4$, but we will need formal results for $d = 4-\varepsilon$ when we dimensionally regulate loop diagrams involving fermions.

We move on to evaluate

$$\begin{aligned}\gamma^\mu \not{a} \gamma_\mu &= \gamma^\mu(-\gamma_\mu \not{a} - 2a_\mu) \\ &= -\gamma^\mu\gamma_\mu \not{a} - 2\not{a} \\ &= (d{-}2)\not{a}\ .\end{aligned} \tag{47.19}$$

We continue with

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 4(ab) - (d{-}4)\not{a}\not{b} \tag{47.20}$$

and

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = 2\not{c}\not{b}\not{a} + (d{-}4)\not{a}\not{b}\not{c}\ ; \tag{47.21}$$

48 Spin-Averaged Cross Sections

In section 46, we computed $|\mathcal{T}|^2$ for (among other processes) $e^+e^- \to e^+e^-$

We take the incoming and outgoing electrons to have momenta p_1 and p'_1, respectively, and the incoming and outgoing positrons to have momenta p_2 and p'_2, respectively.

We have $p^2_i = p'^2_i = -m^2$, where m is the electron (and positron) mass.

The Mandelstam variables are

$$
\begin{aligned}
s &= -(p_1 + p_2)^2 = -(p_1' + p_2')^2 \,, \\
t &= -(p_1 - p_1')^2 = -(p_2 - p_2')^2 \,, \\
u &= -(p_1 - p_2')^2 = -(p_2 - p_1')^2 \,,
\end{aligned}
\tag{48.1}
$$

and they obey $s + t + u = 4m^2$.

Our result was

$$
|\mathcal{T}|^2 = g^4 \left[\frac{\Phi_{ss}}{(M^2 - s)^2} - \frac{\Phi_{st} + \Phi_{ts}}{(M^2 - s)(M^2 - t)} + \frac{\Phi_{tt}}{(M^2 - t)^2} \right] ,
\tag{48.2}
$$

where M is the scalar mass, and

$$
\begin{aligned}
\Phi_{ss} &= \mathrm{Tr}\Big[u_1 \overline{u}_1 v_2 \overline{v}_2\Big] \, \mathrm{Tr}\Big[v_2' \overline{v}_2' u_1' \overline{u}_1'\Big] \,, \\
\Phi_{tt} &= \mathrm{Tr}\Big[u_1 \overline{u}_1 u_1' \overline{u}_1'\Big] \, \mathrm{Tr}\Big[v_2' \overline{v}_2' v_2 \overline{v}_2\Big] \,, \\
\Phi_{st} &= \mathrm{Tr}\Big[u_1 \overline{u}_1 u_1' \overline{u}_1' v_2' \overline{v}_2' v_2 \overline{v}_2\Big] \,, \\
\Phi_{ts} &= \mathrm{Tr}\Big[u_1 \overline{u}_1 v_2 \overline{v}_2 v_2' \overline{v}_2' u_1' \overline{u}_1'\Big] \,.
\end{aligned}
\tag{48.3}
$$

Next, we average over the two initial spins and sum over the two final spins to get

$$\langle|\mathcal{T}|^2\rangle = \tfrac{1}{4}\sum\nolimits_{s_1,s_2,s_1',s_2'} |\mathcal{T}|^2 \,. \tag{48.4}$$

Then we use

$$\begin{aligned} \sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= -\not{p} + m \,, \\ \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= -\not{p} - m \,, \end{aligned} \tag{48.5}$$

to get

$$\langle\Phi_{ss}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_2-m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2'-m)(-\not{p}_1'+m)\Big]\,, \tag{48.6}$$

$$\langle\Phi_{tt}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_1'+m)\Big]\,\mathrm{Tr}\Big[(-\not{p}_2'-m)(-\not{p}_2-m)\Big]\,, \tag{48.7}$$

$$\langle\Phi_{st}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_1'+m)(-\not{p}_2'-m)(-\not{p}_2-m)\Big]\,, \tag{48.8}$$

$$\langle\Phi_{ts}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_2-m)(-\not{p}_2'-m)(-\not{p}_1'+m)\Big]\,. \tag{48.9}$$

It is now merely tedious to evaluate these traces with the technology of section 47.

For example,

$$\begin{aligned} \mathrm{Tr}\Big[(-\not{p}_1+m)(-\not{p}_2-m)\Big] &= \mathrm{Tr}[\not{p}_1\not{p}_2] - m^2\,\mathrm{Tr}\,1 \\ &= -4(p_1p_2) - 4m^2 \,, \end{aligned} \tag{48.10}$$

It is convenient to write four-vector products in terms of the Mandelstam variables.

We have

$$\begin{aligned} p_1 p_2 = p_1' p_2' &= -\tfrac{1}{2}(s - 2m^2) \,, \\ p_1 p_1' = p_2 p_2' &= +\tfrac{1}{2}(t - 2m^2) \,, \\ p_1 p_2' = p_1' p_2 &= +\tfrac{1}{2}(u - 2m^2) \,, \end{aligned} \tag{48.11}$$

and so

$$\mathrm{Tr}\Big[(-\not{p}_1 + m)(-\not{p}_2 - m)\Big] = 2s - 8m^2 \,. \tag{48.12}$$

Thus, we can easily work out eqs. (48.6) and (48.7):

$$\langle \Phi_{ss} \rangle = (s - 4m^2)^2 \,, \tag{48.13}$$

$$\langle \Phi_{tt} \rangle = (t - 4m^2)^2 \,. \tag{48.14}$$

Obviously, if we start with $\langle \Phi_{ss} \rangle$ and make the swap s ↔ t, we get $\langle \Phi_{tt} \rangle$.

We could have anticipated this from eqs. (48.6) and (48.7): if we start with the right-hand side of eq. (48.6) and make the swap $p_2 \leftrightarrow -p'_1$, we get the right-hand side of eq. (48.7).

But from eq. (48.11), we see that this momentum swap is equivalent to s ↔ t.

Let's move on to $\langle \Phi_{st} \rangle$ and $\langle \Phi_{ts} \rangle$.

These two are also related by $p_2 \leftrightarrow -p'_1$, and so we only need to compute one of them.

We have

$$\begin{aligned}
\langle \Phi_{st} \rangle &= \tfrac{1}{4}\mathrm{Tr}[\not p_1 \not p_1' \not p_2' \not p_2] \\
&\quad + \tfrac{1}{4}m^2\,\mathrm{Tr}[\not p_1 \not p_1' - \not p_1 \not p_2' - \not p_1 \not p_2 - \not p_1' \not p_2' - \not p_1' \not p_2 + \not p_2' \not p_2] + \tfrac{1}{4}m^4\,\mathrm{Tr}\,1 \\
&= (p_1 p_1')(p_2 p_2') - (p_1 p_2')(p_2 p_1') + (p_1 p_2)(p_1' p_2') \\
&\quad - m^2[p_1 p_1' - p_1 p_2' - p_1 p_2 - p_1' p_2' - p_1' p_2 + p_2 p_2'] + m^4 \\
&= -\tfrac{1}{2}st + 2m^2 u\ .
\end{aligned} \tag{48.15}$$

To get the last line, we used eq. (48.11), and then simplified it as much as possible via $s + t + u = 4m^2$.

Since our result is symmetric on $s \leftrightarrow t$, we have $\langle \Phi_{ts} \rangle = \langle \Phi_{st} \rangle$.

Putting all of this together, we get

$$\langle |\mathcal{T}|^2 \rangle = g^4 \left[\frac{(s-4m^2)^2}{(M^2-s)^2} + \frac{st - 4m^2 u}{(M^2-s)(M^2-t)} + \frac{(t-4m^2)^2}{(M^2-t)^2} \right]. \tag{48.16}$$

This can then be converted to a differential cross section (in any frame) via the formulae of section 11.

Let's do one more: $e^- \phi \to e^- \phi$.

We take the incoming and outgoing electrons to have momenta p and p′, respectively, and the incoming and outgoing scalars to have momenta k and k′, respectively.

We then have $p^2 = p'^2 = -m^2$ and $k^2 = k'^2 = -M^2$.

The Mandelstam variables are

$$\begin{aligned} s &= -(p+k)^2 = -(p'+k')^2 \,, \\ t &= -(p-p')^2 = -(k-k')^2 \,, \\ u &= -(p-k')^2 = -(k-p')^2 \,, \end{aligned} \tag{48.17}$$

and they obey $s + t + u = 2m^2 + 2M^2$.

Our result in section 46 was

$$\langle |\mathcal{T}|^2 \rangle = \tfrac{1}{2} \mathrm{Tr}\Big[A(-\not{p} + m) A(-\not{p}' + m) \Big] \,, \tag{48.18}$$

where

$$A = g^2 \left[\frac{-\not{k} + 2m}{m^2 - s} + \frac{\not{k}' + 2m}{m^2 - u} \right] . \tag{48.19}$$

Thus we have

$$\langle |\mathcal{T}|^2 \rangle = g^4 \Big[\frac{\langle \Phi_{ss} \rangle}{(m^2-s)^2} + \frac{\langle \Phi_{su} \rangle + \langle \Phi_{us} \rangle}{(m^2-s)(m^2-u)} + \frac{\langle \Phi_{uu} \rangle}{(m^2-u)^2} \Big] , \tag{48.20}$$

where now

$$\langle \Phi_{ss} \rangle = \tfrac{1}{2} \mathrm{Tr} \Big[(-\not{p}' + m)(-\not{k} + 2m)(-\not{p} + m)(-\not{k} + 2m) \Big] \,, \tag{48.21}$$

$$\langle \Phi_{uu} \rangle = \tfrac{1}{2} \mathrm{Tr} \Big[(-\not{p}' + m)(+\not{k}' + 2m)(-\not{p} + m)(+\not{k}' + 2m) \Big] \,, \tag{48.22}$$

$$\langle \Phi_{su} \rangle = \tfrac{1}{2} \mathrm{Tr} \Big[(-\not{p}' + m)(-\not{k} + 2m)(-\not{p} + m)(+\not{k}' + 2m) \Big] \,, \tag{48.23}$$

$$\langle \Phi_{us} \rangle = \tfrac{1}{2} \mathrm{Tr} \Big[(-\not{p}' + m)(+\not{k}' + 2m)(-\not{p} + m)(-\not{k} + 2m) \Big] \,. \tag{48.24}$$

We can evaluate these in terms of the Mandelstam variables by using our trace technology, along with

$$\begin{aligned} pk = p'k' &= -\tfrac{1}{2}(s - m^2 - M^2) \,, \\ pp' &= +\tfrac{1}{2}(t - 2m^2) \,, \\ kk' &= +\tfrac{1}{2}(t - 2M^2) \,, \\ pk' = p'k &= +\tfrac{1}{2}(u - m^2 - M^2) \,. \end{aligned} \tag{48.25}$$

Examining eqs. (48.21) and (48.22), we see that $\langle \Phi_{ss} \rangle$ and $\langle \Phi_{uu} \rangle$ are transformed into each other by $k \leftrightarrow -k'$.

Examining eqs. (48.23) and (48.24), we see that $\langle \Phi_{su} \rangle$ and $\langle \Phi_{us} \rangle$ are also transformed into each other by $k \leftrightarrow -k'$.

From eq. (48.25), we see that this is equivalent to $s \leftrightarrow u$.

Thus we need only compute $\langle \Phi_{ss} \rangle$ and $\langle \Phi_{su} \rangle$, and then take s $\leftrightarrow$ u to get $\langle \Phi_{uu} \rangle$ and $\langle \Phi_{us} \rangle$.

This is, again, merely tedious, and the results are

$$\langle \Phi_{ss} \rangle = -su + m^2(9s+u) + 7m^4 - 8m^2M^2 + M^4 \,, \tag{48.26}$$

$$\langle \Phi_{uu} \rangle = -su + m^2(9u+s) + 7m^4 - 8m^2M^2 + M^4 \,, \tag{48.27}$$

$$\langle \Phi_{su} \rangle = +su + 3m^2(s+u) + 9m^4 - 8m^2M^2 - M^4 \,, \tag{48.28}$$

$$\langle \Phi_{us} \rangle = +su + 3m^2(s+u) + 9m^4 - 8m^2M^2 - M^4 \,. \tag{48.29}$$

49 The Feynman Rules for Majorana Fields

In this section we will deduce the Feynman rules for Yukawa theory, but with a Majorana field instead of a Dirac field.

We can think of the particles associated with the Majorana field as massive neutrinos.

We have

$$\begin{aligned} \mathcal{L}_1 &= \tfrac{1}{2} g \varphi \overline{\Psi} \Psi \\ &= \tfrac{1}{2} g \varphi \Psi^{\mathrm{T}} \mathcal{C} \Psi \,, \end{aligned} \tag{49.1}$$

where Ψ be a Majorana field (with mass m), ϕ is a real scalar field (with mass M), and g is a coupling constant.

In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

From section 41, we have the LSZ rules appropriate for a Majorana field,

$$b_s^\dagger(\mathbf{p})_{\text{in}} \to -i \int d^4x\, e^{+ipx}\, \overline{v}_s(\mathbf{p})(-i\not{\partial} + m)\Psi(x) \tag{49.2}$$

$$= +i \int d^4x\, \Psi^{\mathrm{T}}(x)\mathcal{C}(+i\overleftarrow{\not{\partial}} + m)u_s(\mathbf{p})e^{+ipx}\ , \tag{49.3}$$

$$b_{s'}(\mathbf{p}')_{\text{out}} \to +i \int d^4x\, e^{-ip'x}\, \overline{u}_{s'}(\mathbf{p}')(-i\not{\partial} + m)\Psi(x)\ , \tag{49.4}$$

$$= -i \int d^4x\, e^{-ip'x}\, \Psi^{\mathrm{T}}(x)\mathcal{C}(+i\overleftarrow{\not{\partial}} + m)v_{s'}(\mathbf{p}')e^{-ip'x}\ . \tag{49.5}$$

Eq. (49.3) follows from eq. (49.2) by taking the transpose of the right-hand side, and using $\overline{v}_{s'}(\mathbf{p}')^{\mathrm{T}} = -\mathcal{C}u_{s'}(\mathbf{p}')$ and $(-i\not{\partial} + m)^{\mathrm{T}} = \mathcal{C}(+i\not{\partial} + m)\mathcal{C}^{-1}$; similarly, eq. (49.5) follows from (49.4).

Which form we use depends on convenience, and is best chosen on a diagram-by-diagram basis, as we will see shortly.

Eqs. (49.2–49.5) lead us to compute correlation functions containing Ψ's, but not Ψ's.

In position space, this leads to Feynman rules where the fermion propagator is $1/iS(x-y)C^{-1}$, and the $\varphi\Psi\Psi$ vertex is igC; the factor of 1/2 in $\mathcal{L}_1$ is canceled by a symmetry factor of 2! that arises from having two identical Ψ fields in $\mathcal{L}_1$.

In a particular diagram, as we move along a fermion line, the C^{-1} in the propagator will cancel against the C in the vertex, leaving over a final C^{-1} at one end.

This C^{-1} can be canceled by a C from eq. (49.3) (for an incoming particle) or eq. (49.5) (for an outgoing particle).

On the other hand, for the other end of the same line, we should use either eq. (49.2) (for an incoming particle) or eq. (49.4) (for an outgoing particle) to avoid introducing an extra C at that end.

In this way, we can avoid ever having explicit factors of C in our Feynman rules.[1]

Using this approach, the Feynman rules for this theory are as follows.

1. The total number of incoming and outgoing neutrinos is always even; call this number $2n$. Draw n solid lines. Connect them with internal dashed lines, using a vertex that joins one dashed and two solid lines. Also, attach an external dashed line for each incoming or outgoing scalar. In this way, draw all possible diagrams that are *topologically inequivalent*.

2. Draw arrows on each segment of each solid line; keep the arrow direction continuous along each line.

3. Label each external dashed line with the momentum of an incoming or outgoing scalar. If the particle is incoming, draw an arrow on the dashed line that points *towards* the vertex; If the particle is outgoing, draw an arrow on the dashed line that points *away* from the vertex.

4. Label each external solid line with the momentum of an incoming or outgoing neutrino, but include a minus sign with the momentum if (a) the particle is incoming and the arrow points *away* from the vertex, or (b) the particle is outgoing and the arrow points *towards* the vertex. Do this labeling of external lines in all possible *inequivalent* ways. Two diagrams are considered *equivalent* if they can be transformed into each other by reversing all the arrows on one or more fermion lines, and correspondingly changing the signs of the external momenta on each arrow-reversed line.

5. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

6. The value of a diagram consists of the following factors:

   for each incoming or outgoing scalar, 1;

   for each incoming neutrino labeled with $+p_i$, $u_{s_i}(\mathbf{p}_i)$;

   for each incoming neutrino labeled with $-p_i$, $\overline{v}_{s_i}(\mathbf{p}_i)$;

   for each outgoing neutrino labeled with $+p'_i$, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

   for each outgoing neutrino labeled with $-p'_i$, $v_{s'_i}(\mathbf{p}'_i)$;

for each vertex, ig;

for each internal scalar, $-i/(k^2 + M^2 - i\epsilon)$;

for each internal fermion, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$.

7. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and writing down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any.

8. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative). To compare two diagrams, it may be necessary to use the arrow-reversing equivalence relation of rule #4; there is then an extra minus sign for each arrow-reversed line.

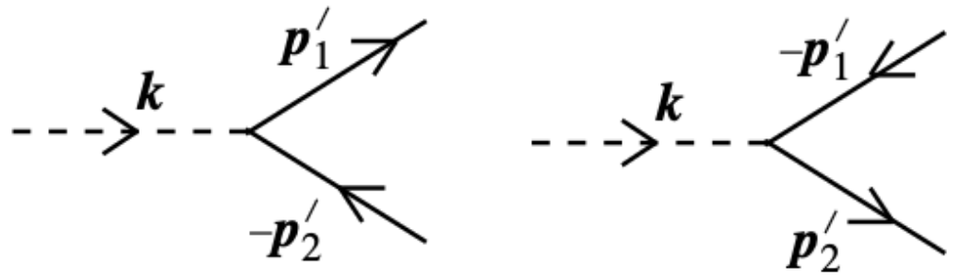
9. The value of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

There are additional rules for counterterms and loops, but we will postpone those to section 51.

Let's look at the simplest process, $\varphi \to \nu\nu$.

There are two possible diagrams for this, shown in fig. (49.1).

Figure 49.1: Two equivalent diagrams for $\varphi \to \nu\nu$.



However, according to rule #4, these two diagrams are equivalent, and we should keep only one of them.

The first diagram yields i $\mathcal{T}_1 = ig\,\overline{v}_2' u_1'$ and the second i $\mathcal{T}_2 = ig\,\overline{v}_1' u_2'$.

Rule #8 then implies that we should have $\mathcal{T}_1 = -\mathcal{T}_2$.

To check this, we note that (after dropping primes to simplify the notation)

$$\begin{aligned} \overline{v}_1 u_2 &= \left[\overline{v}_1 u_2\right]^{\mathrm{T}} \\ &= u_2^{\mathrm{T}}\,\overline{v}_1^{\mathrm{T}} \\ &= \overline{v}_2 \mathcal{C}^{-1}\mathcal{C}^{-1} u_1 \\ &= -\overline{v}_2 u_1 \ , \end{aligned} \tag{49.6}$$

as required.

In general, for processes with a total of just two incoming and outgoing neutrinos, such as $\nu\varphi \to \nu\varphi$ or $\nu\nu \to \varphi\varphi$, these rules give (up to an irrelevant overall sign) the same result for i $\mathcal{T}$ as we would get for the corresponding process in the Dirac case, $e^-\varphi \to e^-\varphi$ or $e^+e^- \to \varphi\varphi$.

(Note, however, that in the Dirac case, we have $\mathcal{L}_1 = g\varphi\overline{\Psi}\Psi$,, as compared with $\mathcal{L}_1 = \frac{1}{2}g\varphi\overline{\Psi}\Psi$ in the Majorana case.)

The differences between Dirac and Majorana fermions become more pronounced for $\nu\nu \to \nu\nu$.

Now there are *three* inequivalent contributing diagrams, shown in fig. (49.2).

The corresponding amplitude can be written as

$$i\mathcal{T} = \tfrac{1}{i}(ig)^2 \left[\frac{(\overline{u}_1' u_1)(\overline{u}_2' u_2)}{-t + M^2} - \frac{(\overline{u}_2' u_1)(\overline{u}_1' u_2)}{-u + M^2} + \frac{(\overline{v}_2 u_1)(\overline{u}_1' v_2')}{-s + M^2} \right], \tag{49.7}$$

where $s = -(p_1+p_2)^2$, $t = -(p_1-p'_1)^2$ and $u = -(p_1-p'_2)^2$.

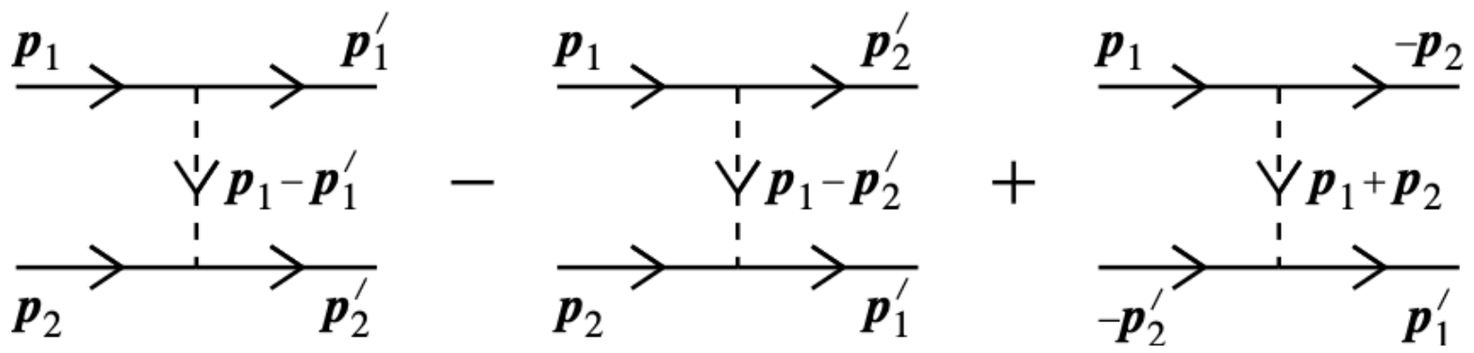


Figure 49.2: Diagrams for $\nu\nu \to \nu\nu$, corresponding to eq. (49.7).

After arbitrarily assigning the first diagram a plus sign, the minus sign of the second diagram follows from rule #8.

To get the sign of the third diagram, we compare it with the first.

To do so, we reverse the arrow direction on the lower line of the first diagram (which yields an extra minus sign), and then redraw it in standard form.

Comparing this modified first diagram with the third diagram (and invoking rule #8) reveals a relative minus sign.

Since the modified first diagram has a minus sign from the arrow reversal, we conclude that the third diagram has an overall plus sign.

After taking the absolute square of eq. (49.7), we can use relations like eq. (49.6) on a term-by-term basis to put everything into a form that allows the spin sums to be performed in the standard way.

In fact, we have already done all the necessary work in the Dirac case.

The s-s, s-t, and t-t terms in $\langle | \mathcal{T} |^2 \rangle$ for $\nu\nu \to \nu\nu$ are the same as those for $e^+e^- \to e^+e^-$, while the t-t, t-u, and u-u terms are the same as those for the crossing-related process $e^-e^- \to e^-e^-$.

Finally, the s-u terms can be obtained from the s-t terms via $t \leftrightarrow u$, or equivalently from the t-u terms via $t \leftrightarrow s$.

Thus the result is

$$\langle|\mathcal{T}|^2\rangle = g^4\Bigg[\frac{(s-4m^2)^2}{(M^2-s)^2} + \frac{st-4m^2u}{(M^2-s)(M^2-t)}$$
$$+\frac{(t-4m^2)^2}{(M^2-t)^2} + \frac{tu-4m^2s}{(M^2-t)(M^2-u)} \tag{49.8}$$
$$+\frac{(u-4m^2)^2}{(M^2-u)^2} + \frac{us-4m^2t}{(M^2-u)(M^2-s)}\Bigg],$$

which is neatly symmetric on permutations of s, t, and u.

50 Massless Particles and Spinor Helicity

Scattering amplitudes often simplify greatly if the particles are massless (or can be approximated as massless because the Mandelstam variables all have magnitudes much larger than the particle masses squared).

In this section we will explore this phenomenon for spin-one-half (and spin-zero) particles.

We will begin developing the technology of spinor helicity, which will prove to be of indispensable utility in Part III.

Recall from section 38 that the u spinors for a massless spin-one-half particle obey

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1+s\gamma_5)(-\not{p}) \ , \tag{50.1}$$

where s = ± specifies the *helicity*, the component of the particle's spin measured along the axis specified by its three-momentum; in this notation the helicity is 1/2 s.

The v spinors obey a similar relation,

$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5)(-\not{p}) \ . \tag{50.2}$$

In fact, in the massless case, with the phase conventions of section 38, we have $v_s(\mathbf{p}) = u_{-s}(\mathbf{p})$.

Thus we can confine our discussion to u-type spinors only, since we need merely change the sign of s to accommodate v-type spinors.

Consider a u spinor for a particle of negative helicity.

We have

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \tfrac{1}{2}(1 - \gamma_5)(-\not{p}) \ . \tag{50.3}$$

Let us define

$$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}} \ . \tag{50.4}$$

The we also have

$$p^{\dot{a}a} = \varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}}p_{c\dot{c}} = p_\mu \bar{\sigma}^{\mu\dot{a}a} \ . \tag{50.5}$$

Then, using

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{50.6}$$

in eq. (50.3), we find

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \begin{pmatrix} 0 & -p_{a\dot{a}} \\ 0 & 0 \end{pmatrix}. \tag{50.7}$$

On the other hand, we know that the lower two components of $u_-(p)$ vanish, and so we can write

$$u_-(\mathbf{p}) = \begin{pmatrix} \phi_a \\ 0 \end{pmatrix}. \tag{50.8}$$

Here ϕ_a is a two-component numerical spinor; it is not an anticommuting object.

Such a commuting spinor is sometimes called a *twistor*.

An explicit numerical formula for it is

$$\phi_a = \sqrt{2\omega} \begin{pmatrix} -\sin(\frac{1}{2}\theta)e^{-i\phi} \\ +\cos(\frac{1}{2}\theta) \end{pmatrix}, \tag{50.9}$$

where θ and ϕ are the polar and azimuthal angles that specify the direction of the three-momentum $\mathbf{p}$, and $\omega = |\mathbf{p}|$.

Barring eq. (50.8) yields

$$\overline{u}_-(\mathbf{p}) = (\,0, \quad \phi^*_{\dot{a}}\,), \tag{50.10}$$

where $\phi^*_{\dot{a}} = (\phi_a)^*$.

Now, combining eqs. (50.8) and (50.10), we get

$$u_-(\mathbf{p})\overline{u}_-(\mathbf{p}) = \begin{pmatrix} 0 & \phi_a\phi^*_{\dot{a}} \\ 0 & 0 \end{pmatrix}. \tag{50.11}$$

Comparing with eq. (50.7), we see that

$$p_{a\dot{a}} = -\phi_a \phi^*_{\dot{a}} \ . \tag{50.12}$$

This expresses the four-momentum of the particle neatly in terms of the twistor that describes its spin state.

The essence of the spinor helicity method is to treat ϕ_a as the fundamental object, and to express the particle's four-momentum in terms of it, via eq. (50.12).

Given eq. (50.8), and the phase conventions of section 38, the positive-helicity spinor is

$$u_+(\mathbf{p}) = \begin{pmatrix} 0 \\ \\ \phi^{*\dot{a}} \end{pmatrix}, \tag{50.13}$$

where $\phi^{*\dot{a}} = \varepsilon^{\dot{a}\dot{c}} \phi^*_{\dot{c}}$.

Barring eq. (50.13) yields

$$\overline{u}_+(\mathbf{p}) = (\, \phi^a, \quad 0\,) \, . \tag{50.14}$$

Computation of $u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$ via eqs. (50.13) and (50.14), followed by comparison with eq. (50.1) with s = +, then reproduces eq. (50.12), but with the indices raised.

In fact, the decomposition of $p_{a\dot{a}}$ into the direct product of a twistor and its complex conjugate is unique (up to an overall phase for the twistor).

To see this, use $\sigma^\mu = (I, \vec{\sigma})$ to write

$$p_{a\dot{a}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} . \tag{50.15}$$

The determinant of this matrix is $-(p_0)^2 + \mathbf{p}^2$, and this vanishes because the particle is (by assumption) massless.

Thus $p_{a\dot{a}}$ has a zero eigenvalue.

Therefore, it can be written as a projection onto the eigenvector corresponding to the nonzero eigenvalue.

That is what eq. (50.12) represents, with the nonzero eigenvalue absorbed into the normalization of the eigenvector ϕ_a.

Let us now introduce some useful notation.

Let p and k be two four-momenta, and ϕ_a and κ_a the corresponding twistors.

We define the twistor product

$$[p\,k] \equiv \phi^a \kappa_a \ . \tag{50.16}$$

Because $\phi^a \kappa_a = \varepsilon^{ac} \phi_c \kappa_a,$ and the twistors commute, we have

$$[k\,p] = -[p\,k] \ . \tag{50.17}$$

From eqs. (50.8) and (50.14), we can see that

$$\overline{u}_+(\mathbf{p})u_-(\mathbf{k}) = [p\,k] \ . \tag{50.18}$$

Similarly, let us define

$$\langle p\,k\rangle \equiv \phi^*_{\dot{a}}\kappa^{*\dot{a}} \ . \tag{50.19}$$

Comparing with eq. (50.16) we see that

$$\langle p\,k\rangle = [k\,p]^* \ , \tag{50.20}$$

which implies that this product is also antisymmetric,

$$\langle k\,p\rangle = -\langle p\,k\rangle \ . \tag{50.21}$$

Also, from eqs. (50.10) and (50.13), we have

$$\overline{u}_-(\mathbf{p})u_+(\mathbf{k}) = \langle p\,k\rangle \ . \tag{50.22}$$

Note that the other two possible spinor products vanish:

$$\overline{u}_+(\mathbf{p})u_+(\mathbf{k}) = \overline{u}_-(\mathbf{p})u_-(\mathbf{k}) = 0 \ . \tag{50.23}$$

The twistor products ⟨pk⟩ and [pk] satisfy another important relation,

$$\begin{aligned}\langle p\,k\rangle[k\,p] &= (\phi^*_{\dot{a}}\kappa^{*\dot{a}})(\kappa^a\phi_a) \\ &= (\phi^*_{\dot{a}}\phi_a)(\kappa^a\kappa^{*\dot{a}}) \\ &= p_{\dot{a}a}k^{a\dot{a}} \\ &= -2p^\mu k_\mu \ ,\end{aligned} \tag{50.24}$$

where the last line follows from $\bar{\sigma}^{\mu\dot{a}a}\sigma^\nu_{a\dot{a}} = -2g^{\mu\nu}$.

Let us apply this notation to the tree-level scattering amplitude for $e^- \varphi \to e^- \varphi$ in Yukawa theory, which we first computed in Section 44, and which reads

$$\mathcal{T}_{s's} = g^2\, \overline{u}_{s'}(\mathbf{p}')\Big[\tilde{S}(p+k) + \tilde{S}(p-k')\Big] u_s(\mathbf{p})\ . \tag{50.25}$$

For a massless fermion, $\tilde{S}(p) = -\not{p}/p^2$.

If the scalar is also massless, then $(p + k)^2 = 2p \cdot k$ and $(p - k')^2 = -2p \cdot k'$.

Also, we can remove the $\not{p}$'s in the propagator numerators in eq. (50.25), because $\not{p} u_s(\mathbf{p}) = 0$. .

Thus we have

$$\mathcal{T}_{s's} = g^2\, \overline{u}_{s'}(\mathbf{p}')\left[\frac{-\not{k}}{2p\cdot k} + \frac{-\not{k}'}{2p\cdot k'}\right] u_s(\mathbf{p})\ . \tag{50.26}$$

Now consider the case $s' = s = +$.

From eqs. (50.13), (50.14), and

$$-\not{k} = \begin{pmatrix} 0 & \kappa_a \kappa^*_{\dot{a}} \\ \kappa^{*\dot{a}} \kappa^a & 0 \end{pmatrix}, \tag{50.27}$$

we get

$$\begin{aligned} \overline{u}_+(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) &= \phi'^a \kappa_a \kappa^*_{\dot{a}} \phi^{*\dot{a}} \\ &= [p'\, k]\, \langle k\, p\rangle\ . \end{aligned} \tag{50.28}$$

Similarly, for s′ = s = −, we find

$$\begin{aligned}\overline{u}_-(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) &= \phi'^*_{\dot{a}}\kappa^{*\dot{a}}\kappa^a\phi_a \\ &= \langle p'\,k\rangle\,[k\,p]\ ,\end{aligned} \tag{50.29}$$

while for s′ ≠ s, the amplitude vanishes:

$$\overline{u}_-(\mathbf{p}')(-\not{k})u_+(\mathbf{p}) = \overline{u}_+(\mathbf{p}')(-\not{k})u_-(\mathbf{p}) = 0\ . \tag{50.30}$$

Then, using eq. (50.24) on the denominators in eq. (50.26), we find

$$\mathcal{T}_{++} = -g^2\left(\frac{[p'\,k]}{[p\,k]} + \frac{[p'\,k']}{[p\,k']}\right), \tag{50.31}$$

$$\mathcal{T}_{--} = -g^2\left(\frac{\langle p'\,k\rangle}{\langle p\,k\rangle} + \frac{\langle p'\,k'\rangle}{\langle p\,k'\rangle}\right),$$

while

$$\mathcal{T}_{+-} = \mathcal{T}_{-+} = 0\ . \tag{50.32}$$

Thus we have rather simple expressions for the fixed-helicity scattering amplitudes in terms of twistor products.

51 Loop Corrections in Yukawa Theory

In this section we will compute the one-loop corrections in Yukawa theory with a Dirac field.

The basic concepts are all the same as for a scalar field, and so we will mainly be concerned with the extra technicalities arising from spin indices and anticommutation.

First let us note that the general discussion of sections 18 and 29 leads us to expect that we will need to add to the lagrangian all possible terms whose coefficients have positive or zero mass dimension, and that respect the symmetries of the original lagrangian.

These include Lorentz symmetry, the U(1) phase symmetry of the Dirac field, and the discrete symmetries of parity, time reversal, and charge conjugation.

The mass dimensions of the fields (in four spacetime dimensions) are $[\varphi] = 1$ and $[\Psi] = 32$.

Thus any power of φ up to φ^4 is allowed.

But there are no additional required terms involving Ψ: the only candidates contain either γ_5 (e.g., $i\overline{\Psi}\gamma_5\Psi$) and are forbidden by parity, or C (e.g, $\Psi^T C\Psi$) and are forbidden by the U(1) symmetry.

Nevertheless, having to deal with the addition of three new terms (φ, φ^3, φ^4) is annoying enough to prompt us to look for a simpler example.

Consider, then, a modified form of the Yukawa interaction,

$$\mathcal{L}_{\rm Yuk} = ig\varphi\overline{\Psi}\gamma_5\Psi \ . \tag{51.1}$$

This interaction will conserve parity if and only if φ is a pseudoscalar:

$$P^{-1}\varphi(\mathbf{x},t)P = -\varphi(-\mathbf{x},t) \ . \tag{51.2}$$

Then, φ and φ^3 are odd under parity, and so we will not need to add them to $\mathcal{L}$.

The one term we will need to add is φ^4.

Therefore, the theory we will consider is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \ , \tag{51.3}$$

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}M^2\varphi^2 \ , \tag{51.4}$$

$$\mathcal{L}_1 = iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \tfrac{1}{24}Z_\lambda\lambda\varphi^4 + \mathcal{L}_{\rm ct} \ , \tag{51.5}$$

$$\begin{aligned}\mathcal{L}_{\rm ct} = {} & i(Z_\Psi{-}1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m{-}1)m\overline{\Psi}\Psi \\ & - \tfrac{1}{2}(Z_\varphi{-}1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_M{-}1)M^2\varphi^2\end{aligned} \tag{51.6}$$

where λ is a new coupling constant.

We will use an on-shell renormalization scheme.

The lagrangian parameter m is then the actual mass of the electron.

We will define the couplings g and λ as the values of appropriate vertex functions when the external four-momenta vanish.

Finally, the fields are normalized according to the requirements of the LSZ formula.

In practice, this means that the scalar and fermion propagators must have appropriate poles with unit residue.

We will assume that $M < 2m$, so that the scalar is stable against decay into an electron-positron pair.

The exact scalar propagator (in momentum space) can be then written in Lehmann-Kallen form as

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + M^2 - i\epsilon} + \int_{M_{\rm th}^2}^{\infty} ds \, \frac{\rho(s)}{k^2 + s - i\epsilon} \, , \tag{51.7}$$

where the spectral density ρ(s) is real and nonnegative.

The threshold mass M_{th} is either 2m (corresponding to the contribution of an electron-positron pair) or 3M (corresponding to the contribution of three scalars; by parity, there is no contribution from two scalars), whichever is less.

We can also write

$$\tilde{\mathbf{\Delta}}(k^2)^{-1} = k^2 + M^2 - i\epsilon - \Pi(k^2) \, , \tag{51.8}$$

where $i\Pi(k^2)$ is given by the sum of one-particle irreducible (1PI for short; see section 14) diagrams with two external scalar lines, and the external propagators removed.

The fact that $\widetilde{\Delta}(k^2)$ has a pole at $k^2 = -M^2$ with residue one implies that $\Pi(-M^2) = 0$ and $\Pi'(-M^2) = 0$; this fixes the coefficients Z_φ and Z_M.

All of this is mimicked for the Dirac field.

When parity is conserved, the exact propagator (in momentum space) can be written in Lehmann-Kallen form as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon} + \int_{m_{\text{th}}^2}^{\infty} ds \, \frac{-\not{p}\rho_1(s) + \sqrt{s}\,\rho_2(s)}{p^2 + s - i\epsilon} \,, \tag{51.9}$$

where the spectral densities $\rho_1(s)$ and $\rho_2(s)$ are both real, and $\rho_1(s)$ is non-negative and greater than $\rho_2(s)$.

The threshold mass m_{th} is m+ M (corresponding to the contribution of a fermion and a scalar), which, by assumption, is less than 3m (corresponding to the contribution of three fermions; by Lorentz invariance, there is no contribution from two fermions).

Since $p^2 = -\not{p}\not{p}$, we can rewrite eq. (51.9) as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{1}{\not{p} + m - i\epsilon} + \int_{m_{\text{th}}^2}^{\infty} ds \, \frac{-\not{p}\rho_1(s) + \sqrt{s}\,\rho_2(s)}{(-\not{p} + \sqrt{s} - i\epsilon)(\not{p} + \sqrt{s} - i\epsilon)} \,, \tag{51.10}$$

with the understanding that 1/(...) refers to the matrix inverse.

However, since $\not{p}$ is the only matrix involved, we can think of $\tilde{\mathbf{S}}(\not{p})$ as an analytic function of the single variable $\not{p}$.

With this idea in mind, we see that $\tilde{\mathbf{S}}(\not{p})$ has an isolated pole at $\not{p}$ =−m with residue one.

This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

We can also write the exact fermion propagator in the form

$$\tilde{\mathbf{S}}(\not{p})^{-1} = \not{p} + m - i\epsilon - \Sigma(\not{p}) \ , \tag{51.11}$$

where iΣ($\not{p}$) is given by the sum of 1PI diagrams with two external fermion lines, and the external propagators removed.

The fact that $\tilde{\mathbf{S}}(\not{p})$ has a pole at $\not{p}$ = −m with residue one implies that Σ(−m) = 0 and Σ′(−m) = 0; this fixes the coefficients Z_Ψ and Z_m.

We proceed to the diagrams.

The Yukawa vertex carries a factor of $i(iZ_g g)\gamma_5 = -Z_g g\gamma_5$. Since $Z_g = 1 + O(g^2)$, we can set $Z_g = 1$ in the one-loop diagrams.

Consider first $\Pi(k^2)$, which receives the one-loop (and counterterm) corrections shown in fig. (51.1).

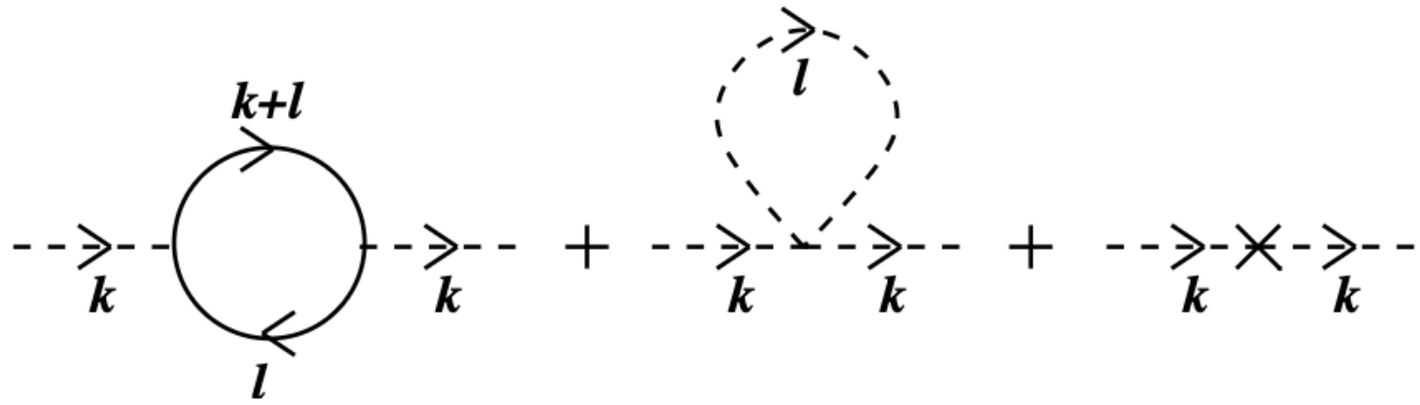


Figure 51.1: The one-loop and counterterm corrections to the scalar propagator in Yukawa theory

The first diagram has a closed fermion loop.

As we will see in section 53, anticommutation of the fermion fields results in an extra factor of minus one for each closed fermion loop.

The spin indices on the propagators and vertices are contracted in the usual way, following the arrows backwards.

Since the loop closes on itself, we end up with a trace over the spin indices.

Thus we have

$$i\Pi_{\Psi\,\mathrm{loop}}(k^2) = (-1)(-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde{S}(\not{\ell}+\not{k})\gamma_5\tilde{S}(\not{\ell})\gamma_5\Big]\,, \tag{51.12}$$

where

$$\tilde{S}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon} \tag{51.13}$$

is the free fermion propagator in momentum space.

We now proceed to evaluate eq. (51.12).

We have

$$\begin{aligned}\mathrm{Tr}[(-\not{\ell}-\not{k}+m)\gamma_5(-\not{\ell}+m)\gamma_5] &= \mathrm{Tr}[(-\not{\ell}-\not{k}+m)(+\not{\ell}+m)] \\ &= 4[(\ell+k)\ell+m^2] \\ &\equiv 4N\ .\end{aligned} \tag{51.14}$$

The first equality follows from $\gamma_5{}^2 = 1$ and $\gamma_5\ \not{p}\ \gamma_5 = -\ \not{p}$.

Next we combine the denominators with Feynman's formula. Suppressing the $i\epsilon$'s, we have

$$\frac{1}{(\ell+k)^2+m^2}\,\frac{1}{\ell^2+m^2} = \int_0^1 dx\,\frac{1}{(q^2+D)^2}\ , \tag{51.15}$$

where $q = \ell + xk$ and $D = x(1-x)k^2 + m^2$.

We then change the integration variable in eq. (51.12) from ℓ to q; the result is

$$i\Pi_{\Psi\,\mathrm{loop}}(k^2) = 4g^2\int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^2}\ , \tag{51.19}$$

where now $N = (q + (1-x)k)(q-xk) + m^2$.

The integral diverges, and so we analytically continue it to $d = 4 - \varepsilon$ spacetime dimensions.

(Here we ignore a subtlety with the definition of γ_5 in d dimensions, and assume that $\gamma_5{}^2 = 1$ and $\gamma_5 \not{p} \gamma_5 = -\not{p}$ continue to hold.)

We also make the replacement $g \to g\ \tilde{\mu}^{\ \varepsilon/2}$, where $\tilde{\mu}$ has dimensions of mass, so that g remains dimensionless.

Expanding out the numerator, we have

$$N = q^2 - x(1-x)k^2 + m^2 + (1-2x)kq \ . \tag{51.17}$$

The term linear in q integrates to zero.

For the rest, we use the general result of section 14 to get

$$\tilde{\mu}^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \ln(D/\mu^2)\right] \ , \tag{51.18}$$

$$\tilde{\mu}^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} + \tfrac{1}{2} - \ln(D/\mu^2)\right](-2D) \ , \tag{51.19}$$

where $\mu^2 = 4\pi e^{-\gamma}\ \tilde{\mu}^{\ 2}$, and we have dropped terms of order ε.

Plugging eqs. (51.18) and (51.19) into eq. (51.16) yields

$$\Pi_{\Psi\,\mathrm{loop}}(k^2) = -\frac{g^2}{4\pi^2}\left[\frac{1}{\varepsilon}(k^2+2m^2) + \tfrac{1}{6}k^2 + m^2 \right. \\ \left. - \int_0^1 dx \left(3x(1-x)k^2 + m^2\right)\ln(D/\mu^2)\right]. \tag{51.20}$$

We see that the divergent term has (as expected) a form that permits cancellation by the counterterms.

We evaluated the second diagram of fig. (51.1) in section 31, with the result

$$\Pi_{\varphi\,\mathrm{loop}}(k^2) = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \tfrac{1}{2} - \tfrac{1}{2}\ln(M^2/\mu^2)\right]M^2\ . \tag{51.21}$$

The third diagram gives the contribution of the counterterms,

$$\Pi_{\mathrm{ct}}(k^2) = -(Z_\varphi - 1)k^2 - (Z_M - 1)M^2\ . \tag{51.22}$$

Adding up eqs. (51.20–51.22), we see that finiteness of $\Pi(k^2)$ requires

$$Z_\varphi = 1 - \frac{g^2}{4\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{51.23}$$

$$Z_M = 1 + \left(\frac{\lambda}{16\pi^2} - \frac{g^2}{2\pi^2}\frac{m^2}{M^2}\right)\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{51.24}$$

plus higher-order (in g and/or λ) corrections.

Note that, although there is an $O(\lambda)$ correction to Z_M, there is not an $O(\lambda)$ correction to $Z\varphi$.

We can impose $\Pi(-M^2) = 0$ by writing

$$\Pi(k^2) = \frac{g^2}{4\pi^2}\left[\int_0^1 dx \left(3x(1-x)k^2 + m^2\right)\ln(D/D_0) + \kappa_\varphi(k^2 + M^2)\right], \tag{51.25}$$

where $D_0 = -x(1-x)M^2 + m^2$, and κ_φ is a constant to be determined.

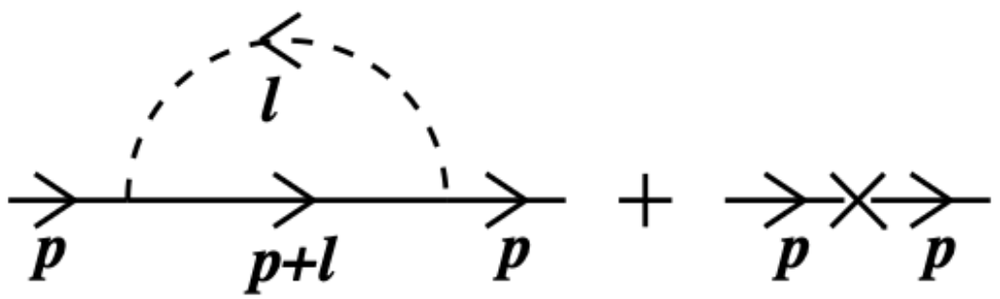
We fix κ_φ by imposing $\Pi'(-M^2) = 0$, which yields

$$\kappa_\varphi = \int_0^1 dx\, x(1-x)[3x(1-x)M^2 - m^2]/D_0\ . \tag{51.26}$$

Note that, in this on-shell renormalization scheme, there is no $O(\lambda)$ correction to $\Pi(k^2)$.

Next we turn to the Ψ propagator, which receives the one-loop (and counterterm) corrections shown in fig. (51.2).

Figure 51.2: The one-loop and counterterm corrections to the fermion prop-agator in Yukawa theory.



The spin indices are contracted in the usual way, following the arrows backwards.

We have

$$i\Sigma_{1\,\mathrm{loop}}(\not{p}) = (-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma_5 \tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)\ , \tag{51.27}$$

where $\tilde{\mathbf{S}}(\not{p})$ is given by eq. (51.13), and

$$\tilde{\Delta}(\ell^2) = \frac{1}{\ell^2 + M^2 - i\epsilon} \tag{51.28}$$

is the free scalar propagator in momentum space.

We evaluate eq. (51.27) with the usual bag of tricks.

The result is

$$i\Sigma_{1\,\mathrm{loop}}(\not{p}) = -g^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^2}\ , \tag{51.29}$$

where q = ℓ + xp and

$$N = \not{q} + (1-x)\not{p} + m\ , \tag{51.30}$$

$$D = x(1-x)p^2 + xm^2 + (1-x)M^2\ . \tag{51.31}$$

The integral diverges, and so we analytically continue it to d = 4 − ε spacetime dimensions, make the replacement g → g $\tilde{\mu}^{\varepsilon/2}$, and take the limit as ε → 0.

The term linear in q in eq. (51.30) integrates to zero.

Using eq. (51.18), we get

$$\Sigma_{1\,\mathrm{loop}}(\not p) = -\frac{g^2}{16\pi^2}\left[\frac{1}{\varepsilon}(\not p + 2m) - \int_0^1 dx\,\Big((1-x)\not p + m\Big)\ln(D/\mu^2)\right]. \tag{51.32}$$

We see that the divergent term has (as expected) a form that permits cancellation by the counterterms, which give

$$\Sigma_{\mathrm{ct}}(\not p) = -(Z_\Psi - 1)\not p - (Z_m - 1)m\ . \tag{51.33}$$

Adding up eqs. (51.32) and (51.33), we see that finiteness of Σ( ) requires

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{51.34}$$

$$Z_m = 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{51.35}$$

plus higher-order corrections.

We can impose Σ(−m) = 0 by writing

$$\Sigma(\not p) = \frac{g^2}{16\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not p + m\Big)\ln(D/D_0) + \kappa_\Psi(\not p + m)\right], \tag{51.36}$$

where D_0 is D evaluated at $p^2 = -m^2$, and κ_Ψ is a constant to be determined.

We fix κ_Ψ by imposing $\Sigma'(-m) = 0$. In differentiating with respect to $\not p$, we take the p^2 in D, eq. (51.31), to be - $\not p^2$ we find

$$\kappa_\Psi = -2\int_0^1 dx\; x^2(1-x)m^2/D_0\;. \tag{51.37}$$

Next we turn to the correction to the Yukawa vertex.

We define the vertex function $i\mathbf{V}_Y(p', p)$ as the sum of one-particle irreducible diagrams with one incoming fermion with momentum p, one outgoing fermion with momentum p', and one incoming scalar with momentum $k = p' - p$.

The original vertex $-Z_g g\gamma_5$ is the first term in this sum, and the diagram of fig. (51.3) is the second.

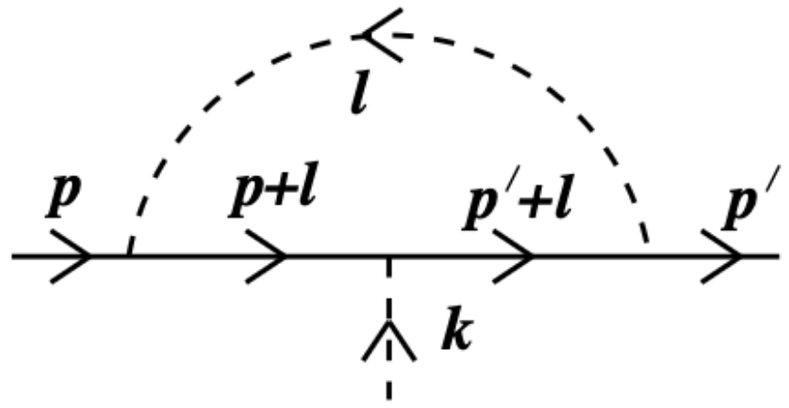


Figure 51.3: The one-loop correction to the scalar-fermion-fermion vertex in Yukawa theory.

Thus we have

$$i\mathbf{V}_Y(p', p) = -Z_g g\gamma_5 + i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p', p) + O(g^5)\;, \tag{51.38}$$

where

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p', p) = (-g)^3\left(\tfrac{1}{i}\right)^3\int\frac{d^d\ell}{(2\pi)^d}\left[\gamma_5\tilde{S}(\not{p}'+\not{\ell})\gamma_5\tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)\;. \tag{51.39}$$

The numerator can be written as

$$N = (\not{p}' + \not{\ell} + m)(-\not{p} - \not{\ell} + m)\gamma_5\;, \tag{51.40}$$

and the denominators combined in the usual way.

We then get

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) = -ig^3 \int dF_3 \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^3}\ , \tag{51.41}$$

where the integral over Feynman parameters was defined in section 16, and now

$$q = \ell + x_1 p + x_2 p'\ , \tag{51.42}$$

$$N = [\not{q} - x_1\not{p} + (1-x_2)\not{p}' + m][-\not{q} - (1-x_1)\not{p} + x_2\not{p}' + m]\gamma_5\ , \tag{51.43}$$

$$\begin{aligned} D = {} & x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2 p\cdot p' \\ & + (x_1+x_2)m^2 + x_3M^2\ . \end{aligned} \tag{51.44}$$

Using $\not{q}\not{q} = -q^2$, we can write N as

$$N = q^2\gamma_5 + \widetilde{N} + (\text{linear in } q)\ , \tag{51.45}$$

where

$$\widetilde{N} = [-x_1\not{p} + (1-x_2)\not{p}' + m][-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma_5\ . \tag{51.46}$$

The terms linear in q in eq. (51.45) integrate to zero, and only the first term is divergent.

Performing the usual manipulations, we find

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) = \frac{g^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - \tfrac{1}{4} - \tfrac{1}{2}\int dF_3\,\ln(D/\mu^2)\right)\gamma_5 + \tfrac{1}{4}\int dF_3\,\frac{\widetilde{N}}{D}\right]. \tag{51.47}$$

From eq. (51.38), we see that finiteness of $\mathbf{V}_Y(p',p)$ requires

$$Z_g = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right), \tag{51.48}$$

plus higher-order corrections.

To fix the finite part of Z_g, we need a condition to impose on $\mathbf{V}_Y(p',p)$.

We will mimic what we did in φ^3 theory in section 16, and require $\mathbf{V}_Y(0,0)$ to have the tree-level value $ig\gamma_5$.

We leave the details.

Next we turn to the corrections to the φ^4 vertex $i\mathbf{V}_4(k_1,k_2,k_3,k_4)$; the tree-level contribution is $-iZ_\lambda\lambda$.

There are diagrams with a closed fermion loop, as shown in fig. (51.4), plus one-loop diagrams with φ particles only that we evaluated in section 31.

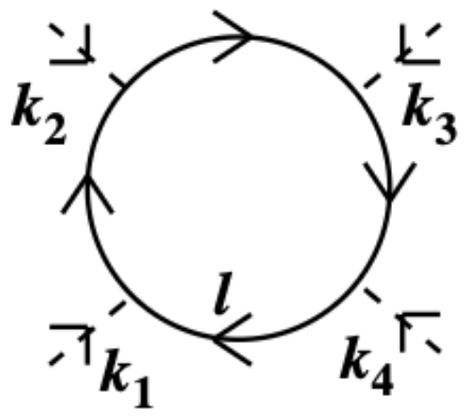


Figure 51.4: One of six diagrams with a closed fermion loop and four external scalar lines; the other five are obtained by permuting the external momenta in all possible inequivalent ways.

We have

$$
\begin{aligned}
i\mathbf{V}_{4,\,\Psi\,\text{loop}} = (-1)(-g)^4\left(\tfrac{1}{i}\right)^4 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde{S}(\not{\ell})\gamma_5\tilde{S}(\not{\ell}-\not{k}_1)\gamma_5 \\
\times \tilde{S}(\not{\ell}+\not{k}_2+\not{k}_3)\gamma_5\tilde{S}(\not{\ell}+\not{k}_2)\gamma_5\Big] \\
+ 5 \text{ permutations of } (k_2, k_3, k_4)\ .
\end{aligned}
\tag{51.49}
$$

Again we can employ the standard methods; there are no unfamiliar aspects.

This being the case, let us concentrate on obtaining the divergent part; this will give us enough information to calculate the one-loop contributions to the beta functions for g and λ.

To obtain the divergent part of eq. (51.49), it is sufficient to set $k_i = 0$.

The term in the numerator that contributes to the divergent part is $\mathrm{Tr}\,(\not{\ell}\gamma_5)^4 = 4(\ell^2)^2$, and the denominator is $(\ell^2 + m^2)^4$.

Then we find, after including identical contributions from the other five permutations of the external momenta,

$$
\mathbf{V}_{4,\,\Psi\,\text{loop}} = -\frac{3g^4}{\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right). \tag{51.50}
$$

From section 31, we have

$$
\mathbf{V}_{4,\,\varphi\,\text{loop}} = \frac{3\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right). \tag{51.51}
$$

Then, using

$$\mathbf{V}_4 = -Z_\lambda \lambda + \mathbf{V}_{4,\,\Psi\,\text{loop}} + \mathbf{V}_{4,\,\varphi\,\text{loop}} + \cdots \,, \tag{51.52}$$

we see that finiteness of $\mathbf{V}_4$ requires

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2 \lambda} \right) \left(\frac{1}{\varepsilon} + \text{finite} \right), \tag{51.53}$$

plus higher-order corrections.

52 Beta Functions in Yukawa Theory

In this section we will compute the beta functions for the Yukawa coupling g and the φ^4 coupling λ in Yukawa theory, using the methods of section 28.

The relations between the bare and renormalized couplings are

$$g_0 = Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g \tilde{\mu}^{\varepsilon/2} g \,, \tag{52.1}$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \,. \tag{52.2}$$

Let us define

$$\ln\left(Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g \right) = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} \,, \tag{52.3}$$

$$\ln\left(Z_\varphi^{-2} Z_\lambda \right) = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} \,. \tag{52.4}$$

From our results in section 51, we have

$$G_1(g,\lambda) = \frac{5g^2}{16\pi^2} + \dots \,, \tag{52.5}$$

$$L_1(g,\lambda) = \frac{3\lambda}{16\pi^2} + \frac{g^2}{2\pi^2} - \frac{3g^4}{\pi^2\lambda} + \dots \,, \tag{52.6}$$

where the ellipses stand for higher-order (in g^2 and/or λ) corrections.

Taking the logarithm of eqs. (52.1) and (52.2), and using eqs. (52.3) and (52.4), we get

$$\ln g_0 = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} + \ln g + \tfrac{1}{2}\varepsilon \ln \tilde{\mu} \,, \tag{52.7}$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} + \ln \lambda + \varepsilon \ln \tilde{\mu} \,. \tag{52.8}$$

We now use the fact that g_0 and λ_0 must be independent of μ.

We differentiate eqs. (52.7) and (52.8) with respect to $\ln \mu$; the left-hand sides vanish, and we multiply the right-hand sides by g and λ, respectively.

The result is

$$0 = \sum_{n=1}^{\infty} \left(g \frac{\partial G_n}{\partial g} \frac{dg}{d\ln\mu} + g \frac{\partial G_n}{\partial \lambda} \frac{d\lambda}{d\ln\mu} \right) \frac{1}{\varepsilon^n} + \frac{dg}{d\ln\mu} + \tfrac{1}{2}\varepsilon g \,, \tag{52.9}$$

$$0 = \sum_{n=1}^{\infty} \left(\lambda \frac{\partial L_n}{\partial g} \frac{dg}{d\ln\mu} + \lambda \frac{\partial L_n}{\partial \lambda} \frac{d\lambda}{d\ln\mu} \right) \frac{1}{\varepsilon^n} + \frac{d\lambda}{d\ln\mu} + \varepsilon\lambda \ . \tag{52.10}$$

In a renormalizable theory, dg/dln μ and dλ/dln μ must be finite in the ε⟶ 0 limit.

Thus we can write

$$\frac{dg}{d\ln\mu} = -\tfrac{1}{2}\varepsilon g + \beta_g(g,\lambda) \ , \tag{52.11}$$

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon\lambda + \beta_\lambda(g,\lambda) \ . \tag{52.12}$$

ubstituting these into eqs. (52.9) and (52.10), and matching powers of ε, we find

$$\beta_g(g,\lambda) = g\left(\tfrac{1}{2} g \frac{\partial}{\partial g} + \lambda \frac{\partial}{\partial \lambda} \right) G_1 \ , \tag{52.13}$$

$$\beta_\lambda(g,\lambda) = \lambda\left(\tfrac{1}{2} g \frac{\partial}{\partial g} + \lambda \frac{\partial}{\partial \lambda} \right) L_1 \ . \tag{52.14}$$

The coefficients of all higher powers of 1/ε must also vanish, but this gives us no more information about the beta functions.

Using eqs. (52.5) and (52.6) in eqs. (52.13) and (52.14), we get

$$\beta_g(g,\lambda) = \frac{5g^3}{16\pi^2} + \dots \,, \tag{52.15}$$

$$\beta_\lambda(g,\lambda) = \frac{1}{16\pi^2}\left(3\lambda^2 + 8\lambda g^2 - 48g^4\right) + \dots \,. \tag{52.16}$$

The higher-order corrections have extra factors of g^2 and/or λ.

53 Functional Determinants

In the section we will explore the meaning of the *functional determinants* that arise when doing gaussian path integrals, either bosonic or fermionic.

We will be interested in situations where the path integral over one particular field is gaussian, but generates a functional determinant that depends on some other field.

We will see how to relate this functional determinant to a certain infinite set of Feynman diagrams.

We will need the technology we develop here to compute the path integral for nonabelian gauge theory in section 70.

We begin by considering a theory of a complex scalar field χ with

$$\mathcal{L} = -\partial^\mu \chi^\dagger \partial_\mu \chi - m^2 \chi^\dagger \chi + g\varphi \chi^\dagger \chi \,, \tag{53.1}$$

where φ is a real scalar *background field*.

That is, φ(x) is treated as a fixed function of spacetime.

Next we define the path integral

$$Z(\varphi) = \int \mathcal{D}\chi^\dagger \, \mathcal{D}\chi \, e^{i \int d^4x \, \mathcal{L}} \, , \tag{53.2}$$

where we use the ϵ trick of section 6 to impose vacuum boundary conditions, and the normalization Z(0) = 1 is fixed by hand.

Recall from section 44 that if we have n complex variables z_i, then we can evaluate gaussian integrals by the general formula

$$\int d^n z \, d^n \bar{z} \, \exp\left(-i \bar{z}_i M_{ij} z_j\right) \propto (\det M)^{-1} \, . \tag{53.3}$$

In the case of the functional integral in eq. (53.2), the index i on the integration variable is replaced by the continuous spacetime label x, and the "matrix" M becomes

$$M(x,y) = [-\partial_x^2 + m^2 - g\varphi(x)]\delta^4(x-y) \, . \tag{53.4}$$

In order to apply eq. (53.3), we have to understand what it means to compute the determinant of this expression.

To this end, let us first note that we can write $M = M_0 \widetilde{M}$, which is shorthand for

$$M(x,z) = \int d^4y\, M_0(x,y)\widetilde{M}(y,z)\ , \tag{53.5}$$

where

$$M_0(x,y) = (-\partial_x^2 + m^2)\delta^4(x-y)\ , \tag{53.6}$$

$$\widetilde{M}(y,z) = \delta^4(y-z) - g\Delta(y-z)\varphi(z)\ . \tag{53.7}$$

Here $\Delta(y-z)$ is the Feynman propagator, which obeys

$$(-\partial_y^2 + m^2)\Delta(y-z) = \delta^4(y-z)\ . \tag{53.8}$$

After various integrations by parts, it is easy to see that eqs. (53.5–53.7) reproduce eq. (53.4).

Now we can use the general matrix relation det AB = det A det B to conclude that

$$\det M = \det M_0\ \det \widetilde{M}\ . \tag{53.9}$$

The advantage of this decomposition is that M_0 is independent of the background field φ, and so the resulting factor of $(\det M_0)^{-1}$ in $Z(\varphi)$ can simply be absorbed into the overall normalization.

Furthermore, we have $\widetilde{M} = I - G$, where

$$I(x,y) = \delta^4(x-y) \tag{53.10}$$

is the identity matrix, and

$$G(x,y) = g\Delta(x-y)\varphi(y) \ . \tag{53.11}$$

Thus, for φ(x) = 0, we have $\widetilde{M}$ = I and so det $\widetilde{M}$ = 1.

Then, using eq. (53.3) and the normalization condition Z(0) = 1, we see that for nonzero φ(x) we must have simply

$$Z(\varphi) = (\det \widetilde{M})^{-1} \ . \tag{53.12}$$

Next, we need the general matrix relation det A = exp Tr ln A, which is most easily proved by working in a basis where A is in Jordan form (that is, all entries below the main diagonal are zero).

Thus we can write

$$\begin{aligned} \det \widetilde{M} &= \exp \operatorname{Tr} \ln \widetilde{M} \\ &= \exp \operatorname{Tr} \ln(I-G) \\ &= \exp \operatorname{Tr} \left[-\sum_{n=1}^{\infty} \frac{1}{n} G^n \right]. \end{aligned} \tag{53.13}$$

Combining eqs. (53.12) and (53.13) we get

$$Z(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} G^n \ , \tag{53.14}$$

where

$$\mathrm{Tr}\, G^n = g^n \int d^4x_1 \ldots d^4x_n \, \Delta(x_1 - x_2)\varphi(x_2) \ldots \Delta(x_n - x_1)\varphi(x_1) \; . \tag{53.15}$$

This is our final result for Z(φ).

To better understand what it means, we will rederive it in a different way.

Consider treating the g$\varphi\chi^\dagger\chi$ term in $\mathcal{L}$ as an interaction.

This leads to a vertex that connects two χ propagators; the associated vertex factor is igφ(x).

According to the general analysis of section 9, we have Z(φ) = exp iΓ(φ), where iΓ(φ) is given by a sum of connected diagrams.

(We have called the exponent Γ rather than W because it is naturally interpreted as a quantum action for φ after χ has been integrated out.)

The only connected diagrams we can draw with these Feynman rules are those of fig. (53.1), with n insertions of the vertex, where n ≥ 1.

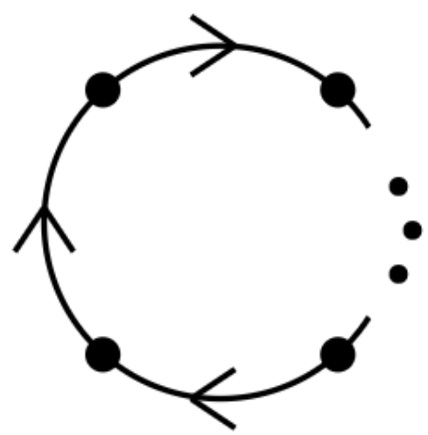


Figure 53.1: All connected diagrams with φ(x) treated as an external field. Each of the n dots represents a factor of igφ(x), and each solid line is a χ or Ψ propagator.

The diagram with n vertices has an n-fold cyclic symmetry, leading to a symmetry factor of S = n.

The factor of i associated with each vertex is canceled by the factor of 1/i associated with each propagator.

Thus the value of the n-vertex diagram is

$$\frac{1}{n}\, g^n \int d^4x_1 \ldots d^4x_n\, \Delta(x_1-x_2)\varphi(x_2) \ldots \Delta(x_n-x_1)\varphi(x_1)\ . \tag{53.16}$$

Summing up these diagrams, and using eq. (53.15), we find

$$i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{Tr}\, G^n\ . \tag{53.17}$$

This neatly reproduces eq. (53.14).

Thus we see that a functional determinant can be represented as an infinite sum of Feynman diagrams.

Next we consider a theory of a Dirac fermion Ψ with

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + g\varphi\overline{\Psi}\Psi\ , \tag{53.18}$$

where φ is again a real scalar background field.

We define the path integral

$$Z(\varphi) = \int \mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi\; e^{i\int d^4x\,\mathcal{L}}\;, \tag{53.19}$$

where we again use the ϱtrick to impose vacuum boundary conditions, and the normalization Z(0) = 1 is fixed by hand.

Recall from section 44 that if we have n complex Grassmann variables ψ_i, then we can evaluate gaussian integrals by the general formula

$$\int d^n\bar{\psi}\,d^n\psi\;\exp\left(-i\bar{\psi}_i M_{ij}\psi_j\right) \propto \det M\;. \tag{53.20}$$

In the case of the functional integral in eq. (53.19), the index i on the integration variable is replaced by the continuous spacetime label x plus the spinor index α, and the "matrix" M becomes

$$M_{\alpha\beta}(x,y) = [-i\not{\partial}_x + m - g\varphi(x)]_{\alpha\beta}\delta^4(x-y)\;. \tag{53.21}$$

In order to apply eq. (53.20), we have to understand what it means to compute the determinant of this expression.

To this end, let us first note that we can write M = $M_0\,\widetilde{M}$, which is shorthand for

$$M_{\alpha\gamma}(x,z) = \int d^4y\, M_{0\alpha\beta}(x,y)\widetilde{M}_{\beta\gamma}(y,z)\;, \tag{53.22}$$

where

$$M_{0\alpha\beta}(x,y) = (-i\partial\!\!\!/_x + m)_{\alpha\beta}\delta^4(x-y)\ , \tag{53.23}$$

$$\widetilde{M}_{\beta\gamma}(y,z) = \delta_{\beta\gamma}\delta^4(y-z) - gS_{\beta\gamma}(y-z)\varphi(z)\ . \tag{53.24}$$

Here $S_{\beta\gamma}(y-z)$ is the Feynman propagator, which obeys

$$(-i\partial\!\!\!/_y + m)_{\alpha\beta}S_{\beta\gamma}(y-z) = \delta_{\alpha\gamma}\delta^4(y-z)\ . \tag{53.25}$$

After various integrations by parts, it is easy to see that eqs. (53.22–53.24) reproduce eq. (53.21).

Now we can use eq. (53.9).

The advantage of this decomposition is that M_0 is independent of the background field φ, and so the resulting factor of det M_0 in $Z(\varphi)$ can simply be absorbed into the overall normalization.

Furthermore, we have $\widetilde{M} = I - G$, where

$$I_{\alpha\beta}(x,y) = \delta_{\alpha\beta}\delta^4(x-y) \tag{53.26}$$

is the identity matrix, and

$$G_{\alpha\beta}(x,y) = gS_{\alpha\beta}(x-y)\varphi(y)\ . \tag{53.27}$$

Thus, for $\varphi(x) = 0$, we have $\widetilde{M} = I$ and so $\det \widetilde{M} = 1$.

Then, using eq. (53.20) and the normalization condition $Z(0) = 1$, we see that for nonzero $\varphi(x)$ we must have simply

$$Z(\varphi) = \det \widetilde{M} \ . \tag{53.28}$$

Next, we use eqs. (53.13) and (53.28) to get

$$Z(\varphi) = \exp -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} G^n \ , \tag{53.29}$$

where now

$$\operatorname{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \operatorname{tr} S(x_1 - x_2)\varphi(x_2) \ldots S(x_n - x_1)\varphi(x_1) \ , \tag{53.30}$$

and "tr" denotes a trace over spinor indices.

This is our final result for $Z(\varphi)$.

To better understand what it means, we will rederive it in a different way.

Consider treating the $g\varphi\,\overline{\Psi}\,\Psi$ term in $\mathcal{L}$ as an interaction.

This leads to a vertex that connects two Ψ propagators; the associated vertex factor is $ig\varphi(x)$.

According to the general analysis of section 9, we have $Z(\varphi) = \exp i\Gamma(\varphi)$, where $i\Gamma(\varphi)$ is given by a sum of connected diagrams.

(We have called the exponent Γ rather than W because it is naturally interpreted as a quantum action for ϕ after Ψ has been integrated out.)

The only connected diagrams we can draw with these Feynman rules are those of fig. (53.1), with n insertions of the vertex, where $n \geq 1$.

The diagram with n vertices has an n-fold cyclic symmetry, leading to a symmetry factor of $S = n$.

The factor of i associated with each vertex is canceled by the factor of 1/i associated with each propagator.

The closed fermion loop implies a trace over the spinor indices.

Thus the value of the n-vertex diagram is

$$\frac{1}{n}\, g^n \int d^4x_1 \ldots d^4x_n \,\mathrm{tr}\, S(x_1 - x_2)\varphi(x_2) \ldots S(x_n - x_1)\varphi(x_1)\ .$$

Summing up these diagrams, we find that we are missing the overall minus sign in eq. (53.29).

The appropriate conclusion is that we must associate an extra minus sign with each closed fermion loop.

Part III

Spin One

φ $'$ τ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ th$\overline{u}$ $\overline{\Psi}$ the the the the $(\overline{\eta}, \eta, J)$ $\overline{v}$

54 Maxwell's Equations

The most common (and important) spin-one particle is the photon.

Emission and absorption of photons by matter is an important phenomenon in many areas of physics, and so that is the context in which most physicists first encounter a serious treatment of photons.

We will use a brief review of this subject (in this section and the next) as our entry point into the theory of quantum electrodynamics.

Let us begin with classical electrodynamics.

Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho \ , \tag{54.1}$$

$$\nabla \times \mathbf{B} - \dot{\mathbf{E}} = \mathbf{J} \ , \tag{54.2}$$

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \ , \tag{54.3}$$

$$\nabla \cdot \mathbf{B} = 0 \ , \tag{54.4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is the charge density, and $\mathbf{J}$ is the current density.

We have written Maxwell's equations in *Heaviside-Lorentz units*, and also set c = 1.

In these units, the magnitude of the force between two charges of magnitude Q is $Q^2/4\pi r^2$.

Maxwell's equations must be supplemented by formulae that give us the dynamics of the charges and currents (such as the Lorentz force law for point particles).

For now, however, we will treat the charges and currents as specified sources, and focus on the dynamics of the electromagnetic fields.

The last two of Maxwell's equations, the ones with no sources on the right-hand side, can be solved by writing the **E** and **B** fields in terms of a scalar potential φ and a vector potential **A**,

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}} \,, \tag{54.5}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \,. \tag{54.6}$$

The potentials uniquely determine the fields, but the fields do not uniquely determine the potentials. Given a particular φ and **A** that result in a particular **E** and **B**, we will get the same **E** and **B** from any other potentials φ' and $\mathbf{A}'$ that are related by

$$\varphi' = \varphi + \dot{\Gamma} \,, \tag{54.7}$$

$$\mathbf{A}' = \mathbf{A} - \nabla\Gamma \,, \tag{54.8}$$

where Γ is an arbitrary function of spacetime.

A change of potentials that does not change the fields is called a *gauge transformation*.

The E and B fields are *gauge invariant*.

All this becomes more compact and elegant in a relativistic notation.

We define the four-vector potential or *gauge field*

$$A^\mu \equiv (\varphi, \mathbf{A}) \ . \tag{54.9}$$

We also define the *field strength*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{54.10}$$

Obviously, $F^{\mu\nu}$ is antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$.

Comparing eqs. (54.5) and (54.6) with eqs. (54.9) and (54.10), we see that

$$F^{0i} = E^i \ , \tag{54.11}$$

$$F^{ij} = \varepsilon^{ijk} B_k \ . \tag{54.12}$$

The first two of Maxwell's equations can now be written as

$$\partial_\nu F^{\mu\nu} = J^\mu \ , \tag{54.13}$$

where

$$J^\mu \equiv (\rho, \mathbf{J}) \tag{54.14}$$

is the charge-current density four-vector.

If we take the four-divergence of eq. (54.13), we get $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu J^\mu$.

The left-hand side of this equation vanishes, because $\partial_\mu \partial_\nu$ is symmetric on exchange of μ and ν, while $F^{\mu\nu}$ is antisymmetric.

We conclude that we must have

$$\partial_\mu J^\mu = 0 \,, \tag{54.15}$$

or equivalently

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0 \,; \tag{54.16}$$

that is, the electromagnetic current must be conserved.

The last two of Maxwell's equations can be written as

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0 \,, \tag{54.17}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor; see section 34.

Plugging in eq. (54.10), we see that eq. (54.17) is automatically satisfied, since the antisymmetric combination of two derivatives vanishes.

Eqs. (54.7) and (54.8) can be combined into

$$A'^\mu = A^\mu - \partial^\mu \Gamma \,. \tag{54.18}$$

Setting $F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$ and using eq. (54.18), we get

$$F'^{\mu\nu} = F^{\mu\nu} - (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\Gamma \ . \tag{54.19}$$

The last term vanishes because derivatives commute; thus the field strength is gauge invariant,

$$F'^{\mu\nu} = F^{\mu\nu} \ . \tag{54.20}$$

Next we will find an action that results in Maxwell's equations as the equations of motion.

We will treat the current as an external source.

The action we seek should be Lorentz invariant, gauge invariant, parity and time-reversal invariant, and no more than second order in derivatives.

The only candidate is $S = \int d^4x\, \mathcal{L}$, where

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \ . \tag{54.21}$$

The first term is obviously gauge invariant, because $F^{\mu\nu}$ is.

After a gauge transformation, eq. (54.18), the second term becomes $J^\mu A'_\mu$, and the difference is

$$\begin{aligned} J^\mu(A'_\mu - A_\mu) &= -J^\mu\partial_\mu\Gamma \\ &= -(\partial_\mu J^\mu)\Gamma - \partial_\mu(J^\mu\Gamma) \ . \end{aligned} \tag{54.22}$$

The first term in eq. (54.22) vanishes because the current is conserved.

The second term is a total divergence, and its integral over d^4x vanishes (assuming suitable boundary conditions at infinity).

Thus the action specified by eq. (54.21) is gauge invariant.

Setting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and multiplying out the terms, eq. (54.21) becomes

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu \tag{54.23}$$

$$= +\tfrac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu A_\mu - \partial^\mu K_\mu \ , \tag{54.24}$$

where $K_\mu = 12\ A^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$.

The last term is a total divergence, and can be dropped.

From eq. (54.24), we can see that varying A^μ while requiring S to be unchanged yields the equation of motion

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu = 0 \ . \tag{54.25}$$

Noting that $\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)A_\nu$, we see that eq. (54.25) is equivalent to eq. (54.13), and hence to Maxwell's equations.

55 Electrodynamics in Coulomb Gauge

Next we would like to construct the hamiltonian, and quantize the electromagnetic field.

There is an immediate difficulty, caused by the gauge invariance: we have too many degrees of freedom.

This problem manifests itself in several ways.

For example, the lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^{\mu}A_{\mu} \tag{55.1}$$

$$= -\tfrac{1}{2}\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} + \tfrac{1}{2}\partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu} + J^{\mu}A_{\mu} \tag{55.2}$$

does not contain the time derivative of A^0.

Thus, this field has no canonically conjugate momentum and no dynamics.

To deal with this problem, we must eliminate the gauge freedom.

We do this by *choosing a gauge*.

We choose a gauge by imposing a *gauge condition*.

This is a condition that we require $A^{\mu}(x)$ to satisfy.

The idea is that there should be only one $A^{\mu}(x)$ that results in a given $F^{\mu\nu}(x)$ and that also satisfies the gauge condition.

One possible class of gauge conditions is $n^\mu A_\mu(x) = 0$, where $n\mu$ is a constant four-vector.

If n is spacelike ($n^2 > 0$), then we have chosen *axial gauge*; if n is lightlike, ($n^2 = 0$), it is *lightcone gauge*; and if n is timelike, ($n^2 < 0$), it is *temporal gauge*.

Another gauge is *Lorenz gauge*, where the condition is $\partial^\mu A_\mu = 0$.

We will meet a family of closely related gauges in section 62.

In this section, we will work in *Coulomb gauge*, also known as *radiation gauge* or *transverse gauge*.

The condition for Coulomb gauge is

$$\nabla \cdot \mathbf{A}(x) = 0 \ . \tag{55.3}$$

We can impose eq. (55.3) by acting on $A_i(x)$ with a projection operator,

$$A_i(x) \to \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) A_j(x) \ . \tag{55.4}$$

We construct the right-hand side of eq. (55.4) by Fourier-transforming $A_i(x)$ to $A_i(k)$, multiplying $A_i(k)$ by the matrix $\delta_{ij} - k_i k_j / k^2$, and then Fourier-transforming back to position space.

From now on, whenever we write A_i, we will implicitly mean the right-hand side of eq. (55.4).

Now let us write out the lagrangian in terms of the scalar and vector potentials, $\varphi = A^0$ and A_i, with A_i obeying the Coulomb gauge condition.

Starting from eq. (55.2), we get

$$\begin{aligned}\mathcal{L} = & \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i \\ & + \tfrac{1}{2}\nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi \\ & + \tfrac{1}{2}\nabla_i\varphi\nabla_i\varphi - \rho\varphi \ .\end{aligned} \tag{55.5}$$

In the second line of eq. (55.5), the ∇_i in each term can be integrated by parts; in the first term, we will then get a factor of $\nabla_j(\nabla_i A_i)$, and in the second term, we will get a factor of $\nabla_i \dot{A}_i$. .

Both of these vanish by virtue of the gauge condition $\nabla_i A_i = 0$, and so both of these terms can simply be dropped.

If we now vary φ (and require $S = \int d^4x\, \mathcal{L}$ to be stationary), we find that φ obeys Poisson's equation,

$$-\nabla^2\varphi = \rho \ . \tag{55.6}$$

The solution is

$$\varphi(\mathbf{x},t) = \int d^3y \ \frac{\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} \ . \tag{55.7}$$

This solution is unique if we impose the boundary conditions that φ and ρ both vanish at spatial infinity.

Eq. (55.7) tells us that $\varphi(x,t)$ is given entirely in terms of the charge density at the same time, and so has no dynamics of its own.

It is therefore legitimate to plug eq. (55.7) back into the lagrangian.

After an integration by parts to turn $\nabla_i\varphi\nabla_i\varphi$ into $-\varphi\nabla^2\varphi = \varphi\rho$, the result is

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i + \mathcal{L}_{\text{coul}} \ , \tag{55.8}$$

where

$$\mathcal{L}_{\text{coul}} = -\frac{1}{2}\int d^3y \ \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} \ . \tag{55.9}$$

We can now vary A_i; keeping proper track of the implicit projection operator in eq. (55.4), we find that A_i obeys the massless Klein-Gordon equation with the projected current as a source,

$$-\partial^2 A_i(x) = \left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right) J_j(x) \ . \tag{55.10}$$

For a free field ($J_i = 0$), the general solution is

$$\mathbf{A}(x) = \sum_{\lambda=\pm}\int \widetilde{dk}\left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k})a^\dagger_\lambda(\mathbf{k})e^{-ikx}\right], \tag{55.11}$$

where $k^0 = \omega = |\mathbf{k}|$, $\widetilde{dk} = d^3k/(2\pi)^3 2\omega$, and $\boldsymbol{\varepsilon}_+(\mathbf{k})$ and $\boldsymbol{\varepsilon}_-(\mathbf{k})$ are polarization vectors.

In order to satisfy the Coulomb gauge condition, the polarization vectors must be orthogonal to the wave vector **k**.

We will choose them to correspond to right- and left-handed circular polarizations; for **k** = (0,0,k), we then have

$$\begin{aligned} \boldsymbol{\varepsilon}_+(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,-i,0)\ , \\ \boldsymbol{\varepsilon}_-(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,+i,0)\ . \end{aligned} \tag{55.12}$$

More generally, the two polarization vectors along with the unit vector in the **k** direction form an orthonormal and complete set,

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0\ , \tag{55.13}$$

$$\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k})\cdot\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) = \delta_{\lambda'\lambda}\ , \tag{55.14}$$

$$\sum_{\lambda=\pm}\varepsilon^*_{i\lambda}(\mathbf{k})\varepsilon_{j\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\ . \tag{55.15}$$

The coefficients $a_\lambda(\mathbf{k})$ and $a^\dagger{}_\lambda(\mathbf{k})$ will become operators after quantization, which is why we have used the dagger symbol for conjugation.

In complete analogy with the procedure used for a scalar field in section 3, we can invert eq. (55.11) and its time derivative to get

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x\ e^{-ikx}\overset{\leftrightarrow}{\partial_0}\mathbf{A}(x)\ , \tag{55.16}$$

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x\ e^{+ikx}\overset{\leftrightarrow}{\partial_0}\mathbf{A}(x)\ , \tag{55.17}$$

where $f\overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f)g$.

Now we can proceed to the hamiltonian formalism. First, we compute the canonically conjugate momentum to A_i,

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i \ . \tag{55.18}$$

Note that $\nabla_i A_i = 0$ implies $\nabla_i \Pi_i = 0$.

The hamiltonian density is then

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{A}_i - \mathcal{L} \\ &= \tfrac{1}{2}\Pi_i \Pi_i + \tfrac{1}{2}\nabla_j A_i \nabla_j A_i - J_i A_i + \mathcal{H}_{\text{coul}} \ , \end{aligned} \tag{55.19}$$

the where $\mathcal{H}_{\text{coul}} = -\mathcal{L}_{\text{coul}}$.

To quantize the field, we impose the canonical commutation relations.

Keeping proper track of the implicit projection operator in eq. (55.4), we have

$$\begin{aligned} [A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] &= i\left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{y}) \\ &= i\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \end{aligned} \tag{55.20}$$

The commutation relations of the $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ operators follow from eq. (55.20) and $[A_i, A_j] = [\Pi_i, \Pi_j] = 0$ (at equal times).

The result is

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0 \,, \tag{55.21}$$

$$[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = 0 \,, \tag{55.22}$$

$$[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = (2\pi)^3 2\omega\, \delta^3(\mathbf{k}' - \mathbf{k})\delta_{\lambda\lambda'} \,. \tag{55.23}$$

We interpret $a^\dagger_\lambda(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ as creation and annihilation operators for photons of definite helicity, with helicity $+1$ corresponding to right-circular polarization and helicity -1 to left-circular polarization.

It is now straightfoward to write the hamiltonian explicitly in terms of these operators.

We find

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\; \omega\; a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) + 2\mathcal{E}_0 V - \int d^3x\; \mathbf{J}(x)\cdot\mathbf{A}(x) + H_{\text{coul}} \,, \tag{55.24}$$

where $\mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3}\int d^3k\; \omega$ is the zero-point energy per unit volume that we found for a real scalar field in section 3, V is the volume of space, the Coulomb hamiltonian is

$$H_{\text{coul}} = \frac{1}{2}\int d^3x\, d^3y\; \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} \,, \tag{55.25}$$

and we use eq. (55.11) to express $A_i(x)$ in terms of $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ at any one particular time (say, $t = 0$).

This is sufficient, because H itself is time independent.

This form of the hamiltonian of electrodynamics is often used as the starting point for calculations of atomic transition rates, with the charges and currents treated via the nonrelativistic Schrodinger equation.

The Coulomb interaction appears explicitly, and the $\mathbf{J} \cdot \mathbf{A}$ term allows for the creation and annihilation of photons of definite polarization.

56 LSZ Reduction for Photons

In section 55, we found that the creation and annihilation operators for free photons could be written as

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) \cdot \int d^3x\; e^{+ikx} \overleftrightarrow{\partial_0}\, \mathbf{A}(x)\;, \tag{56.1}$$

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \int d^3x\; e^{-ikx} \overleftrightarrow{\partial_0}\, \mathbf{A}(x)\;, \tag{56.2}$$

where $\varepsilon_\lambda(\mathbf{k})$ is a polarization vector.

From here, we can follow the analysis of section 5 line by line to deduce the LSZ reduction formula for photons.

The result is that the creation operator for each incoming photon should be replaced by

$$a^\dagger_\lambda(\mathbf{k})_{\rm in} \to i\,\varepsilon^{\mu *}_\lambda(\mathbf{k}) \int d^4x\; e^{+ikx}(-\partial^2)A_\mu(x)\;, \tag{56.3}$$

and the destruction operator for each outgoing photon should be replaced by

$$a_\lambda(\mathbf{k})_{\rm out} \to i\,\varepsilon^{\mu}_\lambda(\mathbf{k}) \int d^4x\; e^{-ikx}(-\partial^2)A_\mu(x)\;, \tag{56.4}$$

and then we should take the vacuum expectation value of the time-ordered product.

Note that, in writing eqs. (56.3) and (56.4), we have made them look nicer by introducing $\varepsilon^0{}_\lambda(\mathbf{k}) \equiv 0$, and then using four-vector dot products rather than three-vector dot products.

The LSZ formula is valid provided the field is normalized according to the free-field formulae

$$\langle 0|A^i(x)|0\rangle = 0\;, \tag{56.5}$$

$$\langle k,\lambda|A^i(x)|0\rangle = \varepsilon^i_\lambda(\mathbf{k})e^{ikx}\;, \tag{56.6}$$

where $|k,\lambda\rangle$ is a single photon state, normalized according to

$$\langle k',\lambda'|k,\lambda\rangle = (2\pi)^3 2\omega\,\delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda\lambda'}\;. \tag{56.7}$$

The zero on the right-hand side of eq. (56.5) is required by rotation invariance, and only the overall scale of the right-hand side of eq. (56.6) might be different in an interacting theory.

The renormalization of A_i necessitates including appropriate Z factors in the lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_1 J^\mu A_\mu \ . \tag{56.8}$$

Here Z_3 and Z_1 are the traditional names; we will meet Z_2 in section 62.

We must choose Z_3 so that eq. (56.6) holds. We will fix Z_1 by requiring the corresponding vertex function to take on a certain value for a particular set of external momenta.

Next we must compute the correlation functions $\langle 0|\mathrm{T}A_i(x)...|0\rangle$.

As usual, we begin by working with free field theory.

The analysis is again almost identical to the case of a scalar field.

We find that, in free field theory,

$$\langle 0|\mathrm{T}A^i(x)A^j(y)|0\rangle = \tfrac{1}{i}\Delta^{ij}(x-y) \ , \tag{56.9}$$

where the propagator is

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\mathbf{k}) \varepsilon_\lambda^j(\mathbf{k}) \ . \tag{56.10}$$

As with a free scalar field, correlations of an odd number of fields vanish, and correlations of an even number of fields are given in terms of the propagator by Wick's theorem; see section 8.

We would now like to evaluate the path integral for the free electromagnetic field

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A \; e^{i\int d^4x\,[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+J^\mu A_\mu]} \; . \tag{56.11}$$

Here we treat the current $J^\mu(x)$ as an external source.

We will evaluate $Z_0(J)$ in Coulomb gauge.

This means that we will integrate over only those field configurations that satisfy $\nabla\cdot\mathbf{A} = 0$.

We begin by integrating over A^0.

Because the action is quadratic in A^μ, this is equivalent to solving the variational equation for A^0, and then substituting the solution back into the lagrangian.

The result is that we have the Coulomb term in the action,

$$S_{\text{coul}} = -\frac{1}{2}\int d^4x\; d^4y\; \delta(x^0-y^0)\; \frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}-\mathbf{y}|} \; . \tag{56.12}$$

Since this term does not depend on the vector potential, we simply get a factor of $\exp(iS_{\text{coul}})$ in front of the remaining path integral over A_i.

We will perform this integral formally (as we did for fermion fields in section 43) by requiring it to yield the correct results for the correlation functions of A_i when we take functional derivatives with respect to J_i.

In this way we find that

$$Z_0(J) = \exp\left[iS_{\rm coul} + \frac{i}{2}\int d^4x\, d^4y\, J_i(x)\Delta^{ij}(x-y)J_j(y)\right] . \tag{56.13}$$

We can make $Z_0(J)$ look prettier by writing it as

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right] , \tag{56.14}$$

where we have defined

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k) , \tag{56.15}$$

$$\tilde{\Delta}^{\mu\nu}(k) \equiv -\frac{1}{\mathbf{k}^2}\,\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\epsilon}\sum_{\lambda=\pm}\varepsilon^{\mu *}_\lambda(\mathbf{k})\varepsilon^\nu_\lambda(\mathbf{k}) . \tag{56.16}$$

The first term on the right-hand side of eq. (56.16) reproduces the Coulomb term in eq. (56.13) by virtue of the facts that

$$\int_{-\infty}^{+\infty}\frac{dk^0}{2\pi}\, e^{-ik^0(x^0-y^0)} = \delta(x^0-y^0) , \tag{56.17}$$

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} . \tag{56.18}$$

The second term on the right-hand side of eq. (56.16) reproduces the second term in eq. (56.13) by virtue of the fact that $\varepsilon^0{}_\lambda(\mathbf{k}) = 0$.

Next we will simplify eq. (56.16). We begin by introducing a unit vector in the time direction,

$$\hat{t}^\mu = (1, \mathbf{0}) \ . \tag{56.19}$$

Next we need a unit vector in the **k** direction, which we will call $\hat{z}^\mu$. .

We first note that $\hat{t}\cdot k$ = –k^0, and so we can write

$$(0, \mathbf{k}) = k^\mu + (\hat{t}\cdot k)\hat{t}^\mu \ . \tag{56.20}$$

The square of this four-vector is

$$\mathbf{k}^2 = k^2 + (\hat{t}\cdot k)^2 \ , \tag{56.21}$$

where we have used $\hat{t}^2$= –1.

Thus the unit vector that we want is

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{56.22}$$

Now we recall from section 55 that

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\mathbf{k})\varepsilon_\lambda^j(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \ . \tag{56.23}$$

This can be extended to i $\rightarrow \mu$ and j $\rightarrow \nu$ by writing

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{\mu*}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k}) = g^{\mu\nu} + \hat{t}^\mu \hat{t}^\nu - \hat{z}^\mu \hat{z}^\nu \ . \tag{56.24}$$

It is not hard to check that the right-hand side of eq. (56.24) vanishes if $\mu = 0$ or $\nu = 0$, and agrees with eq. (56.23) for $\mu = i$ and $\nu = j$.

Putting all this together, we can now write eq. (56.16) as

$$\tilde{\Delta}^{\mu\nu}(k) = \ -\frac{\hat{t}^\mu \hat{t}^\nu}{k^2 + (\hat{t}\cdot k)^2} + \frac{g^{\mu\nu} + \hat{t}^\mu \hat{t}^\nu - \hat{z}^\mu \hat{z}^\nu}{k^2 - i\epsilon} \ . \tag{56.25}$$

The next step is to consider the terms in this expression that contain factors of k^μ or k^ν; from eq. (56.22), we see that these will arise from the $\hat{z}^\mu \hat{z}^\nu$ term.

In eq. (56.15), a factor of k^μ can be written as a derivative with respect to x^μ acting on $e^{ik(x-y)}$.

This derivative can then be integrated by parts in eq. (56.14) to give a factor of $\partial^\mu J_\mu(x)$.

But $\partial^\mu J_\mu(x)$ vanishes, because the current must be conserved.

Similarly, a factor of k^ν can be turned into $\partial^\nu J_\nu(y)$, and also leads to a vanishing contribution.

Therefore, we can *ignore any terms* in $\widetilde{\Delta}^{\mu\nu}(k)$ *that contain factors* of k^μ or k^ν.

From eq. (56.22), we see that this means we can make the substitution

$$\hat{z}^\mu \to \frac{(\hat{t}\cdot k)\hat{t}^\mu}{[k^2+(\hat{t}\cdot k)^2]^{1/2}}\ . \tag{56.26}$$

Then eq. (56.25) becomes

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left[g^{\mu\nu} + \left(-\frac{k^2}{k^2+(\hat{t}\cdot k)^2} + 1 - \frac{(\hat{t}\cdot k)^2}{k^2+(\hat{t}\cdot k)^2}\right)\hat{t}^\mu\hat{t}^\nu\right], \tag{56.27}$$

where the three coefficients of $\hat{t}^\mu\hat{t}^\nu$ come from the Coulomb term, the $\hat{t}^\mu\hat{t}^\nu$ term in the polarization sum, an $\hat{z}^\mu\hat{z}^\nu$ term, respectively.

A bit of algebra now reveals that the net coefficient of $\hat{t}^\mu\hat{t}^\nu$ vanishes, leaving us with the elegant expression

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon}\ . \tag{56.28}$$

Written in this way, the photon propagator is said to be in Feynman gauge.

(It would still be in Coulomb gauge if we had retained the k^μ and k^ν terms that we previously dropped.)

In the next section, we will rederive eq. (56.28) from a more explicit path-integral point of view.

57 The Path Integral for Photons

In this section, in order to get a better understanding of the photon path integral, we will evaluate it directly, using the methods of section 8.

We begin with

$$Z_0(J) = \int \mathcal{D}A \, e^{iS_0} \, , \tag{57.1}$$

$$S_0 = \int d^4x \left[-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right] \, . \tag{57.2}$$

Following section 8, we Fourier-transform to momentum space, where we find

$$\begin{aligned} S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[& -\widetilde{A}_\mu(k) \Big(k^2 g^{\mu\nu} - k^\mu k^\nu \Big) \widetilde{A}_\nu(-k) \\ & + \widetilde{J}^\mu(k) \widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k) \widetilde{A}_\mu(k) \Big] \, . \end{aligned} \tag{57.3}$$

The next step is to shift the integration variable $\widetilde{A}$ so as to "complete the square".

This involves inverting the 4×4 matrix $k^2 g^{\mu\nu} - k^\mu k^\nu$.

However, this matrix has a zero eigenvalue, and cannot be inverted.

To see this, let us write

$$k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k) \, , \tag{57.4}$$

where we have defined

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2 \ . \tag{57.5}$$

This is a projection matrix because, as is easily checked,

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k) \ . \tag{57.6}$$

Thus the only allowed eigenvalues of P are zero and one.

There is at least one zero eigenvalue, because

$$P^{\mu\nu}(k) k_\nu = 0 \ . \tag{57.7}$$

On the other hand, the sum of the eigenvalues is given by the trace

$$g_{\mu\nu} P^{\mu\nu}(k) = 3 \ . \tag{57.8}$$

Thus the remaining three eigenvalues must all be one.

Now let us imagine carrying out the path integral of eq. (57.1), with S_0 given by eq. (57.3).

Let us decompose the field $\widetilde{A}_\mu(k)$ into components aligned along a set of linearly independent four-vectors, one of which is k^μ.

(It will not matter whether or not this basis set is orthonormal.)

Because the term quadratic in $\tilde{A}_\mu$ involves the matrix $k^2 P^{\mu\nu}(k)$, and $P^{\mu\nu}(k)k_\nu = 0$, the component of $\tilde{A}_\mu(k)$ that lies along k_μ does not contribute to this either, because $\partial^\mu J_\mu(x) = 0$ implies $k^\mu J_\mu(k) = 0$.

Thus this component does not appear in the path integral at all!

It then makes no sense to integrate over it.

We therefore define $\int \mathcal{D}A$ to mean integration over only those components that are spanned by the remaining three basis vectors, and therefore satisfy $k^\mu \tilde{A}_\mu(k) = 0$.

This is equivalent to imposing Lorenz gauge, $\partial^\mu A_\mu(x) = 0.$.

The matrix $P^{\mu\nu}(k)$ is simply the matrix that projects a four-vector into the subspace orthogonal to k_μ.

Within the subspace, $P^{\mu\nu}(k)$ is equivalent to the identity matrix.

Therefore, within the subspace, the inverse of $k^2 P^{\mu\nu}(k)$ is $(1/k^2)P^{\mu\nu}(k)$.

Employing the ϵ trick to pick out vacuum boundary conditions replaces k^2 with $k^2 - i\epsilon$.

We can now continue following the procedure of section 8, with the result that

$$Z_0(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\widetilde{J}_\mu(k)\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\,\widetilde{J}_\nu(-k)\right] \tag{57.9}$$

$$= \exp\left[\frac{i}{2}\int d^4x\,d^4y\,J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right],$$

where

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon} \tag{57.10}$$

is the photon propagator in *Lorenz gauge* (also known as *Landau gauge*).

Of course, because the current is conserved, the $k^\mu k^\nu$ term in $P^{\mu\nu}(k)$ does not contribute, and so the result is equivalent to that of Feynman gauge, where $P^{\mu\nu}(k)$ is replaced by $g^{\mu\nu}$.

58 Spinor Electrodynamics

In the section, we will study spinor electrodynamics: the theory of photons interacting with the electrons and positrons of a Dirac field.

(We will use the term quantum electrodynamics to denote any theory of photons, irrespective of the kinds of particles with which they interact.)

We construct spinor electrodynamics by taking the electromagnetic current $j^\mu(x)$ to be proportional to the Noether current corresponding to the U(1) symmetry of a Dirac field; see section 36.

Specifically,

$$j^{\mu}(x) = e\overline{\Psi}(x)\gamma^{\mu}\Psi(x) \ . \tag{58.1}$$

Here e = –0.302822 is the charge of the electron in Heaviside-Lorentz units, with $\hbar = c = 1$.

(We will rely on context to distinguish this e from the base of natural logarithms.)

In these units, the fine-structure constant is $\alpha = e^2/4\pi = 1/137.036$.

With the normalization of eq. (58.1), $Q = \int d^3x j^0(x)$ is the electric charge operator.

Of course, when we specify a number in quantum field theory, we must always have a renormalization scheme in mind; e= –0.302822 corresponds to a specific version of on-shell renormalization that we will explore in sections 62 and 63.

The value of e is different in other renormalization schemes, such as $\overline{MS}$, as we will see in section 66.

The complete lagrangian of our theory is thus

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma^{\mu}\Psi A_{\mu} \ . \tag{58.2}$$

In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

We have a problem, though.

A Noether current is conserved only when the fields obey the equations of motion, or, equivalently, only at points in field space where the action is stationary.

On the other hand, in our development of photon path integrals in sections 56 and 57, we assumed that the current was always conserved.

This issue is resolved by enlarging the definition of a gauge transformation to include a transformation on the Dirac field as well as the electromagnetic field.

Specifically, we define a gauge transformation to consist of

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \Gamma(x) \ , \tag{58.3}$$

$$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x) \ , \tag{58.4}$$

$$\overline{\Psi}(x) \to \exp[+ie\Gamma(x)]\overline{\Psi}(x) \ . \tag{58.5}$$

It is not hard to check that $\mathcal{L}$(x) is *invariant* under this transformation, whether or not the fields obey their equations of motion.

To perform this check most easily, we first rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\not{D}\Psi - m\overline{\Psi}\Psi \ , \tag{58.6}$$

where we have defined the *gauge covariant derivative* (or just *covariant derivative* for short)

$$D_\mu \equiv \partial_\mu - ieA_\mu \ . \tag{58.7}$$

In the last section, we found that $F^{\mu\nu}$ is invariant under eq. (58.3), and so the FF term in $\mathcal{L}$ is obviously invariant as well.

It is also obvious that the m $\overline{\Psi}\Psi$ term in L is invariant under eqs. (58.4) and (58.5).

This leaves the $\overline{\Psi}\not{D}\Psi$ term.

This term will also be invariant if, under the gauge transformation, the covariant derivative of Ψ transforms as

$$D_\mu \Psi(x) \to \exp[-ie\Gamma(x)] D_\mu \Psi(x) \ . \tag{58.8}$$

To see if this is true, we note that

$$\begin{aligned} D_\mu \Psi &\to \Big(\partial_\mu - ie[A_\mu - \partial_\mu \Gamma]\Big)\Big(\exp[-ie\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu \Psi - ie(\partial_\mu \Gamma)\Psi - ie[A_\mu - \partial_\mu \Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu - ieA_\mu\Big)\Psi \\ &= \exp[-ie\Gamma] D_\mu \Psi \ . \end{aligned} \tag{58.9}$$

So eq. (58.8) holds, and $\overline{\Psi}\not{D}\Psi$ is gauge invariant.

We can also write the transformation rule for D_μ a little more abstractly as

$$D_\mu \to e^{-ie\Gamma} D_\mu \, e^{+ie\Gamma} \ , \tag{58.10}$$

where the ordinary derivative in D_μ is defined to act on anything to its right, including any fields that are left unwritten in eq. (58.10).

Thus we have

$$\begin{aligned} D_\mu \Psi \ &\to \ \left(e^{-ie\Gamma} D_\mu e^{+ie\Gamma} \right) \left(e^{-ie\Gamma} \Psi \right) \\ &= \ e^{-ie\Gamma} D_\mu \Psi \ , \end{aligned} \tag{58.11}$$

which is, of course, the same as eq. (58.9).

We can also express the field strength in terms of the covariant derivative by noting that

$$[D^\mu, D^\nu] \Psi(x) = -ieF^{\mu\nu}(x) \Psi(x) \ . \tag{58.12}$$

We can write this more abstractly as

$$F^{\mu\nu} = \frac{i}{e} [D^\mu, D^\nu] \ , \tag{58.13}$$

where, again, the ordinary derivative in each covariant derivative acts on anything to its right.

From eqs. (58.10) and (58.13), we see that, under a gauge transformation,

$$
\begin{aligned}
F^{\mu\nu} \;\to\; & \frac{i}{e}\Big[e^{-ie\Gamma} D^\mu \, e^{+ie\Gamma}, e^{-ie\Gamma} D^\nu \, e^{+ie\Gamma}\Big] \\
= \; & e^{-ie\Gamma}\Big(\frac{i}{e}[D^\mu, D^\nu]\Big)e^{+ie\Gamma} \\
= \; & e^{-ie\Gamma} F^{\mu\nu} e^{+ie\Gamma} \\
= \; & F^{\mu\nu} \; .
\end{aligned}
\tag{58.14}
$$

In the last line, we are able to cancel the $e^{\pm ie\Gamma}$ factors against each other because no derivatives act on them.

Eq. (58.14) shows us that (as we already knew) $F^{\mu\nu}$ is gauge invariant.

It is interesting to note that the gauge transformation on the fermion fields, eqs. (58.4–58.5), is a generalization of the U(1) transformation

$$\Psi \to e^{-i\alpha}\Psi \; , \tag{58.15}$$

$$\overline{\Psi} \to e^{+i\alpha}\overline{\Psi} \; , \tag{58.16}$$

that is a symmetry of the free Dirac lagrangian.

The difference is that, in the gauge transformation, the phase factor is allowed to be a function of spacetime, rather than a constant that is the same everywhere.

Thus, the gauge transformation is also called a local U(1) transformation, while eqs. (58.15–58.16) correspond to a *global* U(1) transformation.

We say that, in a gauge theory, the global U(1) symmetry is promoted to a local U(1) symmetry, or that we have *gauged* the U(1) symmetry.

In section 57, we argued that the path integral over A_μ should be restricted to those components of $\tilde{A}_\mu(k)$ that are orthogonal to k_μ, because the component parallel to k_μ did not appear in the integrand.

Now we must make a slightly more subtle argument.

We argue that the path integral over the parallel component is redundant, because the fermionic path integral over Ψ and $\overline{\Psi}$ already includes all possible values of $\Gamma(x)$.

Therefore, as in section 57, we should not integrate over the parallel component. (We will make a more precise and careful version of this argument when we discuss the quantization of nonabelian gauge theories in section 71.)

By the standard procedure, this leads us to the following form of the path integral for spinor electrodynamics:

$$Z(\overline{\eta},\eta,J) \propto \exp\left[ie\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right] \times Z_0(\overline{\eta},\eta,J)\ , \tag{58.17}$$

where

$$Z_0(\overline{\eta},\eta,J) = \exp\left[i\int d^4x\, d^4y\, \overline{\eta}(x)S(x-y)\eta(y)\right]$$
$$\times \exp\left[\frac{i}{2}\int d^4x\, d^4y\; J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right], \tag{58.18}$$

and

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{(-\not{p}+m)}{p^2+m^2-i\epsilon}\, e^{ip(x-y)}\ , \tag{58.19}$$

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \frac{g_{\mu\nu}}{k^2-i\epsilon}\, e^{ik(x-y)} \tag{58.20}$$

are the appropriate Feynman propagators for the corresponding free fields, with the photon propagator in Feynman gauge.

We impose the normalization Z(0,0,0) = 1, and write

$$Z(\overline{\eta},\eta,J) = \exp[iW(\overline{\eta},\eta,J)]\ . \tag{58.21}$$

Then iW $(\overline{\eta},\eta,J)$ can be expressed as a series of connected Feynman diagrams with sources.

The rules for internal and external Dirac fermions were worked out in the context of Yukawa theory in section 45, and they follow here with no change.

For external photons, the LSZ analysis of section 56 implies that each external photon line carries a factor of the polarization vector $\varepsilon^{\mu}(\mathbf{k})$.

Putting everything together, we get the following set of Feynman rules for tree-level processes in spinor electrodynamics.

1. For each *incoming electron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with the electron's four-momentum, p_i.

2. For each *outgoing electron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with the electron's four-momentum, p'_i.

3. For each *incoming positron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with *minus* the positron's four-momentum, $-p_i$.

4. For each *outgoing positron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with *minus* the positron's four-momentum, $-p'_i$.

5. For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i. (Wavy lines for photons is a standard convention.)

6. For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k'_i.

7. The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one wavy line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent*.

8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9. The value of a diagram consists of the following factors:

   for each incoming photon, $\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$;

   for each outgoing photon, $\varepsilon^{\mu}_{\lambda'_i}(\mathbf{k}'_i)$;

   for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

   for each outgoing electron, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

   for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

   for each outgoing positron, $v_{s'_i}(\mathbf{p}'_i)$;

   for each vertex, $ie\gamma^{\mu}$;

   for each internal photon, $-ig^{\mu\nu}/(k^2 - i\epsilon)$;

   for each internal fermion, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$.

10. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative).

12. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

In the next section, we will do a sample calculation.

59 Scattering in Spinor Electrodynamics

In the last section, we wrote down the Feynman rules for spinor electrodynamics.

In this section, we will compute the scattering amplitude (and its spin-averaged square) at tree level for the process of electron-positron annihilation into a pair of photons, $e^+ e^- \to \gamma\,\gamma$.

The contributing diagrams are shown in fig. (59.1), and the associated expression for the scattering amplitude is

$$\mathcal{T} = e^2\, \varepsilon^{\mu}_{1'} \varepsilon^{\nu}_{2'}\, \overline{v}_2 \left[\gamma_\nu \left(\frac{-\not{p}_1 + \not{k}'_1 + m}{-t + m^2} \right) \gamma_\mu + \gamma_\mu \left(\frac{-\not{p}_1 + \not{k}'_2 + m}{-u + m^2} \right) \gamma_\nu \right] u_1 \,, \tag{59.1}$$

where $\varepsilon^{\mu}{}_{1'}$ is shorthand for $\varepsilon^{\mu}_{\lambda'_1}(\mathbf{k}'_1)$, $\overline{v}_2$ is shorthand for $\overline{v}_{s_2}(\mathbf{p}_2)$, and so on.

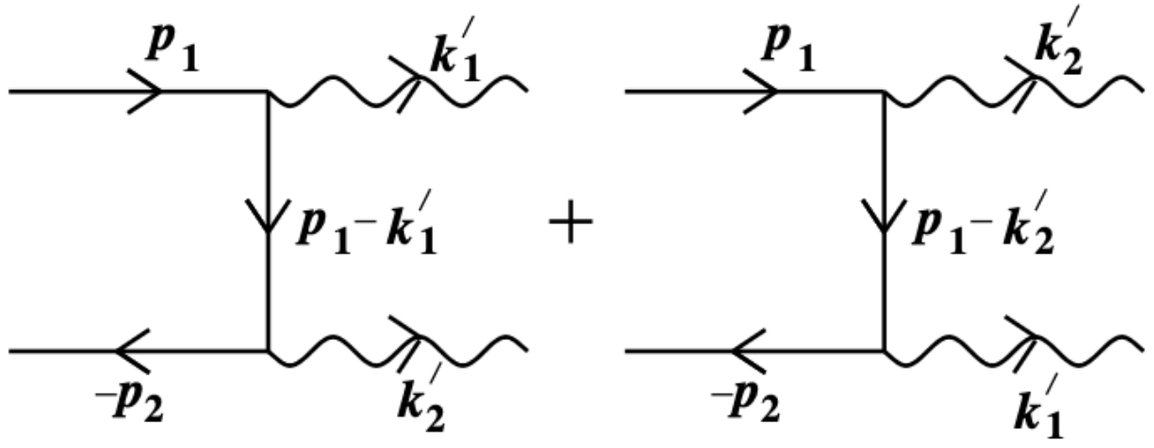


Figure 59.1: Diagrams for $e^+ e^- \to \gamma\,\gamma$, corresponding to eq. (59.1).

The Mandelstam variables are

$$\begin{aligned} s &= -(p_1 + p_2)^2 = -(k'_1 + k'_2)^2 \,, \\ t &= -(p_1 - k'_1)^2 = -(p_2 - k'_2)^2 \,, \\ u &= -(p_1 - k'_2)^2 = -(p_2 - k'_1)^2 \,, \end{aligned} \tag{59.2}$$

and they obey $s + t + u = 2m^2$

Following the procedure of section 46, we write eq. (59.1) as

$$\mathcal{T} = \varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\overline{v}_2 A_{\mu\nu} u_1 \ , \tag{59.3}$$

where

$$A_{\mu\nu} \equiv e^2\left[\gamma_\nu\left(\frac{-\not{p}_1 + \not{k}'_1 + m}{-t+m^2}\right)\gamma_\mu + \gamma_\mu\left(\frac{-\not{p}_1 + \not{k}'_2 + m}{-u+m^2}\right)\gamma_\nu\right] . \tag{59.4}$$

We also have

$$\mathcal{T}^* = \overline{\mathcal{T}} = \varepsilon^{\rho *}_{1'}\varepsilon^{\sigma *}_{2'}\,\overline{u}_1\,\overline{A_{\rho\sigma}}\,v_2 \ . \tag{59.5}$$

Using $\overline{\not{a}\not{b}\ldots} = \ldots\not{b}\not{a}$, we see from eq. (59.4) that

$$\overline{A_{\rho\sigma}} = A_{\sigma\rho} \ . \tag{59.6}$$

Thus we have

$$|\mathcal{T}|^2 = \varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\varepsilon^{\rho *}_{1'}\varepsilon^{\sigma *}_{2'}\,(\overline{v}_2 A_{\mu\nu} u_1)(\overline{u}_1 A_{\sigma\rho} v_2) \ . \tag{59.7}$$

Next, we will average over the initial electron and positron spins, using the technology of section 46; the result is

$$\tfrac{1}{4}\sum_{s_1,s_2}|\mathcal{T}|^2 = \tfrac{1}{4}\varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\varepsilon^{\rho *}_{1'}\varepsilon^{\sigma *}_{2'}\,\mathrm{Tr}\Big[A_{\mu\nu}(-\not{p}_1+m)A_{\sigma\rho}(-\not{p}_2-m)\Big] \ . \tag{59.8}$$

We would also like to sum over the final photon polarizations.

From eq. (59.8), we see that we must evaluate

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(\mathbf{k})\varepsilon_\lambda^{\rho *}(\mathbf{k}) \ . \tag{59.9}$$

We did this polarization sum in Coulomb gauge in section 56, with the result that

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(\mathbf{k})\varepsilon_\lambda^{\rho *}(\mathbf{k}) = g^{\mu\rho} + \hat{t}^\mu \hat{t}^\rho - \hat{z}^\mu \hat{z}^\rho \ , \tag{59.10}$$

where $\hat{t}^\mu$ is a unit vector in the time direction, and $\hat{z}^\mu$ is a unit vector in the $\mathbf{k}$ direction that can be expressed as

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{59.11}$$

It is tempting to drop the k^μ and k^ρ terms in eq. (59.10), on the grounds that the photons couple to a conserved current, and so these terms should not contribute. (We indeed used this argument to drop the analogous terms in the photon propagator.)

This also follows from the notion that the scattering amplitude should be invariant under a gauge transformation, as represented by a transformation of the external polarization vectors of the form

$$\varepsilon_\lambda^\mu(\mathbf{k}) \to \varepsilon_\lambda^\mu(\mathbf{k}) - i\tilde{\Gamma}(k)k^\mu \ . \tag{59.12}$$

Thus, if we write a scattering amplitude $\mathcal{T}$ for a process that includes a particular outgoing photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^\mu_\lambda(\mathbf{k})\mathcal{M}_\mu \ , \tag{59.13}$$

or a particular incoming photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^{\mu *}_\lambda(\mathbf{k})\mathcal{M}_\mu \ , \tag{59.14}$$

then in either case we should have

$$k^\mu \mathcal{M}_\mu = 0 \ . \tag{59.15}$$

Eq. (59.15) is in fact valid; we will give a proof of it, based on the Ward identity for the electromagnetic current, in section 67.

For now, we will take eq. (59.15) as given, and so drop the k^μ and k^ρ terms in eq. (59.10).

This leaves us with

$$\sum_{\lambda=\pm} \varepsilon^\mu_\lambda(\mathbf{k})\varepsilon^{\rho *}_\lambda(\mathbf{k}) \to g^{\mu\rho} + \hat{t}^\mu \hat{t}^\rho - \frac{(\hat{t}\cdot k)^2}{k^2 + (\hat{t}\cdot k)^2}\, \hat{t}^\mu \hat{t}^\rho \ . \tag{59.16}$$

But, for an external photon, $k^2 = 0$.

the the $(\overline{\eta}, \eta, J)$ $\overline{v}$

Thus the second and third terms in eq. (59.16) cancel, leaving us with the beautifully simple substitution rule

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(\mathbf{k})\varepsilon_\lambda^{\rho *}(\mathbf{k}) \to g^{\mu\rho} \ . \tag{59.17}$$

Using eq. (59.17), we can sum $|T|^2$ over the polarizations of the outgoing photons, in addition to averaging over the spins of the incoming fermions; the result is

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &\equiv \tfrac{1}{4}\sum_{\lambda_1',\lambda_2'}\sum_{s_1,s_2}|\mathcal{T}|^2 \\
&= \tfrac{1}{4}\mathrm{Tr}\Big[A_{\mu\nu}(-\not{p}_1+m)A^{\nu\mu}(-\not{p}_2-m)\Big] \\
&= e^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2}+\frac{\langle\Phi_{tu}\rangle+\langle\Phi_{ut}\rangle}{(m^2-t)(m^2-u)}+\frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2}\right],
\end{aligned} \tag{59.18}$$

where

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_1'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_1'+m)\gamma^\nu(-\not{p}_2-m)\Big] \ , \\
\langle\Phi_{uu}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_2'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_2'+m)\gamma^\mu(-\not{p}_2-m)\Big] \ , \\
\langle\Phi_{tu}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_1'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_2'+m)\gamma^\mu(-\not{p}_2-m)\Big] \ , \\
\langle\Phi_{ut}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_2'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_1'+m)\gamma^\nu(-\not{p}_2-m)\Big] \ .
\end{aligned} \tag{59.19}$$

Examinging $\langle \Phi_{tt} \rangle$ and $\langle \Phi_{uu} \rangle$, we see that they are transformed into each other by $k'1 \leftrightarrow k'2$, which is equivalent to $t \leftrightarrow u$.

The same is true of $\langle \Phi_{tu} \rangle$ and $\langle \Phi ut \rangle$.

Thus we need only compute $\langle \Phi_{tt} \rangle$ and $\langle \Phi_{tu} \rangle$, and then take $t \leftrightarrow u$ to get $\langle \Phi_{uu} \rangle$ and $\langle \Phi_{ut} \rangle$.

Now we can apply the gamma-matrix technology of section 47.

In particular, we will need the $d = 4$ relations

$$\begin{aligned} \gamma^\mu \gamma_\mu &= -4 \,, \\ \gamma^\mu \not{a} \gamma_\mu &= 2\not{a} \,, \\ \gamma^\mu \not{a} \not{b} \gamma_\mu &= 4(ab) \,, \\ \gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu &= 2\not{c} \not{b} \not{a} \,, \end{aligned} \tag{59.20}$$

in addition to the trace formulae.

We also need

$$\begin{aligned} p_1 p_2 &= -\tfrac{1}{2}(s - 2m^2) \,, \\ k'_1 k'_2 &= -\tfrac{1}{2} s \,, \\ p_1 k'_1 = p_2 k'_2 &= +\tfrac{1}{2}(t - m^2) \,, \\ p_1 k'_2 = p_2 k'_1 &= +\tfrac{1}{2}(u - m^2) \,. \end{aligned} \tag{59.21}$$

which follow from eq. (59.2) plus the mass-shell conditions $p^2_1 = p^2_2 = -m^2$ and $k'^2_1 = k'^2_2 = 0$.

After a lengthy and tedious calculation, we find

$$\langle \Phi_{tt} \rangle = 2[tu - m^2(3t + u) - m^4] \,, \tag{59.22}$$

$$\langle \Phi_{tu} \rangle = 2m^2(s - 4m^2) \,, \tag{59.23}$$

which then implies

$$\langle \Phi_{uu} \rangle = 2[tu - m^2(3u + t) - m^4] \,, \tag{59.24}$$

$$\langle \Phi_{ut} \rangle = 2m^2(s - 4m^2) \,. \tag{59.25}$$

This completes our calculation.

Other tree-level scattering processes in spinor electrodynamics pose no new calculational diffculties.

In the high-energy limit, where the electron can be treated as massless, we can reduce our labor with the method of spinor helicity,

60 Spinor Helicity for Spinor Electrodynamics

In section 50, we introduced a special notation for u and v spinors of definite helicity for massless electrons and positrons.

This notation greatly simplifies calculations in the high-energy limit (s, |t|, and |u| all much greater than m^2).

We define the twistors

$$
\begin{aligned}
|p] &\equiv u_-(p) = v_+(p) \,, \\
|p\rangle &\equiv u_+(p) = v_-(p) \,, \\
[p| &\equiv \overline{u}_+(p) = \overline{v}_-(p) \,, \\
\langle p| &\equiv \overline{u}_-(p) = \overline{v}_+(p) \,.
\end{aligned}
\tag{60.1}
$$

We then have

$$
\begin{aligned}
[k|\,|p] &= [k\,p] \,, \\
\langle k|\,|p\rangle &= \langle k\,p\rangle \,, \\
[k|\,|p\rangle &= 0 \,, \\
\langle k|\,|p] &= 0 \,,
\end{aligned}
\tag{60.2}
$$

where the *twistor products* [kp] and ⟨kp⟩ are antisymmetric,

$$
\begin{aligned}
[k\,p] &= -[p\,k] \,, \\
\langle k\,p\rangle &= -\langle p\,k\rangle \,,
\end{aligned}
\tag{60.3}
$$

and related by complex conjugation, $\langle pk\rangle^* = [kp]$.

They can be expressed explicitly in terms of the components of the massless four-momenta k and p.

However, more useful are the relations

$$\begin{aligned} \langle k\, p\rangle\, [p\, k] &= \mathrm{Tr}\, \tfrac{1}{2}(1-\gamma_5)\not{k}\not{p} \\ &= -2k\cdot p \\ &= -(k+p)^2 \end{aligned} \tag{60.4}$$

and

$$\begin{aligned} \langle p\, q\rangle\, [q\, r]\, \langle r\, s\rangle\, [s\, p] &= \mathrm{Tr}\, \tfrac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} \\ &= 2[(p\cdot q)(r\cdot s) - (p\cdot r)(q\cdot s) + (p\cdot s)(q\cdot r) \\ &\qquad + i\varepsilon^{\mu\nu\rho\sigma} p_\mu q_\nu r_\rho s_\sigma] \,. \end{aligned} \tag{60.5}$$

Finally, for any massless four-momentum p we can write

$$-\not{p} = |p\rangle[p| + |p]\langle p| \,. \tag{60.6}$$

We will quote other results from section 50 as we need them.

To apply this formalism to spinor electrodynamics, we need to write photon polarization vectors in terms of twistors.

The formulae we need are

$$\varepsilon_+^\mu(k) = -\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle q\, k\rangle} \,, \tag{60.7}$$

$$\varepsilon_-^\mu(k) = -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[q\, k]} \,, \tag{60.8}$$

where q is an arbitrary massless *reference momentum*.

We will verify eqs. (60.7) and (60.8) for a specific choice of k, and then rely on the Lorentz transformation properties of twistors to conclude that the result must hold in any frame (and therefore for any massless four-momentum k).

We will choose $k^\mu = (\omega, \omega\hat{\mathbf{z}}) = \omega(1,0,0,1)$.

Then, the most general form of $\varepsilon^\mu{}_+(k)$ is

$$\varepsilon_+^\mu(k) = e^{i\phi}\frac{1}{\sqrt{2}}(0,1,-i,0) + Ck^\mu \ . \tag{60.9}$$

Here $e^{i\phi}$ is an arbitrary phase factor, and C is an arbitrary complex number; the freedom to add a multiple of k comes from the underlying gauge invariance.

To verify that eq. (60.7) reproduces eq. (60.9), we need the explicit form of the twistors $|k]$ and $|k\rangle$ when the three-momentum is in the z direction.

Using results in section 50 we find

$$|k] = \sqrt{2\omega}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |k\rangle = \sqrt{2\omega}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{60.10}$$

For any value of q, the twistor $\langle q|$ takes the form

$$\langle q| = (0,\ 0,\ \alpha,\ \beta)\ , \tag{60.11}$$

where α and β are complex numbers.

Plugging eqs. (60.10) and (60.11) into eq. (60.7), and using

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{60.12}$$

along with $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma}^\mu = (I, -\vec{\sigma})$, we find that we reproduce eq. (60.9) with $e^{i\phi} = 1$ and $C = -\beta/(\sqrt{2}\,\alpha\omega)$.

There is now no need to check eq. (60.8), because $\varepsilon^\mu{}_-(k) = -[\varepsilon^\mu{}_+(k)]^*$, as can be seen by using $\langle qk\rangle^* = -[qk]$ along with another result from section 50, $\langle q|\gamma^\mu|k]^* = \langle k|\gamma^\mu|q]$.

In spinor electodynamics, the vector index on a photon polarization vector is always contracted with the vector index on a gamma matrix.

We can get a convenient formula for $\not{\varepsilon}_\pm(k)$ by using the Fierz identities

$$-\tfrac{1}{2}\gamma^\mu\langle q|\gamma_\mu|k] = |k]\langle q| + |q\rangle[k| \,, \tag{60.13}$$

$$-\tfrac{1}{2}\gamma^\mu[q|\gamma_\mu|k\rangle = |k\rangle[q| + |q]\langle k| \,. \tag{60.14}$$

We then have

$$\not{\varepsilon}_+(k;q) = \frac{\sqrt{2}}{\langle q\,k\rangle}\Big(|k]\langle q| + |q\rangle[k|\Big)\,, \tag{60.15}$$

$$\not{\varepsilon}_-(k;q) = \frac{\sqrt{2}}{[q\,k]}\Big(|k\rangle[q| + |q]\langle k|\Big)\,, \tag{60.16}$$

where we have added the reference momentum as an explicit argument on the left-hand sides.

Now we have all the tools we need for doing calculations.

However, we can simplify things even further by making maximal use of crossing symmetry.

Note from eq. (60.1) that u_- (which is the factor associated with an incoming electron) and v_+ (an outgoing positron) are both represented by the twistor $|p]$, while $\overline{u}_+$ (an outgoing electron) and $\overline{v}_-$ (an incoming positron) are both represented by $[p|$.

Thus the square-bracket twistors correspond to outgoing fermions with positive helicity, and incoming fermions with negative helicity.

Similarly, the angle-bracket twistors correspond to outgoing fermions with negative helicity, and incoming fermions with positive helicity.

Let us adopt a convention in which all particles are assigned four-momenta that are treated as outgoing.

A particle that has an assigned four-momentum p then has physical four-momentum $\epsilon_p p$, where $\epsilon_p = \text{sign}(p^0) = +1$ if the particle is physically outgoing, and $\epsilon_p = \text{sign}(p^0) = -1$ if the particle is physically incoming.

Since the physical three-momentum of an incoming particle is opposite to its assigned three-momentum, a particle with negative helicity relative to its physical three-momentum has positive helicity relative to its assigned three-momentum.

From now on, we will refer to the helicity of a particle relative to its assigned momentum.

Thus a particle that we say has "positive helicity" actually has negative physical helicity if it is incoming, and positive physical helicity if it is outgoing.

With this convention, the square-bracket twistors $|p]$ and $[p|$ represent positive-helicity fermions, and the angle-bracket twistors $|p\rangle$ and $\langle p|$ represent negative-helicity fermions.

When $\epsilon_p = \mathrm{sign}(p^0) = -1$, we analytically continue the twistors by replacing each $\omega^{1/2}$ in eq. (60.10) with $i|\omega|^{1/2}$.

Then all of our formulae for twistors and polarizations hold without change, with the exception of the rule for complex conjugation of a twistor product, which becomes

$$\langle p\, k\rangle^* = \epsilon_p \epsilon_k [k\, p] \ . \tag{60.17}$$

Now we are ready to calculate some amplitudes.

Consider first the process of fermion-fermion scattering.

The contributing tree-level diagrams are shown in fig. (60.1).

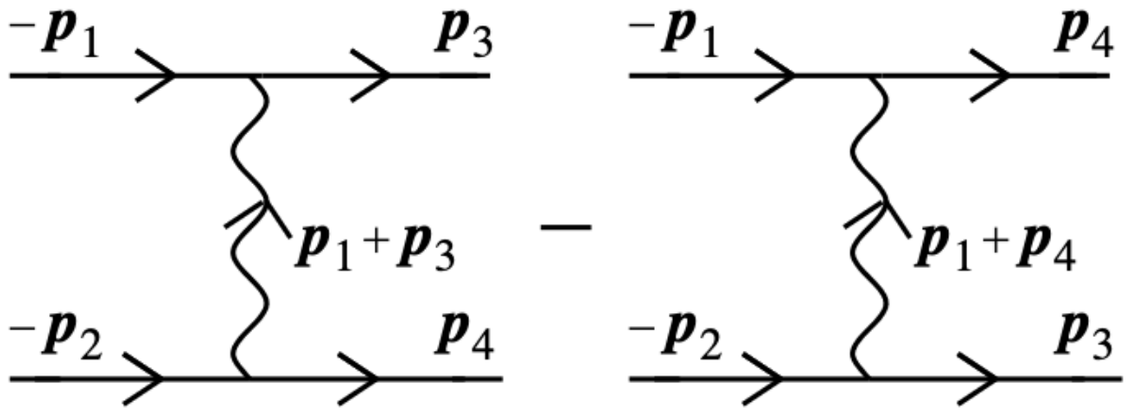


Figure 60.1: Diagrams for fermion-fermion scattering, with all momenta treated as outgoing.

The first thing to notice is that a diagram is zero if two external fermion lines that meet at a vertex have the same helicity.

This is because (as shown in section 50) we get zero if we sandwich the product of an odd number of gamma matrices between two twistors of the same helicity.

In particular, we have $\langle p|\gamma^\mu|k\rangle = 0$ and $[p|\gamma^\mu|k] = 0$.

Thus, we will get a nonzero result for the tree-level amplitude only if two of the helicities are positive, and two are negative.

This means that, of the $2^4 = 16$ possible combinations of helicities, only six give a nonzero tree-level amplitude: $\mathcal{T}_{++--}, \mathcal{T}_{+-+-}, \mathcal{T}_{+--+}, \mathcal{T}_{--++}, \mathcal{T}_{-+-+}$, and $\mathcal{T}_{-++-}$, where the notation is $\mathcal{T}_{s1s2s3s4}$.

Furthermore, the last three of these are related to the first three by complex conjugation, so we only have three amplitudes to compute.

Let us begin with $\mathcal{T}_{+--+}$. Only the first diagram of fig. (60.1) contributes, because the second has two postive-helicity lines meeting at a vertex.

To evaluate the first diagram, we note that the two vertices contribute a factor of $(ie)^2 = -e^2$, and the internal photon line contributes a factor of $ig_{\mu\nu}/s_{13}$, where we have defined the Mandelstam variable

$$s_{ij} \equiv -(p_i + p_j)^2 \ . \tag{60.18}$$

Following the charge arrows backwards on each fermion line, and dividing by i to get $\mathcal{T}$ (rather than $i\mathcal{T}$), we find

$$\begin{aligned} \mathcal{T}_{+--+} &= -e^2 \langle 3|\gamma^\mu|1] [4|\gamma_\mu|2\rangle / s_{13} \\ &= +2e^2 [1\,4] \langle 2\,3\rangle / s_{13} \ , \end{aligned} \tag{60.19}$$

where $\langle 3|$ is short for $\langle p_3|$, etc, and we have used yet another form of the Fierz identity to get the second line.

The computation of $\mathcal{T}_{+-+-}$ is exactly analogous, except that now it is only the second diagram of fig. (60.1) that contributes.

According to the Feynman rules, this diagram comes with a relative minus sign, and so we have

$$\mathcal{T}_{+-+-} = -2e^2 [1\,3] \langle 2\,4\rangle / s_{14} \ . \tag{60.20}$$

Finally, we turn to $\mathcal{T}_{++--}$.

Now both diagrams contribute, and we have

$$
\begin{aligned}
\mathcal{T}_{++--} &= -e^2 \left(\frac{\langle 3|\gamma^\mu|1]\,\langle 4|\gamma_\mu|2]}{s_{13}} - \frac{\langle 4|\gamma^\mu|1]\,\langle 3|\gamma_\mu|2]}{s_{14}} \right) \\
&= -2e^2\,[1\,2]\,\langle 3\,4\rangle \left(\frac{1}{s_{13}} + \frac{1}{s_{14}} \right) \\
&= +2e^2\,[1\,2]\,\langle 3\,4\rangle \left(\frac{s_{12}}{s_{13}s_{14}} \right) \,,
\end{aligned}
\tag{60.21}
$$

where we used the Mandelstam relation $s_{12} + s_{13} + s_{14} = 0$ to get the last line.

To get the cross section for a particular set of helicities, we must take the absolute squares of the amplitudes.

These follow from eqs. (60.4), (60.17), and (60.18) :

$$
|\langle 1\,2\rangle|^2 = |[1\,2]|^2 = \epsilon_1\epsilon_2 s_{12} = |s_{12}| \,. \tag{60.22}
$$

We can then compute the spin-averaged cross section by summing the absolute squares of eqs. (60.19–60.21), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities.

Making use of $s_{34} = s1_2$ and its permutations, we find

$$\langle|\mathcal{T}|^2\rangle = 2e^4\left(\frac{s_{14}^2}{s_{13}^2}+\frac{s_{13}^2}{s_{14}^2}+\frac{s_{12}^4}{s_{13}^2 s_{14}^2}\right)$$
$$= 2e^4\left(\frac{s_{12}^4+s_{13}^4+s_{14}^4}{s_{13}^2 s_{14}^2}\right). \tag{60.23}$$

For the processes of $e^-e^- \to e^-e^-$ and $e^+e^+ \to e^+e^+$, we have $s_{12} = s$, $s_{13} = t$, and $s_{14} = u$, for $e^+e^- \to e^+e^-$, we have $s_{13} = s$, $s_{14} = t$, and $s_{12} = u$,

Now we turn to processes with two external fermions and two external photons, as shown in fig. (60.2).

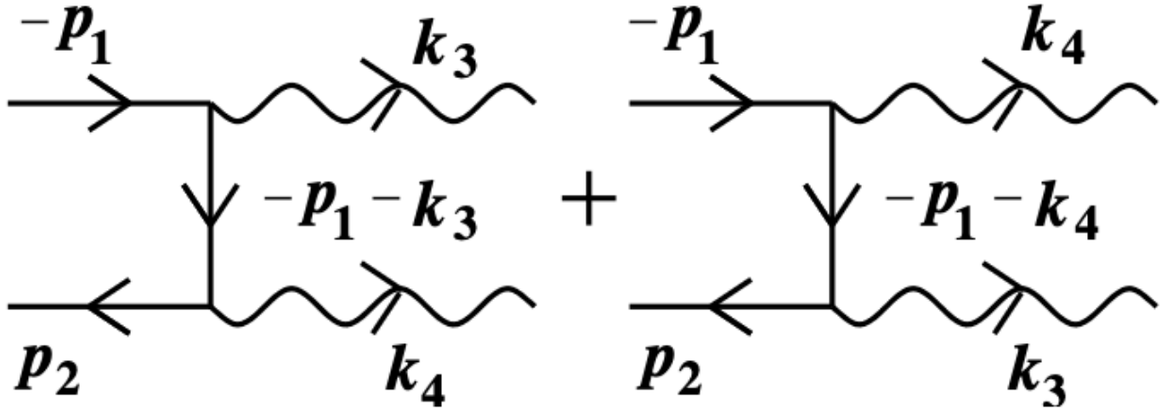


Figure 60.2: Diagrams for fermion-photon scattering, with all momenta treated as outgoing.

The first thing to notice is that a diagram is zero if the two external fermion lines have the same helicity.

This is because the corresponding twistors sandwich an odd number of gamma matrices: one from each vertex, and one from the massless fermion propagator .

Thus we need only compute $\mathcal{T}_{+-\lambda 3\lambda 4}$ since $\mathcal{T}_{-+\lambda 3\lambda 4}$ is related by complex conjugation.

Next we use eqs. (60.15–60.16) and (60.2–60.3) to get

$$\not{\epsilon}_-(k;p)|p] = 0 \,, \tag{60.24}$$

$$[p|\not{\epsilon}_-(k;p) = 0 \,. \tag{60.25}$$

$$\not{\epsilon}_+(k;p)|p\rangle = 0 \,, \tag{60.26}$$

$$\langle p|\not{\epsilon}_+(k;p) = 0 \,, \tag{60.27}$$

Thus we can get some amplitudes to vanish with appropriate choices of the reference momenta in the photon polarizations.

So, let us consider

$$\begin{aligned} \mathcal{T}_{+-\lambda_3\lambda_4} = & -e^2 \langle 2|\not{\epsilon}_{\lambda_4}(k_4;q_4)(\not{p}_1 + \not{k}_3)\not{\epsilon}_{\lambda_3}(k_3;q_3)|1] / s_{13} \\ & -e^2 \langle 2|\not{\epsilon}_{\lambda_3}(k_3;q_3)(\not{p}_1 + \not{k}_4)\not{\epsilon}_{\lambda_4}(k_4;q_4)|1] / s_{14} \,. \end{aligned} \tag{60.28}$$

If we take $\lambda_3 = \lambda_4 = -$, then we can get both terms in eq. (60.28) to vanish by choosing $q_3 = q_4 = p_1$, and using eq. (60.24).

If we take $\lambda_3 = \lambda_4 = +$, then we can get both terms in eq. (60.28) to vanish by choosing $q_3 = q_4 = p_2$, and using eq. (60.27).

Thus, we need only compute $\mathcal{T}_{+--+}$ and $\mathcal{T}_{+-+-}$.

For $\mathcal{T}_{+-+-}$, we can get the second term in eq. (60.28) to vanish by choosing $q_3 = p_2$, and using eq. (60.27).

Then we have

$$\begin{aligned}\mathcal{T}_{+-+-} &= -e^2 \langle 2|\not{\epsilon}_-(k_4;q_4)(\not{p}_1 + \not{k}_3)\not{\epsilon}_+(k_3;p_2)|1]\,/s_{13} \\ &= -e^2 \frac{\sqrt{2}}{[q_4 4]} \langle 2\,4\rangle\, [q_4|(\not{p}_1 + \not{k}_3)|2\rangle\, [3\,1]\, \frac{\sqrt{2}}{\langle 2\,3\rangle}\, \frac{1}{s_{13}}\,.\end{aligned} \tag{60.29}$$

Next we note that $[p|\not{p} = 0$, and so it is useful to choose either $q_4 = p_1$ or $q_4 = k_3$.

There is no obvious advantage in one choice over the other, and they must give equivalent results, so let us take $q_4 = k_3$.

Then, using eq. (60.6) for $\not{k}_3$, we get

$$\mathcal{T}_{+-+-} = 2e^2\, \frac{\langle 2\,4\rangle\, [3\,1]\, \langle 1\,2\rangle\, [3\,1]}{[3\,4]\, \langle 2\,3\rangle\, s_{13}} \tag{60.30}$$

Now we use $[3\ 1]\ \langle 1\ 2\rangle = -[3\ 4]\ \langle 4\ 2\rangle$ in the numerator, and set $s_{13} = \langle 1\ 3\rangle\ [3\ 1]$ in the denominator.

Canceling common factors and using antisymmetry of the twistor product then yields

$$\mathcal{T}_{+-+-} = 2e^2 \frac{\langle 2\,4\rangle^2}{\langle 1\,3\rangle\langle 2\,3\rangle} \,. \tag{60.31}$$

We can now get $\mathcal{T}_{+--+}$ simply by exchanging the labels 3 and 4,

$$\mathcal{T}_{+--+} = 2e^2 \frac{\langle 2\,3\rangle^2}{\langle 1\,4\rangle\langle 2\,4\rangle} \,. \tag{60.32}$$

We can compute the spin-averaged cross section by summing the absolute squares of eqs. (60.31) and (60.32), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities.

The result is

$$\langle|\mathcal{T}|^2\rangle = 2e^4\left(\left|\frac{s_{13}}{s_{14}}\right| + \left|\frac{s_{14}}{s_{13}}\right|\right) . \tag{60.33}$$

For the processes of $e^-\gamma \to e^-\gamma$ and $e^+\gamma \to e^+\gamma$, we have $s_{13} = s$, $s_{12} = t$, and $s_{14} = u$; for $e^+e^- \to \gamma\gamma$ and $\gamma\gamma \to e^+e^-$ we have $s_{12} = s$, $s_{13} = t$, and $s_{14} = u$.

61 Scalar Electrodynamics

In this section, we will consider how charged spin-zero particles interact with photons.

We begin with the lagrangian for a complex scalar field with a quartic interaction,

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2 \ . \tag{61.1}$$

This lagrangian is obviously invariant under the global U(1) symmetry

$$\begin{aligned} \varphi(x) &\to e^{-i\alpha}\varphi(x) \ , \\ \varphi^\dagger(x) &\to e^{+i\alpha}\varphi^\dagger(x) \ . \end{aligned} \tag{61.2}$$

We would like to promote this global symmetry to a local symmetry,

$$\varphi(x) \to \exp[-ie\Gamma(x)]\varphi(x) \ , \tag{61.3}$$

$$\varphi^\dagger(x) \to \exp[+ie\Gamma(x)]\varphi^\dagger(x) \ . \tag{61.4}$$

To do so, we must replace each ordinary derivative in eq. (61.1) with a covariant derivative

$$D_\mu \equiv \partial_\mu - ieA_\mu \ , \tag{61.5}$$

where A_μ transforms as

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \Gamma(x) \ , \tag{61.6}$$

which implies that D_μ transforms as

$$D_\mu \to \exp[-ie\Gamma(x)] D_\mu \exp[+ie\Gamma(x)] \ . \tag{61.7}$$

Our complete lagrangian for scalar electrodynamics is then

$$\mathcal{L} = -(D^\mu \varphi)^\dagger D_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2 - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \ . \tag{61.8}$$

We have added the usual gauge-invariant kinetic term for the gauge field.

The quartic interaction term has a dimensionless coefficient, and so is necessary for renormalizability.

For now, we omit the renormalizing Z factors.

Of course, eq. (61.8) is invariant under a global U(1) transformation as well as a local U(1) transformation: we simply set Γ(x) to a constant.

Then we can find the conserved Noether current corresponding to this symmetry, following the procedure of section 22.

In the case of spinor electrodynamics, this current is same as it is for a free Dirac field, $j^\mu = \overline{\Psi}\gamma^\mu\Psi$.

In the case of a complex scalar field, we find

$$j^\mu = -i[\varphi^\dagger D^\mu \varphi - (D^\mu \varphi)^\dagger \varphi] \ . \tag{61.9}$$

With a factor of e, this current should be identified as the electromagnetic current.

Because the covariant derivative appears in eq. (61.9), the electromagnetic current depends explicitly on the gauge field.

We had not previously contemplated this possibility, but in scalar electrodynamics it arises naturally, and is essential for gauge invariance.

It also poses no special problem in the quantum theory.

We will make the same assumption that we did for spinor electrodynamics: namely, that the correct procedure is to omit integration over the component of $\tilde{A}_\mu(k)$ that is parallel to k_μ, on the grounds that this integration is redundant.

This leads to the same Feynman rules for internal and external photons as in section 58.

We will call the spin-zero particle with electric charge +e a scalar electron or selectron (recall that our convention is that e is negative), and the spin-zero particle with electric charge –e a scalar positron or s positron.

Scalar lines (traditionally drawn as dashed in scalar electrodynamics) carry a charge arrow whose direction must be preserved when lines are joined by vertices.

To determine the kinds of vertices we have, we first write out the interaction terms in the lagrangian of eq. (61.8):

$$\mathcal{L}_1 = ieA^\mu[(\partial_\mu\varphi^\dagger)\varphi - \varphi^\dagger\partial_\mu\varphi] - e^2A^\mu A_\mu\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 \ . \tag{61.10}$$

This leads to the vertices shown in fig. (61.1).

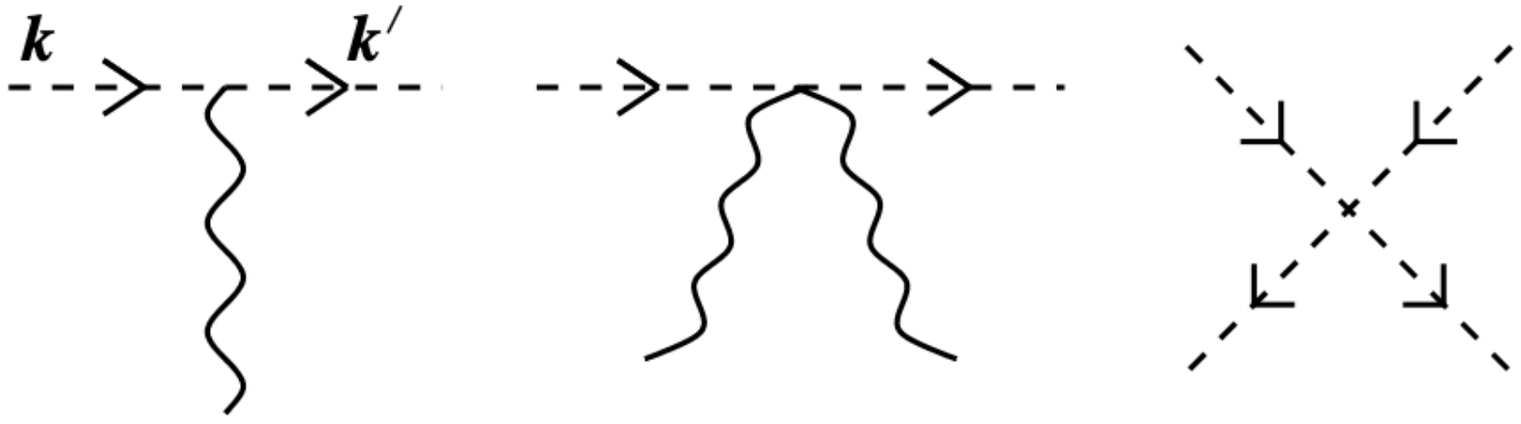


Figure 61.1: The three vertices of scalar electrodynamics; the corresponding vertex factors are $ie(k+k')_\mu$, $-2ie^2g_{\mu\nu}$, and $-i\lambda$.

The vertex factors associated with the last two terms are $-2ie^2g_{\mu\nu}$ and $-i\lambda$.

To get the vertex factor for the first term, we note that if $|k\rangle$ is an incoming selectron state, then $\langle 0|\varphi(x)|k\rangle = e^{ikx}$ and $\langle 0|\varphi^\dagger(x)|k\rangle = 0$; and if $\langle k'|$ is an outgoing selectron state, then $\langle k'|\varphi^\dagger(x)|0\rangle = e^{-ik'x}$ and $\langle 0|\varphi(x)|k\rangle = 0$.

Therefore, in free field theory,

$$\langle k'|(\partial_\mu\varphi^\dagger)\varphi|k\rangle = -ik'_\mu e^{-i(k'-k)x} \ , \tag{61.11}$$

$$\langle k'|\varphi^\dagger\partial_\mu\varphi|k\rangle = +ik_\mu e^{-i(k'-k)x} \ . \tag{61.12}$$

This implies that the vertex factor for the first term in eq. (61.10) is given by $i(ie)[(-ik'_\mu) - (ik_\mu)] = ie(k + k')_\mu$.

Putting everything together, we get the following set of Feynman rules for tree-level processes in scalar electrodynamics.

1. For each *incoming selectron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with the selectron's four-momentum, k_i.

2. For each *outgoing selectron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with the selectron's four-momentum, k'_i.

3. For each *incoming spositron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with *minus* the spositron's four-momentum, $-k_i$.

4. For each *outgoing spositron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with *minus* the spositron's four-momentum, $-k'_i$.

5. For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i.

6. For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k'_i.

7. There are three allowed vertices, shown in fig. (61.1). Using these vertices, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent.*

8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9. The value of a diagram consists of the following factors:

   for each incoming photon, $\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$;

   for each outgoing photon, $\varepsilon^{\mu}_{\lambda_i}(\mathbf{k}_i)$;

   for each incoming or outgoing selectron or spositron, 1;

   for each scalar-scalar-photon vertex, $ie(k + k')_\mu$;

   for each scalar-scalar-photon-photon vertex, $-2ie^2 g_{\mu\nu}$;

   for each four-scalar vertex, $-i\lambda$;

   for each internal photon, $-ig^{\mu\nu}/(k^2 - i\epsilon)$;

   for each internal scalar, $-i/(k^2 + m^2 - i\epsilon)$.

10. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

Let us compute the scattering amplitude for a particular process, $\tilde{e}^+\tilde{e}^- \to \gamma\gamma$, where e–denotes a selectron.

We have the diagrams of fig. (61.2)

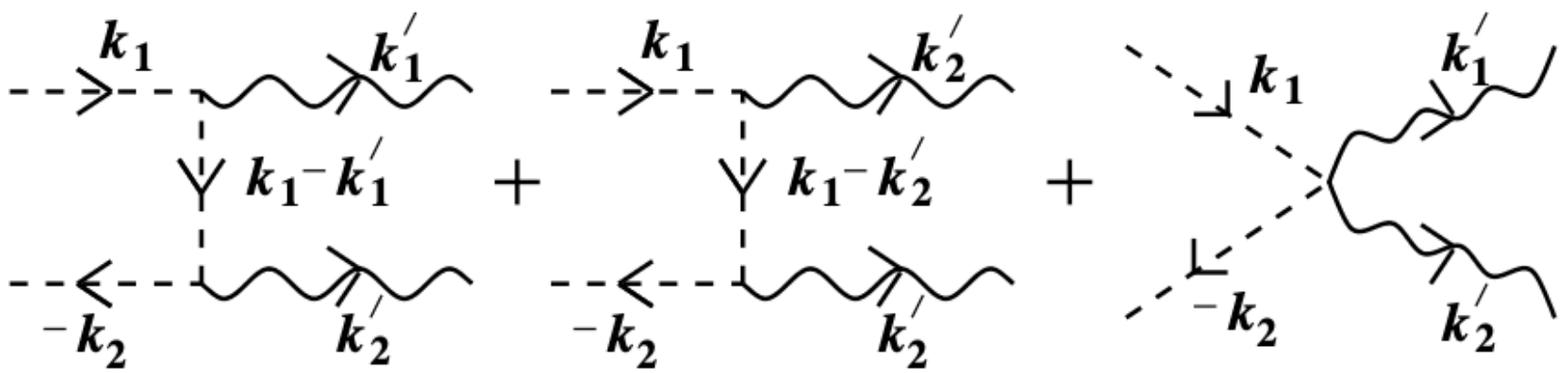


Figure 61.2: Diagrams for $e_+e_- \to \gamma\gamma$.

The amplitude is

$$i\mathcal{T} = (ie)^2 \frac{1}{i} \frac{(2k_1 - k_1')_\mu \varepsilon_{1'}^\mu (k_1 - k_1' - k_2)_\nu \varepsilon_{2'}^\nu}{m^2 - t} + (1' \leftrightarrow 2') \\ - 2ie^2 g_{\mu\nu} \varepsilon_{1'}^\mu \varepsilon_{2'}^\nu \ , \tag{61.13}$$

where $t = -(k_1 - k'_1)^2$ and $u = -(k_1 - k'_2)^2$.

This expression can be simplified by noting that $k_1 - k'_1 - k_2 = k'_2 - 2\,k_2$, and that $k'_i \cdot \varepsilon'_i = 0$.

Then we have

$$\mathcal{T} = -e^2 \left[\frac{4(k_1 \cdot \varepsilon_{1'})(k_2 \cdot \varepsilon_{2'})}{m^2 - t} + \frac{4(k_1 \cdot \varepsilon_{2'})(k_2 \cdot \varepsilon_{1'})}{m^2 - u} + 2(\varepsilon_{1'} \cdot \varepsilon_{2'}) \right] . \tag{61.14}$$

To get the polarization-summed cross section, we take the absolute square of eq. (61.14), and use the substitution rule

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(\mathbf{k}) \varepsilon_\lambda^{\rho *}(\mathbf{k}) \to g^{\mu\rho} \ . \tag{61.15}$$

This is now a straightforward calculation.

62 Loop Corrections in Spinor Electrodynamics

In this section we will compute the one-loop corrections in spinor electrodynamics.

First let us note that the general discussion of sections 18 and 29 leads us to expect that we will need to add to the free lagrangian

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{62.1}$$

all possible terms whose coefficients have positive or zero mass dimension, and that respect the symmetries of the original lagrangian.

These include Lorentz symmetry, the U(1) gauge symmetry, and the discrete symmetries of parity, time reversal, and charge conjugation.

The mass dimensions of the fields (in four spacetime dimensions) are $[A^\mu] = 1$ and $[\Psi] = 3/2$.

Gauge invariance requires that A^μ appear only in the form of a covariant derivative D^μ.

(Recall that the field strength $F^{\mu\nu}$ can be expressed as the commutator of two covariant derivatives.)

Thus, the only possible term we could add to $\mathcal{L}_0$ that does not involve the Ψ field, and that has mass dimension four or less, is $\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$.

This term, however, is odd under parity and time reversal.

Similarly, there are no terms meeting all the requirements that involve Ψ: the only candidates contain either γ_5 (e.g., $i\,\bar{\Psi}\gamma_5\Psi$) and are forbidden by parity, or C (e.g, $\bar{\Psi}^{\mathrm{T}}C\Psi$) and are forbidden by the U(1) symmetry.

Therefore, the theory we will consider is specified by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is given by eq. (62.1), and

$$\mathcal{L}_1 = Z_1 e\bar{\Psi}\not{A}\Psi + \mathcal{L}_{\mathrm{ct}} \ , \tag{62.2}$$

$$\mathcal{L}_{\mathrm{ct}} = i(Z_2-1)\bar{\Psi}\not{\partial}\Psi - (Z_m-1)m\bar{\Psi}\Psi - \tfrac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu} \ . \tag{62.3}$$

We will use an on-shell renormalization scheme.

We can write the exact photon propagator (in momentum space) as a geometric series of the form

$$\tilde{\mathbf{\Delta}}_{\mu\nu}(k) = \tilde{\Delta}_{\mu\nu}(k) + \tilde{\Delta}_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\tilde{\Delta}_{\sigma\nu}(k) + \dots \,, \tag{62.4}$$

where $i\Pi^{\mu\nu}(k)$ is given by a sum of one-particle irreducible (1PI for short; see section 14) diagrams with two external photon lines (and the external propagators removed), and $\widetilde{\Delta}_{\mu\nu}(k)$ is the free photon propagator,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{62.5}$$

Here we have used the freedom to add k_μ or k_ν terms to put the propagator into *generalized Feynman gauge* or R_ξ *gauge*.

(The name R_ξ *gauge* has historically been used only in the context of spontaneous symmetry breaking—see section 85—but we will use it here as well; R stands for *renormalizable* and ξ stands for ξ.)

Setting $\xi = 1$ gives Feynman gauge, and setting $\xi = 0$ gives Lorenz gauge (also known as Landau gauge).

Observable squared amplitudes should not depend on the value of ξ.

This suggests that $\Pi^{\mu\nu}(k)$ should be *transverse*,

$$k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0 \,, \tag{62.6}$$

so that the ξ dependent term in $\widetilde{\Delta}_{\mu\nu}(k)$ vanishes when an internal photon line is attached to $\Pi^{\mu\nu}(k)$.

Eq. (62.6) is in fact valid; the proof is based on the Ward identity for the electromagnetic current.

Without the proof, we will take eq. (62.6) as given.

This implies that we can write

$$\Pi^{\mu\nu}(k) = \Pi(k^2)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big) \tag{62.7}$$

$$= k^2\,\Pi(k^2) P^{\mu\nu}(k)\ , \tag{62.8}$$

where $\Pi(k^2)$ is a scalar function, and $P^{\mu\nu}(k) = g^{\mu\nu} - k^\mu k^\nu/k^2$ is the projection matrix introduced in section 57.

Note that we can also write

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(P_{\mu\nu}(k) + \xi\,\frac{k_\mu k_\nu}{k^2}\right). \tag{62.9}$$

Then, using eqs. (62.8) and (62.9) in eq. (62.4), and summing the geometric series, we find

$$\tilde{\boldsymbol{\Delta}}_{\mu\nu}(k) = \frac{P_{\mu\nu}(k)}{k^2[1 - \Pi(k^2)] - i\epsilon} + \xi\,\frac{k_\mu k_\nu/k^2}{k^2 - i\epsilon}\ . \tag{62.10}$$

The ξ dependent term should be physically irrelevant (and can be set to zero by the gauge choice $\xi = 0$, corresponding to Lorenz gauge).

The remaining term has a pole at $k^2 = 0$ with residue $P_{\mu\nu}(k)/[1-\Pi(0)]$.

In our on-shell renormalization scheme, we should have

$$\Pi(0) = 0 \ . \tag{62.11}$$

This corresponds to the field normalization that is needed for validity of the LSZ formula.

Let us now turn to the calculation of $\Pi^{\mu\nu}(k)$.

The one-loop and counterterm contributions are shown in fig. (62.1).

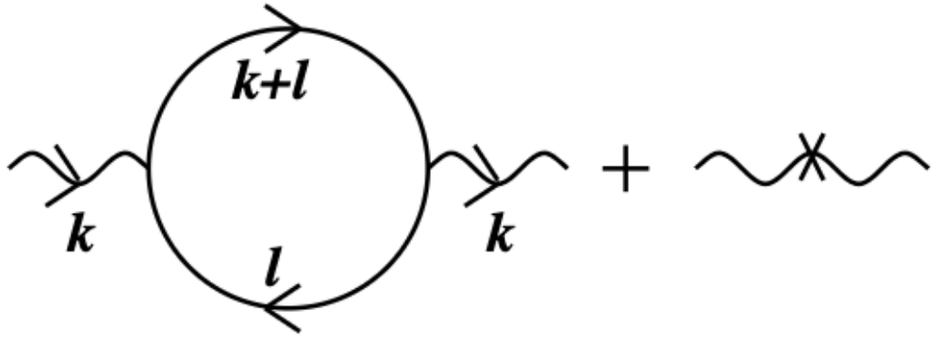


Figure 62.1: The one-loop and counterterm corrections to the photon propagator in spinor electrodynamics.

We have

$$\begin{aligned} i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\tilde{S}(\not{\ell}+\not{k})\gamma^\mu \tilde{S}(\not{\ell})\gamma^\nu\right] \\ -i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4) \ , \end{aligned} \tag{62.12}$$

where the factor of minus one is for the closed fermion loop, and $\tilde{S}(\not{p}) = (-\not{p}+m)/(p^2+m^2-i\epsilon)$ is the free fermion propagator in momentum space.

Anticipating that $Z_1 = 1 + O(e^2)$, we set $Z_1 = 1$ in the first term.

We can write

$$\mathrm{Tr}\Big[\tilde{S}(\not{\ell}+\not{k})\gamma^\mu \tilde{S}(\not{\ell})\gamma^\nu\Big] = \int_0^1 dx\, \frac{4N^{\mu\nu}}{(q^2+D)^2}\ , \tag{62.13}$$

where we have combined denominators in the usual way: q = ℓ+ xk and

$$D = x(1-x)k^2 + m^2 - i\epsilon\ . \tag{62.14}$$

The numerator is

$$4N^{\mu\nu} = \mathrm{Tr}\Big[(-\not{\ell}-\not{k}+m)\gamma^\mu(-\not{\ell}+m)\gamma^\nu\Big] \tag{62.15}$$

Completing the trace, we get

$$N^{\mu\nu} = (\ell+k)^\mu \ell^\nu + \ell^\mu(\ell+k)^\nu - [\ell(\ell+k) + m^2]g^{\mu\nu}\ . \tag{62.16}$$

Setting ℓ = q − xk and and dropping terms linear in q (because they integrate to zero), we find

$$N^{\mu\nu} \to 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu}\ . \tag{62.17}$$

The integrals diverge, and so we analytically continue to d = 4− ε dimensions, and replace e with $e\tilde{\mu}^{\varepsilon/2}$ (so that e remains dimensionless for any d).

Next we use a standard mathematical result

$$\int d^dq\, q^\mu q^\nu f(q^2) = \frac{1}{d}\, g^{\mu\nu} \int d^dq\, q^2 f(q^2)\ . \tag{62.18}$$

This allows the replacement

$$N^{\mu\nu} \to -2x(1-x)k^\mu k^\nu + \left[\left(\tfrac{2}{d}-1\right)q^2 + x(1-x)k^2 - m^2\right] g^{\mu\nu}\ . \tag{62.19}$$

Using the results of section 14, along with a little manipulation of gamma functions, we can show that

$$\left(\tfrac{2}{d}-1\right)\int \frac{d^dq}{(2\pi)^d}\,\frac{q^2}{(q^2+D)^2} = D\int \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2}\ . \tag{62.20}$$

Thus we can make the replacement $(2/d-1)q^2 \to D$ in eq. (62.19), and we find

$$N^{\mu\nu} \to 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu)\ . \tag{62.21}$$

This guarantees that the one-loop contribution to $\Pi^{\mu\nu}(k)$ is transverse (as we expected) in any number of spacetime dimensions.

Now we evaluate the integral over q, using

$$\begin{aligned}\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} &= \frac{i}{16\pi^2}\,\Gamma(\tfrac{\varepsilon}{2})\left(4\pi\tilde{\mu}^2/D\right)^{\varepsilon/2} \\ &= \frac{i}{8\pi^2}\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right]\ ,\end{aligned} \tag{62.22}$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$, and we have dropped terms of order εin the last line.

Combining eqs. (62.7), (62.12), (62.13), (62.21), and (62.22), we get

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right] - (Z_3-1) + O(e^4)\ . \tag{62.23}$$

Imposing $\Pi(0) = 0$ fixes

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\left[\frac{1}{\varepsilon} - \ln(m/\mu)\right] + O(e^4) \tag{62.24}$$

and

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln(D/m^2) + O(e^4)\ . \tag{62.25}$$

Next we turn to the fermion propagator.

The exact propagator can be written in Lehmann-Kallen form as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{1}{\not{p} + m - i\epsilon} + \int_{m_{\rm th}^2}^{\infty} ds\, \frac{\rho_\Psi(s)}{\not{p} + \sqrt{s} - i\epsilon}\ . \tag{62.26}$$

We see that the first term has a pole at $\not{p} = -m$ with residue one.

This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

There is a problem, however: in quantum electrodynamics, the threshold mass m_{th} is m, corresponding to the contribution of a fermion and a zero-energy photon.

Thus the second term has a branch point at $\not{p} = -m$.

The pole in the first term is therefore not isolated, and its residue is ill defined.

This is a reflection of an underlying infrared divergence, associated with the massless photon.

To deal with it, we must impose an infrared cutoff that moves the branch point away from the pole.

The most direct method is to change the denominator of the photon propagator from k^2 to $k^2 + m^2_\gamma$, where m_γ is a fictitious photon mass.

Ultimately, as in section 26, we must deal with this issue by computing cross-sections that take into account detector inefficiencies.

In quantum electrodynamics, we must specify the lowest photon energy ωmin that can be detected.

Only after computing cross sections with extra undedectable photons, and then summing over them, is it safe to take the limit $m_\gamma \to 0$.

It turns out that it is not also necessary to abandon the on-shell shell renormalization scheme (as we were forced to do in massless $\varphi 3$ theory in section 27), as long as the electron is massive.

An alternative is to use dimensional regularization for the infrared divergences as well as the ultraviolet ones.

As discussed in section 25, there are no soft-particle infrared divergences for $d > 4$ (and no colinear divergences at all in quantum electrodynamics with massive charged particles).

In practice, infrared-divergent integrals are finite away from even-integer dimensions, just like ultraviolet-divergent integrals.

Thus we simply keep $d = 4 - \varepsilon$ all the way through to the very end, taking the $\varepsilon \to 0$ limit only after summing over cross sections with extra undetectable photons, all computed in $4 - \varepsilon$ dimensions.

This method is calculationally the simplest, but requires careful bookkeeping to segregate the infrared and ultraviolet singularities.

For that reason, we will not pursue it further.

We can write the exact fermion propagator in the form

$$\tilde{\mathbf{S}}(\not{p})^{-1} = \not{p} + m - i\epsilon - \Sigma(\not{p}) \ , \tag{62.27}$$

where $i\Sigma(\not{p})$ is given by the sum of 1PI diagrams with two external fermion lines (and the external propagators removed).

The fact that $\tilde{\mathbf{S}}(\not{p})$ has a pole at $\not{p} = -m$ with residue one implies that $\Sigma(-m) = 0$ and $\Sigma'(-m) = 0$; this fixes the coefficients Z_2 and Z_m.

As we will see, we must have an infrared cutoff in place in order to have a finite value for $\Sigma'(-m)$.

Let us now turn to the calculation of $\Sigma(\not{p})$.

The one-loop and counterterm contributions are shown in fig. (62.2).

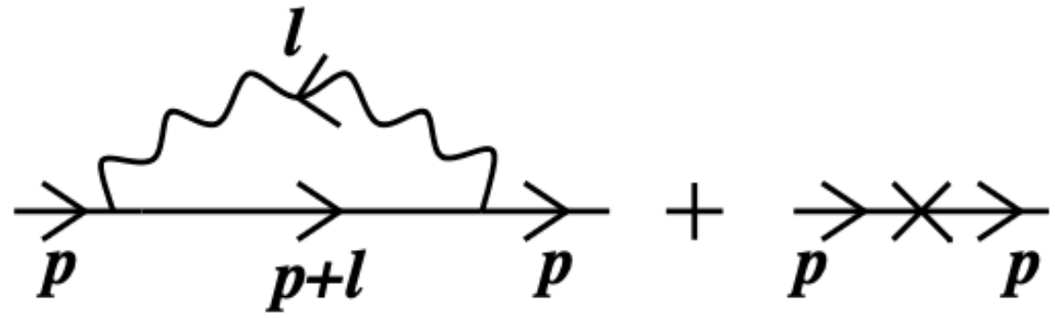


Figure 62.2: The one-loop and counterterm corrections to the fermion propagator in spinor electrodynamics.

We have

$$i\Sigma(\not{p}) = (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \tilde{S}(\not{p}+\not{\ell})\gamma^\mu\right] \tilde{\Delta}_{\mu\nu}(\ell) \qquad (62.28)$$

$$- i(Z_2{-}1)\not{p} - i(Z_m{-}1)m + O(e^4) \ .$$

It is simplest to work in Feynman gauge, where we take

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2 + m_\gamma^2 - i\epsilon} \ ; \qquad (62.29)$$

here we have included the fictitious photon mass m_γ as an infrared cutoff.

We now apply the usual bag of tricks to get

$$i\Sigma(\not{p}) = e^2 \tilde{\mu}^\varepsilon \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{N}{(q^2+D)^2} \qquad (62.30)$$

$$- i(Z_2{-}1)\not{p} - i(Z_m{-}1)m + O(e^4) \ ,$$

where q = ℓ + xp and

$$D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2 \ , \tag{62.31}$$

$$\begin{aligned} N &= \gamma_\mu(-\not{p} - \not{\ell} + m)\gamma^\mu \\ &= -(d-2)(\not{p} + \not{\ell}) - dm \\ &= -(d-2)[\not{q} + (1-x)\not{p}] - dm \ , \end{aligned} \tag{62.32}$$

where we have used (from section 47) $\gamma_\mu\gamma^\mu = -d$ and $\gamma_\mu\not{p}\gamma^\mu = (d-2)\not{p}$.

The term linear in q integrates to zero, and then, using eq. (62.22), we get

$$\begin{aligned} \Sigma(\not{p}) = & -\frac{e^2}{8\pi^2}\int_0^1 dx\Big((2-\varepsilon)(1-x)\not{p} + (4-\varepsilon)m\Big)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right] \\ & -(Z_2-1)\not{p} - (Z_m-1)m + O(e^4) \ . \end{aligned} \tag{62.33}$$

We see that finiteness of $\Sigma(\not{p})$ requires

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ , \tag{62.34}$$

$$Z_m = 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ . \tag{62.35}$$

We can impose $\Sigma(-m) = 0$ by writing

$$\Sigma(\not{p}) = \frac{e^2}{8\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not{p} + 2m\Big)\ln(D/D_0) + \kappa_2(\not{p}+m)\right] + O(e^4)\ , \tag{62.36}$$

where D_0 is D evaluated at $p^2 = -m^2$.

$$D_0 = x^2m^2 + (1-x)m_\gamma^2\ , \tag{62.37}$$

and κ^2 is a constant to be determined.

We fix κ^2 by imposing $\Sigma'(-m) = 0$.

In differentiating with respect to $\not{p}$, we take the p^2 in D, eq. (62.31), to be $-\not{p}^2$; we find

$$\begin{aligned}\kappa_2 &= -2\int_0^1 dx\, x(1-x^2)m^2/D_0 \\ &= -2\ln(m/m_\gamma) + 1\ ,\end{aligned} \tag{62.38}$$

where we have dropped terms that go to zero with the infrared cutoff m_γ.

Next we turn to the loop correction to the vertex.

We define the vertex function $i\mathbf{V}^\mu(p',p)$ as the sum of one-particle irreducible diagrams with one incoming fermion with momentum p, one outgoing fermion with momentum p′, and one incoming photon with momentum $k = p'-p$.

The original vertex $iZ_1e\gamma^\mu$ is the first term in this sum, and the diagram of fig. (62.3) is the second.

Thus we have

$$i\mathbf{V}^\mu(p',p) = iZ_1e\gamma^\mu + i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) + O(e^5)\ , \tag{62.39}$$

where

$$i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = (ie)^3\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho\tilde{S}(\not{p}'+\not{\ell})\gamma^\mu\tilde{S}(\not{p}+\not{\ell})\gamma^\nu\right]\tilde{\Delta}_{\nu\rho}(\ell)\ . \tag{62.40}$$

We again use eq. (62.29) for the photon propagator, and combine denominators in the usual way.

We then get

$$i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = e^3\int dF_3\int\frac{d^4q}{(2\pi)^4}\,\frac{N^\mu}{(q^2+D)^3}\ , \tag{62.41}$$

where the integral over Feynman parameters is

$$\int dF_3 \equiv 2\int_0^1 dx_1dx_2dx_3\,\delta(x_1{+}x_2{+}x_3{-}1)\ , \tag{62.42}$$

and

$$q = \ell + x_1p + x_2p'\ , \tag{62.43}$$

$$\begin{aligned} D = {}& x_1(1{-}x_1)p^2 + x_2(1{-}x_2)p'^2 - 2x_1x_2p{\cdot}p' \\ & + (x_1{+}x_2)m^2 + x_3m_\gamma^2\ , \end{aligned} \tag{62.44}$$

$$
\begin{aligned}
N^\mu &= \gamma_\nu(-\not{p}' - \not{\ell} + m)\gamma^\mu(-\not{p} - \not{\ell} + m)\gamma^\nu \\
&= \gamma_\nu[-\not{q} + x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-\not{q} - (1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu \\
&= \gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu + \widetilde{N}^\mu + (\text{linear in } q) \ , \qquad (62.45)
\end{aligned}
$$

where

$$
\widetilde{N}^\mu = \gamma_\nu[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu \ . \qquad (62.46)
$$

The terms linear in q in eq. (62.45) integrate to zero, and only the first term is divergent.

After continuing to d dimensions, we can use eq. (62.18) to make the replacement

$$
\gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu \to \frac{1}{d}\, q^2\, \gamma_\nu\gamma_\rho\gamma^\mu\gamma^\rho\gamma^\nu \ . \qquad (62.47)
$$

then we use $\gamma_\rho\gamma^\mu\gamma_\rho = (d-2)\gamma^\mu$ twice to get

$$
\gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu \to \frac{(d-2)^2}{d}\, q^2\, \gamma^\mu \ . \qquad (62.48)
$$

Performing the usual manipulations, we find

$$
\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = \frac{e^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - 1 - \tfrac{1}{2}\int dF_3\, \ln(D/\mu^2)\right)\gamma^\mu + \tfrac{1}{4}\int dF_3\, \frac{\widetilde{N}^\mu}{D}\right] . \qquad (62.49)
$$

From eq. (62.39), we see that finiteness of $\mathbf{V}^\mu(p', p)$ requires

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ .$$

To completely fix $\mathbf{V}^\mu(p', p)$, we need a suitable condition to impose on it.

We take this up in the next section.

63 The Vertex Function in Spinor Electrodynamics

In the last section, we computed the one-loop contribution to the vertex function $\mathbf{V}^\mu(p', p)$ in spinor electrodynamics, where p is the four-momentum of an incoming electron (or outgoing positron), and p' is the four-momentum of an outgoing electron (or incoming positron).

We left open the issue of the renormalization condition we wish to impose on $\mathbf{V}^\mu(p', p)$.

For the theories we have studied previously, we have usually made the mathematically convenient (but physically obscure) choice to define the coupling constant as the value of the vertex function when all external four-momenta are set to zero.

However, in the case of spinor electrodynamics, the masslessness of the photon gives us the opportunity to do something more physically meaningful: we can define the coupling constant as the value of the vertex function when all three particles are on shell: $p^2 = p'^2 = -m^2$, and $q^2 = 0$, where $q \equiv p' - p$ is the photon four-momentum.

Because the photon is massless, these three on-shell conditions are compatible with momentum conservation.

To be more precise, let us sandwich $\mathbf{V}^\mu(p', p)$ between the spinor factors that are appropriate for an incoming electron with momentum p and an outgoing electron with momentum p′, impose the on-shell conditions, and define the electron charge e via

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p', p)u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}} = e\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}} \cdot \tag{63.1}$$

This definition is in accord with the usual one provided by Coulomb's law.

To see why, consider the process of electron-electron scattering.

According to the general discussion in section 19, we compute the exact amplitude for this process by using tree diagrams with exact internal propagators and vertices, as shown in fig. (63.1).

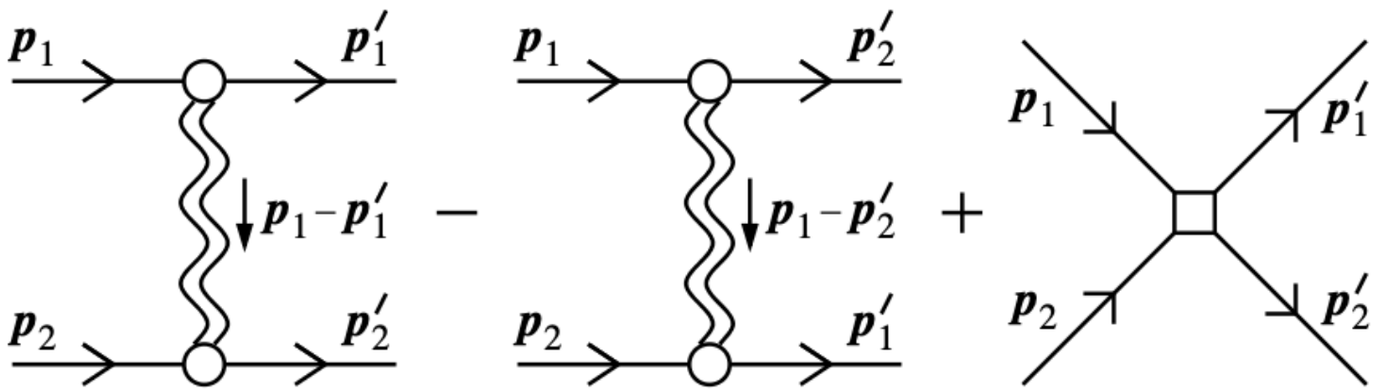


Figure 63.1: Diagrams for the exact electron-electron scattering amplitude.The vertices and photon propagator are exact; external lines stand for the usual u and spinor factors, times the unit residue of the po $\overline{u}$ at $p^2 = -m^2$.

In the last section, we renormalized the photon propagator so that it approaches its tree-level value $\widetilde{\Delta}_{\mu\nu}(q)$ when $q^2 \to 0$.

And we have just chosen to renormalize the electron-photon vertex function by requiring it to approach its tree-level value $e\gamma^\mu$ when $q^2 \to 0$, and when sandwiched between external spinors for on-shell incoming and outgoing electrons.

Therefore, as $q^2 \to 0$, the first two diagrams in fig. (63.1) approach the tree-level scattering amplitude, with the electron charge equal to e.

Furthermore, the third diagram does not have a pole at $q^2 = 0$, and so can be neglected in this limit.

Physically, $q^2 \to 0$ means that the electron's momentum changes very little during the scattering.

Measuring a slight deflection in the trajectory of one charged particle (due to the presence of another) is how we measure the coefficient in Coulomb's law.

Thus, eq. (63.1) corresponds to this traditional definition of the charge of the electron.

We can simplify eq. (63.1) by noting that the on-shell conditions actually enforce $p' = p$.

So we can rewrite eq. (63.1) as

$$\begin{aligned} \overline{u}_s(\mathbf{p})\mathbf{V}^\mu(p,p)u_s(\mathbf{p}) &= e\,\overline{u}_s(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) \\ &= 2ep^\mu \ , \end{aligned} \tag{63.2}$$

where $p^2 = -m^2$ is implicit.

We have taken $s' = s$, because otherwise the right-hand side vanishes (and hence does not specify a value for e).

Now we can use eq. (63.2) to completely determine $\mathbf{V}^{\mu}(p',p)$.

Using the freedom to choose the finite part of Z_1, we write

$$\mathbf{V}^{\mu}(p',p) = e\gamma^{\mu} - \frac{e^3}{16\pi^2}\int dF_3 \left[\Big(\ln(D/D_0)+2\kappa_1\Big)\gamma^{\mu} - \frac{N^{\mu}}{2D}\right] + O(e^5)\ , \tag{63.3}$$

where

$$\begin{aligned} D = {} & x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2 p\cdot p' \\ & + (x_1+x_2)m^2 + x_3 m_{\gamma}^2\ , \end{aligned} \tag{63.4}$$

D_0 is D evaluated at $p' = p$ and $p^2 = -m^2$,

$$\begin{aligned} D_0 &= (x_1+x_2)^2 m^2 + x_3 m_{\gamma}^2 \\ &= (1-x_3)^2 m^2 + x_3 m_{\gamma}^2\ , \end{aligned} \tag{63.5}$$

and

$$N^{\mu} = \gamma_{\nu}[x_1 \not{p} - (1-x_2)\not{p}' + m]\gamma^{\mu}[-(1-x_1)\not{p} + x_2 \not{p}' m]\gamma^{\nu}\ ; \tag{63.6}$$

N^{μ} was called $\widetilde{N}^{\mu}$ in section 62, but we have dropped the tilde for notational convenience.

We fix the constant κ_1 in eq. (63.3) by imposing eq. (63.2).

This yields

$$4\kappa_1 p^\mu = \int dF_3 \, \frac{\overline{u}_s(\mathbf{p}) N_0^\mu u_s(\mathbf{p})}{2D_0} \ , \tag{63.7}$$

where $N^\mu{}_0$ is N^μ with $p' = p$ and $p^2 = p'^2 = -m^2$.

So now we must evaluate $\overline{u}$ $N^\mu{}_0$ u.

To do so, we first write

$$N^\mu = \gamma_\nu (\not{a}_1 + m) \gamma^\mu (\not{a}_2 + m) \gamma^\nu \ , \tag{63.8}$$

where

$$\begin{aligned} a_1 &= x_1 p - (1-x_2) p' \ , \\ a_2 &= x_2 p' - (1-x_1) p \ . \end{aligned} \tag{63.9}$$

Now we use the gamma matrix contraction identities to get

$$N^\mu = 2\not{a}_2 \gamma^\mu \not{a}_1 + 4m(a_1+a_2)^\mu + 2m^2 \gamma^\mu \ . \tag{63.10}$$

Here we have set $d = 4$, because we have already removed the divergence and taken the limit $\varepsilon \to 0$.

Setting $p' = p$, and using $\overline{u}\,u = -mu$ and $u\,\overline{u} = -mu$, along with $\overline{u}\,\gamma^\mu u = 2p^\mu$ and $\overline{u}u = 2m$, and recalling that $x_1+x_2+x_3 = 1$, we find

$$\overline{u}N_0^\mu u = 4(1-4x_3+x_3^2)m^2p^\mu \ . \tag{63.11}$$

Using eqs. (63.5), (63.7), and (63.11), we get

$$\begin{aligned}\kappa_1 &= \frac{1}{2}\int dF_3 \ \frac{1-4x_3+x_3^2}{(1-x_3)^2+x_3m_\gamma^2/m^2} \\ &= \int_0^1 dx_3\,(1-x_3)\,\frac{1-4x_3+x_3^2}{(1-x_3)^2+x_3m_\gamma^2/m^2} \\ &= -2\ln(m/m_\gamma)+\tfrac{5}{2}\end{aligned} \tag{63.12}$$

in the limit of $m_\gamma \to 0$.

We see that an infrared regulator is necessary for the vertex function as well as the fermion propagator.

Now that we have $\mathbf{V}^\mu(p',p)$, we can extract some physics from it.

Consider again the process of electron-electron scattering, shown in fig. (63.1).

In order to compute the contributions of these diagrams, we must evaluate $\overline{u}_{s'}(p')\ \mathbf{V}^\mu(p',p)\ u_s(p)$ with $p^2 = -p'^2 = -m^2$, but with $q^2 = (p'-p)^2$ arbitrary.

To evaluate $\overline{u}'N^\mu u$, we start with eq. (63.10), and use the anticommutation relations of the gamma matrices to move all the $\not{p}$'s in N^μ to the far right (where we can use $\not{p}u = -mu$) and all the $\not{p}'$'s to the far left (where we can use $\overline{u}'\not{p}' = -m\overline{u}'$).

This results in

$$\begin{aligned} N^\mu \rightarrow\ & [4(1-x_1-x_2+x_1x_2)p\cdot p' + 2(2x_1-x_1^2+2x_2-x_2^2)m^2]\gamma^\mu \\ & + 4m(x_1^2-x_2+x_1x_2)p^\mu + 4m(x_2^2-x_1+x_1x_2)p'^\mu \ . \end{aligned} \tag{63.13}$$

Next, replace $p\cdot p'$ with $-1/2\ q^2 - m^2$, group the p^μ and p'^μ terms into $p'+p$ and $p'-p$ combinations, and make use of $x_1+x_2+x_3 = 1$ to simplify some coefficients.

The result is

$$\begin{aligned} N^\mu \rightarrow\ & 2[(1-2x_3-x_3^2)m^2 - (x_3+x_1x_2)q^2]\gamma^\mu \\ & - 2m(x_3-x_3^2)(p'+p)^\mu \\ & - 2m[(x_1+x_1^2) - (x_2+x_2^2)](p'-p)^\mu \ . \end{aligned} \tag{63.14}$$

In the denominator, set $p^2 = p'^2 = -m^2$ and $p\cdot p' = -1/2\ q^2 - m^2$ to get

$$D \rightarrow x_1x_2q^2 + (1-x_3)^2m^2 + x_3m_\gamma^2 \ . \tag{63.15}$$

Note that the right-hand side of eq. (63.15) is symmetric under $x_1 \leftrightarrow x_2$.

Thus the last line of eq. (63.14) will vanish when we integrate $\overline{u}'N^\mu u/D$ over the Feynman parameters.

Finally, we use the Gordon identity from section 38,

$$\overline{u}'(p'+p)^\mu u = \overline{u}'[2m\gamma^\mu + 2iS^{\mu\nu}q_\nu]u \ , \tag{63.16}$$

where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, to get

$$\begin{aligned} N^\mu \to\ & 2[(1-4x_3+x_3^2)m^2 - (x_3+x_1x_2)q^2]\gamma^\mu \\ & - 4im(x_3-x_3^2)S^{\mu\nu}q_\nu \ . \end{aligned} \tag{63.17}$$

So now we have

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p}) = e\overline{u}'\Big[F_1(q^2)\gamma^\mu - \frac{i}{m}F_2(q^2)S^{\mu\nu}q_\nu\Big]u \ , \tag{63.18}$$

where we have defined the *form factors*

$$\begin{aligned} F_1(q^2) = 1 - \frac{e^2}{16\pi^2}\int dF_3 \Bigg[& \ln\left(1+\frac{x_1x_2q^2/m^2}{(1-x_3)^2}\right) + \frac{1-4x_3+x_3^2}{(1-x_3)^2 + x_3m_\gamma^2/m^2} \\ & + \frac{(x_3+x_1x_2)q^2/m^2 - (1-4x_3+x_3^2)}{x_1x_2q^2/m^2 + (1-x_3)^2 + x_3m_\gamma^2/m^2}\Bigg] + O(e^4) \ , \end{aligned} \tag{63.19}$$

$$F_2(q^2) = \frac{e^2}{8\pi^2}\int dF_3 \ \frac{x_3-x_3^2}{x_1x_2q^2/m^2 + (1-x_3)^2} + O(e^4) \ . \tag{63.20}$$

We have set $m_\gamma = 0$ in eq. (63.20), and in the logarithm term in eq. (63.19), because these terms do not suffer from infrared divergences.

We can simplify $F_2(q^2)$ by using the delta function in dF_3 to do the integral over x_2 (which replaces x_2 with $1 - x_3 - x_1$), making the change of variable $x_1 = (1 - x_3)y$, and performing the integral over x_3 from zero to one; the result is

$$F_2(q^2) = \frac{e^2}{8\pi^2} \int_0^1 \frac{dy}{1 - y(1-y)q^2/m^2} + O(e^4) \ . \tag{63.21}$$

This last integral can also be done in closed form, but we will be mostly interested in its value at $q^2 = 0$, corresponding to an on-shell photon:

$$F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \ , \tag{63.22}$$

where $\alpha = e^2/4\pi = 1/137.036$ is the fine-structure constant.

We will explore the physical consequences of eq. (63.22) in the next section.

64 The Magnetic Moment of the Electron

In the last section, we computed the one-loop contribution to the vertex function $V\mu(p',p)$ in spinor electrodynamics, where p is the four-momentum of an incoming electron, and p' is the four-momentum of an outgoing electron.

We found

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^{\mu}(p',p)u_s(\mathbf{p}) = e\overline{u}'\left[F_1(q^2)\gamma^{\mu} - \frac{i}{m}F_2(q^2)S^{\mu\nu}q_{\nu}\right]u \ , \tag{64.1}$$

where $q = p' - p$ is the four-momentum of the photon (treated as incoming), and with complicated expressions for the *form factors* $F_1(q^2)$ and $F_2(q^2)$.

For our purposes in this section, all we will need to know is that

$$\begin{aligned} F_1(0) &= 1 \quad \text{exactly}, \\ F_2(0) &= \frac{\alpha}{2\pi} + O(\alpha^2) \, . \end{aligned} \tag{64.2}$$

Eq. (64.1) follows from a quantum action of the form

$$\Gamma = \int d^4x \, \left[eF_1(0)\overline{\Psi} \not{A} \Psi + \frac{e}{2m} F_2(0) F_{\mu\nu} \overline{\Psi} S^{\mu\nu} \Psi + \ldots \right], \tag{64.3}$$

where the ellipses stand for terms with more derivatives.

The displayed terms yield the vertex factor of eq. (64.1) with $q^2 = 0$.

To see this, recall that an incoming photon translates into a factor of $A_\mu \sim \varepsilon^*{}_\mu e^{iqx}$, and therefore of $F_{\mu\nu} \sim i(q_\mu \varepsilon^*{}_\nu - q_\nu \varepsilon^*{}_\mu) e^{iqx}$; the two terms in $F_{\mu\nu}$ cancel the extra factor of one half in the second term in eq. (64.3).

Now we will see what eq. (64.3) predicts for the *magnetic moment of the electron*.

We define the magnetic moment by the following procedure.

We take the photon field A^μ to be a classical field that corresponds to a constant magnetic field in the z direction: $A^0 = 0$ and $\mathbf{A} = (0,Bx,0)$.

This yields $F_{12} = -F_{21} = B$, with all other components of $F_{\mu\nu}$ vanishing.

Then we define a normalized state of an electron at rest, with spin up along the z axis:

$$|e\rangle \equiv \int \widetilde{dp}\, f(\mathbf{p}) b_+^\dagger(\mathbf{p})|0\rangle \ , \tag{64.4}$$

where the wave packet is rotationally invariant (so that there is no orbital angular momentum) and sharply peaked at $\mathbf{p} = 0$, something like

$$f(\mathbf{p}) \sim \exp(-a^2\mathbf{p}^2/2) \tag{64.5}$$

with $a \ll 1/m$.

We normalize the wave packet by $\int \widetilde{dp}\, |f(\mathbf{p})|^2 = 1$; then we have $\langle \mathrm{ele} \rangle = 1$.

Now we define the interaction hamiltonian as what we get from the two displayed terms in eq. (64.3), using our specified field A^μ, and with the form-factor values of eq. (64.2):

$$H_1 \equiv -eB \int d^3x\, \overline{\Psi} \left[x\gamma^2 + \frac{\alpha}{2\pi m} S^{12} \right] \Psi \ . \tag{64.6}$$

Then the electron's magnetic moment μ is specified by

$$\mu B \equiv -\langle e|H_1|e\rangle \ . \tag{64.7}$$

In quantum mechanics in general, if we identify H_1 as the piece of the hamiltonian that is linear in the external magnetic field, then eq. (64.7) defines the magnetic moment of a normalized quantum state with definite angular momentum in the **B** direction.

Now we turn to the computation.

We need to evaluate $\langle e|\overline{\Psi}_\alpha(x)\Psi_\beta(x)|e\rangle$.

Using the usual plane-wave expansions, we have

$$\langle 0|b_+(\mathbf{p}')\overline{\Psi}_\alpha(x)\Psi_\beta(x)b_+^\dagger(\mathbf{p})|0\rangle = \overline{u}_+(\mathbf{p}')_\alpha u_+(\mathbf{p})_\beta \, e^{i(p-p')x} \ . \tag{64.8}$$

Thus we get

$$\begin{aligned}\langle e|H_1|e\rangle = -eB \int \widetilde{dp}\ \widetilde{dp'}\ d^3x\ e^{i(p-p')x} \\ \times f^*(\mathbf{p}')\overline{u}_+(\mathbf{p}')\Big[x\gamma^2 + \frac{\alpha}{2\pi m}S^{12}\Big]u_+(\mathbf{p})f(\mathbf{p}) \ .\end{aligned} \tag{64.9}$$

We can write the factor of x as $-i\partial_{p1}$ acting on $e^{i(p-p')x}$, and integrate by parts to put this derivative onto $u_+(\mathbf{p})f(\mathbf{p})$; the wave packets kill any surface terms.

Then we can complete the integral over d^3x to get a factor of $(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$, and do the integral over $\widetilde{dp'}$.

The result is

$$\langle e|H_1|e\rangle = -eB\int \frac{\widetilde{dp}}{2\omega}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\Big[\, i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\Big]u_+(\mathbf{p})f(\mathbf{p})\ . \tag{64.11}$$

where $K^j = S^{j0} = i/2\ \gamma^j\gamma^0$ is the boost matrix, $\hat{\mathbf{p}}$ is a unit vector in the $\mathbf{p}$ direction, and $\eta = \sinh^{-1}(|\mathbf{p}|/m)$ is the rapidity.

Since the wave packet is sharply peaked at $\mathbf{p} = 0$, we can expand eq. (64.11) to linear order in $\mathbf{p}$, take the derivative with respect to p_1, and then set $\mathbf{p} = 0$; the result is

$$\begin{aligned}
\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \frac{i}{m}K^1u_+(\mathbf{0}) \\
&= -\frac{1}{2m}\,\gamma^1\gamma^0u_+(\mathbf{0}) \\
&= -\frac{1}{2m}\,\gamma^1u_+(\mathbf{0})\ ,
\end{aligned} \tag{64.12}$$

where we used $\gamma^0 u_s(0) = u_s(0)$ to get the last line.

Then we have

$$\begin{aligned}
\overline{u}_+(\mathbf{p})i\gamma^2\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \overline{u}_+(\mathbf{0})\frac{-i}{2m}\gamma^2\gamma^1u_+(\mathbf{0}) \\
&= \frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})
\end{aligned} \tag{64.13}$$

Plugging this into eq. (64.10) yields

$$\begin{aligned}\langle e|H_1|e\rangle &= -eB\int \frac{\widetilde{dp}}{2\omega}\,|f(\mathbf{p})|^2\left(1+\frac{\alpha}{2\pi}\right)\frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0}) \\ &= -\frac{eB}{2m^2}\left(1+\frac{\alpha}{2\pi}\right)\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})\ .\end{aligned} \tag{64.14}$$

Next we use $S^{12}u_\pm(\mathbf{0}) = \pm 1/2\ u_\pm(\mathbf{0})$ and $\overline{u}_\pm(\mathbf{0})u_\pm(\mathbf{0}) = 2m$ to get

$$\langle e|H_1|e\rangle = -\frac{eB}{2m}\left(1+\frac{\alpha}{2\pi}\right)\ . \tag{64.15}$$

Comparing with eq. (64.7), we see that the magnetic moment of the electron is

$$\mu = g\,\frac{1}{2}\,\frac{e}{2m}\ , \tag{64.16}$$

where e/2m is the *Bohr magneton*, the extra factor of one-half is for the electron's spin (a classical spinning ball of charge would have a magnetic moment equal to the Bohr magneton times its angular momentum), and g is the *Lande g factor*, given by

$$g = 2\left(1+\frac{\alpha}{2\pi}+O(\alpha^2)\right)\ . \tag{64.17}$$

Since g can be measured to high precision, calculations of μ provide a stringent test of spinor electrodynamics.

Corrections up through the α^4 term have been computed; the result is currently in good agreement with experiment.

65 Loop Corrections in Scalar Electrodynamics

In this section we will compute the one-loop corrections in scalar electrodynamics.

We will concentrate on the divergent parts of the diagrams, enabling us to compute the renormalizing Z factors in the $\overline{MS}$ scheme, and hence the beta functions.

This gives us the most important qualitative information about the theory: whether it becomes strongly coupled at high or low energies.

Our lagrangian for scalar electrodynamics in section 61 already includes all possible terms whose coefficients have positive or zero mass dimension, and that respect Lorentz symmetry, the U(1) gauge symmetry, parity, time reversal, and charge conjugation.

Therefore, the theory we will consider is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \ , \tag{65.1}$$

$$\mathcal{L}_0 = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \ , \tag{65.2}$$

$$\begin{aligned} \mathcal{L}_1 = {} & i Z_1 e [\varphi^\dagger \partial^\mu \varphi - (\partial^\mu \varphi^\dagger) \varphi] A_\mu - Z_4 e^2 \varphi^\dagger \varphi A^\mu A_\mu \\ & - \tfrac{1}{4} Z_\lambda \lambda (\varphi^\dagger \varphi)^2 + \mathcal{L}_{\rm ct} \ , \end{aligned} \tag{65.3}$$

$$\mathcal{L}_{\rm ct} = -(Z_2 - 1) \partial^\mu \varphi^\dagger \partial_\mu \varphi - (Z_m - 1) m^2 \varphi^\dagger \varphi - \tfrac{1}{4} (Z_3 - 1) F^{\mu\nu} F_{\mu\nu} \ . \tag{65.4}$$

We will use the $\overline{MS}$ renormalization scheme to fix the values of the Z's.

We begin with the photon self-energy, $\Pi^{\mu\nu}$(k).

The one-loop and counterterm contributions are shown in fig. (65.1).

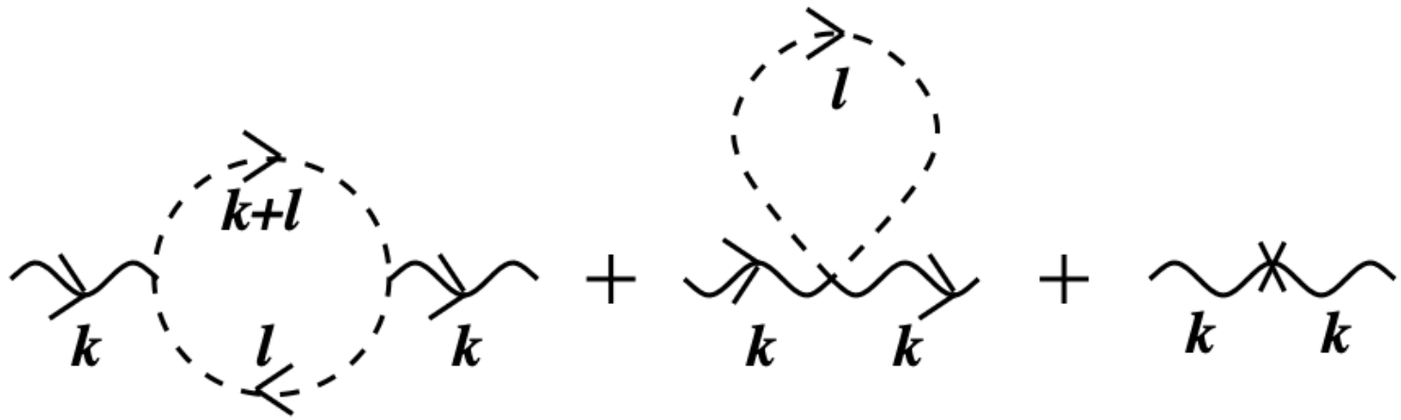


Figure 65.1: The one-loop and counterterm corrections to the photon propagator in scalar electrodynamics.

We have

$$
\begin{aligned}
i\Pi^{\mu\nu}(k) \;=\; & (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \, \frac{(2\ell+k)^\mu (2\ell+k)^\nu}{((\ell+k)^2+m^2)(\ell^2+m^2)} \\
& + (-2iZ_4) e^2 g^{\mu\nu} \int \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{\ell^2+m^2} \\
& - i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \dots \;,
\end{aligned}
\tag{65.5}
$$

where the ellipses stand for higher-order (in e^2 and/or λ) terms.

We can set $Z_i = 1 + O(e^2, \lambda)$ in the first two terms.

It will prove convenient to combine these first two terms into

$$i\Pi^{\mu\nu}(k) = e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{((\ell+k)^2+m^2)(\ell^2+m^2)} \qquad (65.6)$$
$$- i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \dots \ ,$$

where

$$N^{\mu\nu} = (2\ell + k)^\mu (2\ell + k)^\nu - 2[(\ell+k)^2 + m^2] g^{\mu\nu} \ . \qquad (65.7)$$

Then we continue to d dimensions, replace e with $e\tilde{\mu}^{\varepsilon/2}$, and combine the denominators with Feynman's formula; the result is

$$i\Pi^{\mu\nu}(k) = e^2 \tilde{\mu}^\varepsilon \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N^{\mu\nu}}{(q^2+D)^2} \qquad (65.8)$$
$$- i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \dots \ ,$$

where $q = \ell + xk$ and $D = x(1-x)k^2 + m^2$.

The numerator is

$$\begin{aligned} N^{\mu\nu} &= (2q + (1-2x)k)^\mu (2q + (1-2x)k)^\nu - 2[(q + (1-x)k)^2 + m^2] g^{\mu\nu} \\ &= 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu - 2[q^2 + (1-x)^2 k^2 + m^2] g^{\mu\nu} \\ &\quad + (\text{linear in } q) \qquad (65.9) \\ &\to 4d^{-1} g^{\mu\nu} q^2 + (1-2x)^2 k^\mu k^\nu - 2[q^2 + (1-x)^2 k^2 + m^2] g^{\mu\nu} \ , \end{aligned}$$

where we used the symmetric-integration identity from problem 14.3 to get the last line.

We can rearrange eq. (65.9) into

$$N^{\mu\nu} = 2\Big(\tfrac{2}{d} - 1\Big) g^{\mu\nu} q^2 + (1-2x)^2 k^\mu k^\nu - 2[(1-x)^2 k^2 + m^2] g^{\mu\nu} \ . \tag{65.10}$$

Now we recall from section 62 that, when q^2 is integrated against $(q^2+D)^{-2}$, we can make the replacement $(\ 2/d - 1)q^2 \to D$; thus we have

$$\begin{aligned} N^{\mu\nu} &\to 2D g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu - 2[(1-x)^2 k^2 + m^2] g^{\mu\nu} \\ &= (1-2x)^2 k^\mu k^\nu - 2(1-2x)(1-x) k^2 g^{\mu\nu} \ . \end{aligned} \tag{65.11}$$

Next we note that if we make the change of variable $x = y + 1/2$, then we have $D = (1-1/4\ y^2)k^2 + m^2$, and y is integrated from $-1/2$ to $+ 1/2$.

Therefore, any term in $N^{\mu\nu}$ that is even in y will integrate to zero.

We then get

$$\begin{aligned} N^{\mu\nu} &= 4y^2 k^\mu k^\nu - 2(2y^2 - y) k^2 g^{\mu\nu} \\ &\to -4y^2 (k^2 g^{\mu\nu} - k^\mu k^\nu) \ . \end{aligned} \tag{65.12}$$

Thus we see that $\Pi^{\mu\nu}(k)$ is transverse, as expected.

Performing the integral over q in eq. (65.8), and focusing on the divergent part, we get

$$\tilde{\mu}^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2} \frac{1}{\varepsilon} + O(\varepsilon^0) \ . \tag{65.13}$$

Then performing the integral over y yields

$$\int_{-1/2}^{1/2} dy \, N^{\mu\nu} = -\tfrac{1}{3}(k^2 g^{\mu\nu} - k^\mu k^\nu) \ . \tag{65.14}$$

Combining eqs. (65.8), (65.13), and (65.14), we get

$$\Pi^{\mu\nu}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu) \ , \tag{65.15}$$

where

$$\Pi(k^2) = \ -\frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \text{finite} - (Z_3 - 1) + \dots \ . \tag{65.16}$$

Thus we find, in the $\overline{MS}$ scheme,

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \dots \ . \tag{65.17}$$

Now we turn to the one-loop corrections to the scalar propagator, shown in fig. (65.2).

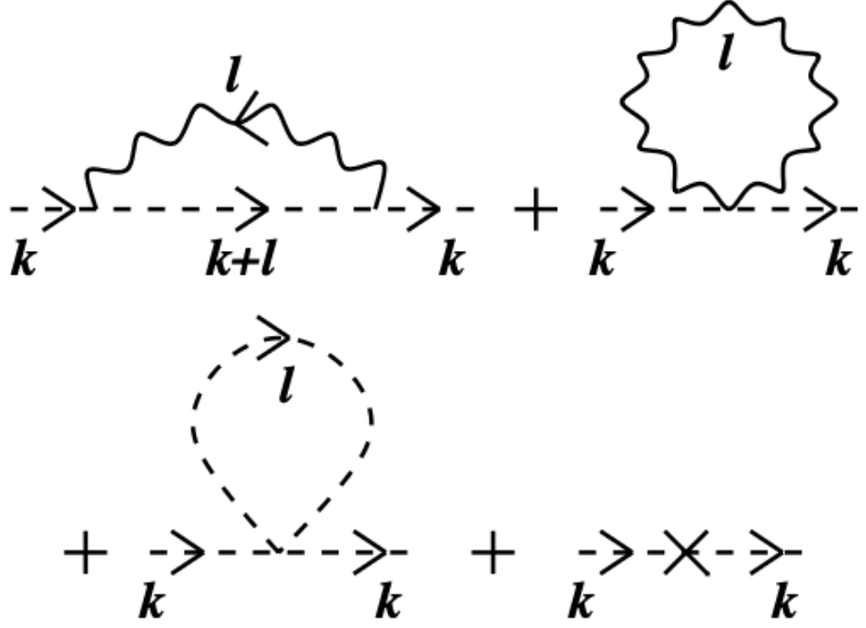


Figure 65.2: The one-loop and counterterm corrections to the scalar propagator in scalar electrodynamics.

It will prove very convenient to work in Lorenz gauge, where the photon propagator is

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{P_{\mu\nu}(\ell)}{\ell^2 - i\epsilon}\,, \tag{65.18}$$

with $P_{\mu\nu}(\ell) = g_{\mu\nu} - \ell_\mu \ell_\nu/\ell^2$.

The diagrams in fig. (65.2) then yield

$$\begin{aligned} i\Pi_\varphi(k^2) = {} & (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{P_{\mu\nu}(\ell)(\ell+2k)^\mu(\ell+2k)^\nu}{\ell^2((\ell+k)^2+m^2)} \\ & + (-2iZ_4 e^2 g^{\mu\nu})\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \frac{P_{\mu\nu}(\ell)}{\ell^2+m_\gamma^2} \\ & + (-i\lambda)\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2} \\ & - i(Z_2-1)k^2 - i(Z_m-1)m^2 + \ldots\,. \end{aligned} \tag{65.19}$$

In the second line, m_γ is a fictitious photon mass; it appears here as an infrared regulator.

We can set $Z_i = 1 + O(e^2, \lambda)$ in the first three lines.

Continuing to d dimensions, making the replacements $e \to e\tilde{\mu}^{\varepsilon/2}$ and $\lambda \to \lambda\tilde{\mu}^{\varepsilon}$, , and using the relations $\ell^\mu P_{\mu\nu}(\ell) = \ell^\nu P_{\mu\nu}(\ell) = 0$ and $g^{\mu\nu}P_{\mu\nu}(\ell) = d-1$, we get

$$
\begin{aligned}
i\Pi_\varphi(k^2) = {} & 4e^2\tilde{\mu}^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d} \frac{P_{\mu\nu}(\ell)k^\mu k^\nu}{\ell^2((\ell+k)^2+m^2)} \\
& - 2(d-1)e^2\tilde{\mu}^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2+m_\gamma^2} \\
& - \lambda\tilde{\mu}^{\varepsilon} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2} \\
& - i(Z_2-1)k^2 - i(Z_m-1)m^2 + \dots \ .
\end{aligned}
\tag{65.20}
$$

We evaluate the second and third lines via

$$
\tilde{\mu}^{\varepsilon} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2+m^2} = -\frac{i}{8\pi^2}\frac{1}{\varepsilon}m^2 + O(\varepsilon^0) \ . \tag{65.21}
$$

Then, taking the limit $m^2 \to 0$ (with ε fixed) in eq. (65.21) shows that the second line of eq. (65.20) vanishes when the infrared regulator is removed.

To evaluate the first line of eq. (65.20), we multiply the numerator and denominator by ℓ^2 and use Feynman's formula to get

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2 P_{\mu\nu}(\ell) k^\mu k^\nu}{\ell^2\ell^2((\ell+k)^2+m^2)} = \int dF_3 \int \frac{d^dq}{(2\pi)^d} \frac{N}{(q^2+D)^3} \,, \tag{65.22}$$

where $q = \ell + x_3 k$, $D = x_3(1-x_3)k^2 + x_3 m^2$, and

$$\begin{aligned} N &= \ell^2 k^2 - (\ell\cdot k)^2 \\ &= (q - x_3 k)^2 k^2 - (q\cdot k - x_3 k^2)^2 \\ &= q^2 k^2 - (q\cdot k)^2 + (\text{linear in } q) \\ &\to q^2 k^2 - d^{-1} q^2 k^2 \,. \end{aligned} \tag{65.23}$$

Now we use

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} = \frac{i}{8\pi^2} \frac{1}{\varepsilon} + O(\varepsilon^0) \,. \tag{65.24}$$

Combining eqs. (65.20–65.24), and requiring $\Pi_\varphi(k^2)$ to be finite, we find

$$Z_2 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \dots \,, \tag{65.25}$$

$$Z_m = 1 + \frac{\lambda}{8\pi^2} \frac{1}{\varepsilon} + \dots \tag{65.26}$$

in the $\overline{MS}$ scheme.

Now we turn to the one-loop corrections to the three-point (scalar– scalar–photon) vertex, shown in fig. (65.3).

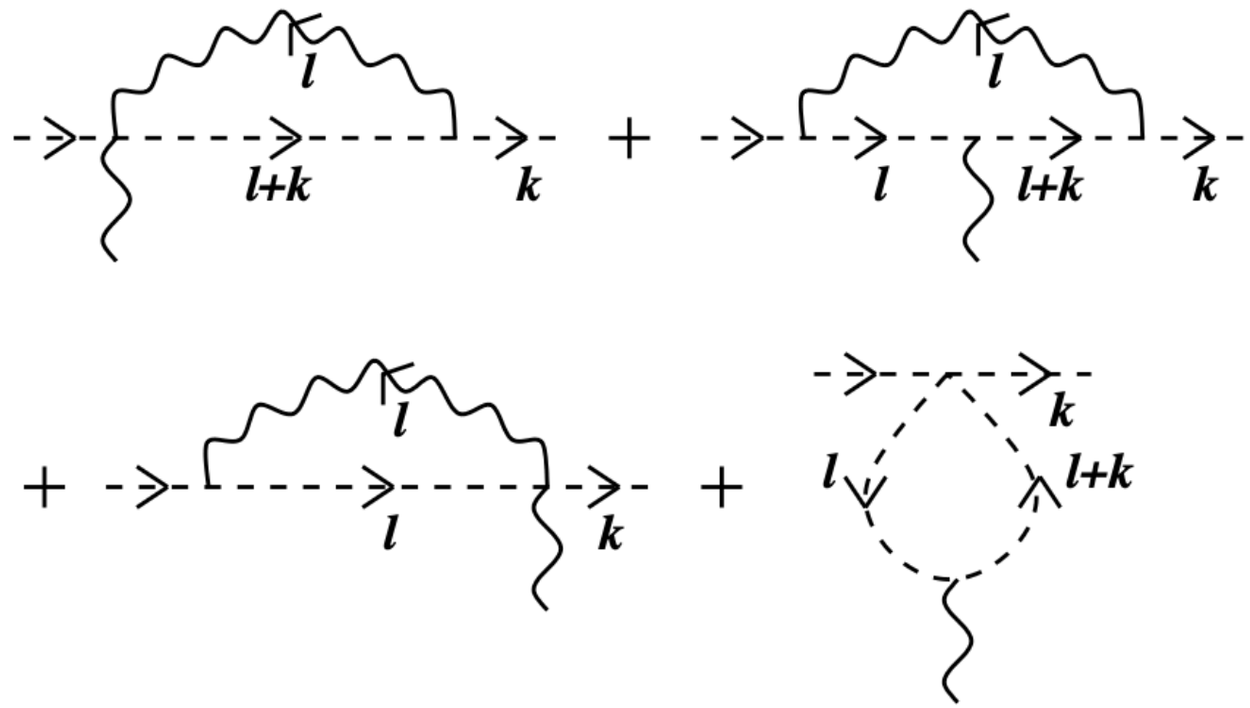


Figure 65.3: The one-loop corrections to the three-point vertex in scalar electrodynamics.

In order to simplify the calculation of the divergent terms as much as possible, we have chosen a special set of external momenta.

(If we wanted the complete vertex function, including the finite terms, we would need to use a general set of external momenta.)

We take the incoming scalar to have zero four-momentum, and the photon (treated as incoming) to have four-momentum k; then, by momentum conservation, the outgoing scalar also has four-momentum k.

We take the internal photon to have four-momentum ℓ.

Now comes the magic of Lorenz gauge: in the second and third diagrams of fig. (65.3), the vertex factor for the leftmost vertex is $ie\ell^\mu$, and this is zero when contracted with the $P_{\mu\nu}(\ell)$ of the internal photon propagator.

Thus the second and third diagrams vanish.

Alas, we will have to do some more work to evaluate the first and fourth diagrams.

We have

$$
\begin{aligned}
i\mathbf{V}_3^\mu(k,0) = {} & ieZ_1 k^\mu \\
& + (iZ_1 e)(-2iZ_4 e^2 g^{\mu\nu})\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\, \frac{P_{\nu\rho}(\ell)(\ell+2k)^\rho}{\ell^2((\ell+k)^2+m^2)} \\
& + (-iZ_\lambda \lambda)(iZ_1 e)\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\, \frac{(2\ell+k)^\mu}{(\ell^2+m^2)((\ell+k)^2+m^2)} \\
& + \ldots .
\end{aligned}
\tag{65.27}
$$

We can set Zi = 1 + O(e^2 , λ) in the second and third lines.

We then do the usual manipulations; the integral in the third line becomes

$$
\int_0^1 dx \int \frac{d^d q}{(2\pi)^d}\, \frac{2q^\mu + (1-2x)k^\mu}{(q^2+D)^2} \ , \tag{65.28}
$$

where q = ℓ + xk and D = x(1−x)k^2 + m^2.

The term linear in q vanishes upon integration over q, and the term linear in k vanishes upon integration over x.

Thus the third line of eq. (65.27) evaluates to zero.

To evaluate the second line of eq. (65.27), we note that, since $P_{\nu\rho}(\ell)\ell^\rho = 0$, it already has an overall factor of k.

We can then treat k as infinitesimal, and set k = 0 in the denominator.

We can then use the symmetric-integration identity to make the replacement $\ell_\nu\ell_\rho/\ell 2 \to d{-}1 g_{\nu\rho}$ in the numerator.

Putting all of this together, and using eq. (65.13), we find

$$\mathbf{V}_3^\mu(k,0)/e = Z_1 k^\mu - \frac{3e^2}{8\pi^2}\,\frac{1}{\varepsilon}\,k^\mu + O(\varepsilon^0) + \dots\ , \tag{65.29}$$

and so

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\,\frac{1}{\varepsilon} + \dots\ , \tag{65.30}$$

in the $\overline{MS}$ scheme.

Next up is the four-point, scalar–scalar–photon–photon vertex.

Because the tree-level vertex factor,$-2iZ_4e^2g^{\mu\nu}$, does not depend on the external four-momenta, we can simply set them all to zero.

Then, whenever an internal photon line attaches to an external scalar with a three-point vertex, the diagram is zero, for the same reason that the second and third diagrams of fig. (65.3) were zero.

This kills a lot of diagrams; the survivors are shown in fig. (65.4).

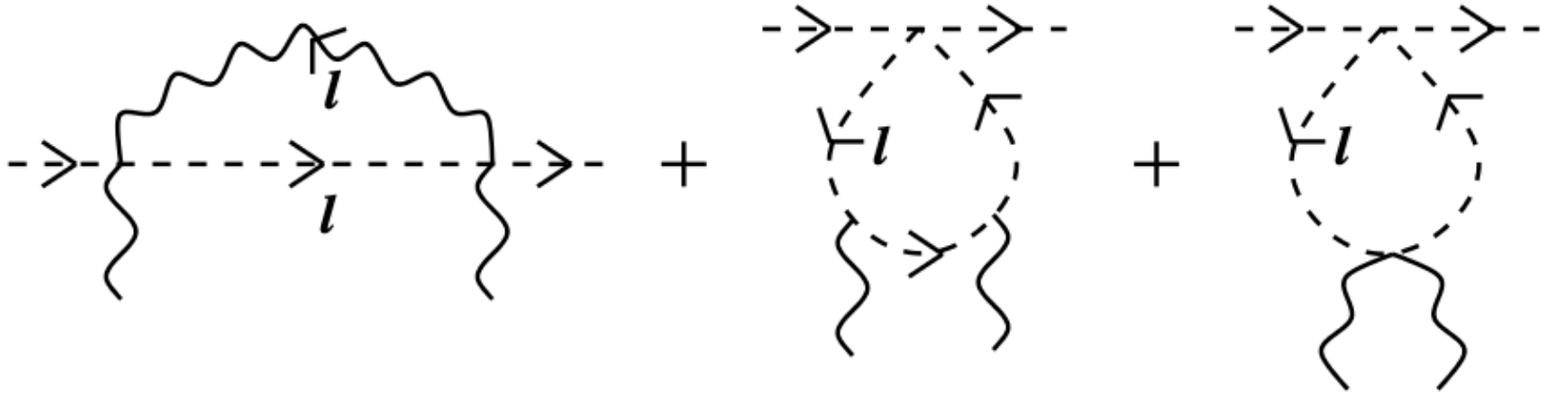


Figure 65.4: The nonvanishing one-loop corrections to the scalar–scalar–photon–photon vertex in scalar electrodynamics (in Lorenz gauge with vanishing external momenta).

We have

$$
\begin{aligned}
i\mathbf{V}_4^{\mu\nu}(0,0,0) &= -2iZ_4e^2g^{\mu\nu} \\
&\quad + (-2iZ_4e^2)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{g^{\mu\rho}P_{\rho\sigma}(\ell)g^{\sigma\nu}}{\ell^2(\ell^2+m^2)} + (\mu\leftrightarrow\nu) \\
&\quad + (iZ_1e)^2(-iZ_\lambda\lambda)\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\,\frac{(2\ell)^\mu(2\ell)^\nu}{(\ell^2+m^2)^3} + (\mu\leftrightarrow\nu) \qquad (65.31) \\
&\quad + (-iZ_\lambda\lambda)(-2iZ_4e^2g^{\mu\nu})\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2+m^2)^2} \\
&\quad + \ldots\ .
\end{aligned}
$$

The notation $+(\mu\leftrightarrow\nu)$ in the second and third lines means that we must add the same expression with these indices swapped; this is because the original and swapped versions of each diagram are topologically distinct, and contribute separately to the vertex function.

As usual, we set $Z_i = 1 + O(e^2, \lambda)$ in the second through fourth lines.

After using the symmetric-integration identity, along with eqs. (65.13) and (65.24), we can see that the divergent parts of the third and fourth lines cancel each other.

The first line is easily evaluated with symmetric integration and eq. (65.13).

Then we have

$$\mathbf{V}_4^{\mu\nu}(0,0,0)/e^2 = -2Z_4 g^{\mu\nu} + \frac{3e^2}{4\pi^2}\,\frac{1}{\varepsilon}\,g^{\mu\nu} + O(\varepsilon^0) + \dots\ , \tag{65.32}$$

and so

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\,\frac{1}{\varepsilon} + \dots \tag{65.33}$$

in the $\overline{MS}$ scheme.

Finally, we have the one-loop corrections to the four-scalar vertex.

Once again, because the tree-level vertex factor, $-iZ_\lambda\lambda$, does not depend on the external four-momenta, we can set them all to zero.

Then, whenever an internal photon line attaches to an external scalar with a three-point vertex, the diagram is zero.

The remaining diagrams are shown in fig. (65.5).

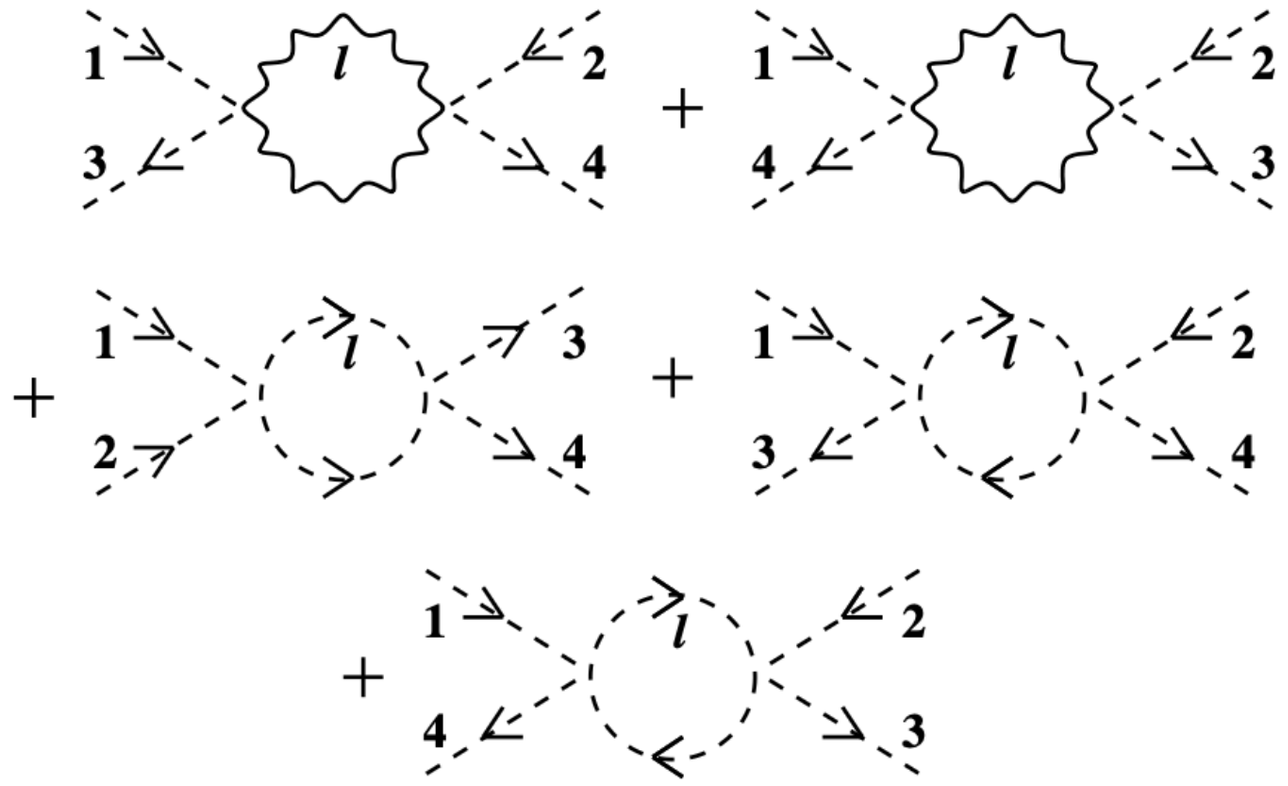


Figure 65.5: The nonvanishing one-loop corrections to the four-scalar vertex in scalar electrodynamics (in Lorenz gauge with vanishing external momenta).

Even though we have set the external momenta to zero, we still have to keep track of which particle is which, in order to count the diagrams correctly; thus the external lines are labelled 1 through 4.

Lines 1 and 2 have arrows pointing towards their vertices, and 3 and 4 have arrows pointing away from their vertices.

The symmetry factor for each of the first three diagrams is S = 2; for each of the last two, it is S = 1.

The difference arises because the last two diagrams have the charge arrows pointing in opposite directions on the two internal propagators, and so these propagators cannot be exchanged.

It is clear that the first two diagrams will yield identical contributions to the vertex function (when the external momenta are all zero).

Similarly, except for symmetry factors, the contributions of the last three diagrams are also identical.

Thus we have

$$
\begin{aligned}
i\mathbf{V}_{4\varphi}(0,0,0) = & -iZ_\lambda\lambda \\
& + \left(\tfrac{1}{2}+\tfrac{1}{2}\right)(-2iZ_4e^2)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{g^{\mu\nu}P_{\nu\rho}(\ell)g^{\rho\sigma}P_{\rho\mu}(\ell)}{(\ell^2+m_\gamma^2)^2} \\
& + \left(\tfrac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2+m^2)^2} \\
& + \dots\ .
\end{aligned}
\tag{65.34}
$$

Using the familiar techniques, we find

$$
\mathbf{V}_{4\varphi}(0,0,0) = -iZ_\lambda\lambda + \frac{3e^4}{2\pi^2}\,\frac{1}{\varepsilon} + \frac{5\lambda^2}{16\pi^2}\,\frac{1}{\varepsilon} + O(\varepsilon^0) + \dots\ , \tag{65.35}
$$

and so

$$
Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\varepsilon} + \dots \tag{65.36}
$$

in the $\overline{MS}$ scheme.

66 Beta Functions in Quantum Electrodynamics

In this section we will compute the beta function for the electromagnetic coupling e in spinor electrodynamics and scalar electrodynamics.

We will also compute the beta function for the φ^4 coupling λ in scalar electrodynamics.

In spinor electrodynamics, the relation between the bare and renormalized couplings is

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \, \tilde{\mu}^{\varepsilon/2} e \; . \tag{66.1}$$

It is convenient to recast this formula in terms of the fine-structure constant $\alpha = e^2/4\pi$ and its bare counterpart $\alpha_0 = e^2{}_0/4\pi$,

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha \; . \tag{66.2}$$

From section 62, we have

$$Z_1 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.3}$$

$$Z_2 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.4}$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi} \frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.5}$$

in the $\overline{MS}$ scheme.

Let us write

$$\ln\left(Z_3^{-1}Z_2^{-2}Z_1^2\right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} \ . \tag{66.6}$$

Then we have

$$\ln \alpha_0 = \sum^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu} \ . \tag{66.7}$$

From eqs. (66.3–66.5), we get

$$E_1(\alpha) = \frac{2\alpha}{3\pi} + O(\alpha^2) \ . \tag{66.8}$$

Then, the general analysis of section 28 yields

$$\beta(\alpha) = \alpha^2 E_1'(\alpha) \ , \tag{66.9}$$

where the prime denotes differentiation with respect to α.

Thus we find

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3) \tag{66.10}$$

in spinor electrodynamics.

We can, if we like, restate this in terms of e as

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5) \ . \tag{66.11}$$

To go from eq. (66.10) to eq. (66.11), we use $\alpha = e^2/4\pi$ and $\dot{\alpha} = e\dot{e}/2\pi$, where the dot denotes d/dln μ.

The most important feature of either eq. (66.10) or eq. (66.11) is that the beta function is positive: the electromagnetic coupling in spinor electrodynamics gets stronger at high energies, and weaker at low energies.

It is easy to generalize eqs. (66.10) and (66.11) to the case of N Dirac fields with electric charges $Q_i e$.

There is now a factor of Z_{2i} for each field, and of Z_{1i} for each interaction.

These are found by replacing α in eqs. (66.3) and (66.4) with $Q^2_i\alpha$.

Then we find $Z_{1i}/Z_{2i} = 1 + O(\alpha^2)$, so that this ratio is universal, at least through $O(\alpha)$.

In fact, as we will see in section 67, Z_{1i}/Z_{2i} is always exactly equal to one, and so it always cancels in eq. (66.6).

As for Z_3, now each Dirac field contributes separately to the fermion loop in the photon self-energy, and so we should replace α in eq. (66.5) with $iQ^2_i\alpha$.

Thus we find that the generalization of eq. (66.11) is

$$\beta(e) = \frac{\sum_{i=1}^{N} Q_i^2}{12\pi^2}\, e^3 + O(e^5)\ . \qquad (66.12)$$

Now we turn to scalar electrodynamics.

The relations between the bare and renormalized couplings are

$$e_0 = Z_3^{-1/2} Z_\varphi^{-1} Z_1 \, \tilde{\mu}^{\varepsilon/2} e \, . \tag{66.13}$$

$$e_0^2 = Z_3^{-2} Z_\varphi^{-2} Z_4 \, \tilde{\mu}^{\varepsilon} e^2 \, . \tag{66.14}$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \, . \tag{66.15}$$

We have two different relations between e and e_0, coming from the two types of vertices.

We can guess (and will demonstrate in section 67) that these two renormalizations must work out to give the same answer. I

ndeed, from section 65, we have

$$Z_1 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \dots \, , \tag{66.16}$$

$$Z_2 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \dots \, , \tag{66.17}$$

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \dots \, , \tag{66.18}$$

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\,\frac{1}{\varepsilon} + \dots\ , \tag{66.19}$$

$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\varepsilon} + \dots\ , \tag{66.20}$$

in Lorenz gauge in the $\overline{MS}$ scheme; the ellipses stand for higher powers of e^2 and/or λ.

We see that $Z_1 = Z_2 = Z_4$, at least through $O(e^2, \lambda)$.

The correct guess is that this is true exactly.

Thus eqs. (66.13) and (66.14) both collapse to $e_0 = Z_3^{-1/2}\, e$, just as in spinor electrodynamics.

Thus we can write

$$\ln\left(Z_3^{-1/2}\right) = \sum_{n=1}^{\infty} \frac{E_n(e,\lambda)}{\varepsilon^n}\ , \tag{66.21}$$

$$\ln\left(Z_2^{-2} Z_\lambda\right) = \sum_{n=1}^{\infty} \frac{L_n(e,\lambda)}{\varepsilon^n}\ . \tag{66.22}$$

Then we have

$$\ln e_0 = \sum_{n=1}^{\infty} \frac{E_n(e,\lambda)}{\varepsilon^n} + \ln e + \tfrac{1}{2} e \ln\tilde{\mu}\ , \tag{66.23}$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(e,\lambda)}{\varepsilon^n} + \ln\lambda + e\ln\tilde{\mu}\ . \tag{66.24}$$

Using eqs. (66.17), (66.18) and (66.20), we have

$$E_1(e,\lambda) = \frac{e^2}{24\pi^2} + \dots \,, \tag{66.25}$$

$$L_1(e,\lambda) = \frac{1}{16\pi^2}\left(5\lambda + 24e^4/\lambda - 12e^2\right) + \dots \,, \tag{66.26}$$

Now applying the general analysis of section 52 yields

$$\beta_e(e,\lambda) = \frac{e^3}{48\pi^2} + \dots \,, \tag{66.27}$$

$$\beta_\lambda(e,\lambda) = \frac{1}{16\pi^2}\left(5\lambda^2 - 6\lambda e^2 + 24e^4\right) + \dots \,. \tag{66.28}$$

Both right-hand sides are strictly positive, and so both e and λ become large at high energies, and small at low energies.

Generalizing eq. (66.25) to the case of several complex scalar fields with charges $Q_i e$ works in the same way as it does in spinor electrodynamics.

67 Ward Identities in Quantum Electrodynamics I

In section 59, we assumed that scattering amplitudes would be gauge invariant, in the sense that they would be unchanged if we replaced any photon polarization vector ε^μ with $^\mu + ck^\mu$, where k^μ is the photon's four-momentum and c is an arbitrary constant.

Thus, if we write a scattering amplitude $\mathcal{T}$ for a process that includes an external photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^\mu \mathcal{M}_\mu \ , \tag{67.1}$$

then we should have

$$k^\mu \mathcal{M}_\mu = 0 \ . \tag{67.2}$$

In this section, we will use the Ward identity for the electromagnetic current to prove eq. (67.2).

We begin by recalling the LSZ formula for scalar fields,

$$\langle f|i\rangle = i \int d^4x_1 \, e^{-ik_1 x_1} (-\partial_1^2 + m^2) \ldots \langle 0|\mathrm{T}\varphi(x_1) \ldots |0\rangle \ . \tag{67.3}$$

We have treated all external particles as outgoing; an incoming particle has $k_i^0 < 0$.

can rewrite eq. (67.3) as

$$\langle f|i\rangle = \lim_{k_i^2 \to -m^2} (k_1^2 + m^2) \ldots \langle 0|\mathrm{T}\tilde{\varphi}(k_1) \ldots |0\rangle \ . \tag{67.4}$$

Here $\tilde{\varphi}(k) = i\int d^4x\, e^{-ikx}\varphi(x)$ is the field in momentum space (with an extra factor of i), and we do not fix $k^2 = -m^2$.

We know that the right-hand side of eq. (67.4) must include an overall energy-momentum delta function, so let us write

$$\langle 0|\mathrm{T}\tilde{\varphi}(k_1)\ldots|0\rangle = (2\pi)^4\delta^4(\textstyle\sum_i k_i)\mathcal{F}(k_i^2, k_i\cdot k_j)\ , \tag{67.5}$$

where $\mathcal{F}(k_i^2, k_i\cdot k_j)$ is a function of the Lorentz scalars k_i^2 and $k_i\cdot k_j$.

Then, since

$$\langle f|i\rangle = i(2\pi)^4\delta^4(\textstyle\sum_i k_i)\mathcal{T}\ , \tag{67.6}$$

eq. (67.4) tells us that $\mathcal{F}$ should have a multivariable pole as each k_i^2 approaches $-m^2$, and that i $\mathcal{T}$ is the residue of this pole.

That is, near $k_i^2 = -m^2$, $\mathcal{F}$ takes the form

$$\mathcal{F}(k_i^2, k_i\cdot k_j) = \frac{i\mathcal{T}}{(k_1^2+m^2)\ldots(k_n^2+m^2)} + \text{nonsingular}\ . \tag{67.7}$$

The key point is this: contributions to $\mathcal{F}$ that do not have this multivariable pole do not contribute to $\mathcal{T}$.

We have framed this discussion in terms of scalar fields in order to keep the notation as simple as possible, but the general point holds for fields of any spin.

In section 22, we analyzed how various classical field equations apply to quantum correlation functions.

For example, we derived the *Schwinger-Dyson equations*

$$\begin{aligned}\langle 0|\mathrm{T}\frac{\delta S}{\delta\phi_a(x)}\,\phi_{a_1}(x_1)\dots\phi_{a_n}(x_n)|0\rangle \\ = i\sum_{j=1}^{n}\langle 0|\mathrm{T}\phi_{a_1}(x_1)\dots\delta_{aa_j}\delta^4(x-x_j)\dots\phi_{a_n}(x_n)|0\rangle\ .\end{aligned} \tag{67.8}$$

Here we have used $\varphi_a(x)$ to denote any kind of field, not necessarily a scalar field, carrying any kind of index or indices.

The classical equation of motion for the field $\varphi_a(x)$ is $\delta S/\delta\varphi_a(x) = 0$.

Thus, eq. (67.8) tells us that the classical equation of motion holds for a field inside a quantum correlation function, as long as its spacetime argument and indices do not match up exactly with those of any other field in the correlation function.

These matches, which constitute the right-hand side of eq. (67.8), are called contact terms.

Suppose we have a correlation function that, for whatever reason, includes a contact term with a factor of, say, $\delta^4(x_1-x_2)$.

φ $'$ τ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the $\mathcal{F}$ the the the $(\overline{\eta}, \eta, J)$ $\overline{v}$

After Fourier-transforming to momentum space, this contact term is a function of $k_1 + k_2$, but is independent of $k_1 - k_2$; hence it cannot take the form of the singular term in eq. (67.7).

Therefore, contact terms in a correlation function $\mathcal{F}$ do not contribute to the scattering amplitude $\mathcal{T}$.

Now let us consider a scattering process in quantum electrodyamics that involves an external photon with four-momentum k.

In Lorenz gauge (the simplest for this analysis), the LSZ formula reads

$$\langle f|i\rangle = i\varepsilon^\mu \int d^4x\, e^{-ikx}(-\partial^2)\ldots\langle 0|\mathrm{T}A_\mu(x)\ldots|0\rangle\ , \tag{67.9}$$

and the classical equation of motion for A^μ is

$$-Z_3\,\partial^2 A_\mu = \frac{\partial\mathcal{L}}{\partial A^\mu}\ . \tag{67.10}$$

In spinor electrodynamics, the right-hand side of eq. (67.10) is $Z_1 j^\mu$, where j^μ is the electromagnetic current.

(This is also true for scalar electrodynamics if $Z_4 = Z_1^2/Z_2$; we saw in problem 65.2 that this condition is necessary for gauge invariance.)

We therefore have

$$\langle f|i\rangle = iZ_3^{-1}Z_1\varepsilon^\mu \int d^4x\, e^{-ikx}\ldots\Big[\langle 0|\mathrm{T}j_\mu(x)\ldots|0\rangle + \text{contact terms}\Big]. \tag{67.11}$$

The contact terms arise because, as we saw in eq. (67.8), the classical equations of motion hold inside quantum correlation functions only up to contact terms.

However, the contact terms cannot generate singularities in the k^2's of the other particles, and so they do not contribute to the left-hand side.

(Remember that, for each of the other particles, there is still an appropriate wave operator, such as the Klein-Gordon wave operator for a scalar, acting on the correlation function. These wave operators kill any term that does not have an appropriate singularity.)

Now let us try replacing ε^μ in eq. (67.11) with k^μ.

We are attempting to prove that the result is zero, and we are almost there.

We can write the factor of ik^μ as $-\partial^\mu$ acting on the e^{-ikx}, and then we can integrate by parts to get ∂^μ acting on the correlation function.

(Strictly speaking, we need a wave packet for the external photon to kill surface terms.)

Then we have $\partial^\mu \langle 0|Tj_\mu(x)...|0\rangle$ on the right-hand side.

Now we use another result from section 22, namely that a Noether current for an exact symmetry obeys $\partial^\mu j\mu = 0$ classically, and

$$\partial^\mu \langle 0|\mathrm{T} j_\mu(x) \dots |0\rangle = \text{contact terms} \tag{67.12}$$

quantum mechanically; this is the *Ward (or Ward-Takahashi) identity*.

But once again, the contact terms do not have the right singularities to contribute to $\langle f \mid i \rangle$.

Thus we conclude that $\langle f \mid i \rangle$ vanishes if we replace an external photon's polarization vector ε^μ with its four-momentum k^μ, quo erat demonstratum

68 Ward Identities in Quantum Electrodynamics II

In this section, we will show that $Z_1 = Z_2$ in spinor electrodynamics, and that $Z_1 = Z_2 = Z_4$ in scalar electrodyanmics (in the OS and $\overline{MS}$ renormalization schemes).

Let us specialize to the case of spinor electrodynamics with a single Dirac field, and consider the correlation function

$$C^\mu_{\alpha\beta}(k,p',p) \equiv iZ_1 \int d^4x\, d^4y\, d^4z\, e^{ikx-ip'y+ipz} \langle 0|\mathrm{T} j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle \ , \qquad (68.1)$$

where $j^\mu = e\overline{\Psi}\gamma^\mu\Psi$ is the electromagnetic current.

As we saw in section 67, including $Z_1\, j^\mu(x)$ inside a correlation function adds a vertex for an external photon; the factor of Z_1 provides the necessary renormalization of this vertex.

The explicit fermion fields on the right-hand side of eq. (68.1) combine with the fermion fields in the current to generate propagators.

Thus we have

$$C^\mu_{\alpha\beta}(k,p',p) = (2\pi)^4\delta^4(k+p-p')\left[\tfrac{1}{i}\tilde{\mathbf{S}}(p')i\mathbf{V}^\mu(p',p)\tfrac{1}{i}\tilde{\mathbf{S}}(p)\right]_{\alpha\beta} \ , \qquad (68.2)$$

where is the exact fermion propagator, and $V\mu(p',p)$ is the exact 1PI photon–fermion–fermion vertex function.

Now let us consider $k_\mu C^\mu{}_{\alpha\beta}(k, p', p)$.

Using eq. (68.1), we can write thefactor of ik_μ on the right-hand side as ∂_μ acting on e^{ikx}, and then integrate by parts to get $-\partial_\mu$ acting on $j^\mu(x)$.

(Strictly speaking, we need a wave packet for the external photon to kill surface terms.)

Thus we have

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -\int d^4x\, d^4y\, d^4z\, e^{ikx-ip'y+ipz}\, \partial_\mu \langle 0|\mathrm{T} j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle\ . \tag{68.3}$$

Now we use the Ward identity from section 22, which in general reads

$$\begin{aligned} &-\partial_\mu \langle 0|\mathrm{T} J^\mu(x)\phi_{a_1}(x_1)\dots\phi_{a_n}(x_n)|0\rangle \\ &\quad = i\sum_{j=1}^{n}\langle 0|\mathrm{T}\phi_{a_1}(x_1)\dots\delta\phi_{a_j}(x)\delta^4(x-x_j)\dots\phi_{a_n}(x_n)|0\rangle\ . \end{aligned} \tag{68.4}$$

Here $\delta\varphi_a(x)$ is the change in a field $\varphi_a(x)$ under an infinitesimal transformation that leaves the action invariant, and

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \tag{68.5}$$

is the corresponding Noether current.

In the case of spinor electrodynamics,

$$\begin{aligned} \delta\Psi(x) &= -ie\Psi(x)\ , \\ \delta\overline{\Psi}(x) &= +ie\overline{\Psi}(x)\ , \end{aligned} \tag{68.6}$$

where we have dropped an infinitesimal parameter on the right-hand sides, but included a factor of the electron charge e.

Using $\partial\mathcal{L}/\partial(\partial_\mu\Psi) = iZ_2\overline{\Psi}\gamma^\mu$ and $\partial\mathcal{L}/\partial(\partial_\mu\overline{\Psi}) = 0$ we find that, with these conventions, the Noether current is $J^\mu = Z_2 e\overline{\Psi}\gamma^\mu\Psi = Z_2 j^\mu$.

Thus the Ward identity becomes

$$\begin{aligned} -Z_2\partial_\mu\langle 0|\mathrm{T}j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle \;=\; &+e\,\delta^4(x-y)\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle \\ &-e\,\delta^4(x-z)\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle\ . \end{aligned} \tag{68.7}$$

Recall that

$$\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle = \frac{1}{i}\int\frac{d^4q}{(2\pi)^4}\,e^{iq(y-z)}\,\tilde{\mathbf{S}}(q)_{\alpha\beta}\ . \tag{68.8}$$

Using eqs. (68.7) and (68.8) in eq. (68.3), and carrying out the coordinate integrals, we get

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -iZ_2^{-1}Z_1(2\pi)^4\delta^4(k+p-p')\Big[e\tilde{\mathbf{S}}(p) - e\tilde{\mathbf{S}}(p')\Big]_{\alpha\beta} \,. \tag{68.9}$$

On the other hand, from eq. (68.2) we have

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -i(2\pi)^4\delta^4(k+p-p')\Big[\tilde{\mathbf{S}}(p')k_\mu\mathbf{V}^\mu(p',p)\tilde{\mathbf{S}}(p)\Big]_{\alpha\beta} \,. \tag{68.10}$$

Comparing eqs. (68.9) and (68.10) shows that

$$(p'-p)_\mu\tilde{\mathbf{S}}(p')\mathbf{V}^\mu(p',p)\tilde{\mathbf{S}}(p) = Z_2^{-1}Z_1 e\Big[\tilde{\mathbf{S}}(p) - \tilde{\mathbf{S}}(p')\Big] \,, \tag{68.11}$$

where we have dropped the spin indices.

We can simplify eq. (68.11) by multiplying on the left by $\tilde{\mathbf{S}}(p')^{-1}$, and on the right by $\tilde{\mathbf{S}}(p)^{-1}$, to get

$$(p'-p)_\mu\mathbf{V}^\mu(p',p) = Z_2^{-1}Z_1 e\Big[\tilde{\mathbf{S}}(p')^{-1} - \tilde{\mathbf{S}}(p)^{-1}\Big] \,. \tag{68.12}$$

Thus we find a relation between the exact photon–fermion–fermion vertex function $\mathbf{V}^\mu(p', p)$ and the exact fermion propagator $\tilde{\mathbf{S}}(p)$.

Since both $\tilde{\mathbf{S}}(p)$ and $\mathbf{V}^\mu(p', p)$ are finite, eq. (68.12) implies that Z_1/Z_2 must be finite as well.

In the $\overline{MS}$ scheme (where all corrections to $Z_i = 1$ are divergent), this immediately implies that

$$Z_1 = Z_2 \,. \tag{68.13}$$

In the OS scheme, we recall that near the on-shell point $p^2 = p'^2 = -m^2$ and $(p' - p)^2 = 0$ we have $\tilde{\mathbf{S}}(p) = \not{p} + m$ and $\mathbf{V}^\mu(p', p) = e\gamma^\mu$.

Plugging these expressions into eq. (68.12) then yields $Z_1 = Z_2$ for the OS scheme.

To better understand this result, we note that when $Z_1 = Z_2$, we can combine the fermion kinetic term $iZ_2\overline{\Psi}\not{\partial}\Psi$ and the interaction term $Z_1 e\overline{\Psi}\not{A}\Psi$ into $iZ_2\overline{\Psi}\not{D}\Psi$, , where $D^\mu = \partial^\mu - ieA^\mu$ is the covariant derivative.

Recall that it is D^μ that has a simple gauge transformation, and so we might expect the lagrangian, written in terms of renormalized fields, to include ∂^μ and A^μ only in the combination D^μ.

It is still necessary to go through the analysis that led to eq. (68.12), however, because quantization requires fixing a gauge, and this renders suspect any naive arguments based on gauge invariance.

Still, in this case, those arguments yield the correct result.

A similar analysis is made in scalar electrodynamics.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\boldsymbol{\mathcal{L}}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the the the the the $(\overline{\eta}, \eta, J)$ $\boldsymbol{\mathcal{F}}\overline{v}$

69 Nonabelian Gauge Theory

Consider a lagrangian with N scalar or spinor fields $\phi_i(x)$ that is invariant under a continuous SU(N) or SO(N) symmetry,

$$\phi_i(x) \to U_{ij}\phi_j(x) \ , \tag{69.1}$$

where U_{ij} is an N × N special unitary matrix in the case of SU(N), or an N × N special orthogonal matrix in the case of SO(N).

(Special means that the determinant of U is one.)

Eq. (69.1) is called a global symmetry transformation, because the matrix U does not depend on the spacetime label x.

In section 58, we saw that quantum electrodynamics could be understood as having a *local* U(1) symmetry,

$$\phi(x) \to U(x)\phi(x) \ , \tag{69.2}$$

where $U(x) = \exp[-ie\Gamma(x)]$ can be thought of as a 1 × 1 unitary matrix that *does* depend on the spacetime label x.

Eq. (69.2) can be a symmetry of the lagrangian only if we include a U(1) *gauge field* $A_\mu(x)$, and promote ordinary derivatives ∂_μ of $\phi(x)$ to covariant derivatives $D_\mu = \partial_\mu - ieA_\mu$.

Under the transformation of eq. (69.2), we have

$$D_\mu \to U(x) D_\mu U^\dagger(x) \ . \tag{69.3}$$

With this transformation rule, a scalar kinetic term like $-(D_\mu\phi)^\dagger D_\mu\phi$, or a fermion kinetic term like $i\overline{\Psi}\not{D}\Psi$, is invariant, as are mass terms like $m^2\phi^\dagger\phi$ and $m\ \overline{\Psi}\ \Psi$.

We call eq. (69.3) a *gauge transformation*, and say that the lagrangian is *gauge invariant*.

Eq. (69.3) implies that the gauge field transforms as

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{e} U(x) \partial_\mu U^\dagger(x) \ . \tag{69.4}$$

If we use $U(x) = \exp[-ie\Gamma(x)]$, then eq. (69.4) simplifies to

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Gamma(x) \ , \tag{69.5}$$

which is what we originally had in section 54.

We can now easily generalize this construction of U(1) gauge theory to SU(N) or SO(N).

(We will consider other possibilities later.)

To be concrete, let us consider SU(N).

Recall from section 24 that we can write an infinitesimal SU(N) transformation as

$$U_{jk}(x) = \delta_{jk} - ig\theta^a(x)(T^a)_{jk} + O(\theta^2) \ , \tag{69.6}$$

where we have inserted a coupling constant g for later convenience.

The indices j and k run from 1 to N, the index a runs from 1 to $N^2 - 1$ (and is implicitly summed), and the *generator matrices* T^a are hermitian and traceless.

(These properties of T^a follow immediately from the special unitarity of U.)

The generator matrices obey commutation relations of the form

$$[T^a, T^b] = if^{abc}T^c \ , \tag{69.7}$$

where the real numerical factors f_{abc} are called the structure coefficients of the group.

If f_{abc} does not vanish, the group is *nonabelian*.

We can choose the generator matrices so that they obey the normalization condition

$$\mathrm{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab} \ ; \tag{69.8}$$

then eqs. (69.7) and (69.8) can be used to show that f_{abc} is completely antisymmetric.

For SU(2), we have $T^a = 1/2\ \sigma^a$, where σ^a is a Pauli matrix, and $f_{abc} = \varepsilon_{abc}$, where ε_{abc} is the completely antisymmetric Levi-Civita symbol.

Now we define an SU(N) gauge field $A_\mu(x)$ as a traceless hermitian N × N matrix of fields with the gauge transformation property

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \ . \tag{69.9}$$

Note that this is identical to eq. (69.4), except that now U(x) is a special unitary matrix (rather than a phase factor), and $A_\mu(x)$ is a traceless hermitian matrix (rather than a real number).

(Also, the electromagnetic coupling e has been replaced by g.)

We can write U(x) in terms of the generator matrices as

$$U(x) = \exp[-ig\Gamma^a(x)T^a] \ , \tag{69.10}$$

where the real parameters $\Gamma^a(x)$ are no longer infinitesimal.

The covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu(x) \ , \tag{69.11}$$

where there is an understood N × N identity matrix multiplying ∂_μ.

Acting on the set of N fields $\phi_i(x)$ that transform according to eq. (69.2), the covariant derivative can be written more explicitly as

$$(D_\mu \phi)_j(x) = \partial_\mu \phi_j(x) - igA_\mu(x)_{jk}\phi_k(x) \ , \tag{69.12}$$

with an understood sum over k.

The covariant derivative transforms according to eq. (69.3).

Replacing all ordinary derivatives in $\mathcal{L}$ with covariant derivatives renders $\mathcal{L}$ gauge invariant (assuming, of course, that $\mathcal{L}$ originally had a global SU(N) symmetry).

We still need a kinetic term for $A_\mu(x)$.

Let us define the *field strength*

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] \tag{69.13}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ . \tag{69.14}$$

Because A_μ is a matrix, the final term in eq. (69.14) does not vanish, as it does in U(1) gauge theory.

Eqs. (69.3) and (69.13) imply that, under a gauge transformation,

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^\dagger(x) \ . \tag{69.15}$$

Therefore,

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) \tag{69.16}$$

is gauge invariant, and can serve as a kinetic term for the SU(N) gauge field.

(Note, however, that the field strength itself is not gauge invariant, in contrast to the situation in U(1) gauge theory.)

Since we have taken $A_\mu(x)$ to be hermitian and traceless, we can expand it in terms of the generator matrices

$$A_\mu(x) = A_\mu^a(x)T^a \ . \tag{69.17}$$

Then we can use eq. (69.8) to invert eq. (69.17):

$$A_\mu^a(x) = 2\,\text{Tr}\,A_\mu(x)T^a \ . \tag{69.18}$$

Similarly, we have

$$F_{\mu\nu}(x) = F_{\mu\nu}^a T^a \ , \tag{69.19}$$

$$F_{\mu\nu}^a(x) = 2\,\text{Tr}\,F_{\mu\nu}T^a \ . \tag{69.20}$$

Using eq. (69.19) in eq. (69.14), we get

$$\begin{aligned} F_{\mu\nu}^c T^c &= (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)T^c - igA_\mu^a A_\nu^b[T^a, T^b] \\ &= (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc}A_\mu^a A_\nu^b)T^c \ . \end{aligned} \tag{69.21}$$

Then using eqs. (69.20) and (69.8) yields

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu \ . \tag{69.22}$$

Also, using eqs. (69.19) and (69.8) in eq. (69.16), we get

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{4} F^{c\mu\nu} F^c_{\mu\nu} \ . \tag{69.23}$$

From eq. (69.22), we see that $\mathcal{L}_{\text{kin}}$ includes interactions among the gauge fields.

A theory of this type, with nonzero fabc, is called *nonabelian gauge theory* or *Yang–Mills theory*.

Everything we have just said about SU(N) also goes through for SO(N), with unitary replaced by orthogonal, and traceless replaced by antisymmetric.

There is also another class of compact nonabelian groups called Sp(2N), and five exceptional compact groups: G(2), F(4), E(6), E(7), and E(8).

Compact means that $\mathrm{Tr}(T^a T^b)$ is a positive definite matrix.

Non-abelian gauge theory must be based on a compact group, because otherwise some of the terms in $\mathcal{L}_{\text{kin}}$ would have the wrong sign, leading to a hamiltonian that is unbounded below.

As a specific example, let us consider *quantum chromodynamics*, or QCD, which is based on the gauge group SU(3).

There are several Dirac fields corresponding to *quarks*.

Each quark comes in three *colors*; these are the values of the SU(3) index.

(These colors have nothing to do with ordinary color.)

There are also six *flavors*: up, down, strange, charm, bottom (or beauty) and top (or truth).

Thus we consider the Dirac field $\Psi_{iI}(x)$, where i is the color index and I is the flavor index.

The lagrangian is

$$\mathcal{L} = i\overline{\Psi}_{iI}\not{D}_{ij}\Psi_{jI} - m_I\overline{\Psi}_I\Psi_I - \tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \ , \tag{69.24}$$

where all indices are summed.

The different quark flavors have different masses, ranging from a few MeV for the up and down quarks to 178 GeV for the top quark.

(The quarks also have electric charges: +2/3 lel for the u, c, and t quarks, and −1/3 lel for the d, s, and b quarks. For now, however, we omit the appropriate coupling to the electromagnetic field.)

The covariant derivative in eq. (69.24) is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - igA^a_\mu T^a_{ij} \ . \tag{69.25}$$

The index a on $A^a{}_\mu$ runs from 1 to 8, and the corresponding massless spin-one particles are the eight *gluons*.

In a nonabelian gauge theory in general, we can consider scalar or spinor fields in different *representations* of the group.

A representation of a compact nonabelian group is a set of finite-dimensional hermitian matrices $T^a{}_R$ (the R is part of the name, not an index) that obey that same commutation relations as the original generator matrices T^a.

Given such a set of D(R) × D(R) matrices (where D(R) is the *dimension* of the representation*)*, and a field $\phi(x)$ with D(R) components, we can write its covariant derivative as $D_\mu = \partial_\mu - igA^a{}_\mu T^a{}_R$, with an understood D(R) × D(R) identity matrix multiplying ∂_μ.

Under a gauge transformation, $\phi(x) \rightarrow U_R(x)\phi(x)$, where $U_R(x)$ is given by eq. (69.10) with ^{Ta} replaced by $T^a{}_R$.

The theory will be gauge invariant provided that the transformation rule for $A^a{}_\mu$ is independent of the representation used in eq. (69.9); it is.

We will not need to know a lot of representation theory, but we collect some useful facts in the next section.

70 Group Representations

Given the structure coefficients f_{abc} of a compact nonabelian group, a *representation* of that group is specified by a set of D(R) × D(R) traceless hermitian matrices T^a_R (the R is part of the name, not an index) that obey that same commutation relations as the original generators matrices T^a, namely

$$[T^a_{\rm R}, T^b_{\rm R}] = i f^{abc} T^c_{\rm R} \ . \tag{70.1}$$

The number D(R) is the *dimension* of the representation.

The original T^a's correspond to the *fundamental* or *defining representation*.

Consider taking the complex conjugate of the commutation relations, eq. (70.1).

Since the structure coefficients are real, we see that the matrices $-(T^a_R)^*$ also obey these commutation relations.

If $-(T^a_R)^* = T^a_R$ or if we can find a unitary transformation $T^a_R \to U^{-1} T^a_R\, U$ that makes $-(T^a_R)^* = T^a_R$ for every a, then the representation R is *real*.

If such a unitary transformation does not exist, but we can find a unitary matrix $V \neq I$ such that $-(T^a_R)^* = V^{-1}\, T^a_R\, V$ for every a, then the representation R is *pseudoreal*.

If such a unitary matrix also does not exist, then the representation R is *complex*.

In this case, the *complex conjugate representation* R is specified by

$$T^a_{\overline{\mathrm{R}}} = -(T^a_{\mathrm{R}})^* \ . \tag{70.2}$$

One way to prove that a representation is complex is to show that at least one generator matrix T^a_R (or a real linear combination of them) has eigenvalues that do not come in plus-minus pairs.

This is the case for the fundamental representation of SU(N) with $N \geq 3$.

For SU(2), the fundamental representation is pseudoreal, because $-(1/2\ \sigma^a)^* \neq 1/2\ \sigma_a$, but $-(1/2\ \sigma_a)^* = V^{-1}(1/2\ \sigma_a)V$ with $V = \sigma_2$.

For SO(N), the fundamental representation is real, because the generator matrices are antisymmetric, and every antisymmetric hermitian matrix is equal to minus its complex conjugate.

An important representation for any compact nonabelian group is the *adjoint representatio*n A.

This is given by

$$(T^a_{\mathrm{A}})^{bc} = -if^{abc} \ . \tag{70.3}$$

Because f_{abc} is real and completely antisymmetric, T^a_A is manifestly hermitian, and also satisfies eq. (70.2); thus the adjoint representation is real.

The dimension of the adjoint representation D(A) is equal to the number of generators of the group; this number is also called the dimension of the group.

To see that the $T^a{}_A$'s satisfy the commutation relations, we use the Jacobi identity

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0 \,, \tag{70.4}$$

which holds for the structure coefficients of any group.

To prove the Jacobi identity, we note that

$$\mathrm{Tr}\, T^e \Big([[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] \Big) = 0 \,, \tag{70.5}$$

where the T^a's are the original generator matrices.

That the left-hand side of eq. (70.5) vanishes can be seen by writing out all the commutators as matrix products, and noting that they cancel in pairs.

Employing the commutation relations twice in each term, followed by

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} \,, \tag{70.6}$$

ultimately yields eq. (70.4).

Then, using the antisymmetry of the structure coefficients, inserting some judicious factors of i, and moving the last term of eq. (70.4) to the right-hand side, we can rewrite it as

$$(-if^{abd})(-if^{cde}) - (-if^{cbd})(-if^{ade}) = if^{acd}(-if^{dbe}) \,. \tag{70.7}$$

Now we use eq. (70.3) in eq. (70.7) to get

$$(T_{\rm A}^a)^{bd}(T_{\rm A}^c)^{de} - (T_{\rm A}^c)^{bd}(T_{\rm A}^a)^{de} = if^{acd}(T_{\rm A}^d)^{be} \ , \tag{70.8}$$

or equivalently $[T^a_A , T^c_A] = if_{acd}T^d_A$.

Thus the T^a_A's satisfy the appropriate commutation relations.

Two related numbers usefully characterize a representation: the *index* T(R) and the *quadratic Casimir* C(R).

The index is defined via

$$\mathrm{Tr}(T_{\rm R}^a T_{\rm R}^b) = T(\mathrm{R})\delta^{ab} \ . \tag{70.9}$$

Now it can be shown that the matrix $T^a_R\ T^a_R$ commutes with every generator, and so must be a number times the identity matrix; this number is the quadratic Casimir C(R).

It is easy to show that

$$T(\mathrm{R})D(\mathrm{A}) = C(\mathrm{R})D(\mathrm{R}) \ . \tag{70.10}$$

With the standard normalization conventions for the generators, we have T(N) = 1/2 for the fundamental representation of SU(N) and T(N) = 2 for the fundamental representation of SO(N).

It can be shown that T(A) = N for the adjoint representation of SU(N), and also shown that T(A) = 2N– 4 for the adjoint representation of SO(N).

A representation R is reducible if there is a unitary transformation $T^a_R \to U^{-1} T^a_R U$ that puts all the nonzero entries into the same diagonal blocks for each a; otherwise it is irreducible.

Consider a reducible representation R whose generators can be put into (for example) two blocks, with the blocks forming the generators of the irreducible representations R_1 and R_2.

Then R is the direct sum representation $R = R_1 \oplus R_2$, and we have

$$D(R_1 \oplus R_2) = D(R_1) + D(R_2) \ , \tag{70.11}$$

$$T(R_1 \oplus R_2) = T(R_1) + T(R_2) \ . \tag{70.12}$$

Suppose we have a field $\varphi_{iI}(x)$ that carries two group indices, one for the representation R_1 and one for the representation R_2, denoted by i and I respectively.

This field is in the direct product representation $R_1 \oplus R_2$. The corresponding generator matrix is

$$(T^a_{R_1 \otimes R_2})_{iI,jJ} = (T^a_{R_1})_{ij} \delta_{IJ} + \delta_{ij} (T^a_{R_2})_{IJ} \ , \tag{70.13}$$

where i and I together constitute the row index, and j and J together constitute the column index.

We then have

$$D(R_1 \otimes R_2) = D(R_1) D(R_2) \ , \tag{70.14}$$

$$T(R_1 \otimes R_2) = T(R_1) D(R_2) + D(R_1) T(R_2) \ . \tag{70.15}$$

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