OPTICS

This study of OPTICS will cover a lot of material that you probably never thought was part of the subject.

There will be all the ordinary stuff that many of you have probably seen before, but there will now be a lot of mathematics to enable us to present the theory behind the standard optics rules.

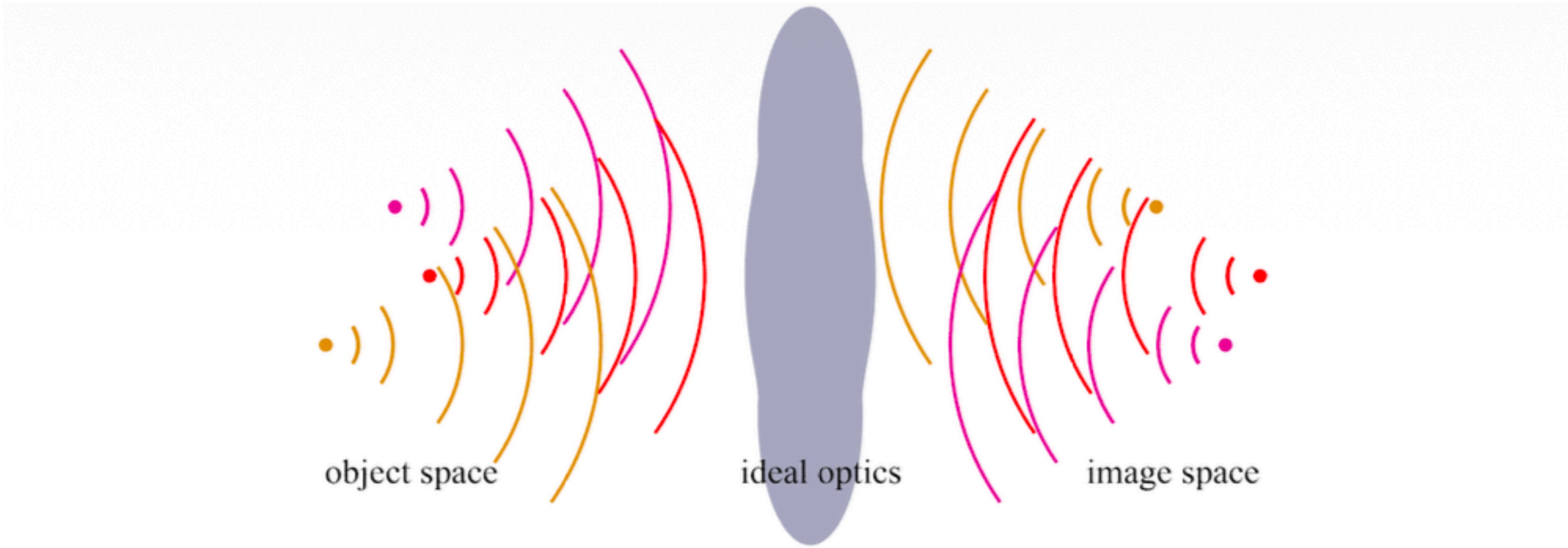
We begin this study with standard geometrical optics and then go to a mathematical approach to apply the standard rules of optics and then step back to go deeper into the mathematics and the physics theory to build up the rules from a set of assumptions.

Geometrical Optics - A quick Visual Tour to Learn Terminology

Outline

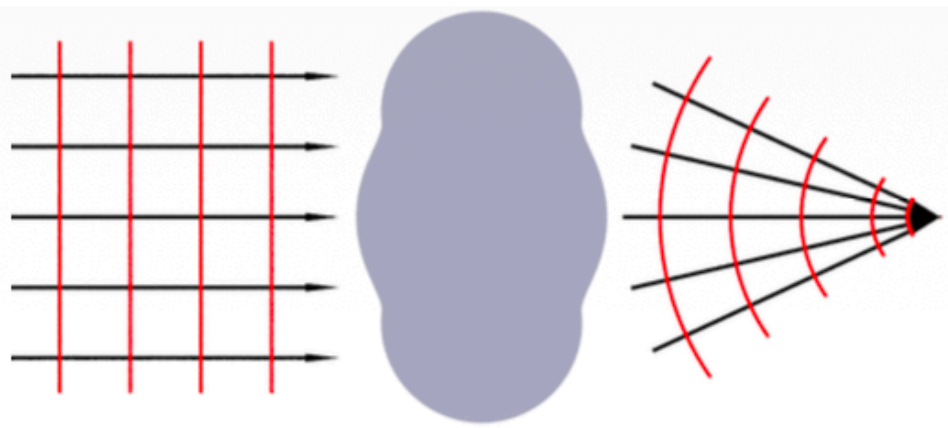
1. Geometrical Approximation
2. Lenses
3. Mirrors
4. Optical Systems
5. Images and Pupils
6. Aberrations

Ideal Optics



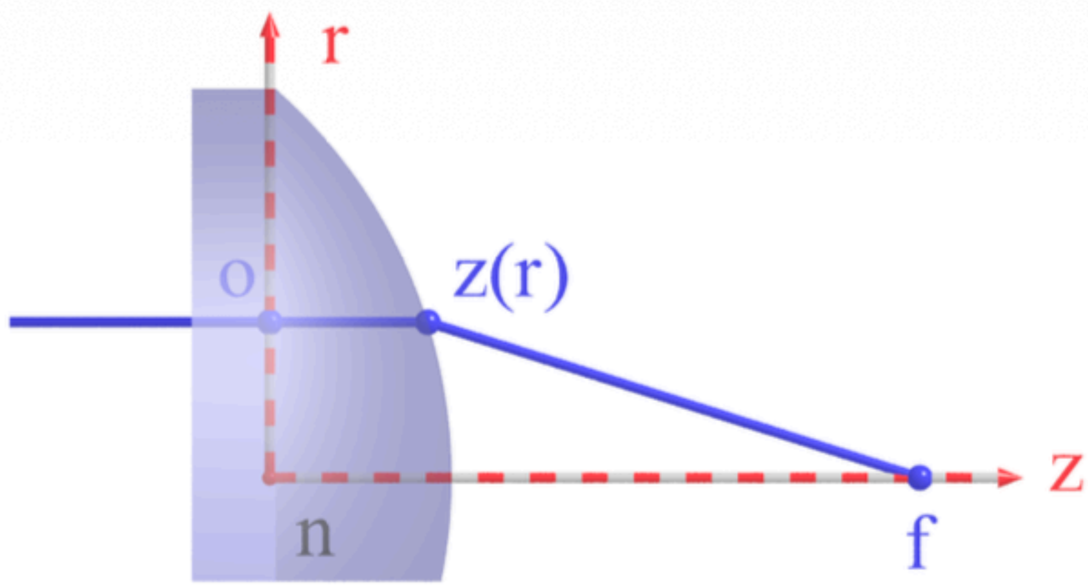
- ideal optics: spherical waves from any point in object space are imaged into points in image space
- corresponding points are called *conjugate points*
- *focal point*: center of converging or diverging spherical wavefront
- object space and image space are reversible

Geometrical Optics



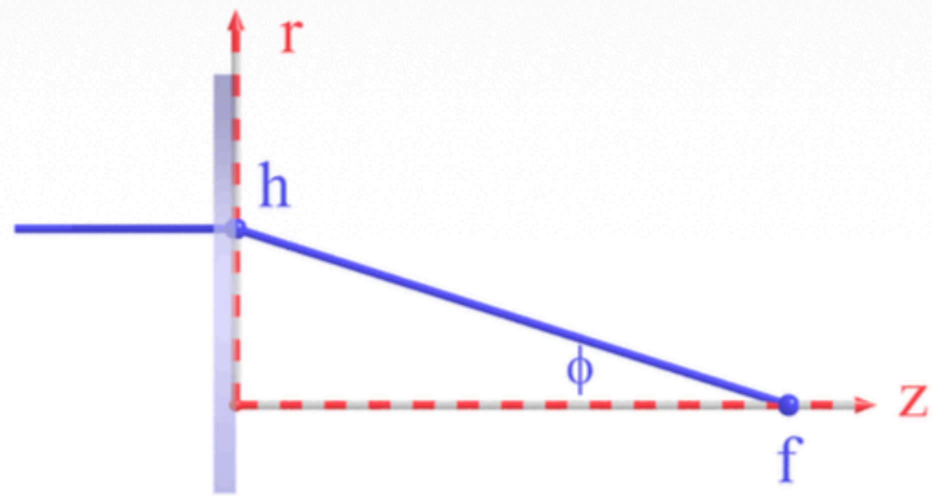
- rays are normal to locally flat wave (locations of constant phase)
- rays are reflected and refracted according to Fresnel equations
- phase is neglected $\Rightarrow$ incoherent sum
- rays can carry polarization information
- optical system is finite $\Rightarrow$ diffraction
- geometrical optics neglects diffraction effects: $\lambda \Rightarrow 0$
- *physical optics* $\lambda > 0$
- simplicity of geometrical optics mostly outweighs limitations

Surface Shape of Perfect Lens



- lens material has index of refraction n
- $\overline{o\,z(r)} \cdot n + \overline{z(r)\,f} = \text{constant}$
- $n \cdot z(r) + \sqrt{r^2 + (f - z(r))^2} = \text{constant}$
- solution $z(r)$ is hyperbola with eccentricity $e = n > 1$

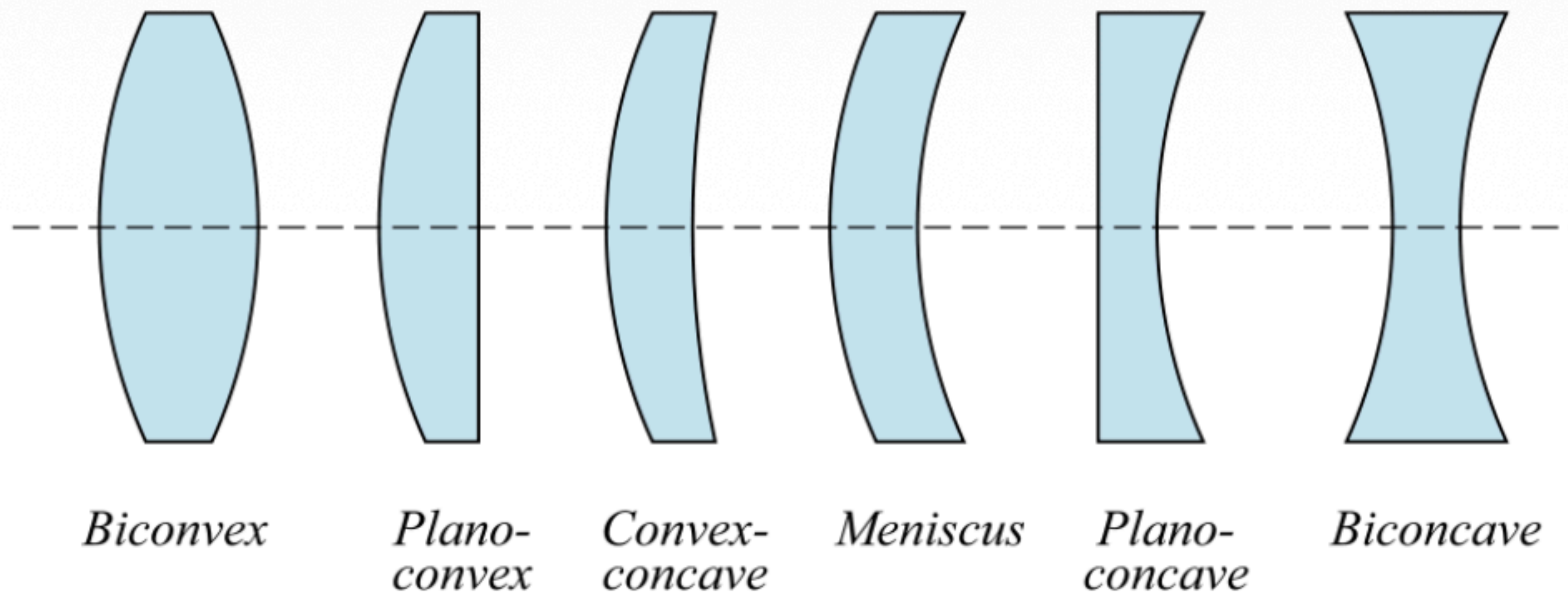
Paraxial Optics



Assumptions:

1. assumption 1: Snell's law for small angles of incidence ($\sin\phi \approx \phi$)
2. assumption 2: ray hight h small so that optics curvature can be neglected (plane optics, ($\cos\phi \approx 1$))
3. assumption 3: $tan\phi \approx \phi = h/f$
4. decent until about 10 degrees

Spherical Lenses



en.wikipedia.org/wiki/File:Lens2.svg

- if two spherical surfaces have same radius, can fit them together
- surface error requirement less than $\lambda/10$
- grinding spherical surfaces is easy $\Rightarrow$ most optical surfaces are spherical

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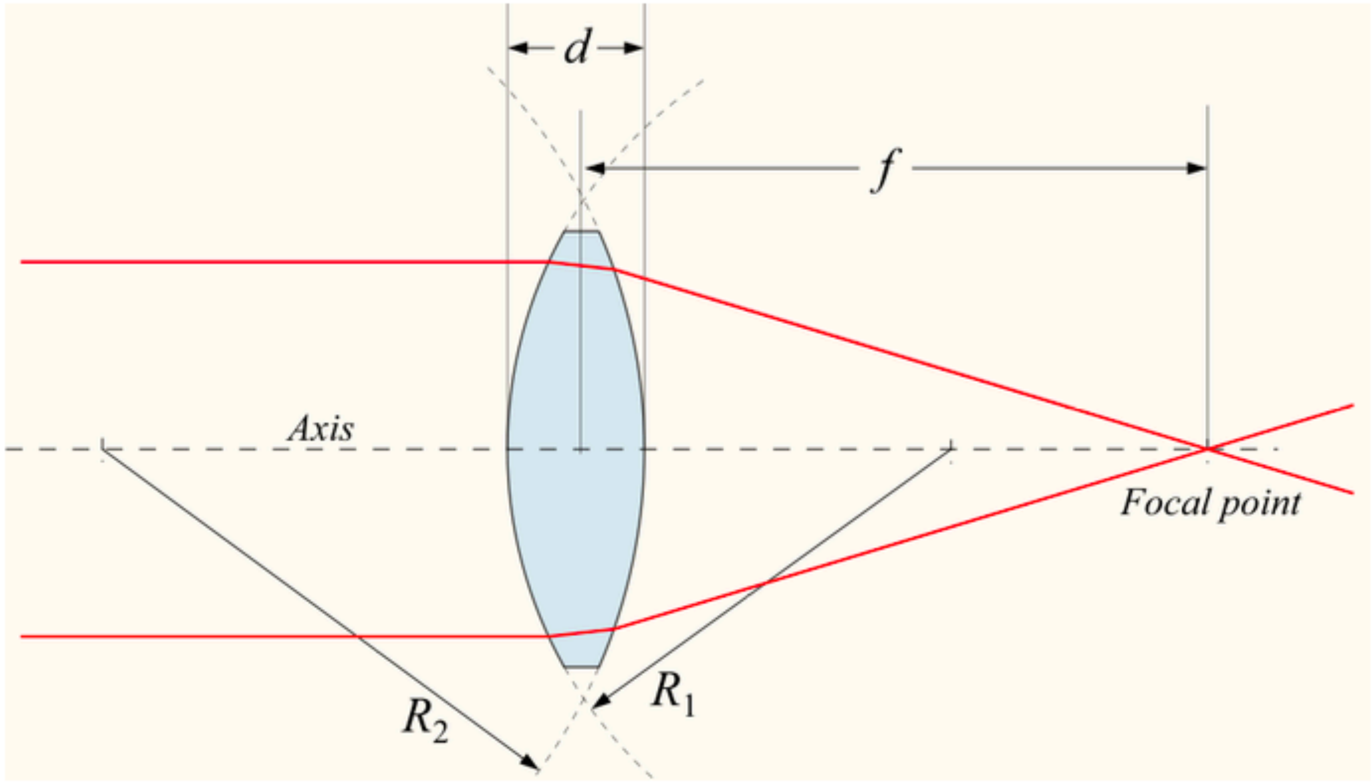
1.1-meter Singlet Lens of Swedish Solar Telescope

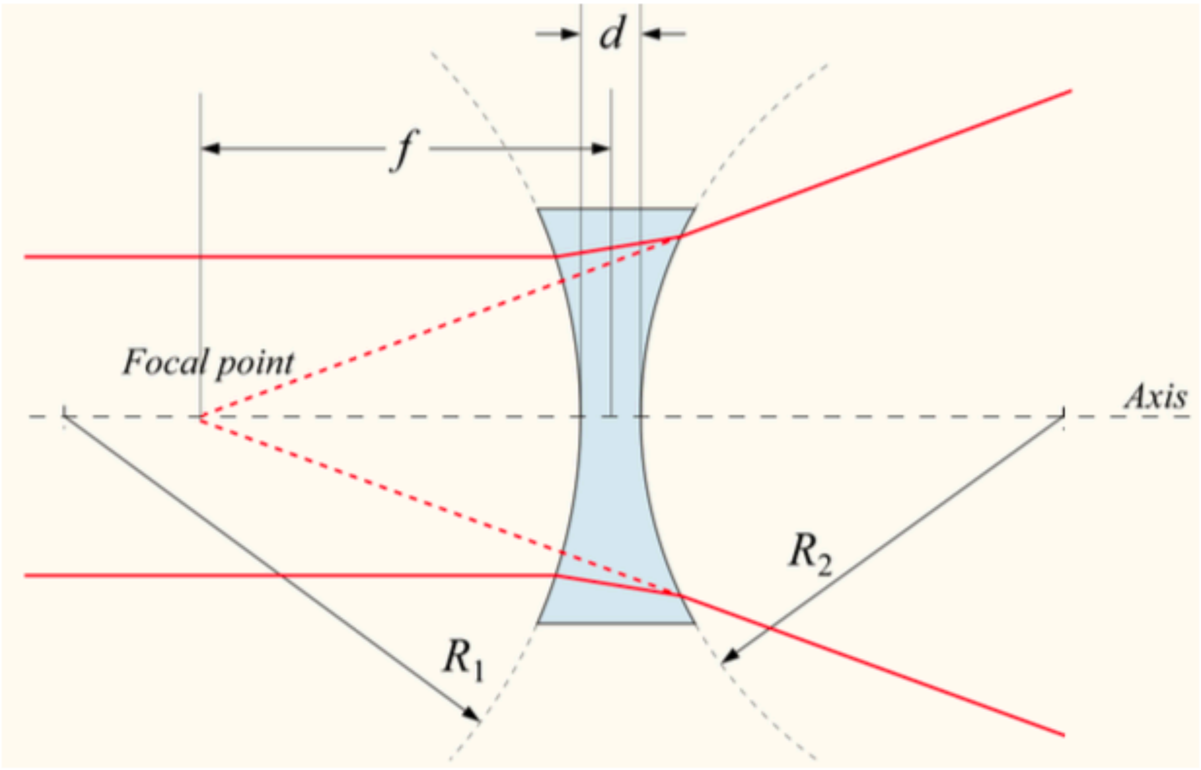
Positive/Converging Spherical Lens Parameters



commons.wikimedia.org/wiki/File:Lens1.svg

- center of curvature and radii with signs: $R_1 > 0, R_2 < 0$
- center thickness: d
- positive focal length $f > 0$

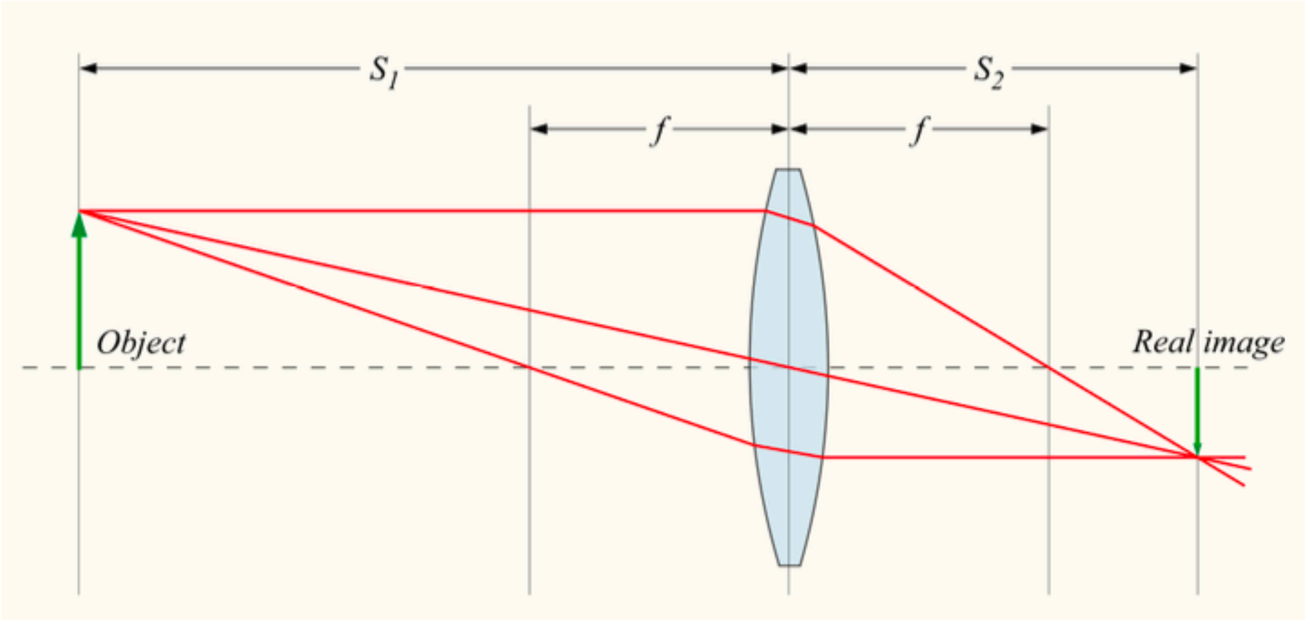
Negative/Diverging Spherical Lens Parameters



commons.wikimedia.org/wiki/File:Lens1b.svg

- note different signs of radii: $R_1 < 0, R_2 > 0$
- virtual focal point
- negative focal length ($f < 0$)

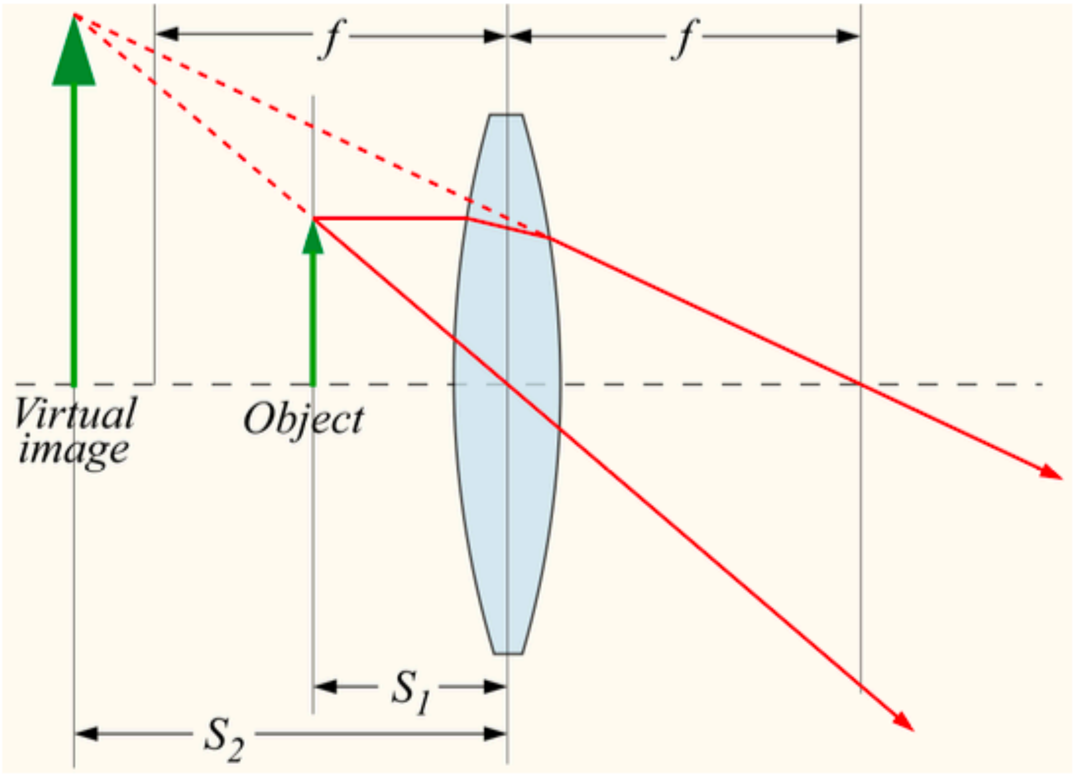
General Lens Setup: Real Image



commons.wikimedia.org/wiki/File:Lens3.svg

- *object distance* S_1, *object height* h_1
- *image distance* S_2, *image height* h_2
- axis through two centers of curvature is *optical axis*
- surface point on optical axis is the *vertex*
- *chief ray* through center maintains direction

General Lens Setup: Virtual Image



commons.wikimedia.org/wiki/File:Lens3b.svg

- note object closer than focal length of lens
- virtual image

Thin Lens Approximation

- thin-lens equation:

$$\frac{1}{S_1} + \frac{1}{S_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

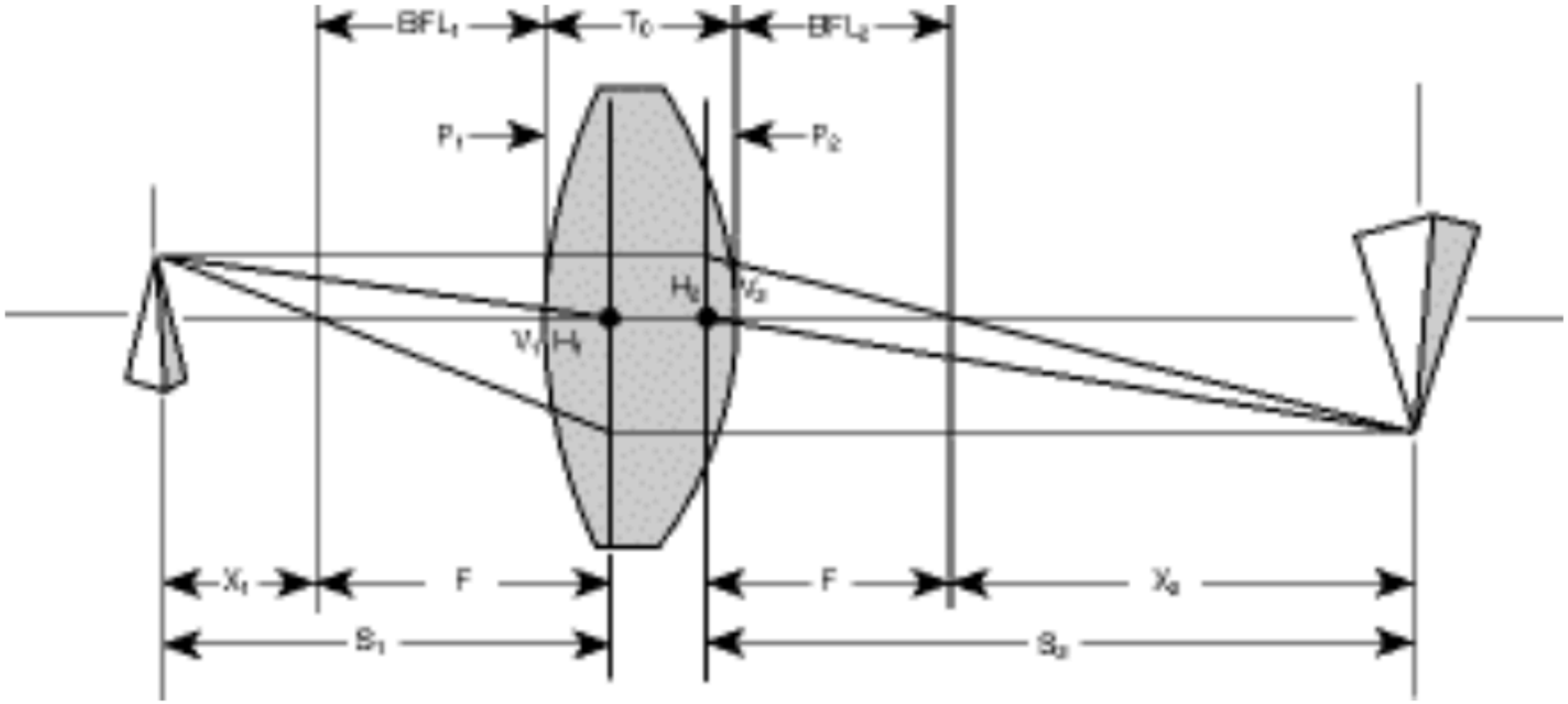
- Gaussian lens formula:

$$\frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{f}$$

Finite Imaging

- rarely image point sources, but extended object
- object and image size are proportional
- orientation of object and image are inverted
- (transverse) magnification perpendicular to optical axis:
  $M = h_2/h_1 = -S_2/S_1$

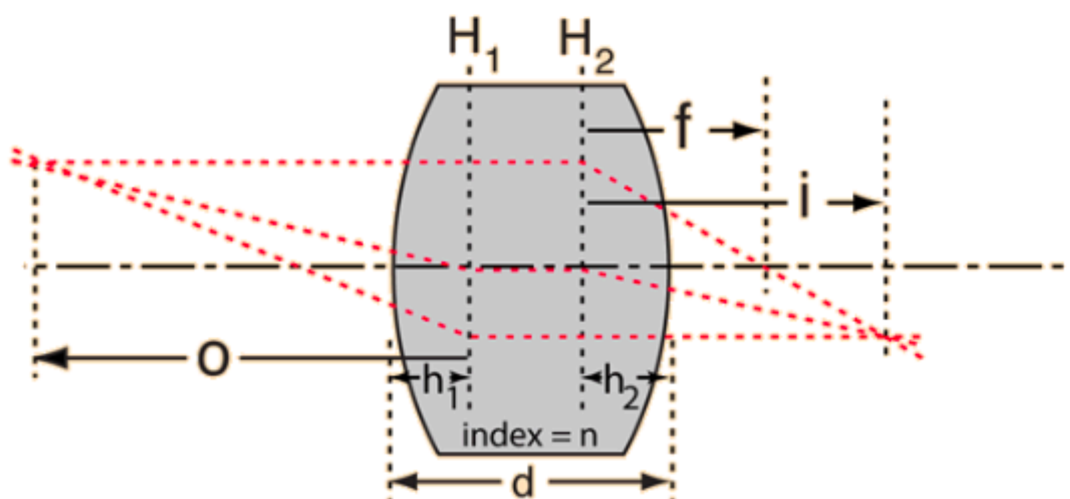
Thick Lenses



www.newport.com/servicesupport/Tutorials/default.aspx?id=169

- basic thick lens equation $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right)$
- reduces to thin lens for $d << R_1 R_2$

Principal Planes

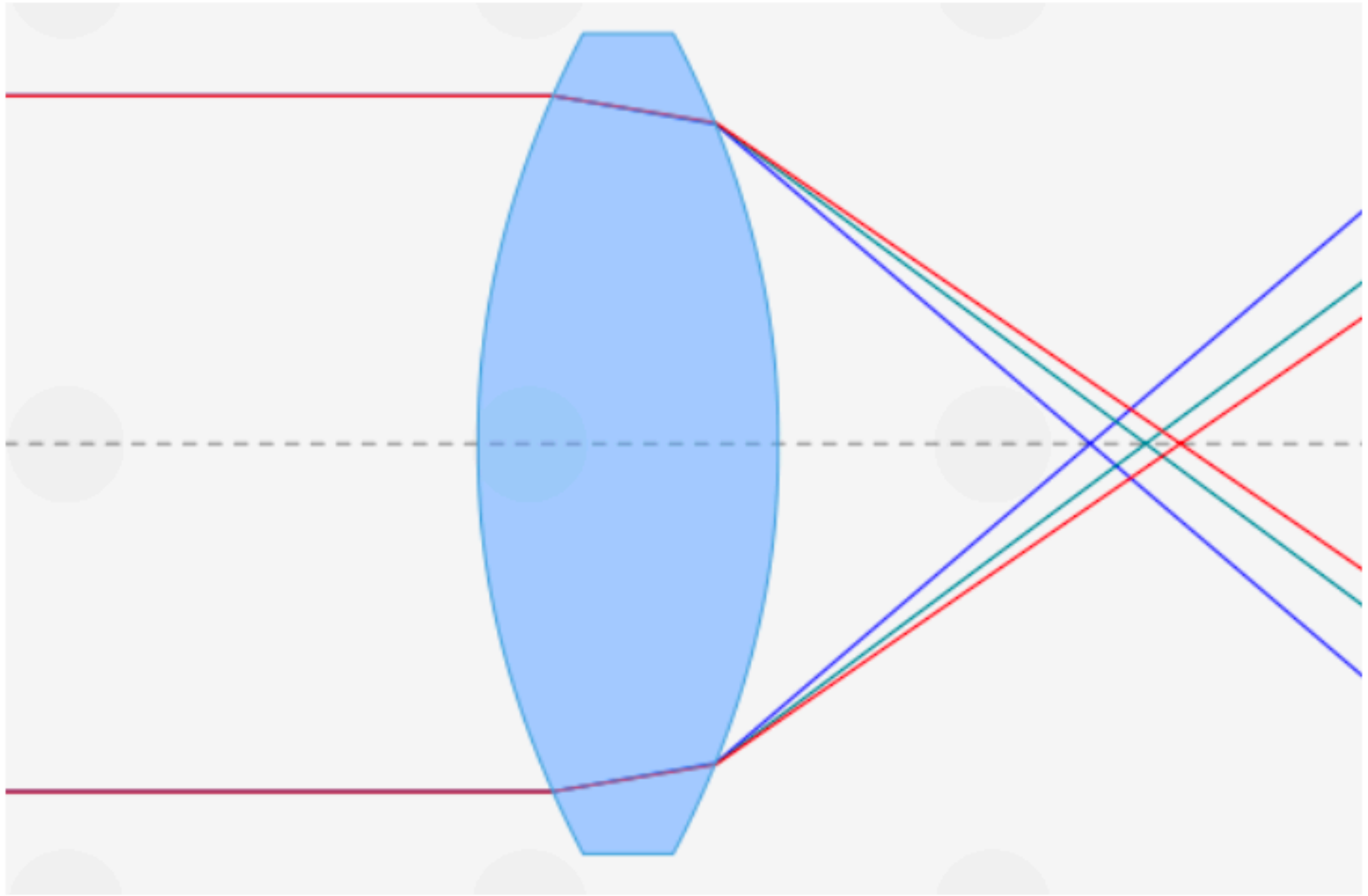


hyperphysics.phy-astr.gsu.edu/hbase/geoopt/priplan.html

- focal lengths, distances measured from *principal planes*
- refraction can be considered to only happen at principle planes
- principle planes only depend on lens properties itself
- thin lens equation neglects finite distance between principle planes
- distance between vertices and principal planes given by

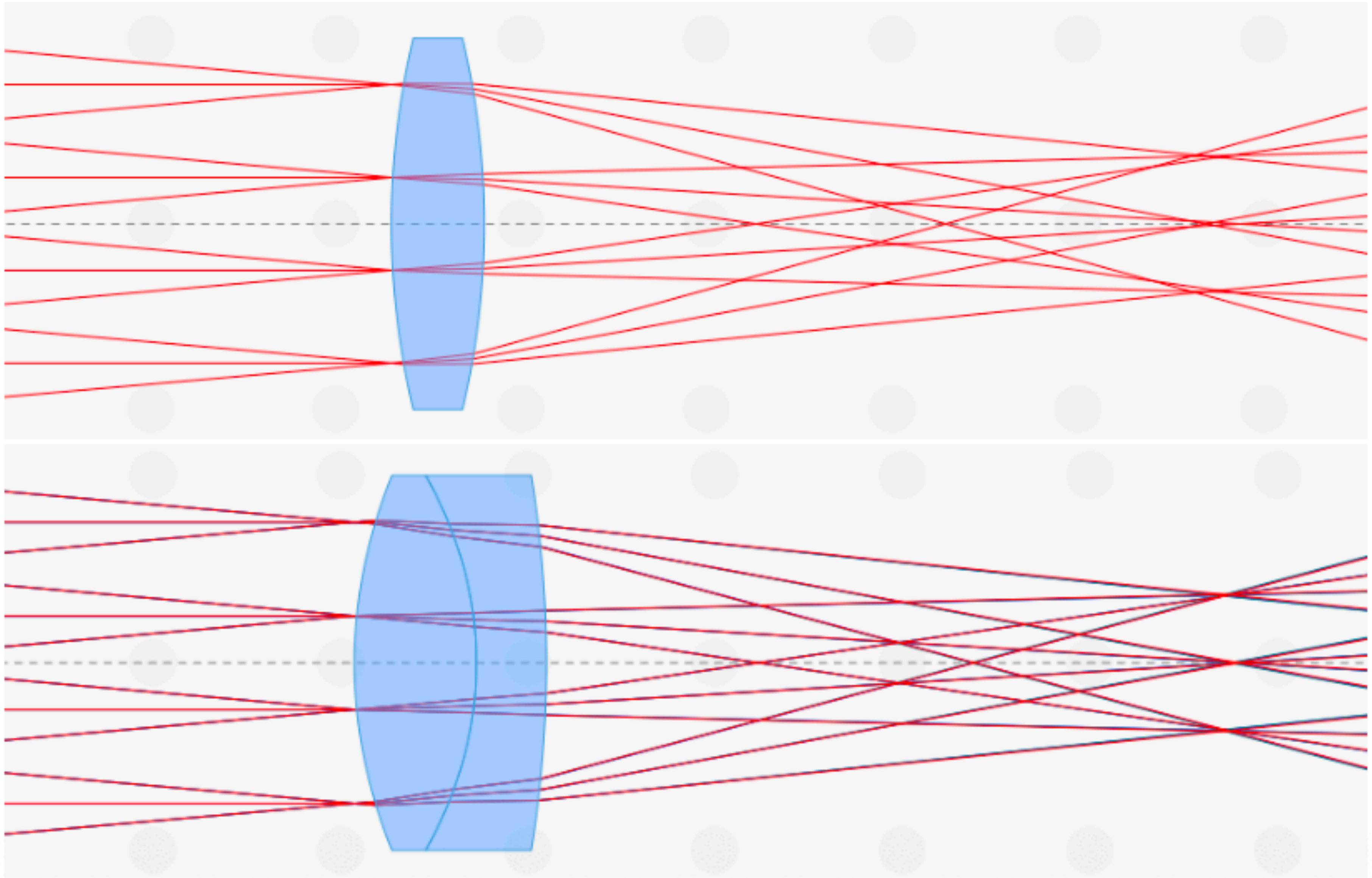
$$H_{1,2} = -\frac{f(n-1)d}{R_{2,1}n}$$

Chromatic Aberration



- due to wavelength dependence of index of refraction
- higher index in the blue $\Rightarrow$ shorter focal length in blue

Achromatic Lens



- combination of 2 lenses, different glass dispersion
- also less spherical aberration

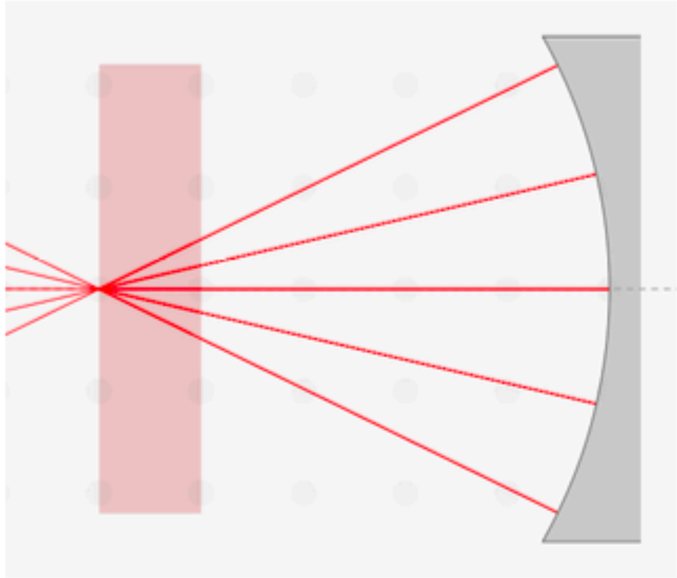
Mirrors

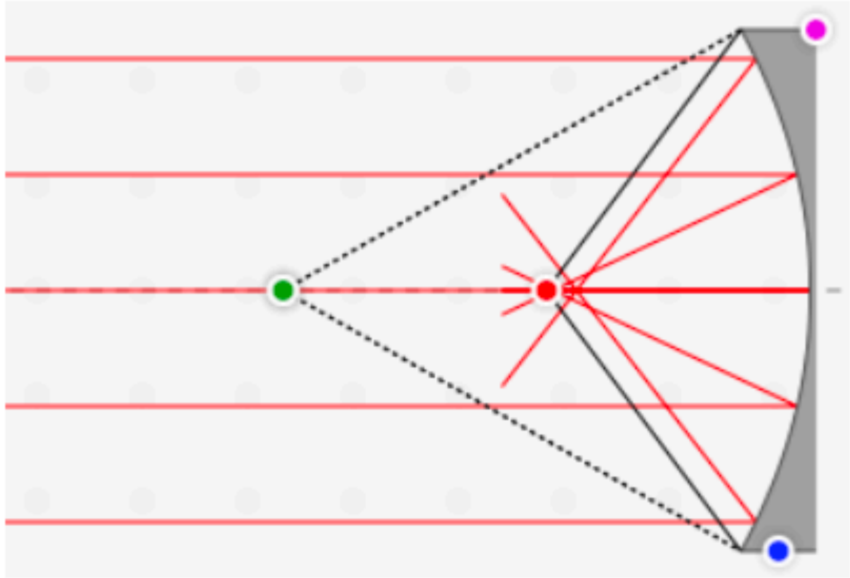
Mirrors vs. Lenses

- mirrors are completely achromatic
- reflective over very large wavelength range (UV to radio)
- can be supported from the back
- can be segmented
- wavefront error is twice that of surface, lens is (n-1) times surface
- only one surface to 'play' with

Plane Mirrors: Fold Mirrors and Beamsplitters

Spherical Mirrors

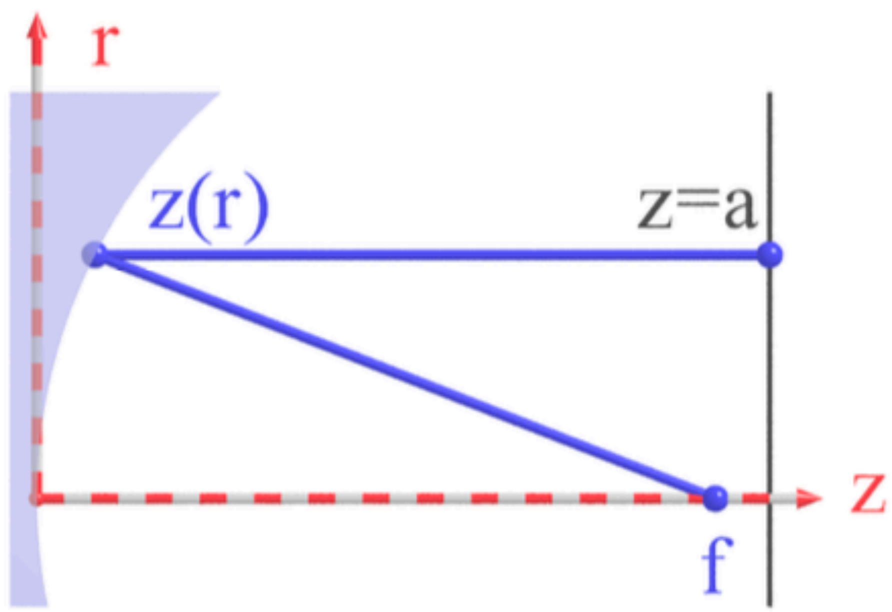




- easy to manufacture
- focuses light from center of curvature onto itself
- focal length is half of curvature: $f = R/2$
- tip-tilt misalignment does not matter
- has no optical axis
- does not image light from infinity correctly (spherical aberration)

2-meter Spherical Primary Mirror in Tautenburg Schmidt Camera

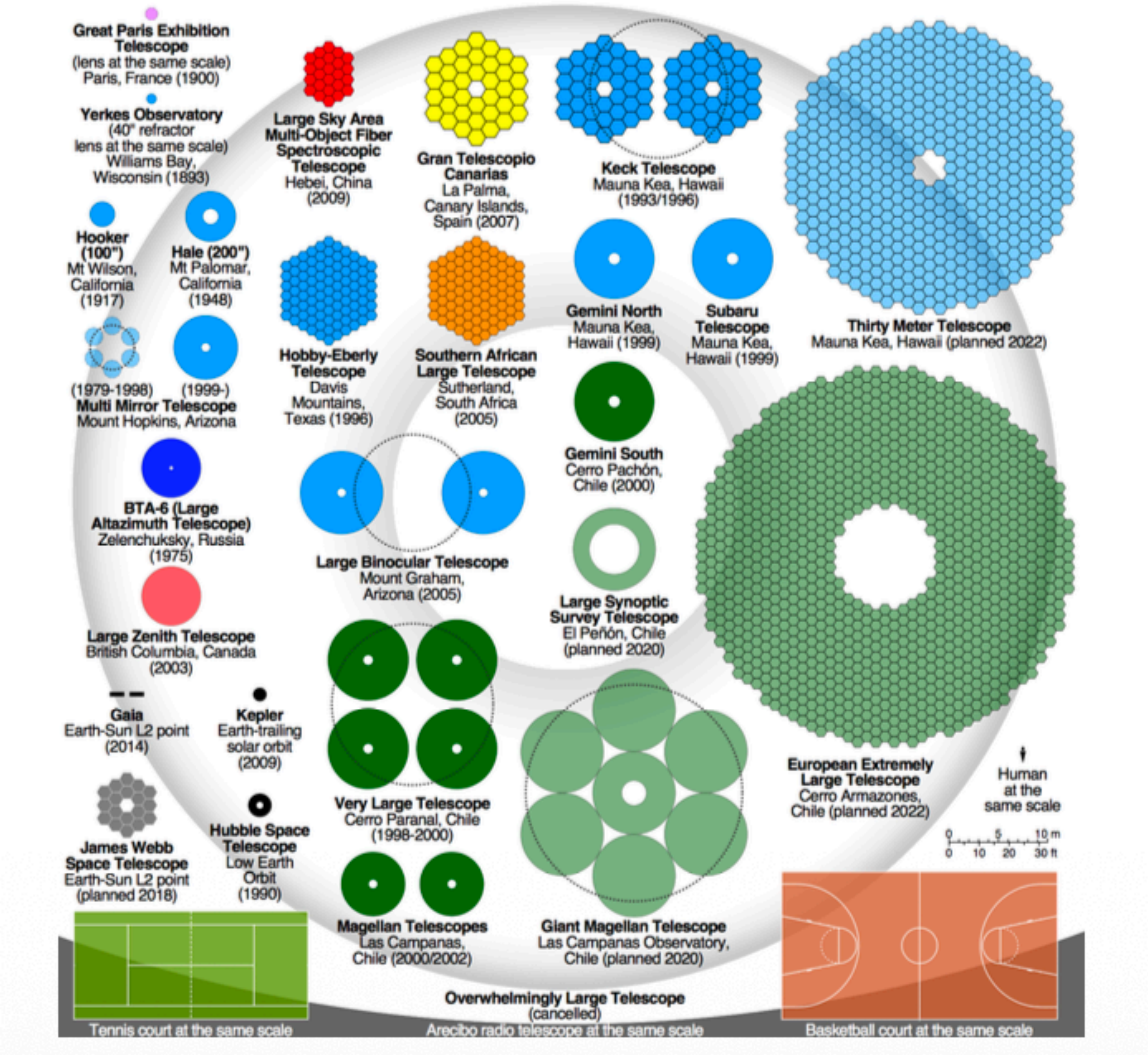
Parabolic Mirrors



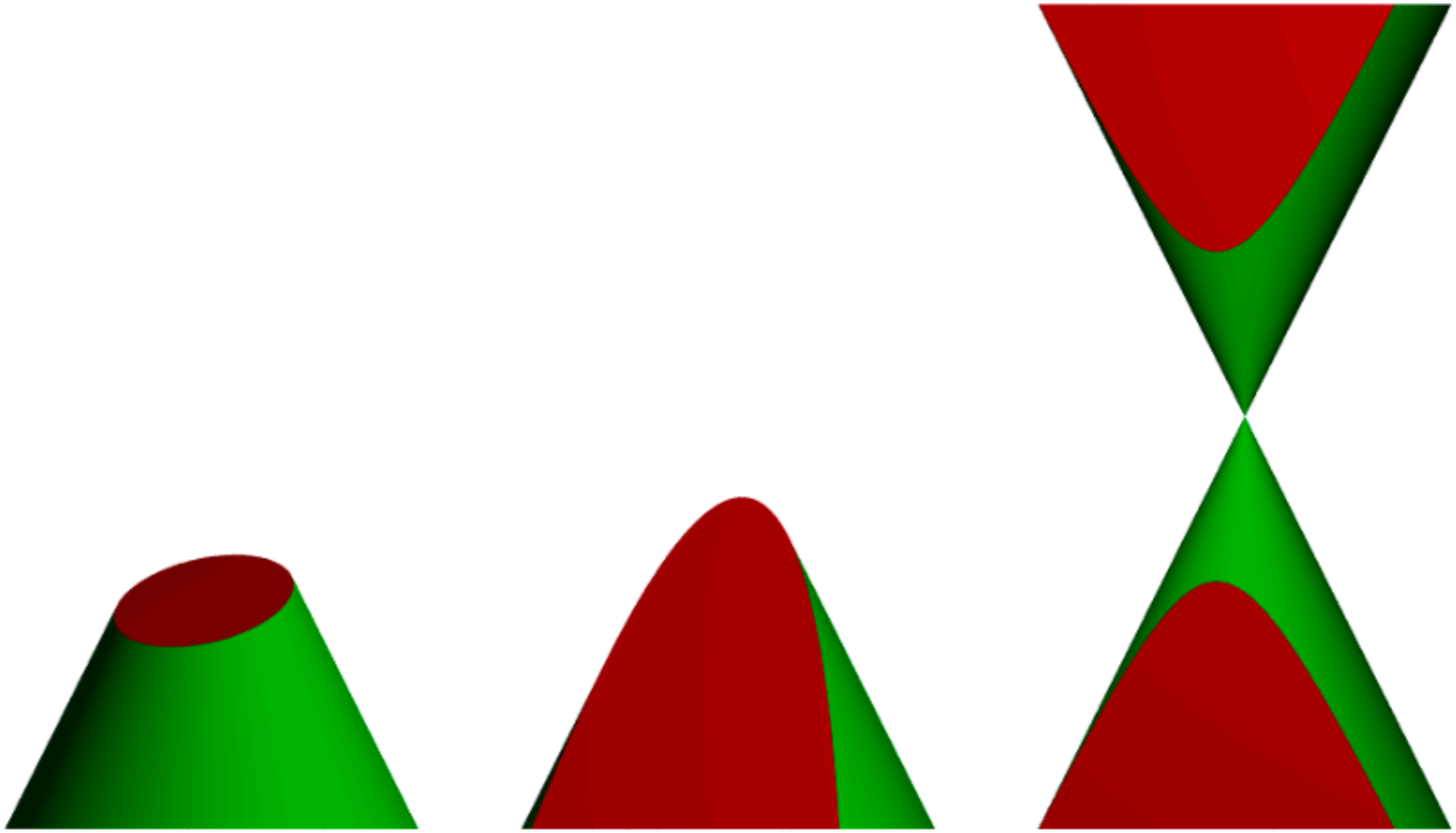
- want to make flat wavefront into spherical wavefront
- distance $\overline{az(r)} + \overline{z(r)f}$ = const.
- $z(r) = r^2/2R$
- perfect image of objects at infinity
- has clear optical axis

8.4-meter Large Binocular Telescope Primary Mirror Nr. 2

Telescope Mirror Comparison

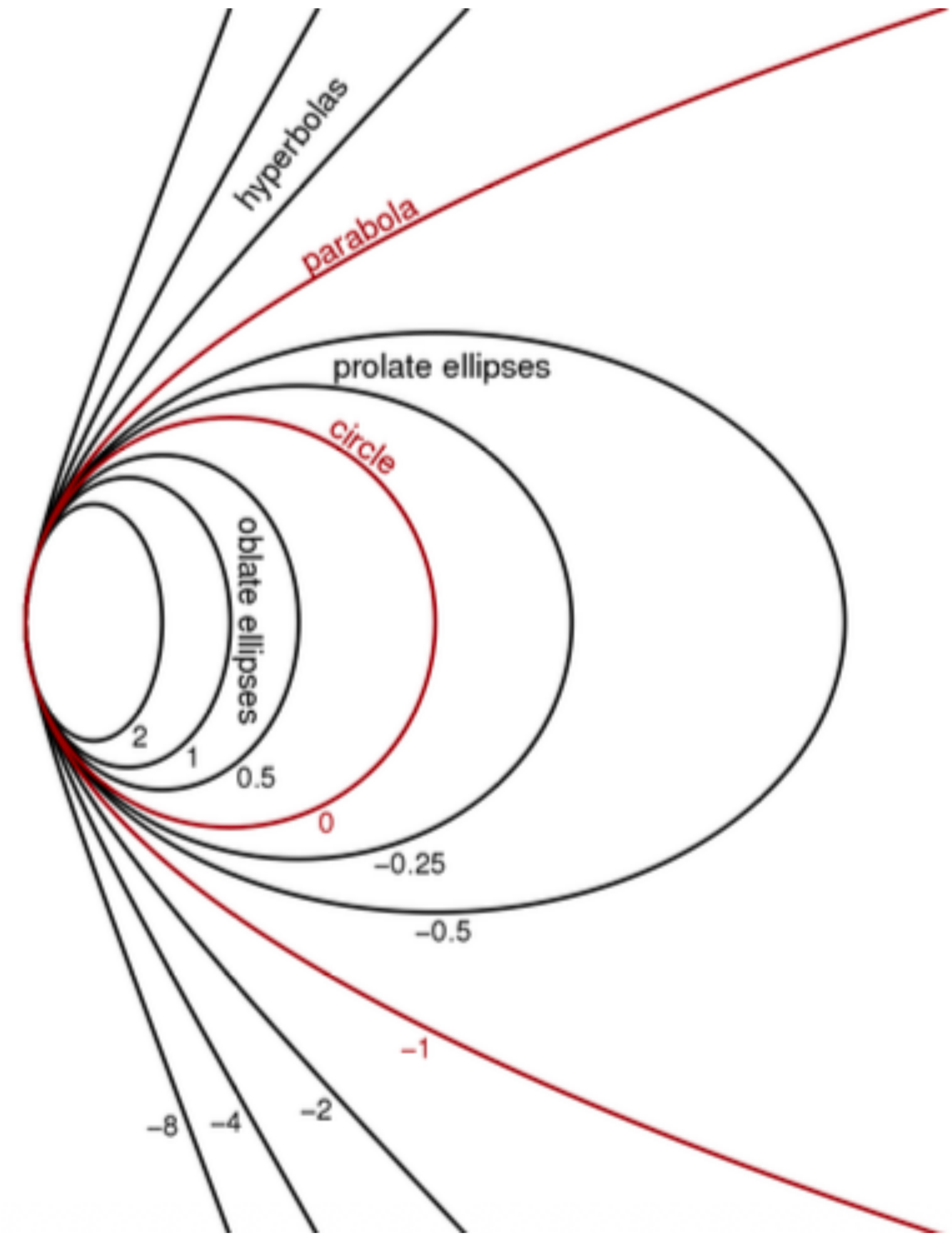


Conic Sections



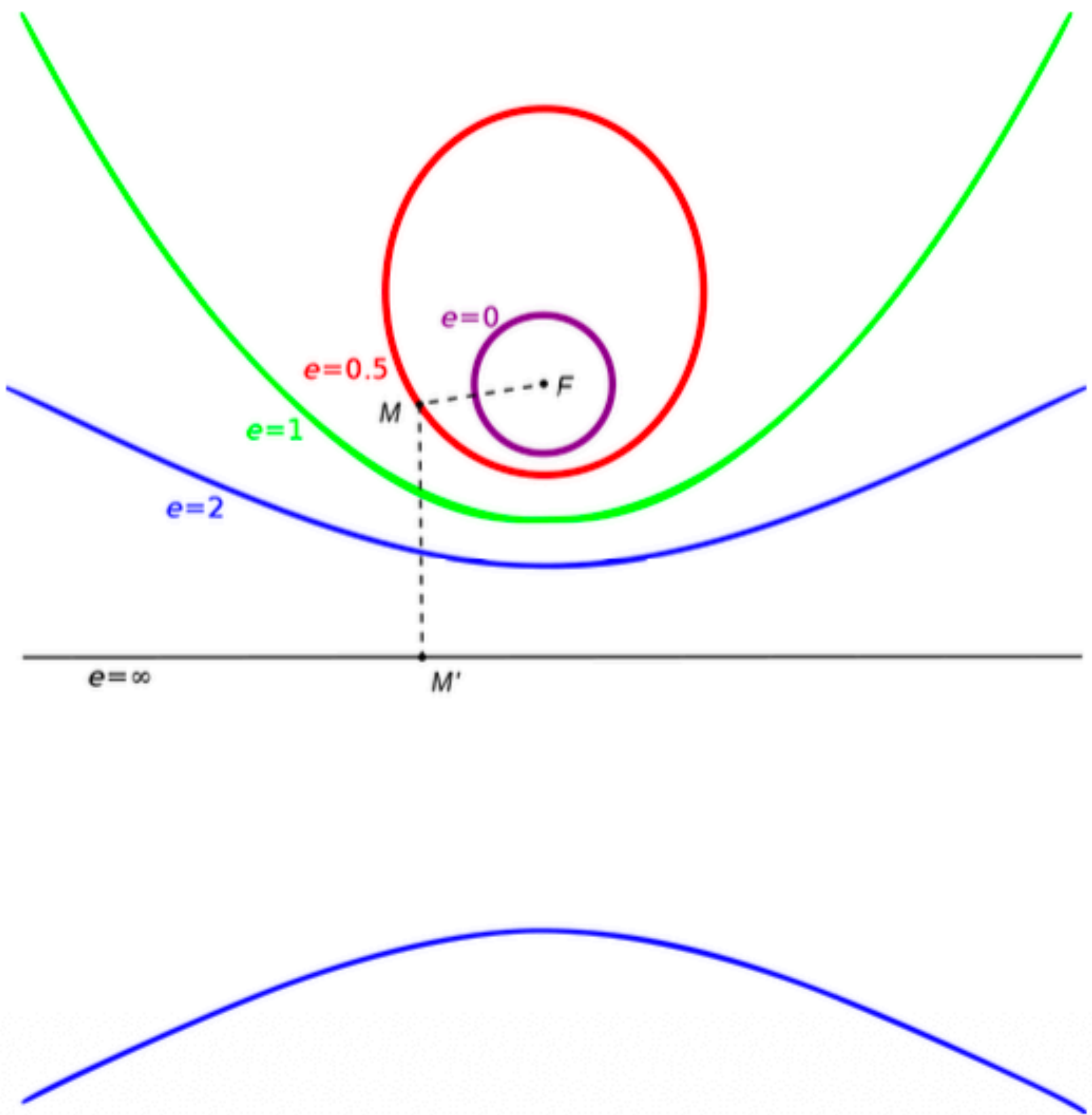
- circle and ellipses: cuts angle < cone angle
- parabola: angle = cone angle
- hyperbola: cut along axis

Conic Constant K



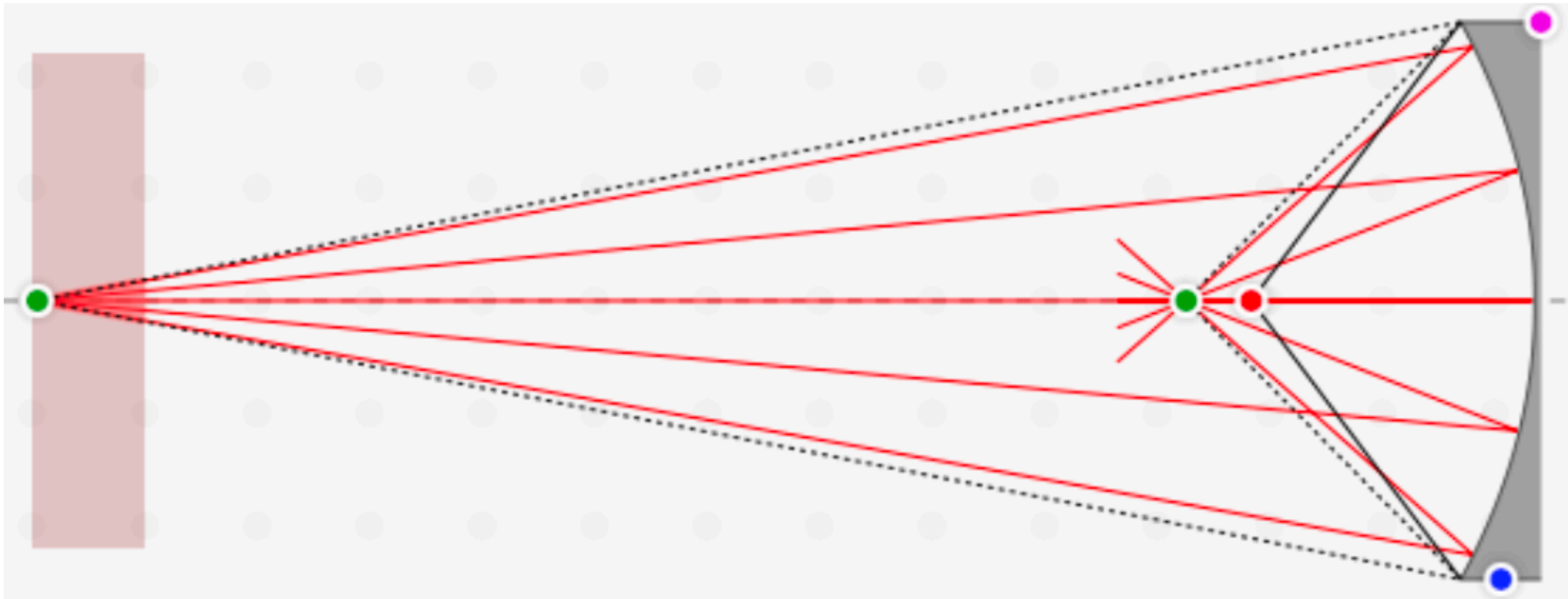
- $r^2 - 2Rz + (1+K)z^2 = 0$ for $z(r=0) = 0$
- $z = \frac{r^2}{R} \frac{1}{1+\sqrt{1-(1+K)\frac{r^2}{R^2}}}$
- R radius of curvature
- $K = -e^2$, e eccentricity
- prolate ellipsoid ($K > 0$)
- sphere ($K = 0$)
- oblate ellipsoid ($0 > K > -1$)
- parabola ($K = -1$)
- hyperbola ($K < -1$)
- all conics are almost spherical close to origin
- analytical ray intersections

Foci of Conic Sections



- sphere has single focus
- ellipse has two foci
- parabola (ellipse with $e = 1$) has one focus (and another one at infinity)
- hyperbola ($e > 1$) has two focal points

Elliptical Mirrors



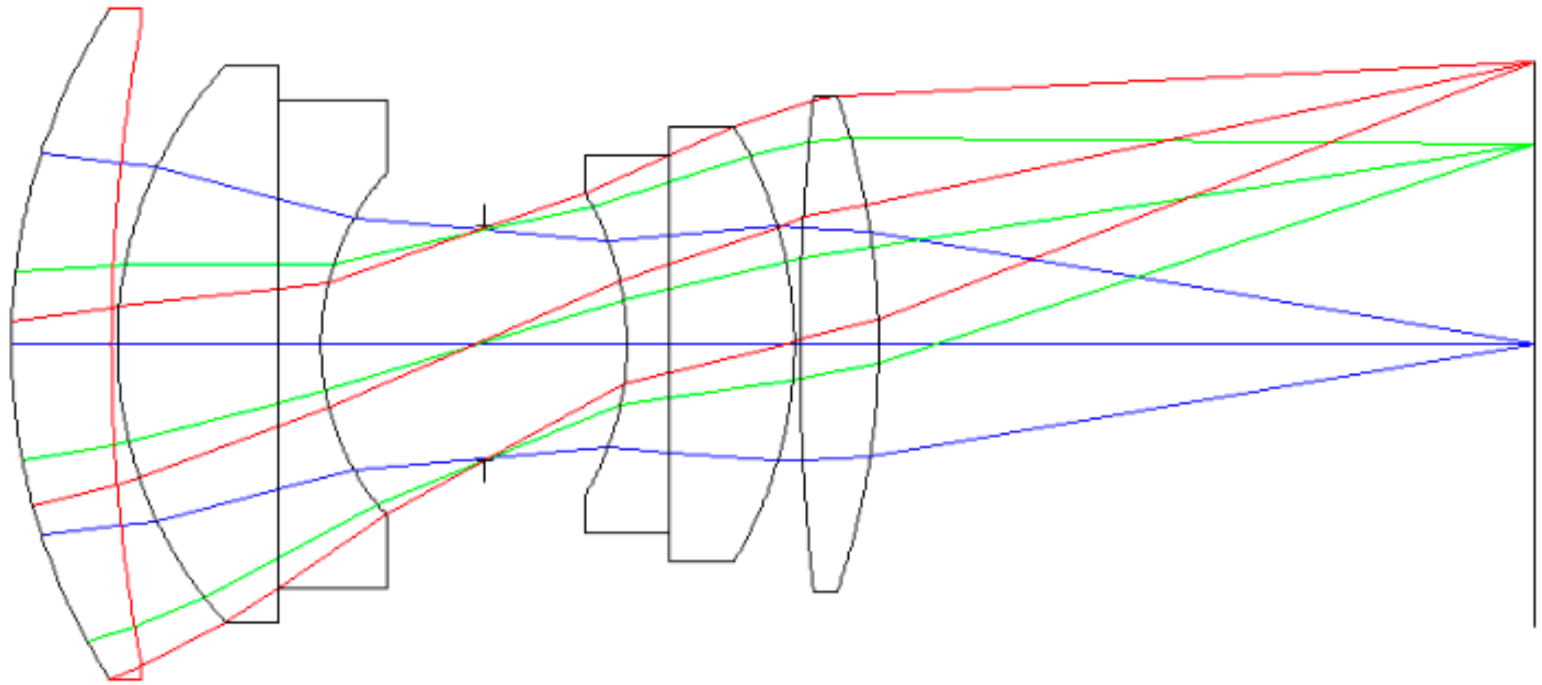
- have two foci at finite distances
- perfectly reimage one focal point into another

Hyperbolic Mirrors

- have a real focus and a virtual focus (behind mirror)
- perfectly reimage one focal point into another

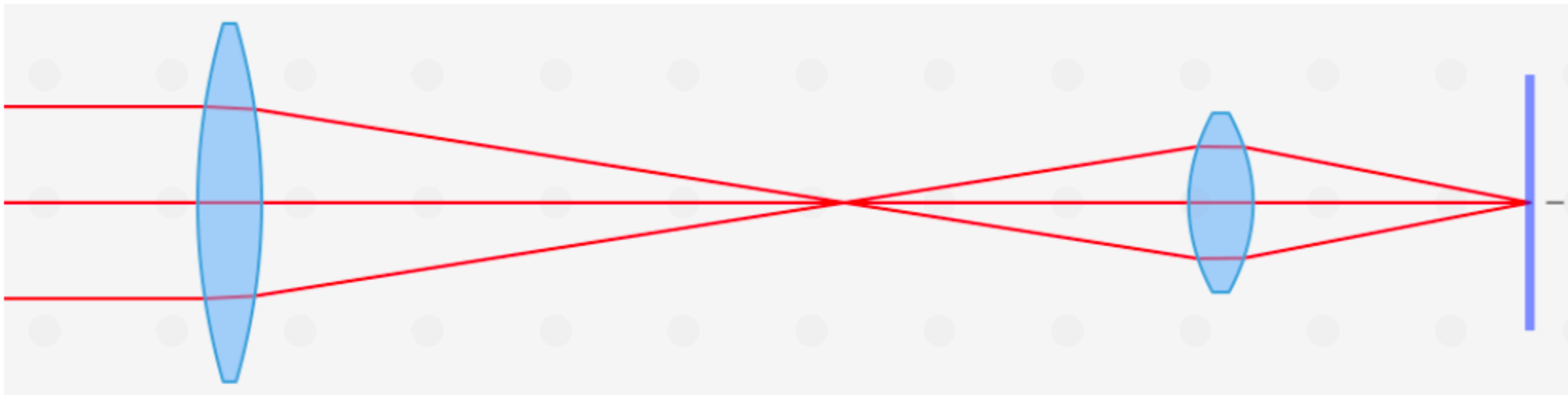
Optical Systems

Overview



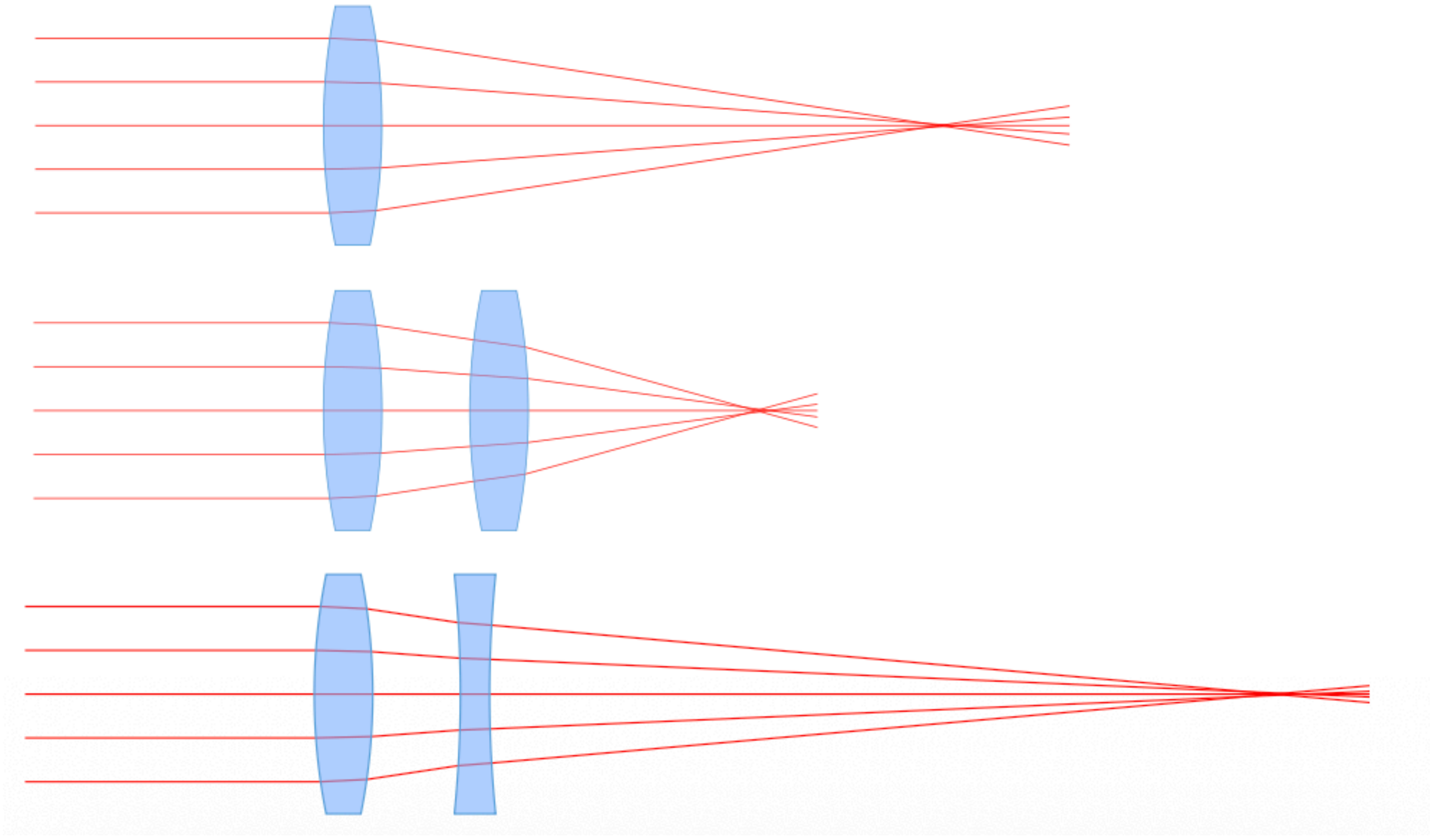
- combinations of several optical elements (lenses, mirrors, stops)
- examples: camera “lens”, microscope, telescopes, instruments
- thin-lens combinations can be treated analytically
- effective focal length: $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$

Simple Thin-Lens Combinations



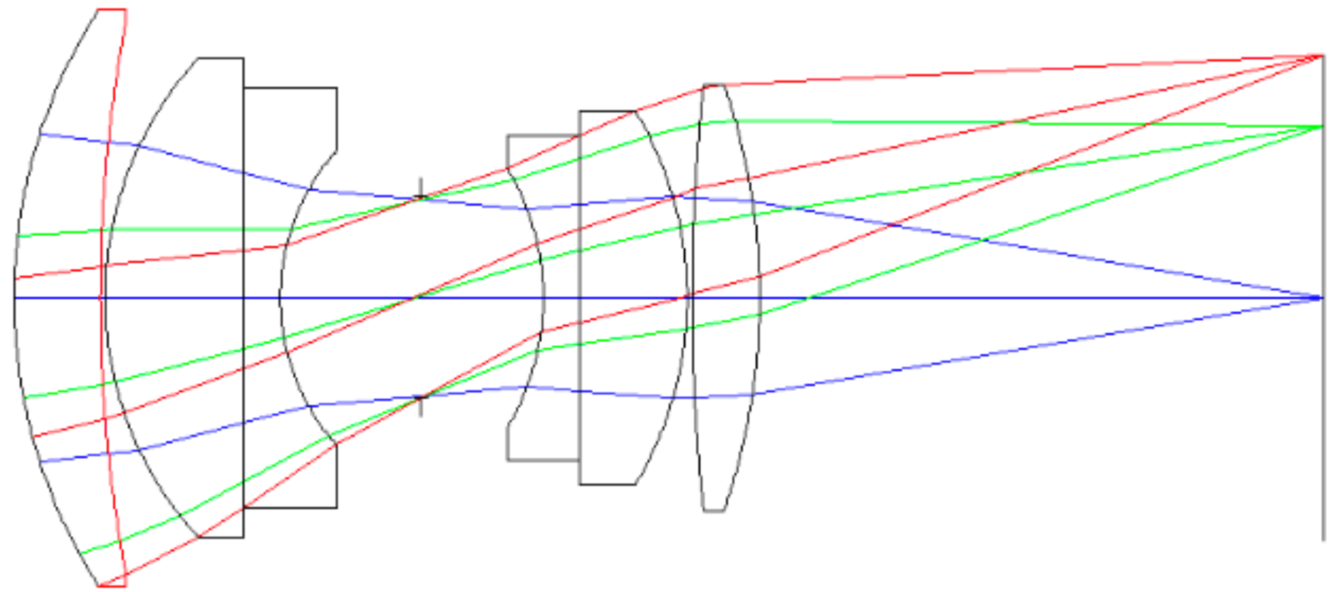
- distance > sum of focal lengths ⇒ real image between lenses
- apply single-lens equation successively

Second Lens Adds Convergence or Divergence



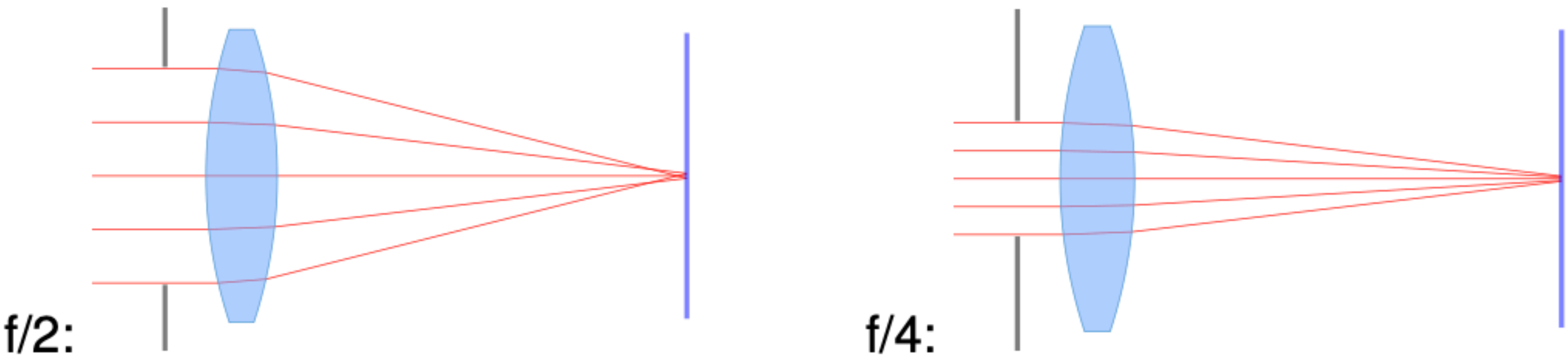
F-number and Numerical Aperture

Aperture



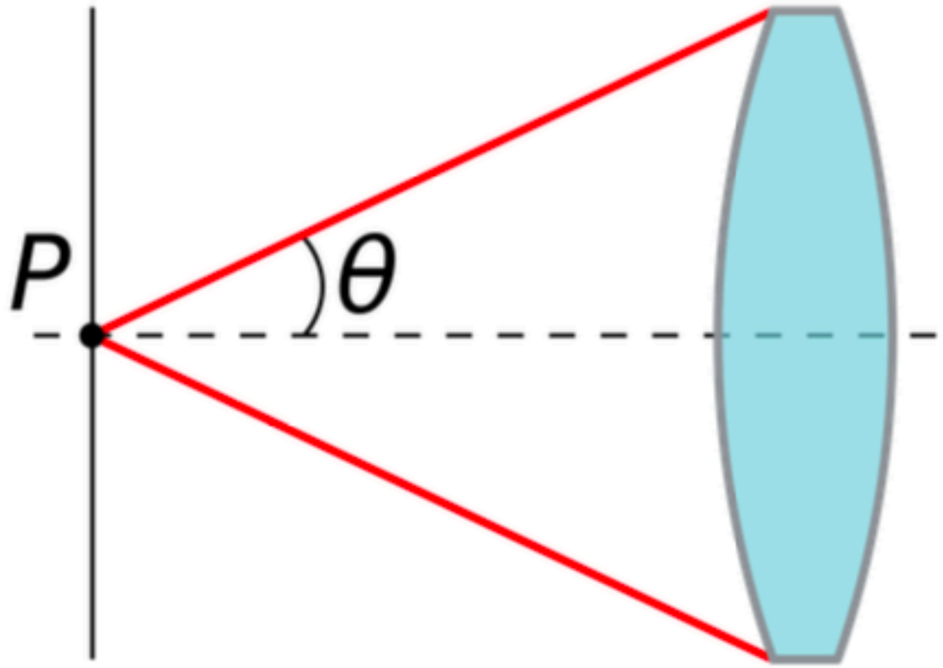
- all optical systems have a place where 'aperture' is limited
- main mirror of telescopes
- aperture stop in photographic lenses
- aperture typically has a maximum diameter
- aperture size is important for diffraction effects

F-number



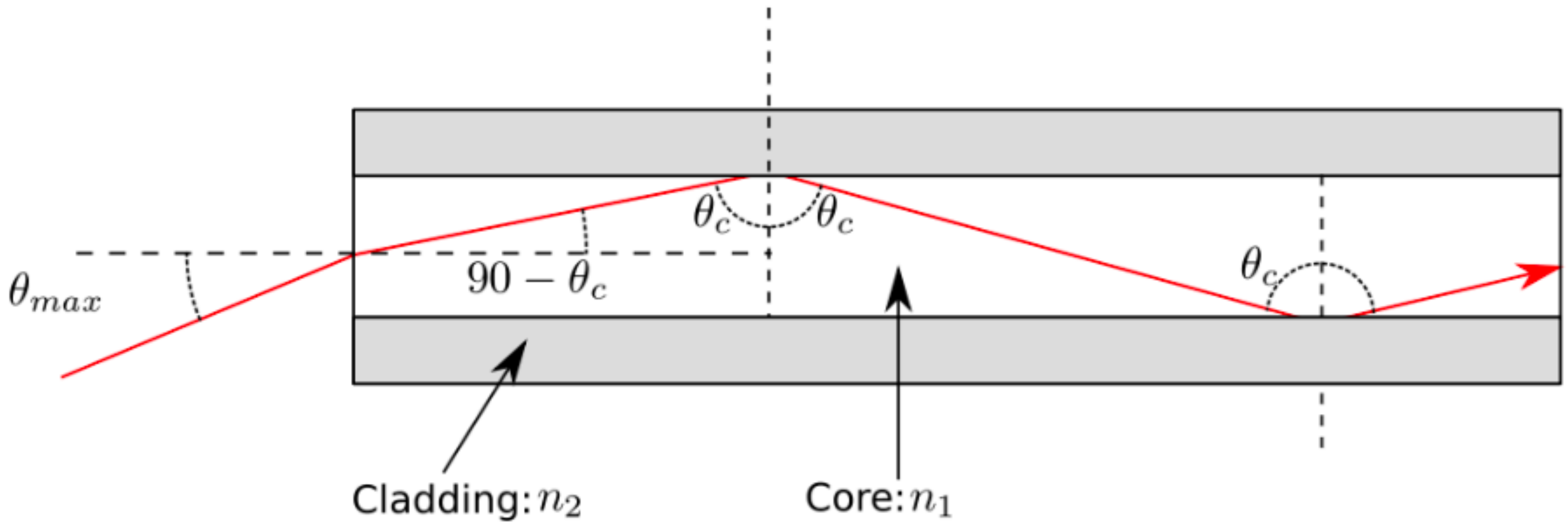
- describes the light-gathering ability of the lens
- f-number given by $F = f/D$
- also called focal ratio or f-ratio, written as: f/F
- the bigger F, the better the paraxial approximation works
- fast system for $F < 2$, slow system for $F > 2$

Numerical Aperture



- numerical aperture (NA): $n \sin \theta$
- n index of refraction of working medium
- θ half-angle of maximum cone of light that can enter or exit lens
- important for microscope objectives (n often not 1)

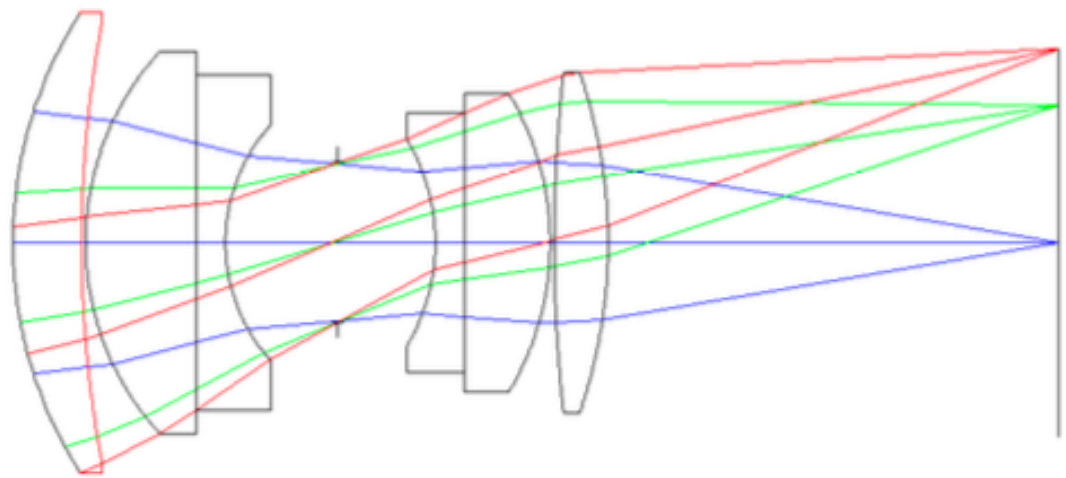
Numerical Aperture in Fibers



en.wikipedia.org/wiki/File:OF-na.svg

- acceptance cone of the fiber determined by materials
- $NA = n\sin\theta = \sqrt{n_1^2 - n_2^2}$
- n index of refraction of working medium

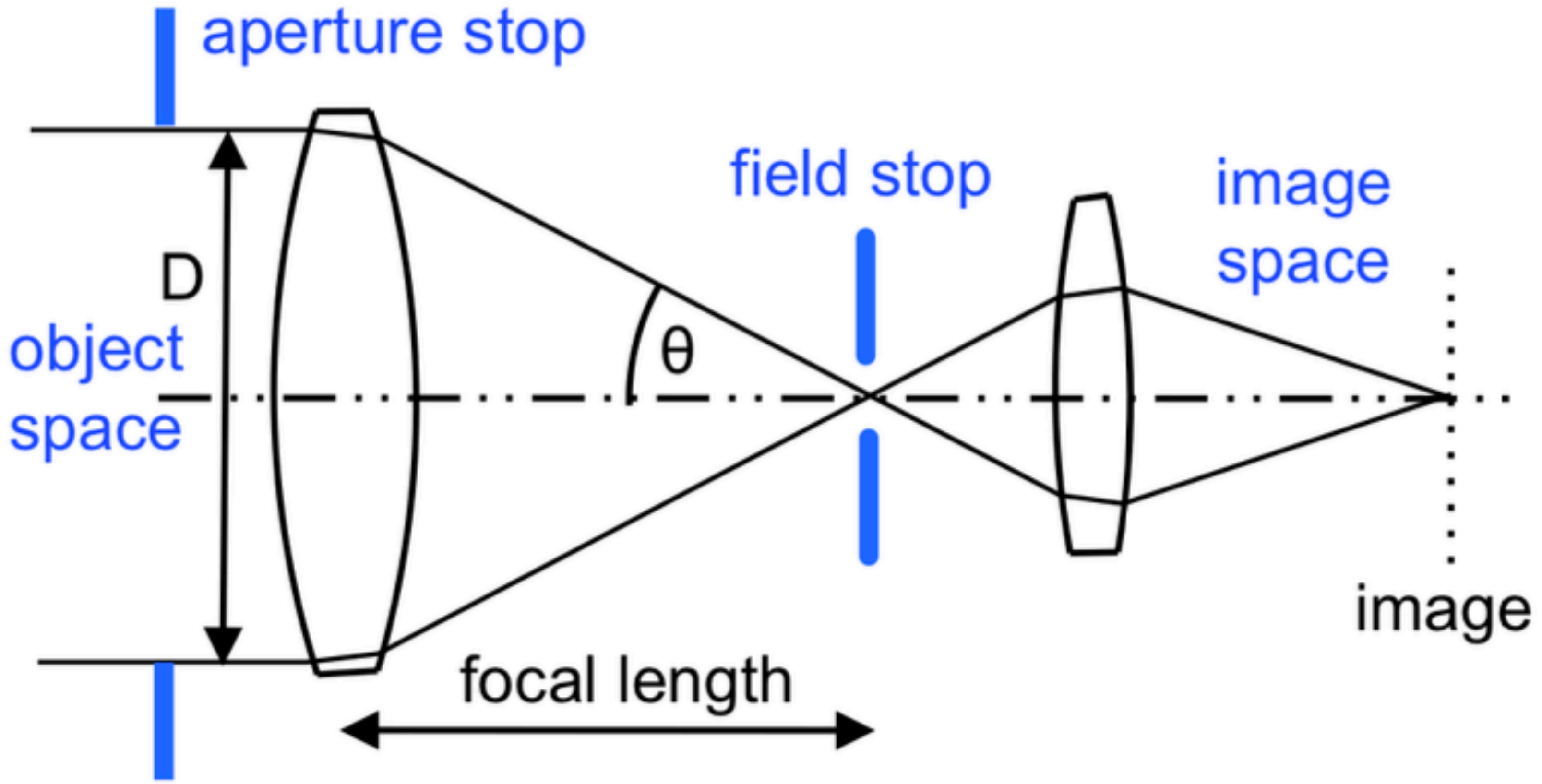
Images and Pupils



Images and Pupils

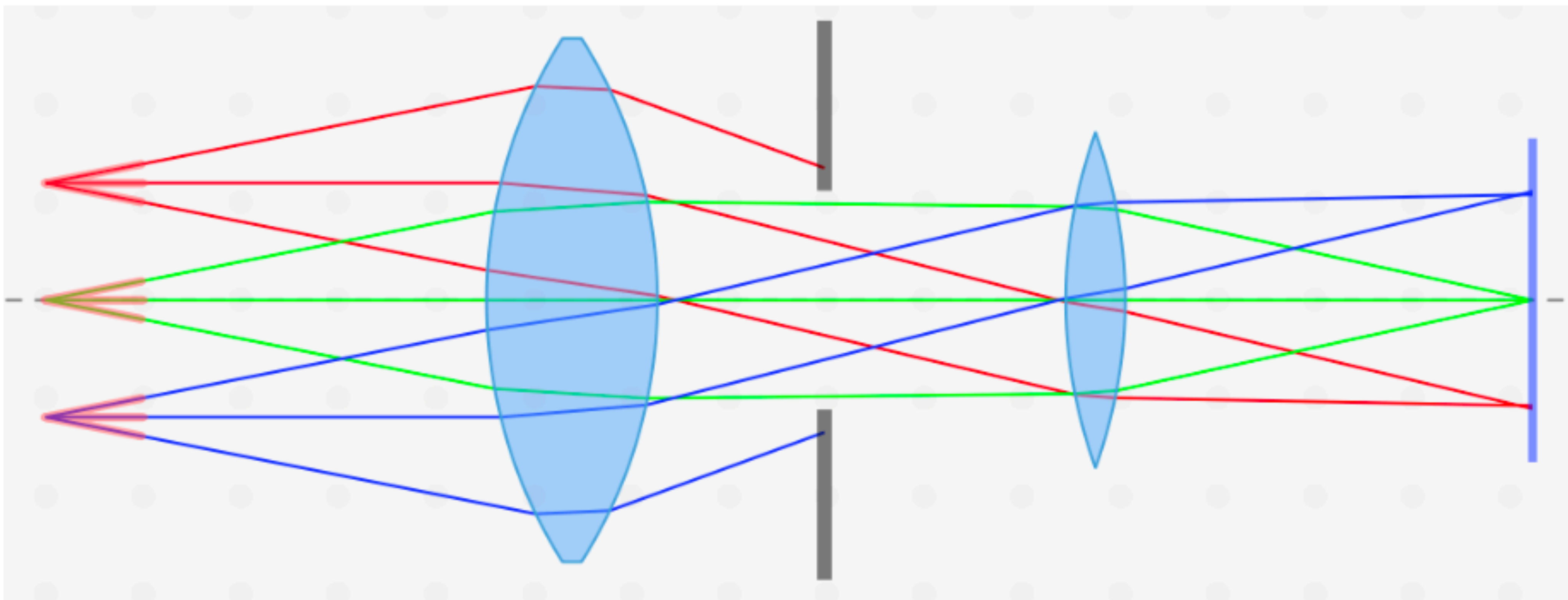
- image
  - every object point comes to a focus in an image plane
  - light in one image point comes from pupil positions
  - object information is encoded in position, not in angle
- pupil
  - all object rays are smeared out over complete aperture
  - light in one pupil point comes from different object positions
  - object information is encoded in angle, not in position

Aperture and Field Stops



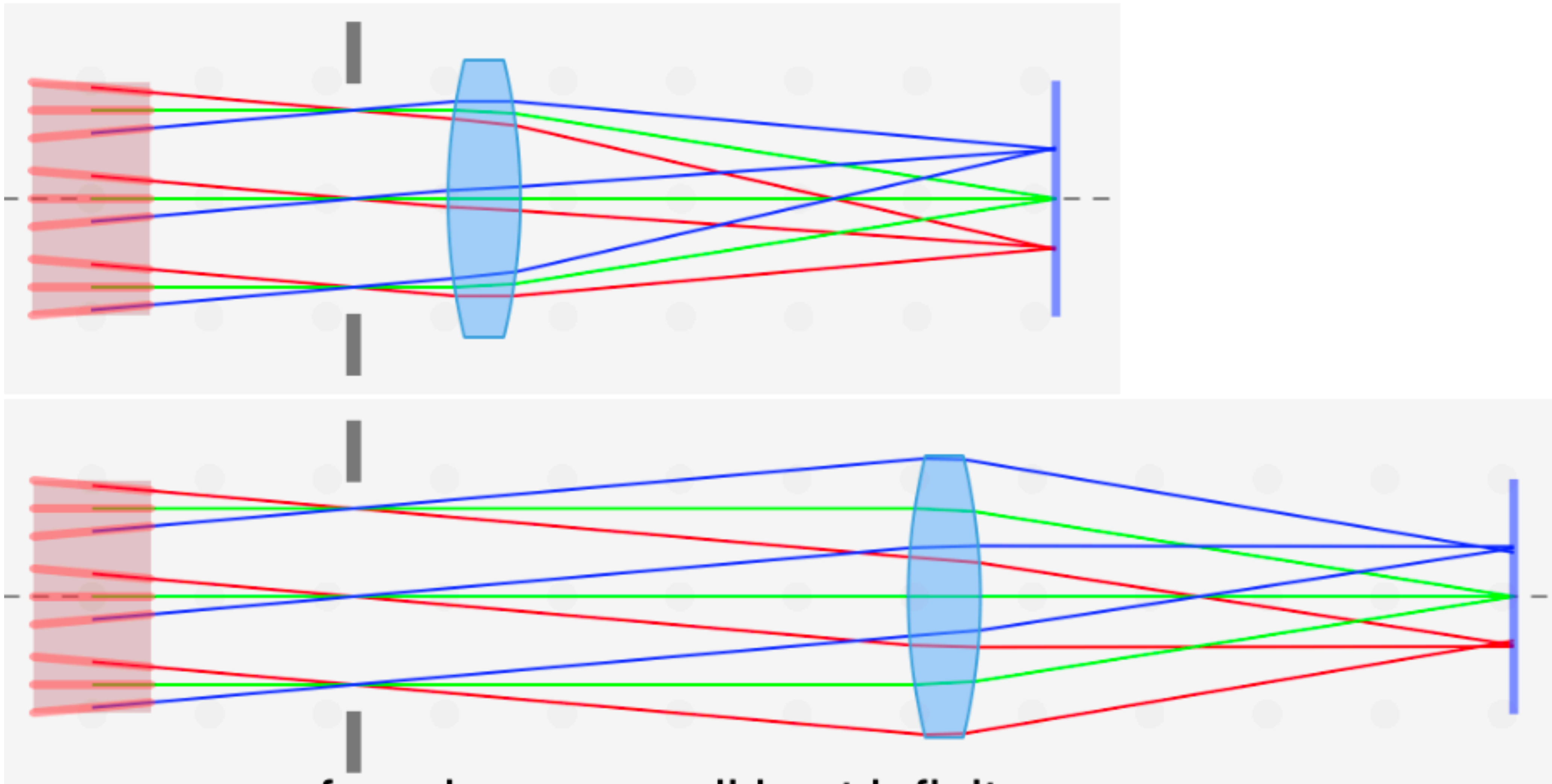
- aperture stop limits the amount of light reaching the image
- aperture stop determines light-gathering ability of optical system
- field stop limits the image size or angle

Vignetting



- effective aperture stop depends on position in object
- image fades toward its edges

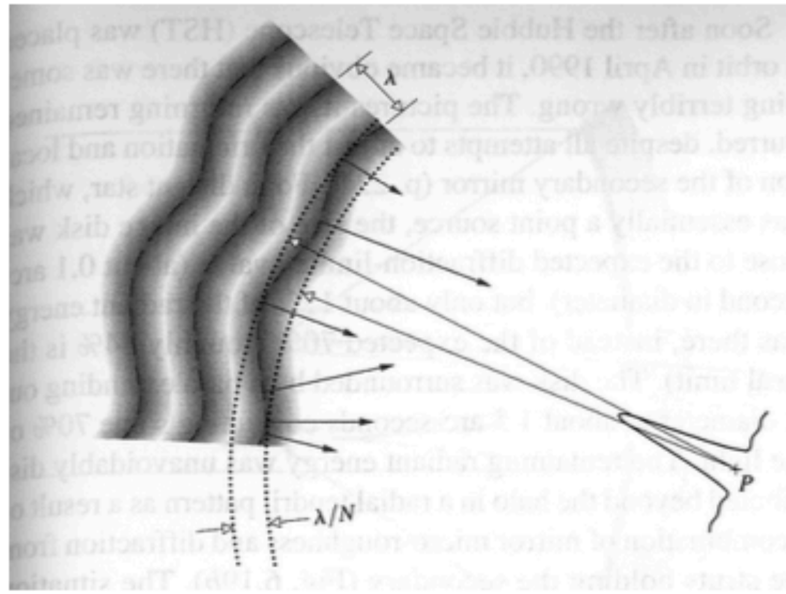
Telecentric Arrangement



- as seen from image, pupil is at infinity
- easy: lens is its focal length away from pupil (image)
- magnification does not change with focus positions
- ray cones for all image points have the same orientation

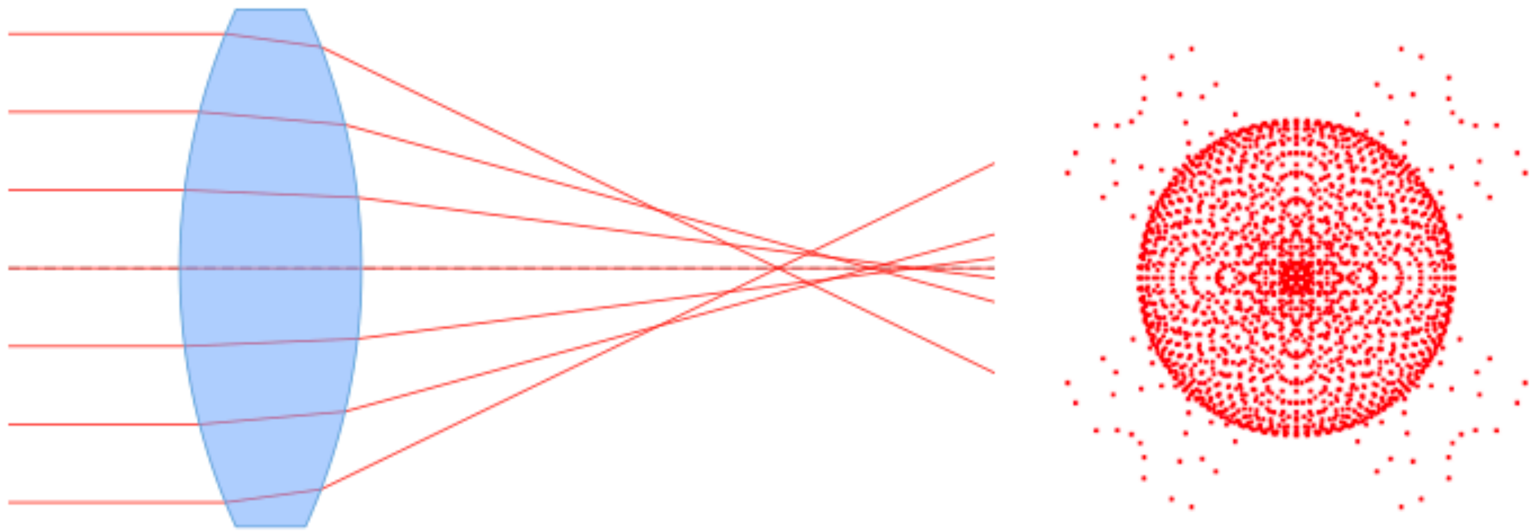
Aberrations

Spot Diagrams and Wavefronts



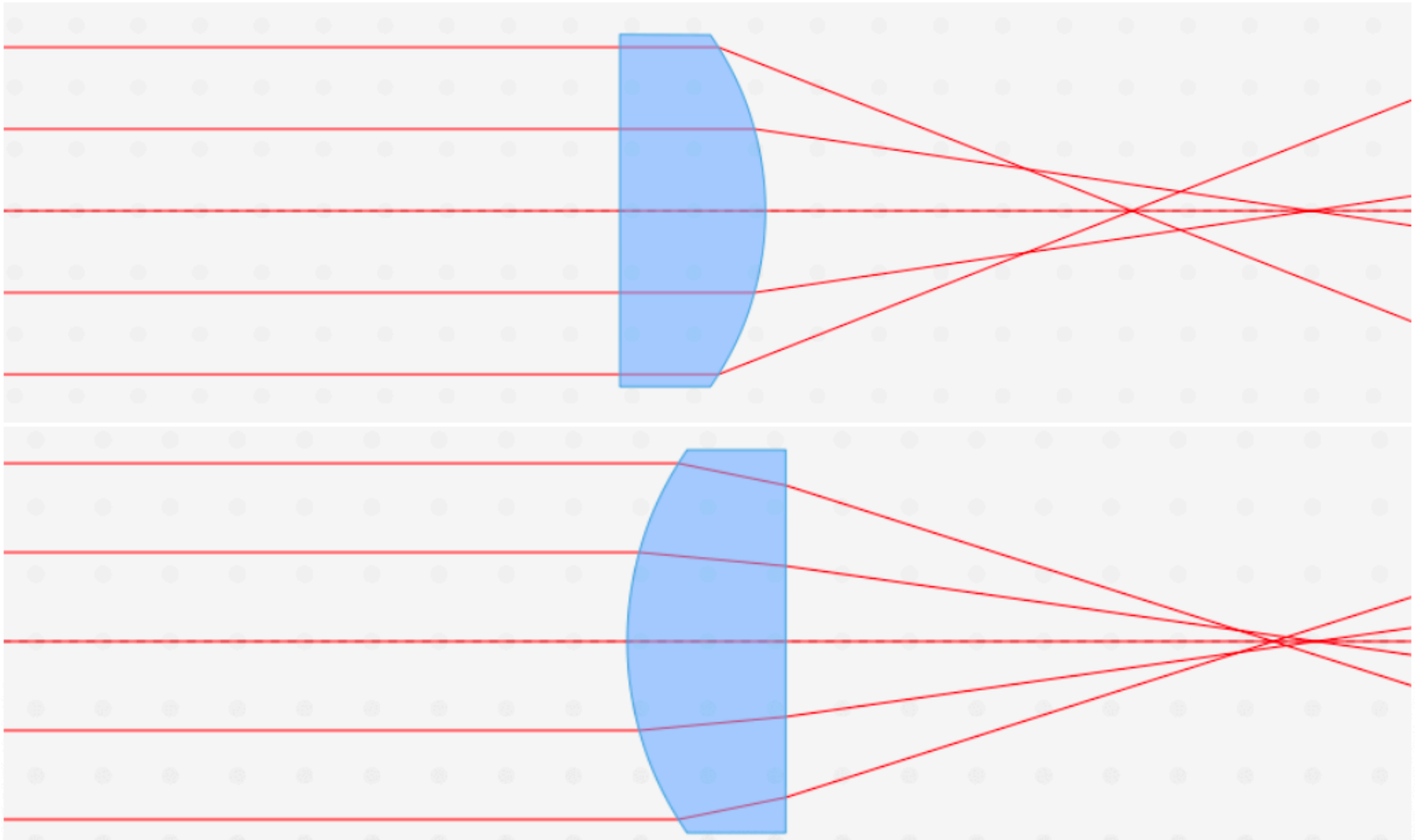
- plane of least confusion is location where image of point source has smallest diameter
- spot diagram: shows ray locations in plane of least confusion
- spot diagrams are closely connected with wavefronts
- aberrations are deviations from spherical wavefront

Spherical Aberration of Spherical Lens

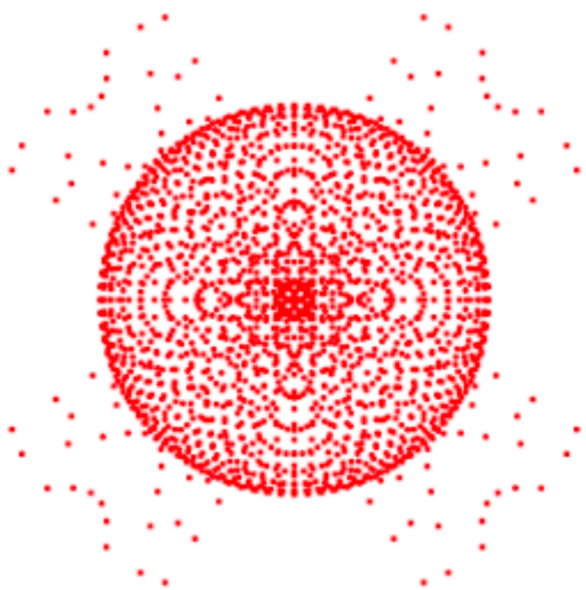


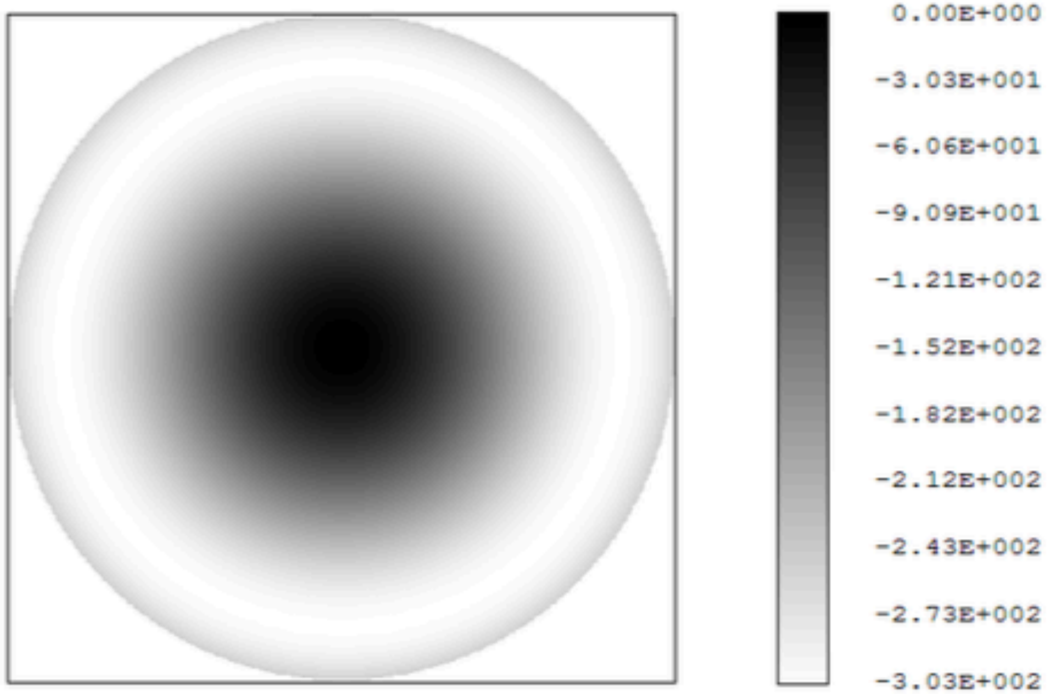
- different focal lengths of paraxial and marginal rays
- longitudinal spherical aberration along optical axis
- transverse (or lateral) spherical aberration in image plane
- much more pronounced for short focal ratios
- foci from paraxial beams are further away than marginal rays
- spot diagram shows central area with fainter disk around it

Minimizing Spherical Aberrations



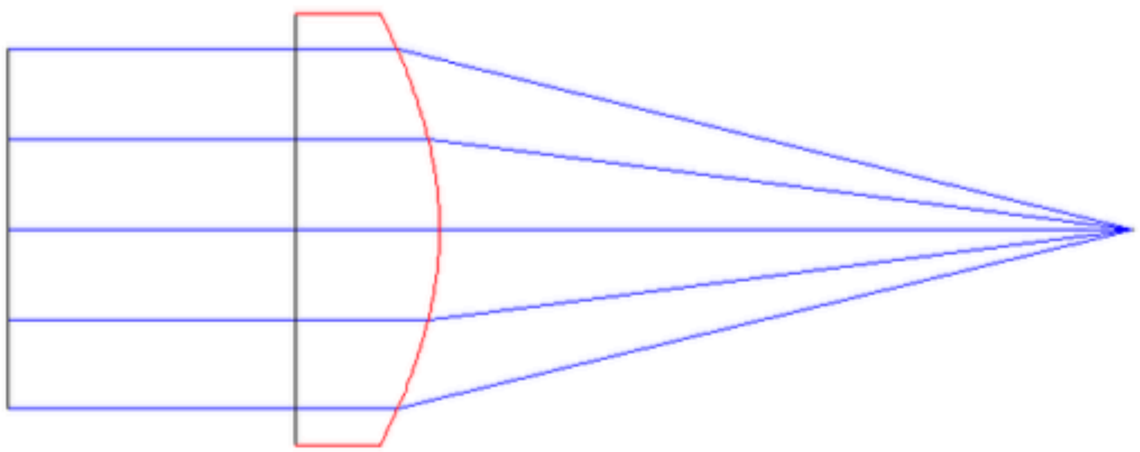
Spherical Aberration Spots and Waves

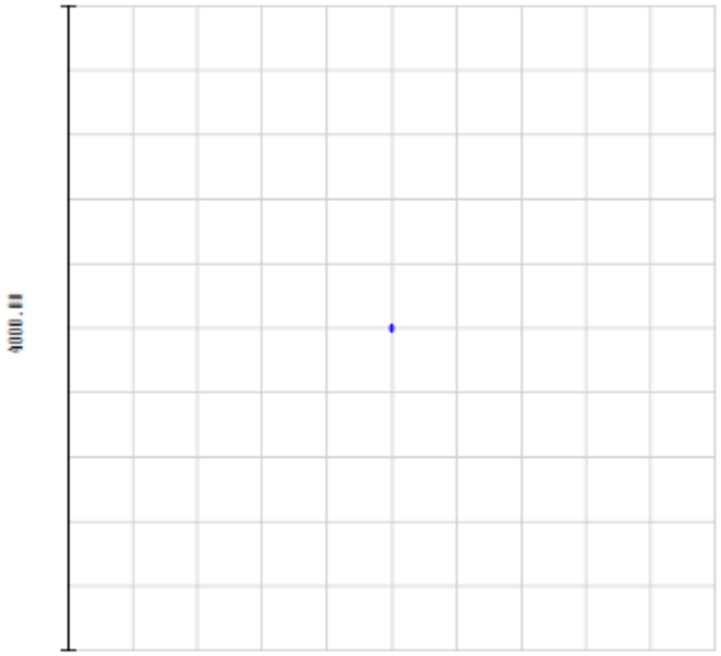




- spot diagram shows central area with fainter disk around it
- wavefront has peak and turned-up edges

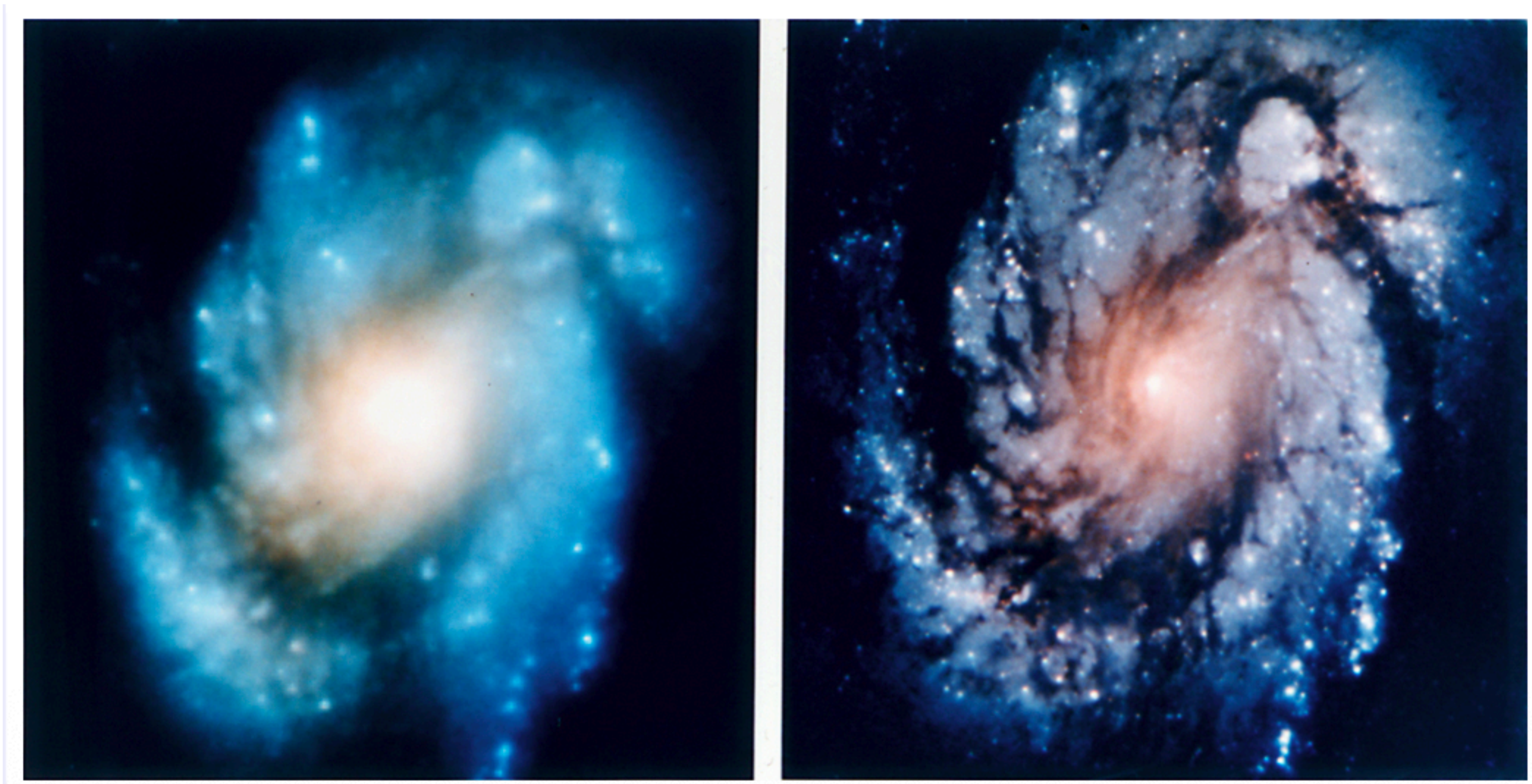
Aspheric Lens



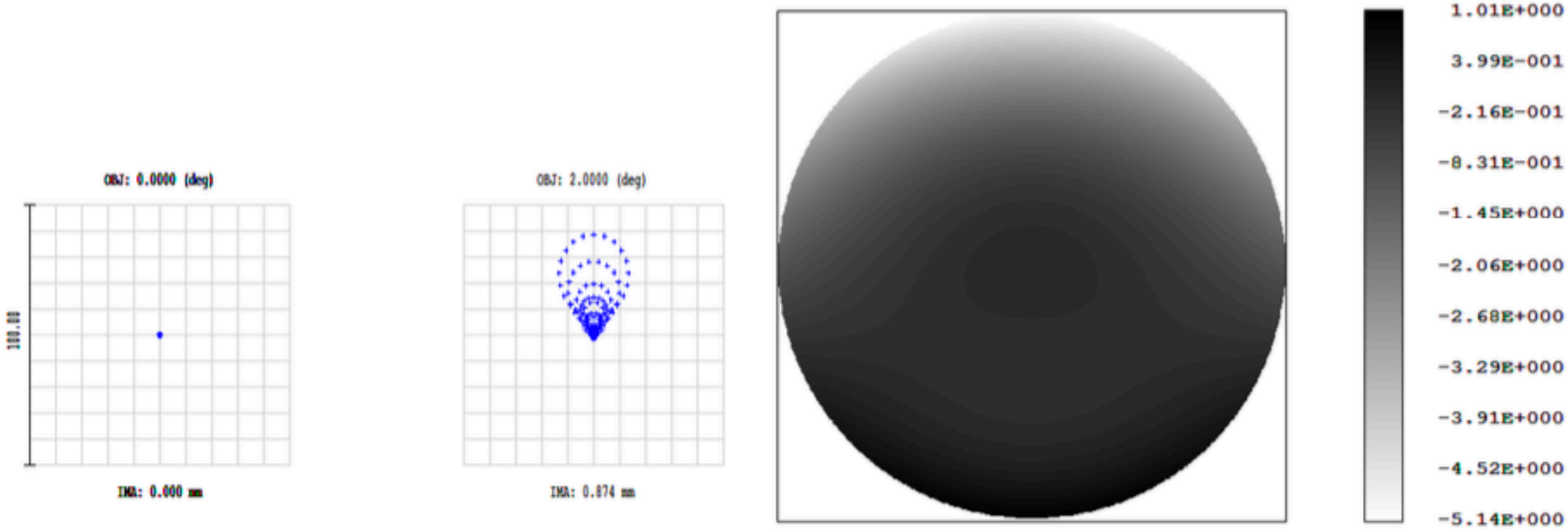


- conic constant $K = -1 - \sqrt{n}$ makes perfect lens
- difficult to manufacture
- but possible these days

HST

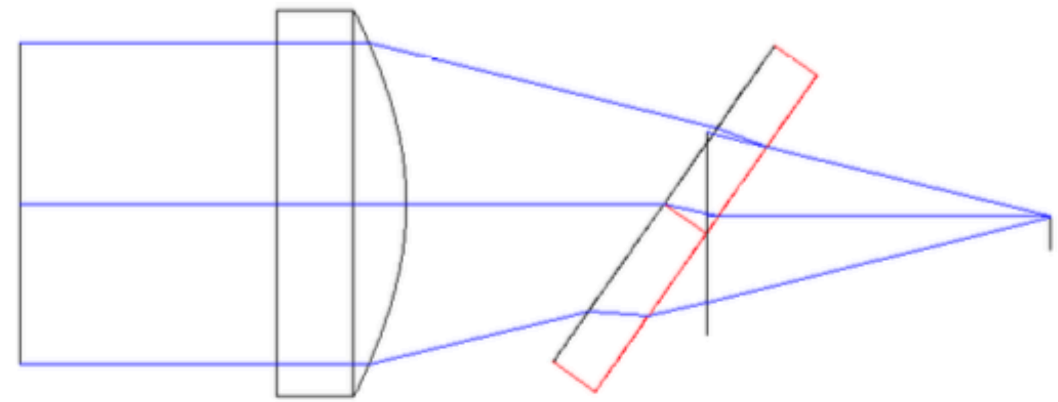


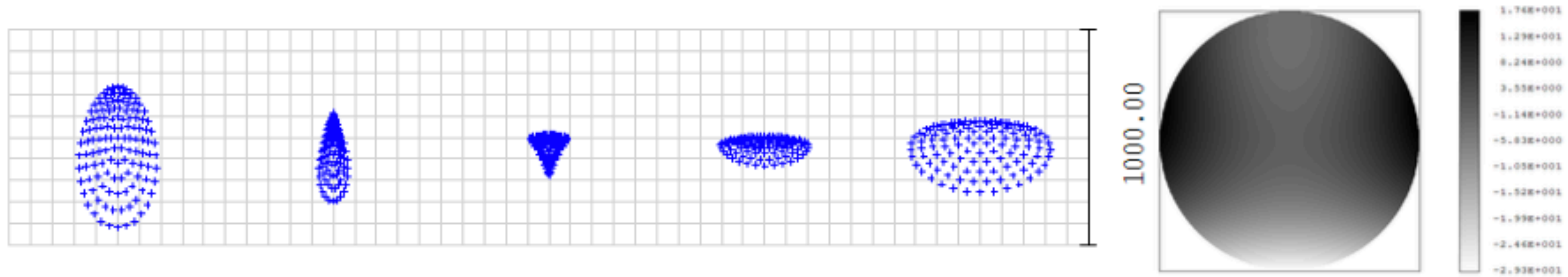
Coma Spots and Waves



- parabolic mirror with perfect on-axis performance
- spots and wavefront for off-axis image points
- wavefront is tilted in inner part

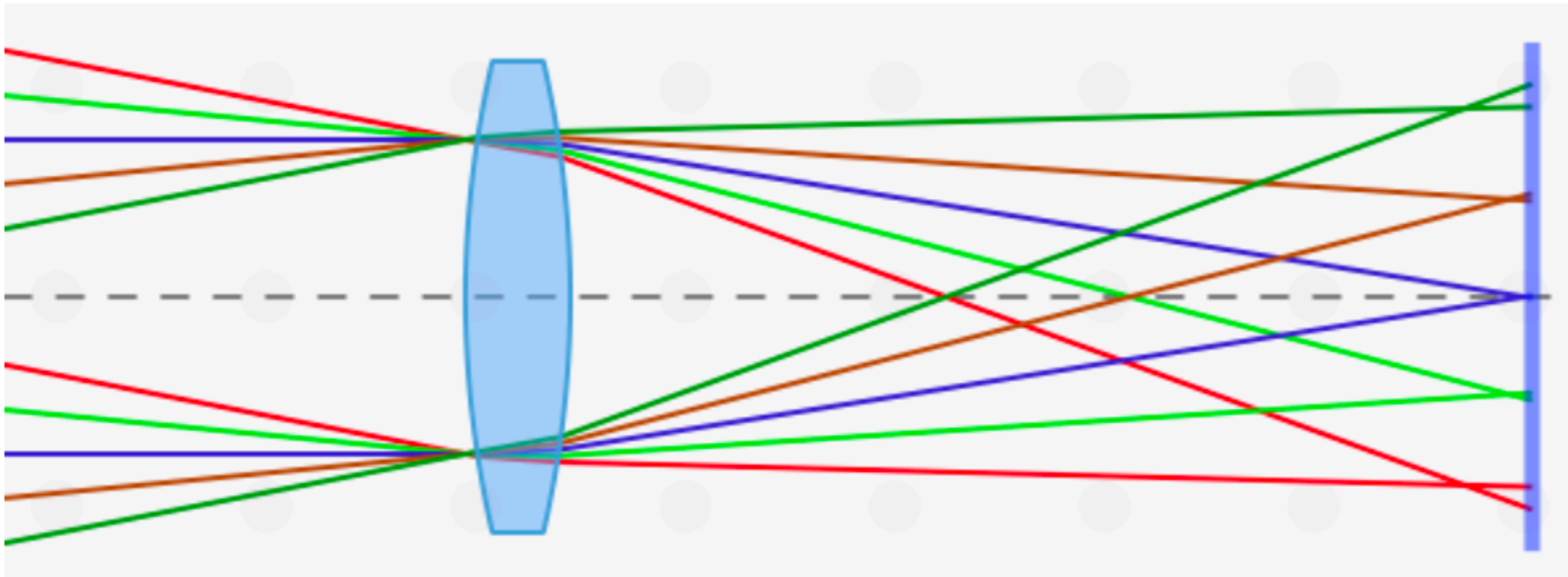
Astigmatism due to Tilted Glass Plate in Converging Beam





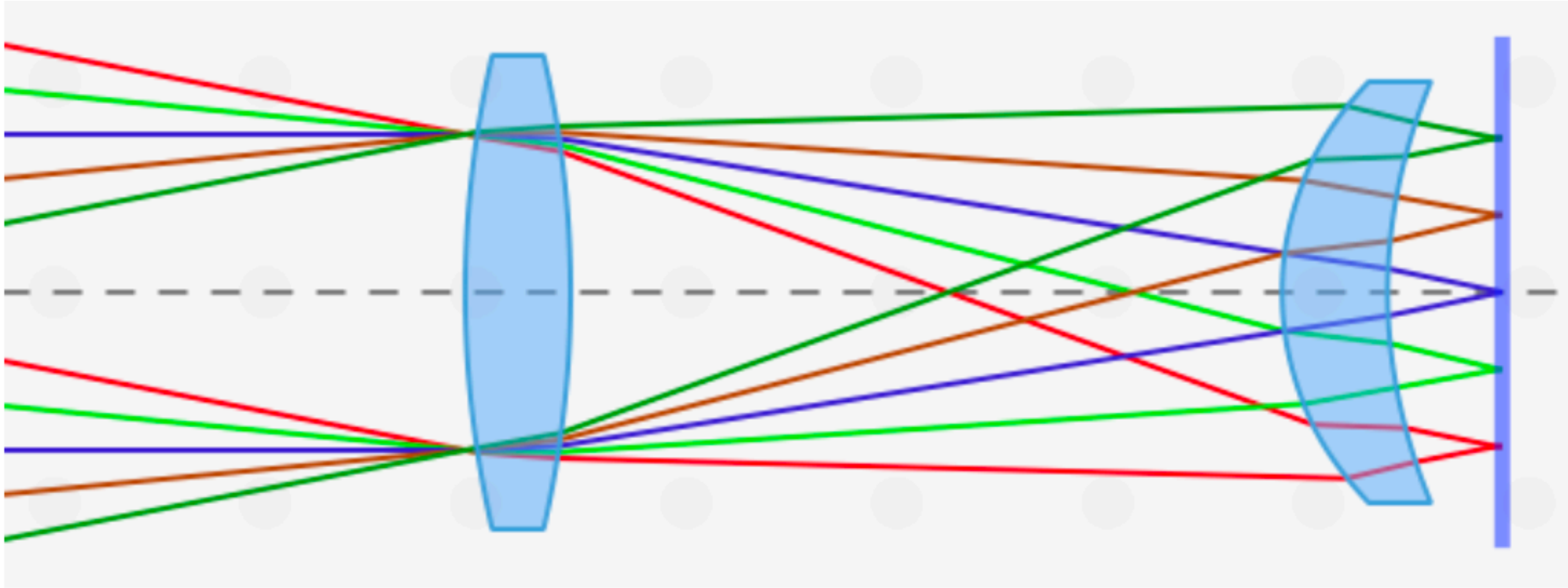
- astigmatism: focus in two orthogonal directions, but not in both at the same time
- tilted glass-plate: astigmatism, spherical aberration, beam shift
- tilted plates: beam shifters, filters, beamsplitters
- difference of two parabolae with different curvatures

Field Curvature



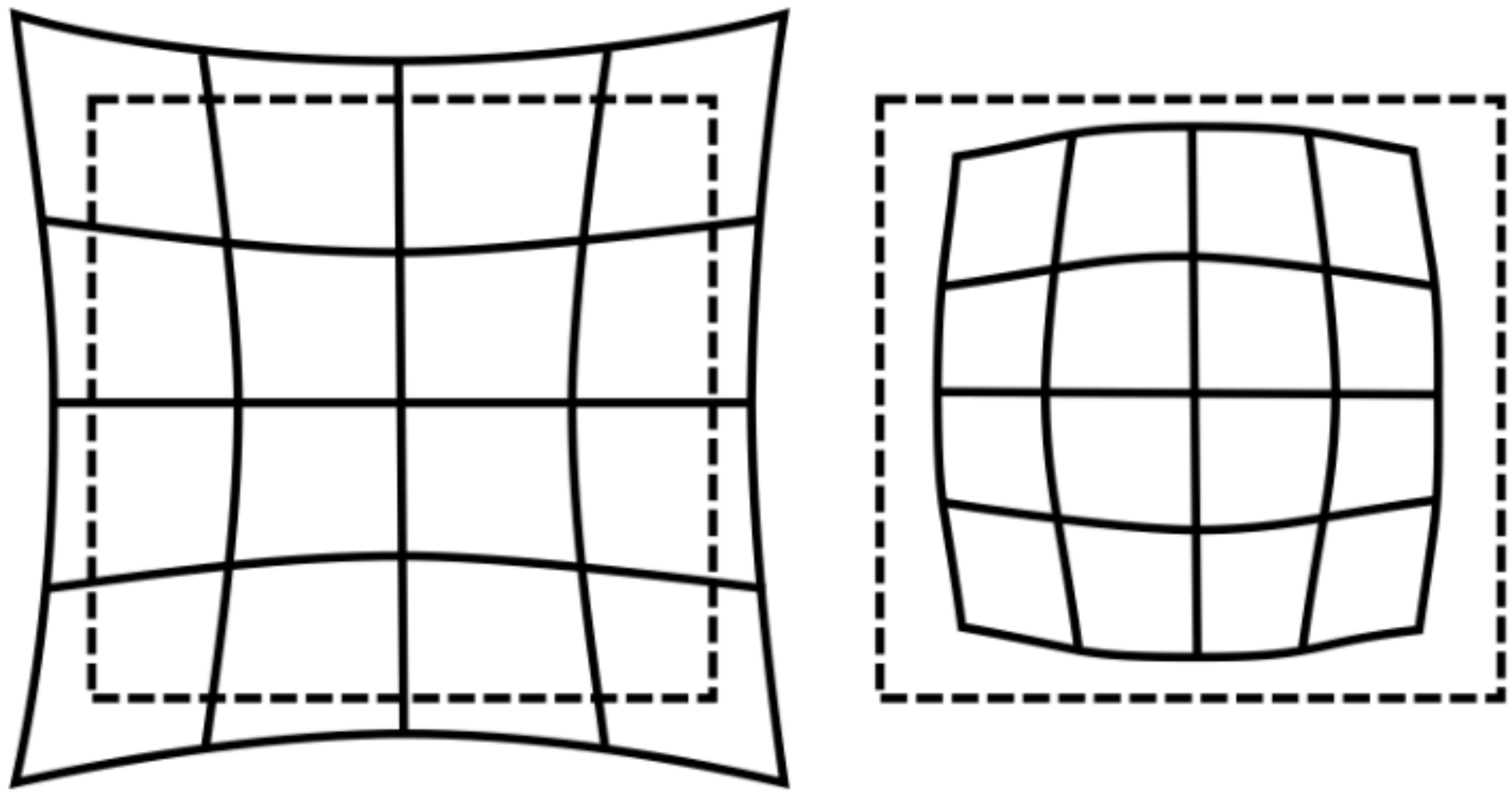
- field (Petzval) curvature: image lies on curved surface
- curvature comes from lens thickness variation across aperture
- problems with flat detectors (e.g. CCDs)
- potential solution: field flattening lens close to focus

Petzval Field Flattening



- Petzval curvature only depends on index of refraction and focal length of lenses
- Petzval curvature is independent of lens position!
- field flattener also makes image much more telecentric

Distortion



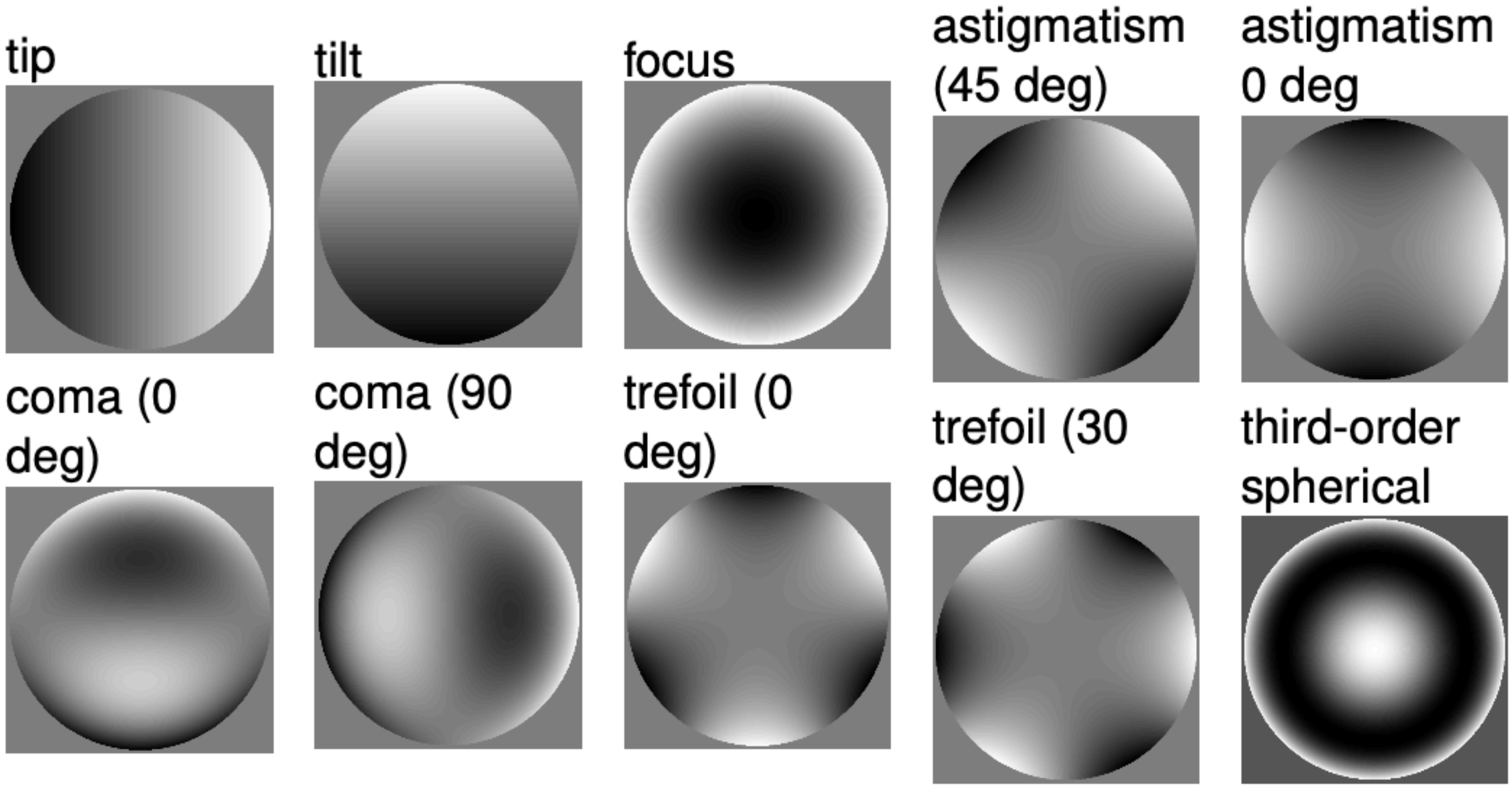
- image is sharp but geometrically distorted
- (a) object
- (b) positive (or pincushion) distortion
- (c) negative (or barrel) distortion

Aberration Descriptions

Seidel Aberrations

- Ludwig von Seidel (1857)
- Taylor expansion of $\sin\phi$
- $\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots$
- paraxial: first-order optics
- Seidel optics: third-order optics
- Seidel aberrations: spherical, astigmatism, coma, field curvature, distortion

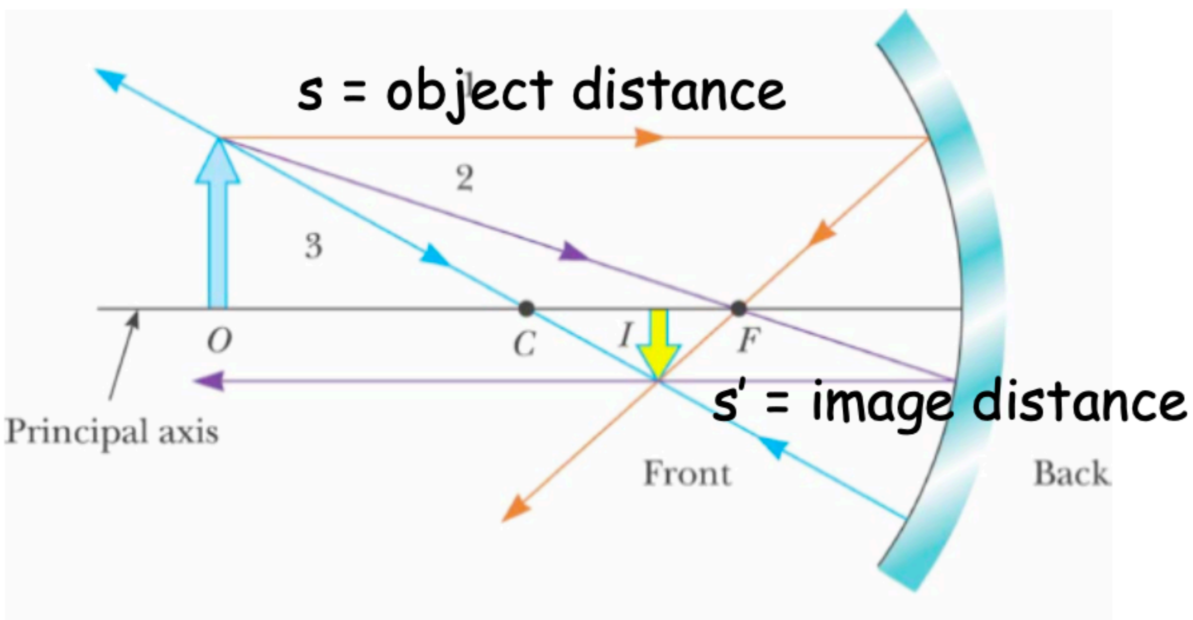
Zernike Polynomials

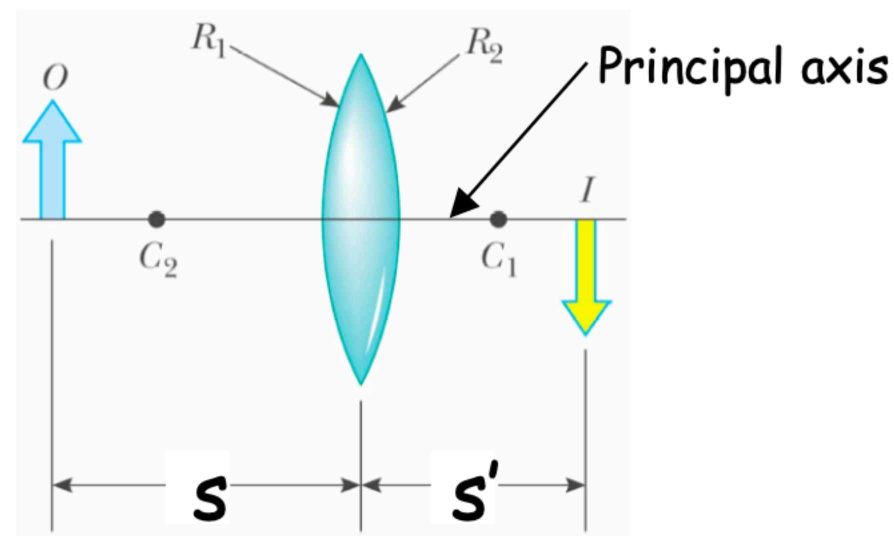


- low orders equal Seidel aberrations
- form orthonormal basis on unit circle

Preview

- We will obtain the equations that describe the images formed by spherical mirrors and thin lenses.





- This is done under the assumption that all rays make a small angle with respect to the principal axis of the mirror/lens:

θ < ~ 0.1 radian => $\sin\theta \sim \tan\theta \sim \theta$ (paraxial rays)

Preview and notation

- The location and character of images can then be calculated form the laws of reflection and refraction.
- We will use the following notation:

  s = object distance $\quad$ s' = image distance

  y = the object height $\quad$ y' = the image height

  f = the focal length of the mirror or lens

  R = the radius of curvature of the spherical surface
- Then for a mirror or a thin lens, for paraxial rays:

  $1/s + 1/s' = 1/f$ and M = the magnification = $y'/y = -s'/s$

The sign convention is important as we will discuss.

Preview and terminology

- The image formed by an object has two important characteristics.

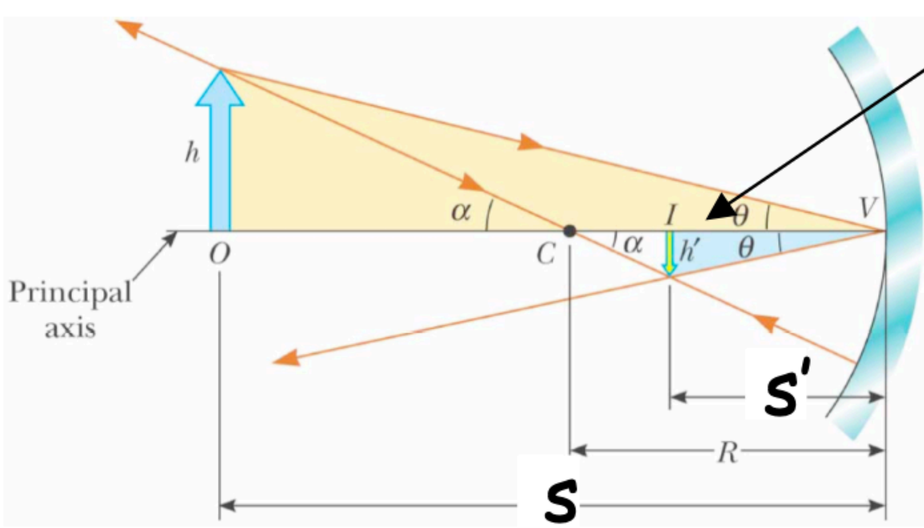
  1. The transverse size of the image relative to the object

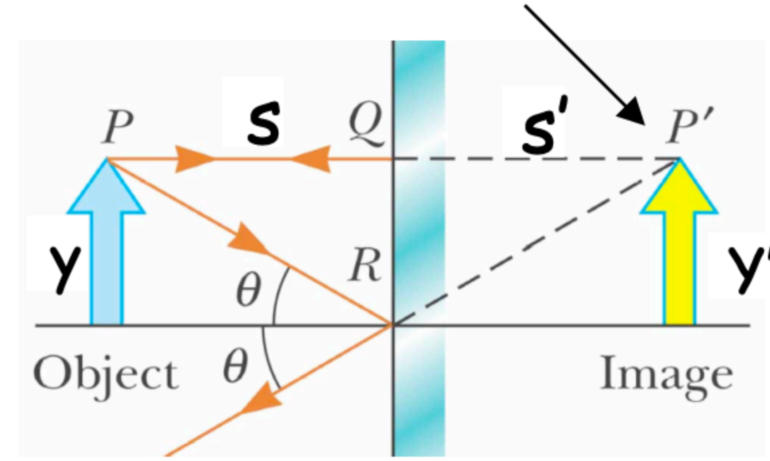
     M = the magnification = y′/y (upright or inverted)

  2. The image formed can be real or virtual

     light rays pass through a real image

     extrapolation of light rays pass through a virtual image

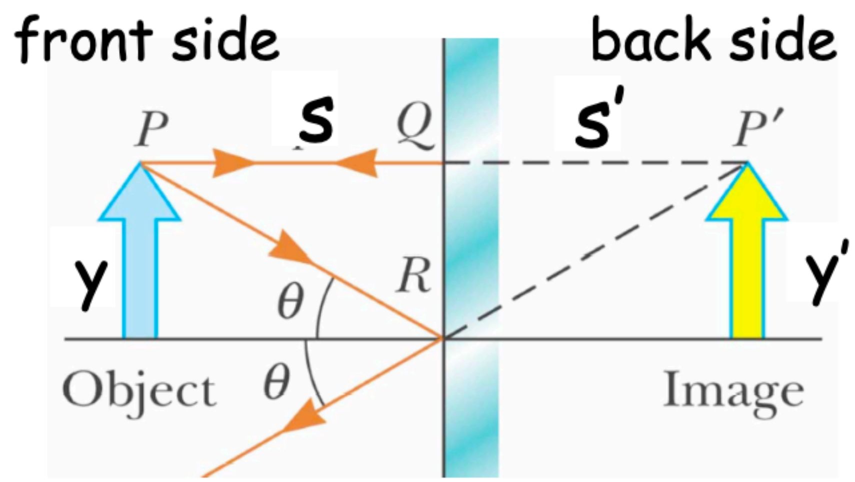




A plane mirror

- A plane (flat) mirror forms an image on the back side of the mirror.

light direction



Sign convention:

s and s' are

+ on the front side

- on the back side

- For an object on the front side the image is always virtual and upright:

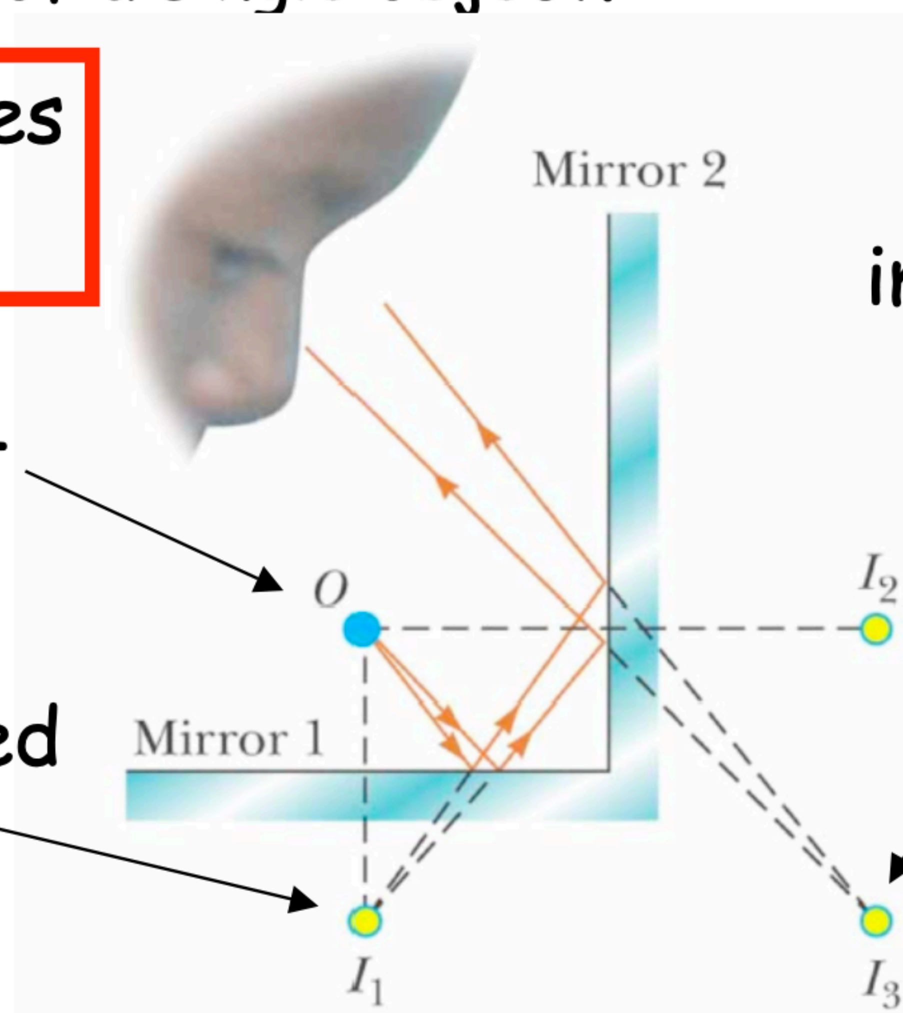
$$s' = -s$$

$$M = y'/y = +1$$

Plane mirrors

- Multiple images of an object are formed by two plane mirrors.
- Consider two at right angles as shown. There will be three virtual images of a single object.

observer sees 3 images



object

image I_1 formed in mirror 1

image I_2 formed in mirror 2

image I_3 of I_2 in mirror 1 and I_1 in mirror 2

Spherical mirrors: the basics

- This analysis will use only " $\theta_1 = \theta_1'$ " and will be valid for paraxial rays forming the image.
- The mirror is described by a single parameter:

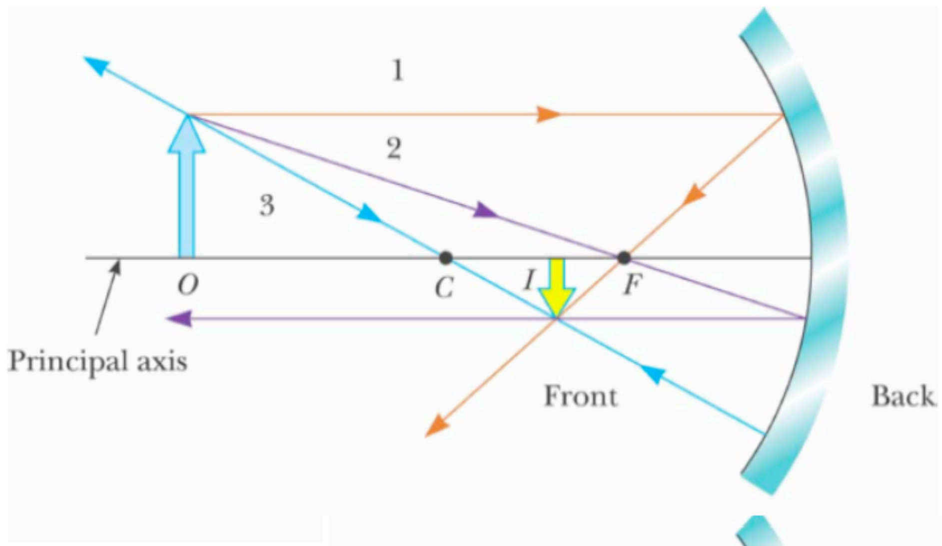
  R = the radius of curvature of the mirror surface
- The sign convention is the following:

  s, s' and R are + (-) on the front (back) side of the mirror

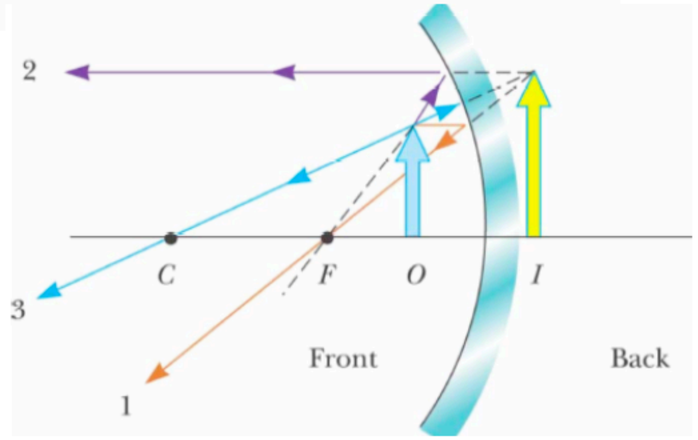
  y and y' are + (-) if upright (inverted)
- The problem is the following:

  Given R (the mirror) and s and y (the object description) what are s' and y' (the image description)?
- Note that the image is real (virtual) for q positive (negative)

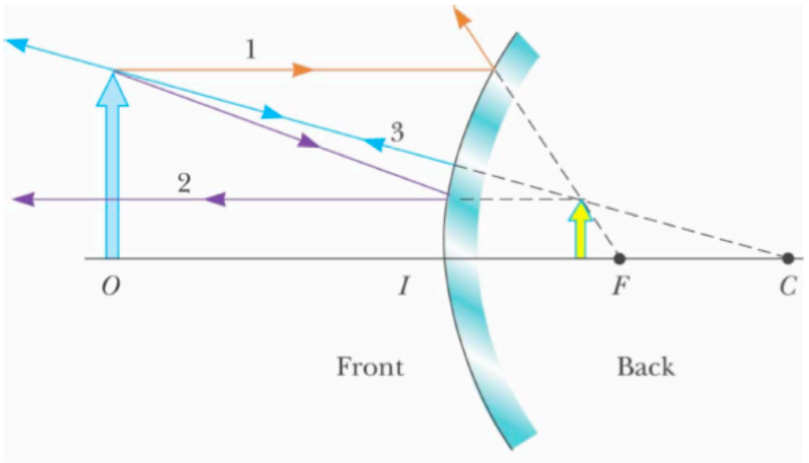
Spherical mirrors: examples



Concave mirror:
real, inverted image



Concave mirror:
virtual, upright image

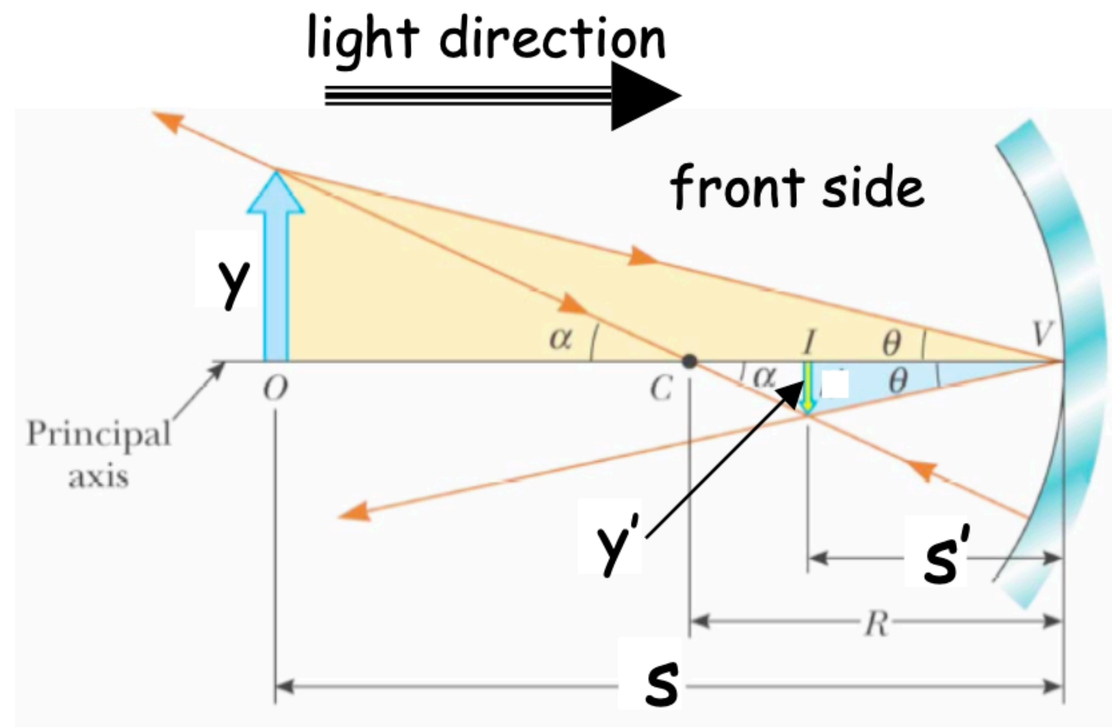


Convex mirror:
virtual, upright image

Spherical mirrors: the details

$\tan\theta = y/s = -y'/s'$

Magnification $M = y'/y = -s'/s$



$\tan\alpha = y/(s-R) = -y'/(R-s')$

$-y'/y = (R-s')/(s-R)$

Substitute $y'/y = -s'/s$

$\Rightarrow s'/s = (R-s)/(s-R)$

Solve to obtain:

$1/s + 1/s' = 2/R$

Valid for any paraxial ray

In this example:

s and s' are +

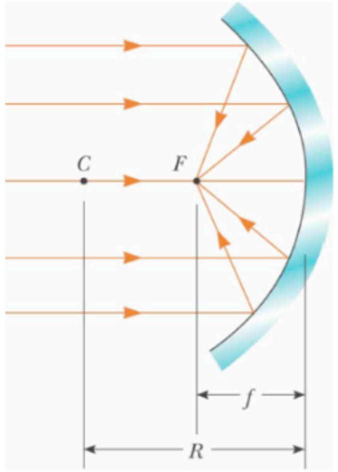
y is + and y' is -

Spherical mirrors: the details

- The derivation was done for the special case of a concave mirror with a real object and image.
- It is not difficult to show that the result is completely general:

  $M = y'/y = -s'/s$ and $1/s + 1/s' = 2/R$

  for any object and image combination for concave and convex mirrors (using the sign convention defined above).
- Note also that if the object is at infinity (parallel rays hitting the mirror) then the image is formed at R/2



Define $R/2 = f$ = focal length of the mirror

Then $1/s + 1/s' = 1/f$

Summary: the mirror equation for paraxial rays

- For spherical mirrors with radius R:

$$\boxed{\begin{array}{l} 1/s + 1/s' = 1/f \\ M = \text{transverse magnification} = y'/y = -s'/s \end{array}}$$

where s (s') = object (image) distance

y (y') = object (image) height

f = focal length = $R/2$

- The sign convention is:

  s, s', R are + (-) on the front (back) side of the mirror

  y and y' are + (-) if upright (inverted)

Ray diagrams for mirrors

- It is often useful to use some simple rays to geometrically reconstruct the location and character of an image.

  Ray 1: a ray entering parallel to the axis will be reflected to pass through the focal point of the mirror

  Ray 2: a ray passing through the focal point will be reflected parallel to the principal axis

  Ray 3: a ray passing through the center of curvature will be reflected back upon itself.

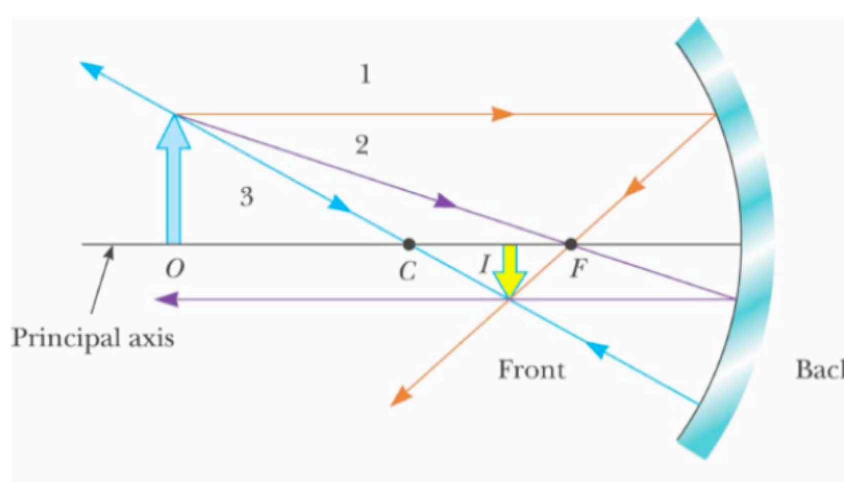
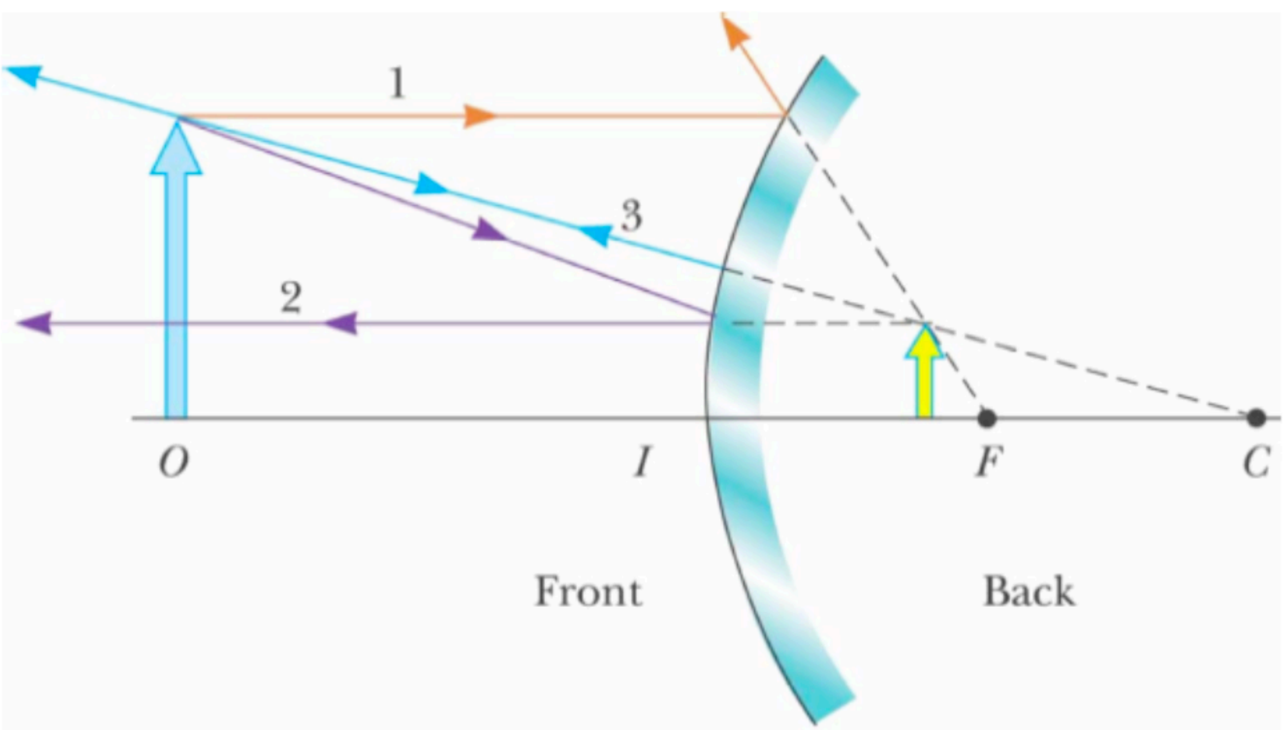


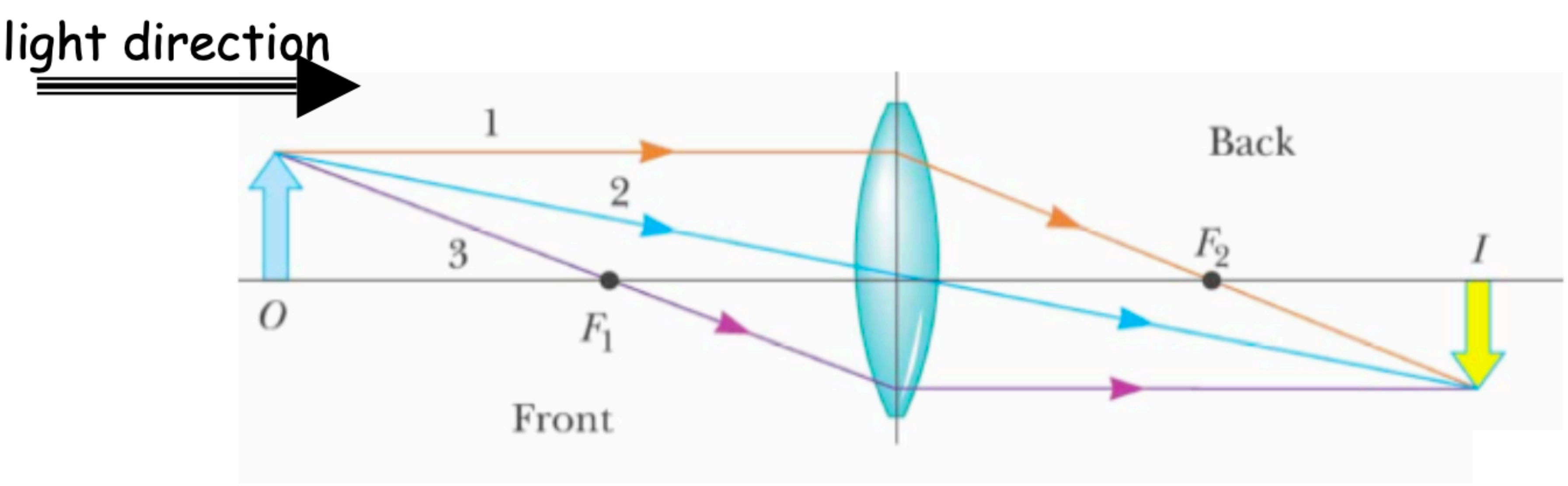
Image reconstruction using simple rays



Convex mirror:
virtual, upright
reduced, image

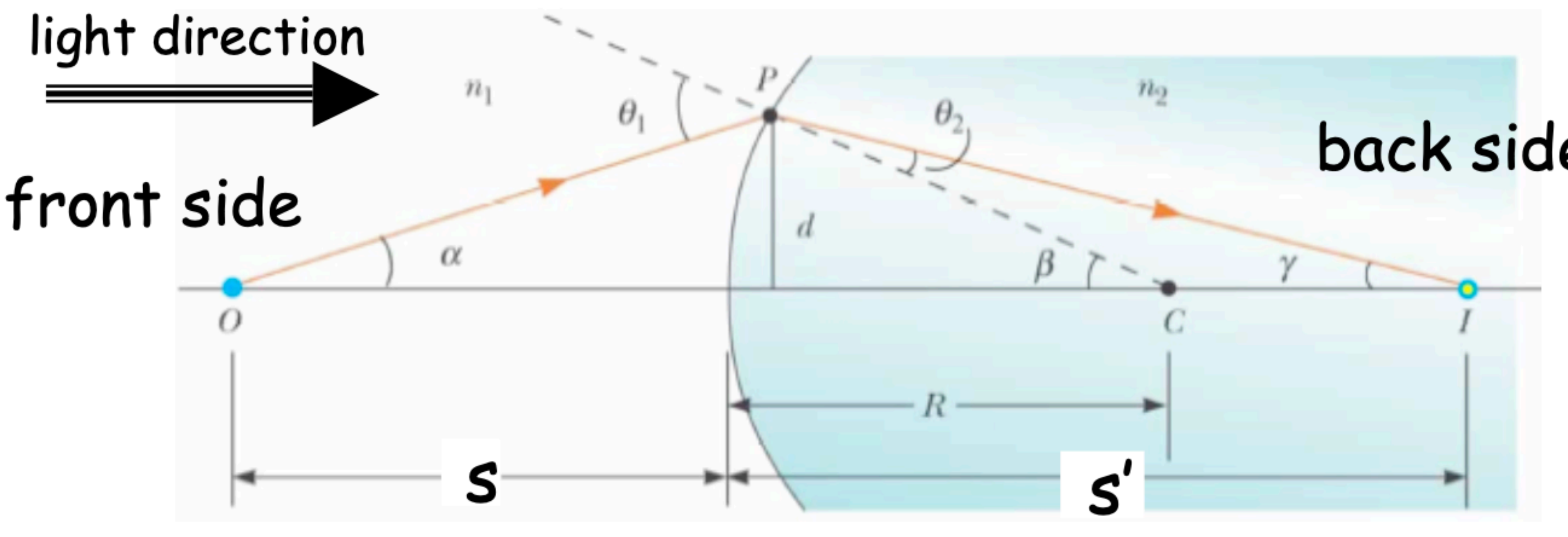
Thin lenses: the basics

- The terminology and conventions applied to mirrors will be used for images formed by lenses.
- In this case the light from an object passes through two refracting surfaces to form the image.
- We assume the rays are paraxial so that we can use $\sin\theta \sim \theta$ and therefore Snell's law is $n_1 \theta_1 \sim n_2 \theta_2$ at each surface.



Refraction from one spherical surface

- First the sign convention.



- s is + (-) on the front (back) side
- y and y' are + (-) if upright (inverted)

same as for mirrors

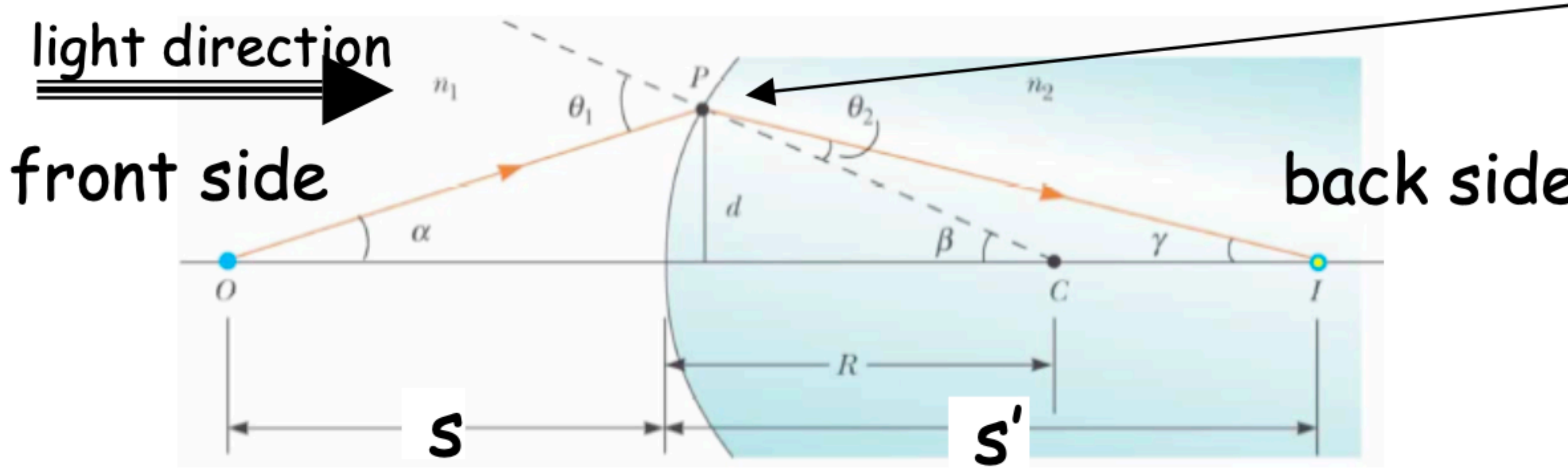
- s' is + (-) on the back (front)side of the surface
- R is + (-) on the back (front) side of the surface

opposite to mirrors

Refraction from one spherical surface

- Now calculate the image distance s'

$n_1\,\theta_1 = n_2\,\theta_2$



$\theta_1 = \alpha + \beta$

$\theta_2 = \beta - \gamma$

- Substitute for θ_1 and θ_2:

$$n_1\,(\alpha + \beta) = n_2\,(\beta - \gamma)$$

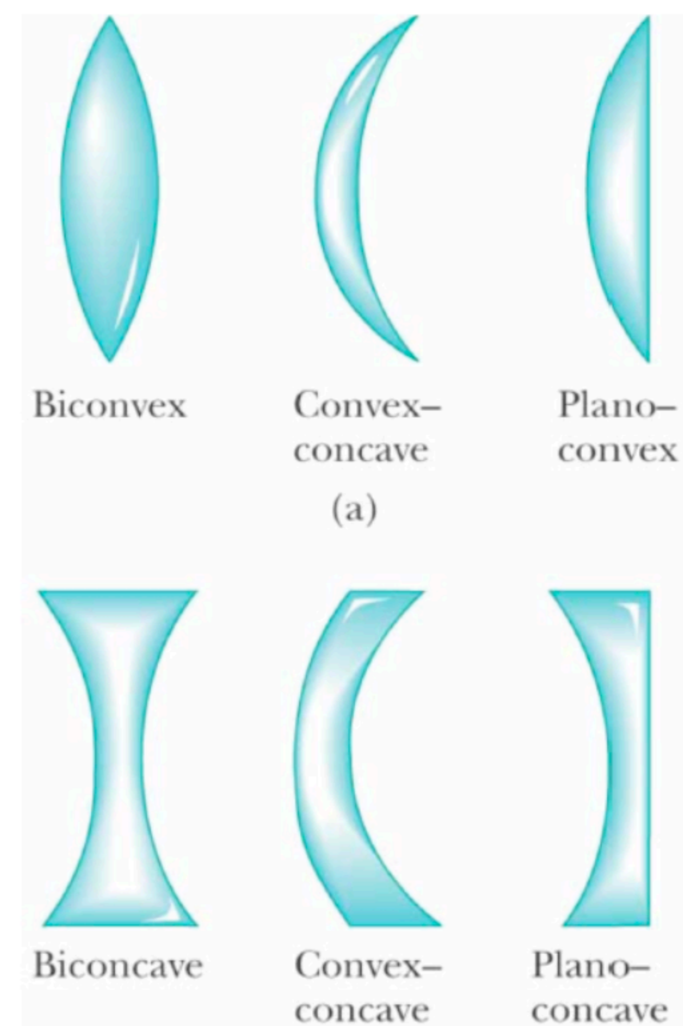
$$n_1\,\alpha + n_2\,\gamma = (n_2 - n_1)\,\beta$$

- For paraxial rays: $\tan\alpha = \alpha = d/s$; $\tan\beta = \beta = d/R$; $\tan\gamma = \gamma = d/s'$

$$\boxed{n_1/s + n_2/s' = (n_2 - n_1)/R}$$

A thin lens: two refracting surfaces

- A lens forms an image as the result of refraction from two surfaces.
- There are six different basic lenses as shown



- For paraxial rays and thin lenses (see derivation below):

$$1/s + 1/s' = 1/f$$

$$M = -\,s'/s$$

A thin lens: two refracting surfaces

Surface 1

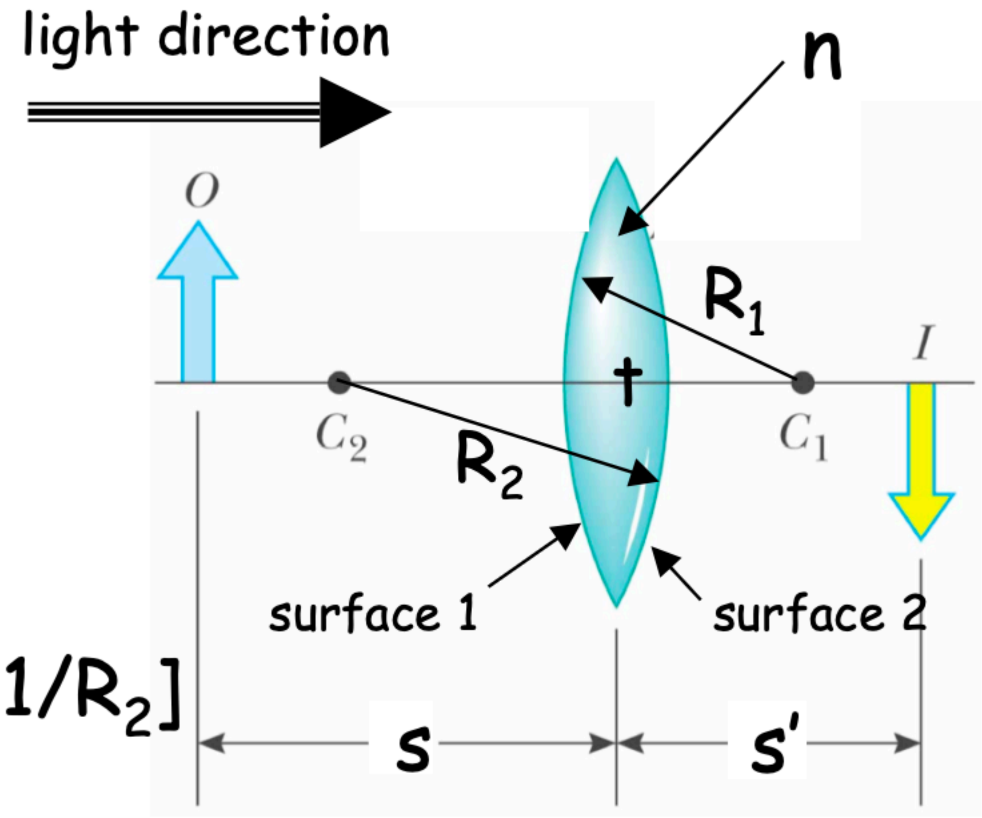
$$1/s + n/s_1' = (n-1)/R_1$$

Surface 2

$$n/s_2 + 1/s' = (1-n)/R_2$$

Add above 2 equations

$$1/s + 1/s' + [n/s_1' + n/s_2] = (n-1)[1/R_1 - 1/R_2]$$



Also $s_2 = t - s_1'$; for a thin lens $t \sim 0$ and $s_2 = - s_1'$

$$\boxed{1/s + 1/s' = (n-1)[1/R_1 - 1/R_2]}$$

The thin lens equation

Images formed by a thin lens

- As for a mirror we define f as the point where parallel rays are focused. f can be + or - depending on the shape of the lens.

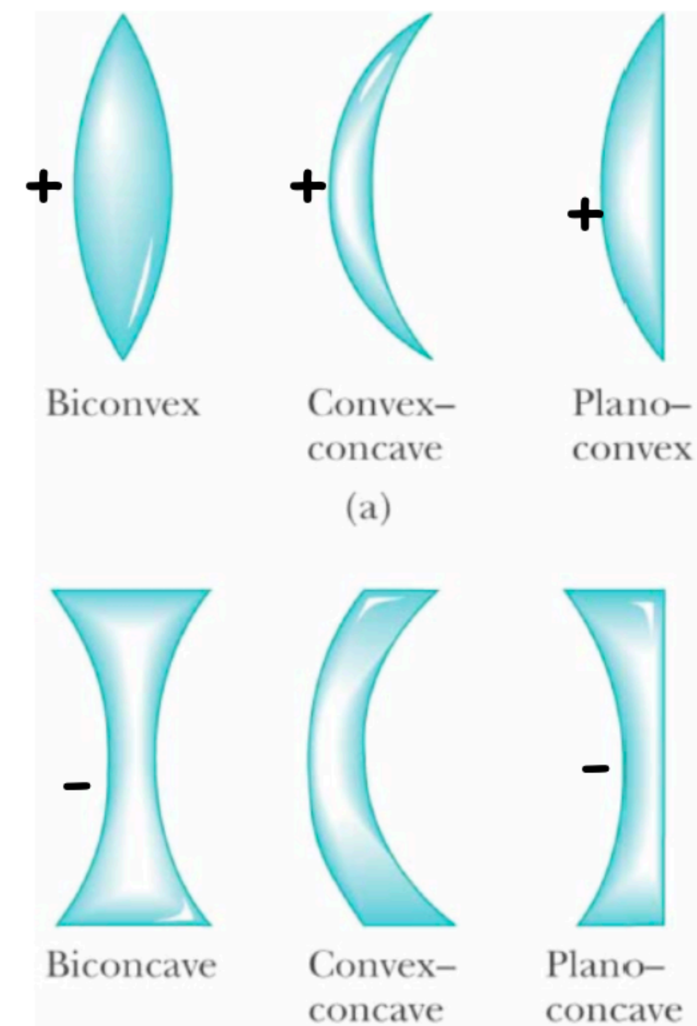
$$\boxed{1/f = (n-1)[1/R_1 - 1/R_2]}$$

Positive f => converging lens

Negative f => diverging lens

- The thin lens equation becomes

$$\boxed{1/s + 1/s' = 1/f}$$



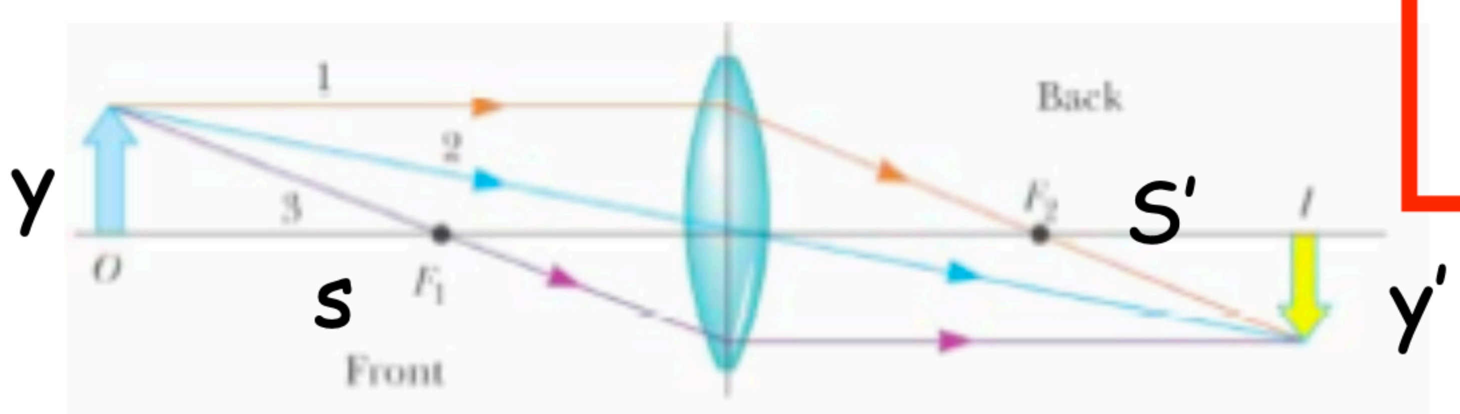
Ray diagrams for lenses

- As for mirrors a few simple light rays can be used to reconstruct the location and character of an image.

  Ray 1: a ray entering parallel to the axis will be refracted to pass through the focal point of the lens.

  Ray 2: a ray passing through the center of the lenses will be undeflected.

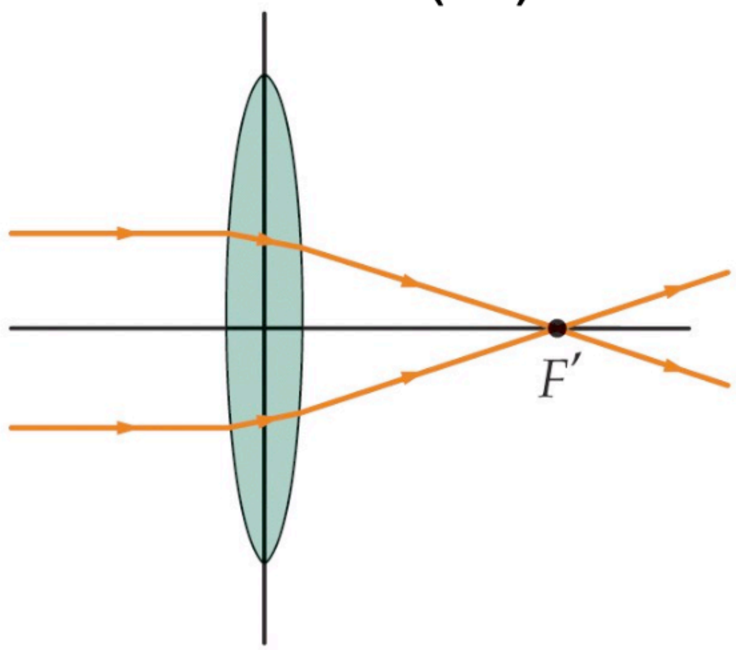
  Ray 3: a ray passing through the focal point will be refracted parallel to the axis.

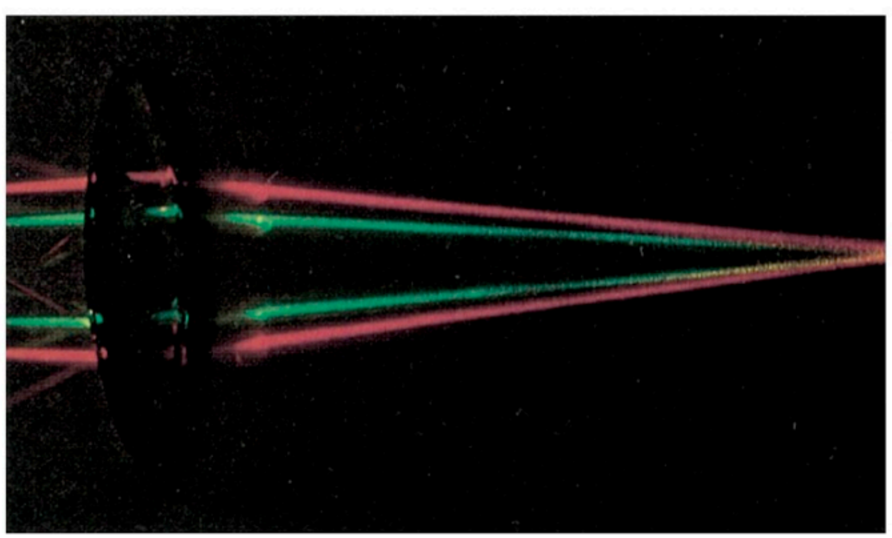


$M = y'/y = -s'/s$

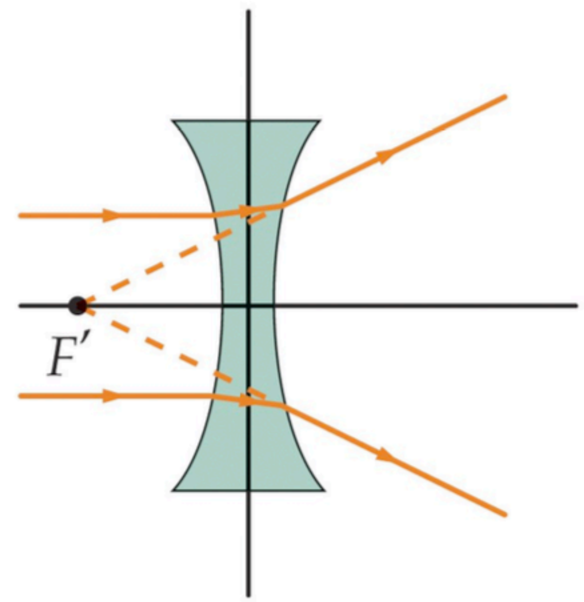
Converging and Diverging Lenses

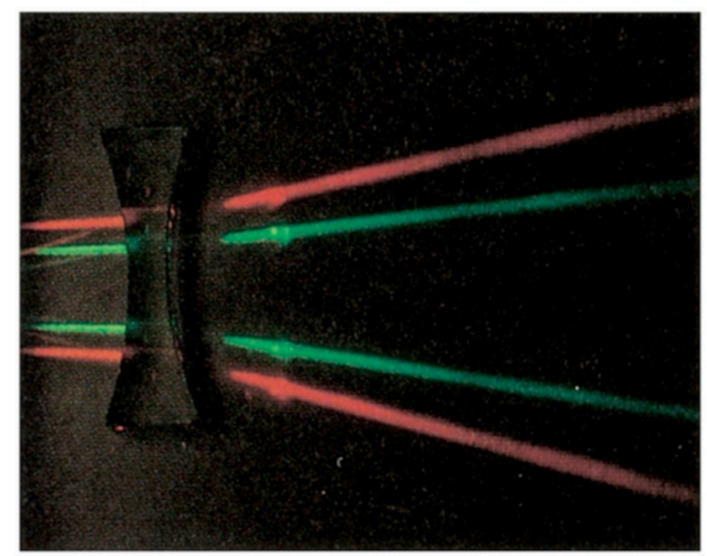
Converging or
Positive lens (f +)





Diverging or
Negative lens (f -)





Ray diagrams for lenses

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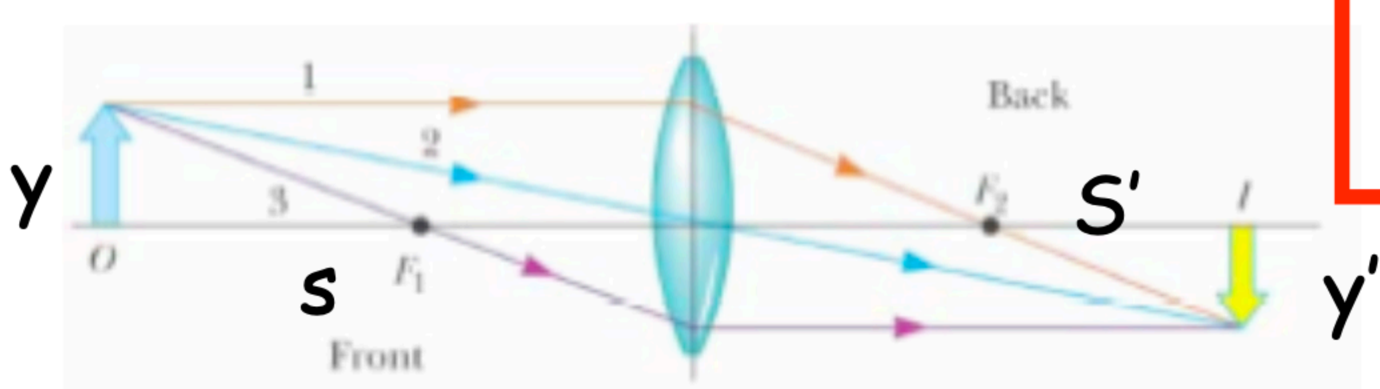


$$\boxed{M = y'/y = -s'/s}$$

Images formed by multiple optical elements

- Optical instruments generally use multiple lenses and/or mirrors to improve the quality of the image.
- The determination of the location and character of the image formed involves repeated application of the basic mirror and thin lens equations.
- We will start with some simple examples, and then move on to specific optical instruments (microscopes, telescopes, etc.)

Images formed by multiple optical elements

- The approach is the following:

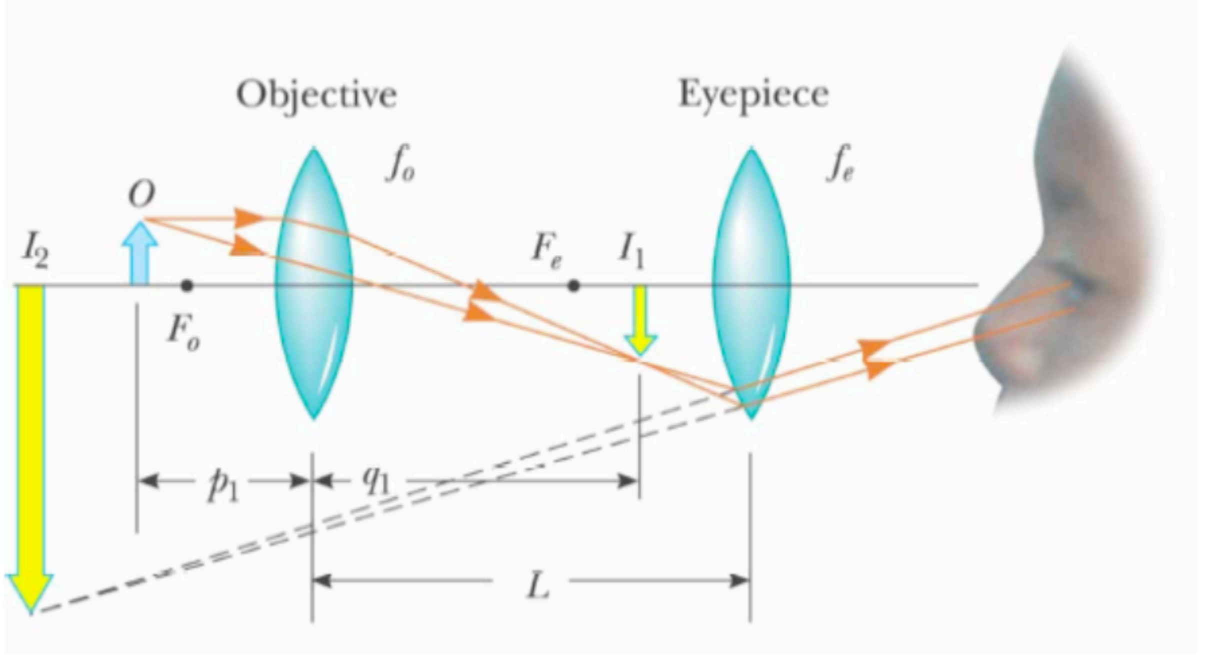
  1. Starting with the object, calculate the location and magnification of the image formed by the first lens/mirror.

  2. Take this image as the object for the second lens/mirror. Redefine the sign convention relative to this second lens/mirror.

  3. Calculate the image formed by the second lens/mirror.

  4. Repeat for any additional optical elements.

Example: compound microscope



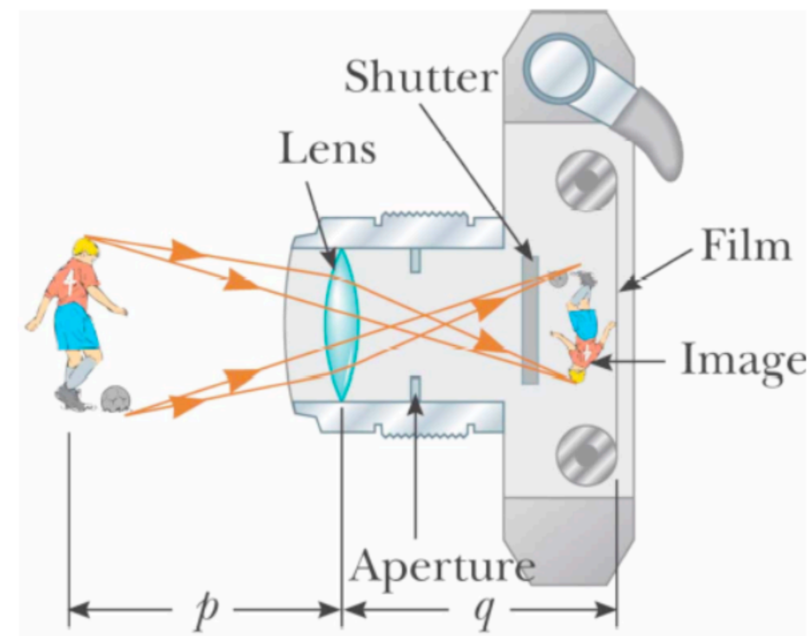
Why optical instruments?

- The purpose of optical instruments is to improve the viewing of objects beyond that possible with your unaided eye.
- The image is enhanced in 3 ways:

1. Make the image sharper: Often the lens of the unaided eye forms a blurred image on the retina. Glasses, contacts or a lens replacement can fix this.
2. Make the object viewed "bigger" so that finer details can be observed.
3. Make the object brighter. The eye has an aperture of about 5mm, and therefore can collect a limited amount of light.

The Camera

- A camera forms the image of an object on film or a charged-coupled device (CCD) to permanently record the image.
- A camera has the following basic parts:
  - a converging lens (focal length f) to form a real image of an object
  - a device to move the position of the lens relative to the film so that the real image is formed on the film (focusing the camera)
  - a variable aperture (D) to control the light intensity
  - a shutter that rapidly opens and closes to "freeze" an object in motion



The Camera

- When taking a picture with a camera you adjust (either manually or using internal electronic processing logic):
  - The lens - film separation: depends on object distance
  - The shutter speed say 1/30, 1/60, … 1/500s: depends on the motion of the object
  - The aperture D: depends on the intensity of the light source
  - The "f-number" is defined to be f/D: the intensity of the light on the film is proportional to $1/(\text{f-number})^2$

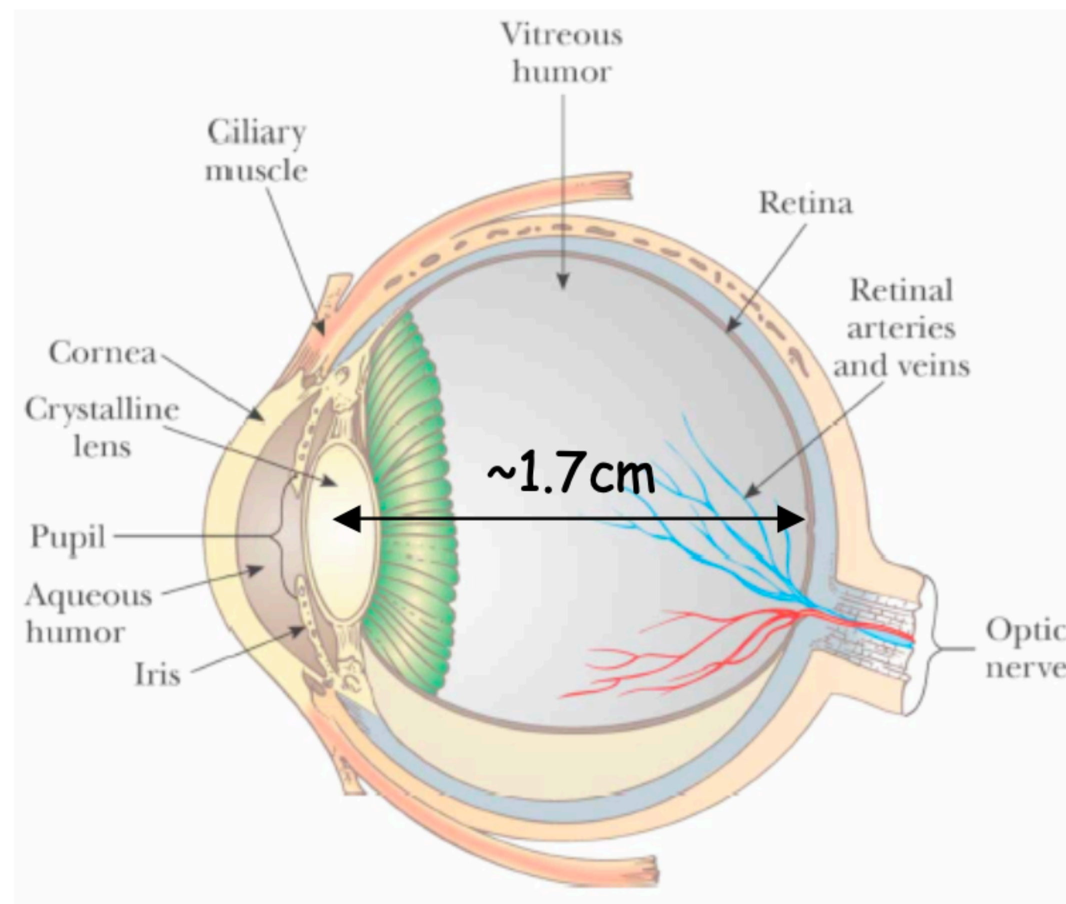
    Standard f-numbers for cameras are 2.8, 4, 5.6, 8, 11, 16.

The operation of the amazing human eye

- Our eyes are the biological optical transducers that take light signals and transform them to electrical impulses that are processed and interpreted by our brain.

- There is some analogy to a camera with a digital CCD read out.

- But the eye + brain's ability to detect and analyze light signals is truly amazing.



The eye

- The intensity of light entering the eye is controlled by a variable aperture called the iris. Intensities from bright sun to few photons can be detected.

- Ciliary muscles are used to modify the shape the lens and change its focus length to form an image on the back readout surface.

- This process is called accommodation. An eye can focus objects at distances from a far point (usually infinity) to a near point as close as about 25 cm for young eyes.

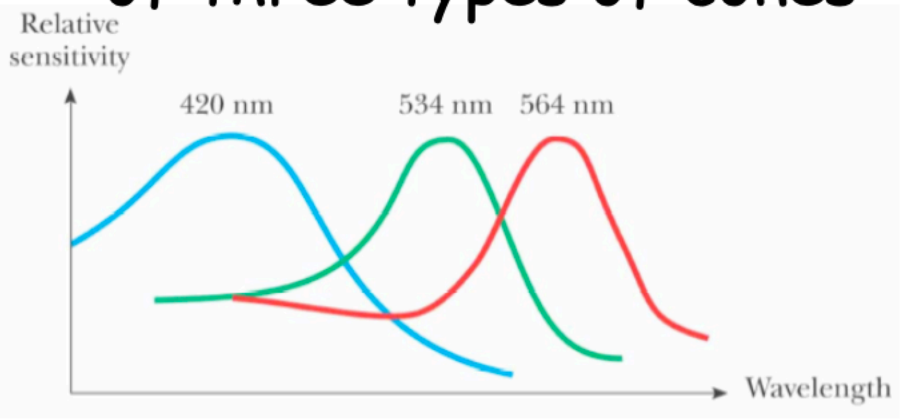
- The retina is the back surface of the eye. It is composed of read out sensors (rods and cones) that transform the light signals to electrical pulses that are sent via the optic nerve to the brain.

The eye

- The cones do the high resolution readout and are located near the center of the of retina. Three different types of cones respond to different wavelength ranges of the light spectrum. This is the information we use to construct the concept of color.

- The rods located on the periphery of the retina, are less dense and do not differentiate colors.

Spectral response of three types of cones



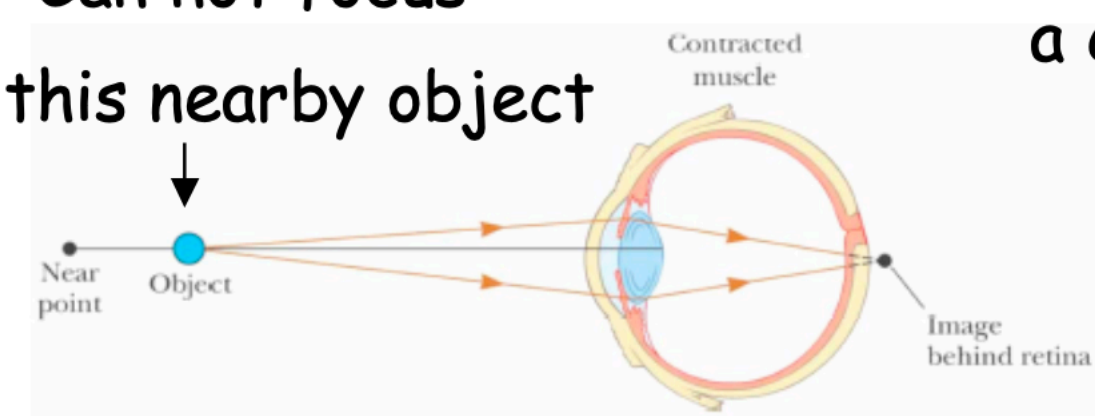
- There are on the order of a 100 million cones/rods used for readout.

Imperfections of the eye

- A common imperfection, especially as people age, is a reduction of the flexibility of the ciliary -lens system. This limits the accommodation of the eye to focus on distant and close objects.
- Farsighted person: can focus on distant objects but not those nearby. Their near point is beyond the "normal" 25 cm.
- This can be corrected by placing a converging lens in front of the eye.

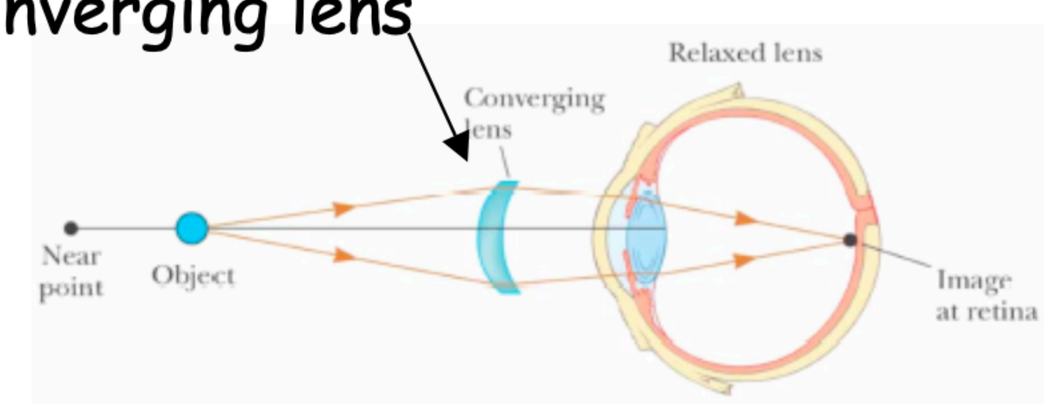
Can not focus

on this nearby object



Correct with

a converging lens



Improvement of the eye's performance

- Let's assume you have normal eyes, or they are corrected to be normal using glasses:

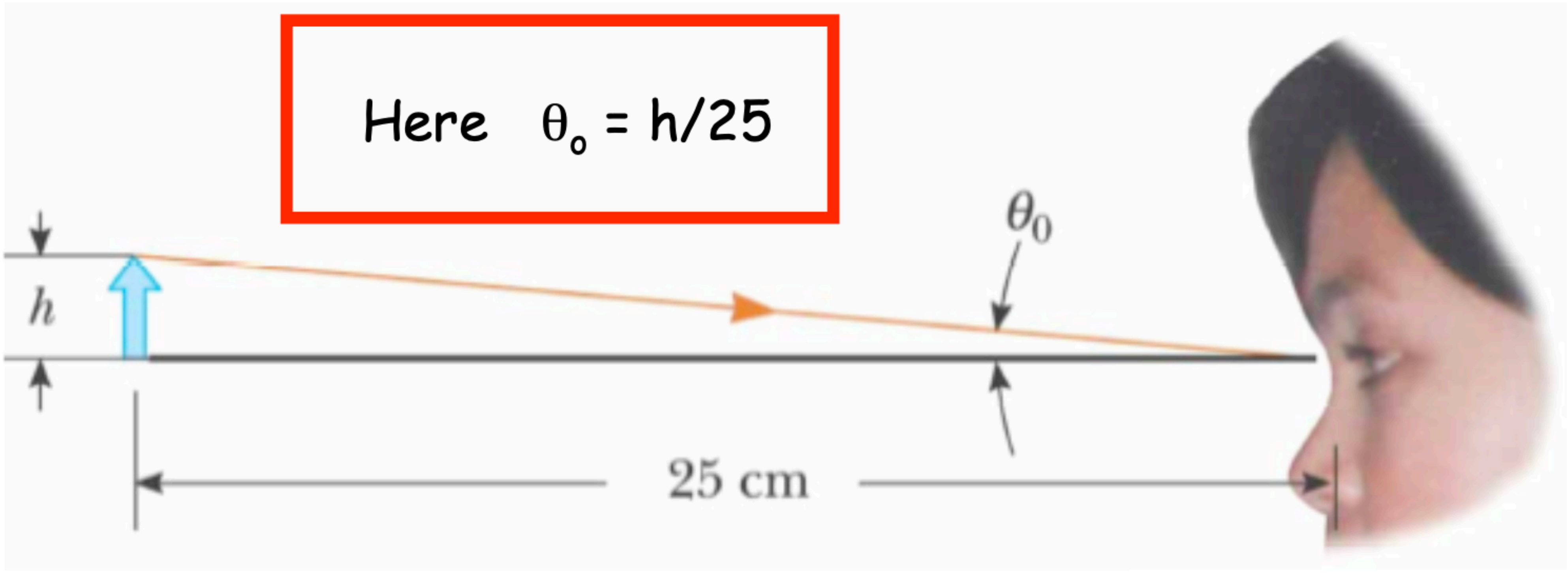
  far point is at infinity

  near point is at ~ 25 cm

- As discussed this "normal" eye is a marvelous instrument to convert light signals to electrical signals for you brain to process and you to "see".
- However it has limitations in recording details of very small objects (say a bacteria) and very distant objects (say craters on the moon).
- Microscopes allow you to see very small objects.
- Telescopes allow you to see very distant objects

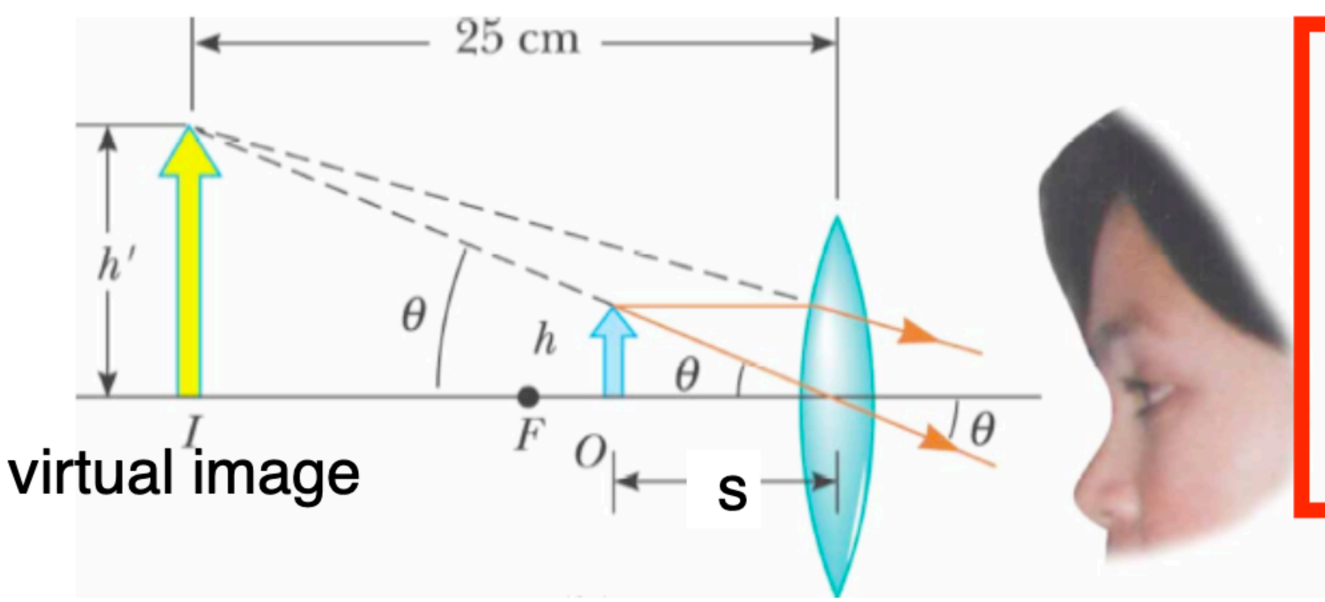
The simple magnifier

- The simplest enhancement to viewing small, close objects is a magnifying glass -- a short focal length converging lens.
- Consider viewing a small object with your unaided eye, with the object brought as close to the eye as possible = the near point of the eye of 25 cm.



The simple magnifier

- Now view the object through a converging lens (the magnifier) with the object placed within the focal point of the lens. The observer then views a virtual image of the object.
- Adjust the object distance so that the virtual image is now at the near point of the eye as shown below ($s' = -25$cm)
- From the figure: $1/s + 1/-25 = 1/f$ or $s = 25f/(25+f)$
- Therefore $\theta = h/s = h(25+f)/25f$ and using $m = \theta/\theta_o$ with $\theta_o = h/25$



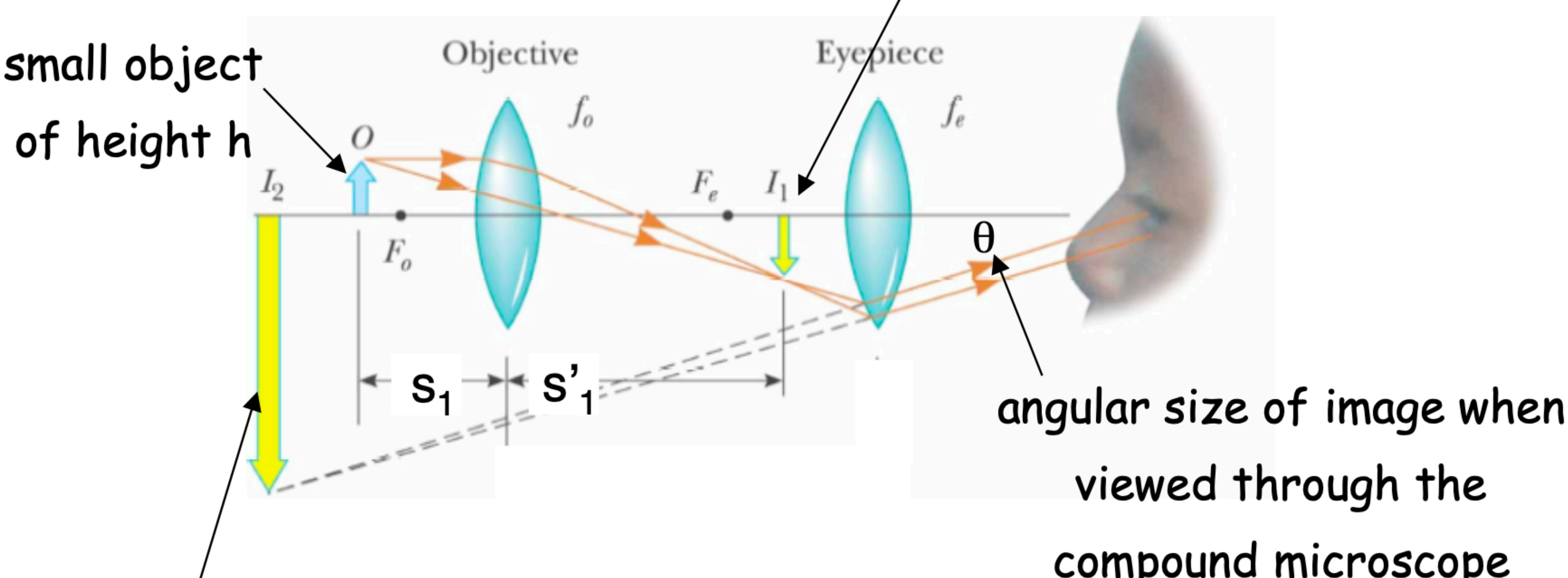
m = angular magnification

$= 1 + 25/f$(cm)

(for the image at the eye's near point)

Operation of a compound microscope

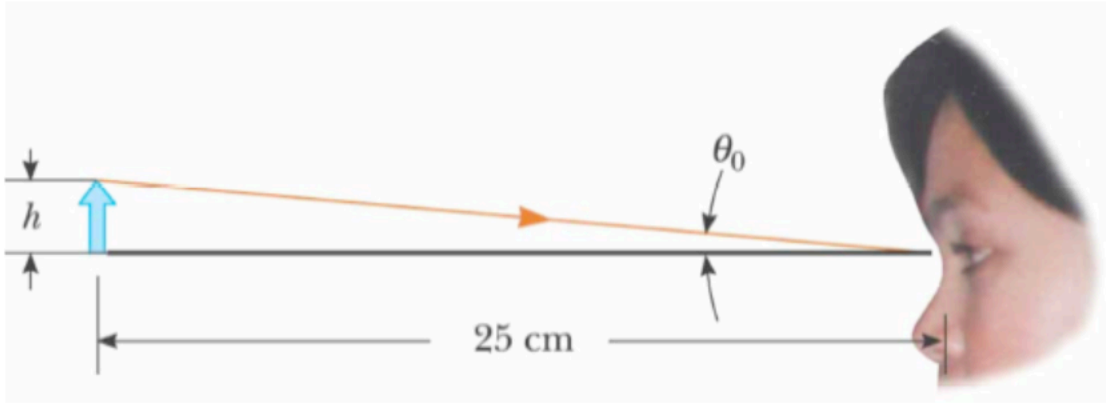
- The objective lens forms a real image of the object.



- The eyepiece views this real image and provides a virtual image to the eye.

Magnification of a compound microscope

- We want to calculate angular magnification = m = θ / θ_o where as defined above θ_o = angular size of the object when viewed with the unaided eye and θ = the angular size when viewed through the microscope.



- Step 1: find θ_o

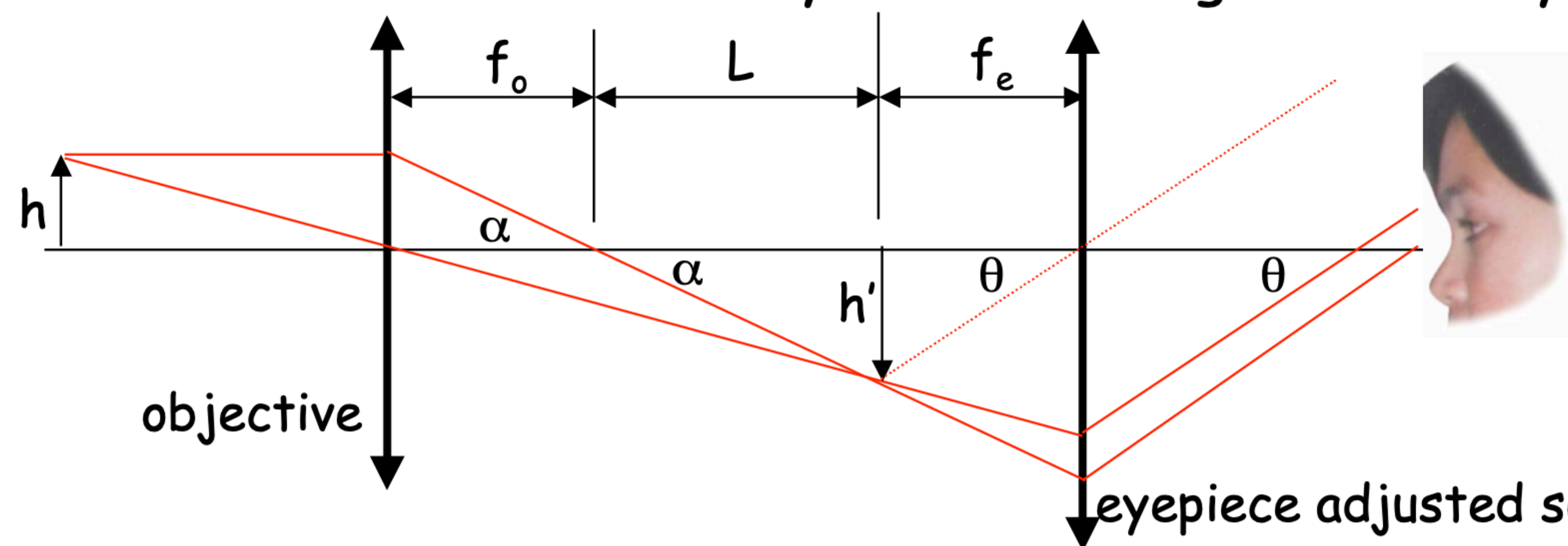
  Using an unaided eye

  with the object place at the near point of the eye:

  θ_o = h/25 cm (assuming a near point of 25 cm)

Magnification of a compound microscope

- Step 2: find the angular size of the final image = θ . Do this for the case of a relaxed eye => final image at infinity.



eyepiece adjusted so that image of h' is at infinity

1. $\alpha = h'/L = h/f_o$

   and $h' = L\, h/f_o$

2. $\theta = h'/f_e$

   and $h' = \theta\, f_e$

3. Therefore $L\, h/f_o = \theta\, f_e$ and $\theta = L\, h/(f_o f_e)$

Magnification of a compound microscope

- From Step 1: $\theta_o = h/25\text{ cm}$
- From Step 2: $\theta = L\,h/(f_o f_e)$
- And finally angular magnification $m = \theta/\theta_o$ is:

$$\boxed{m = -L\,25/(f_o f_e)}$$

Compound microscope adjusted for relaxed eye viewing by an eye with a near point at 25 cm

insert - sign since image inverted

Now for Basic Geometrical Optics - Now How to Calculate in Detail

We now begin multiple passes through these ideas with the math levels always increasing.

Optics is the cornerstone of photonics systems and applications.

Now we will learn about one of the two main divisions of basic optics—geometrical (ray) optics.

We will learn about the other—physical (wave) optics after this Part.

Geometrical optics helps you understand the basics of light reflection and refraction and the use of simple optical elements such as mirrors, prisms, lenses, and fibers.

Physical opticsz(later) will help you understand the phenomena of light wave interference, diffraction, and polarization; the use of thin film coatings on mirrors to enhance or suppress reflection; and the operation of such devices as gratings and quarter-wave plates.

Basic Concepts

1. THE LAWS OF REFLECTION AND REFRACTION

We begin our study of basic geometrical optics by examining how light reflects and refracts at smooth, plane interfaces.

Figure 1a shows ordinary reflection of light at a plane surface, and Figure 1b shows refraction of light at two successive plane surfaces. In each instance, light is pictured simply in terms of straight lines, which we refer to as light rays.

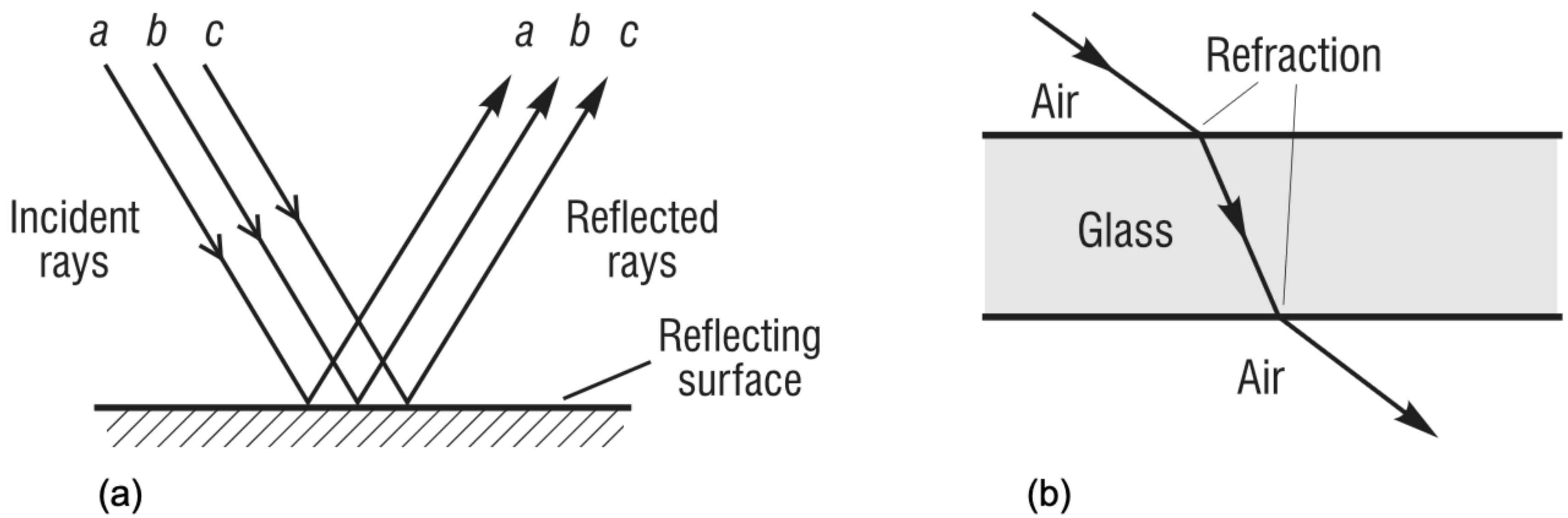


Figure 1 Light rays undergoing reflection and refraction at plane surfaces

After a study of how light reflects and refracts at plane surfaces, we extend our analysis to smooth, curved surfaces, thereby setting the stage for light interaction with mirrors and lenses— the basic elements in many optical systems.

The analysis of how light interacts with plane and curved surfaces is carried out with light rays.

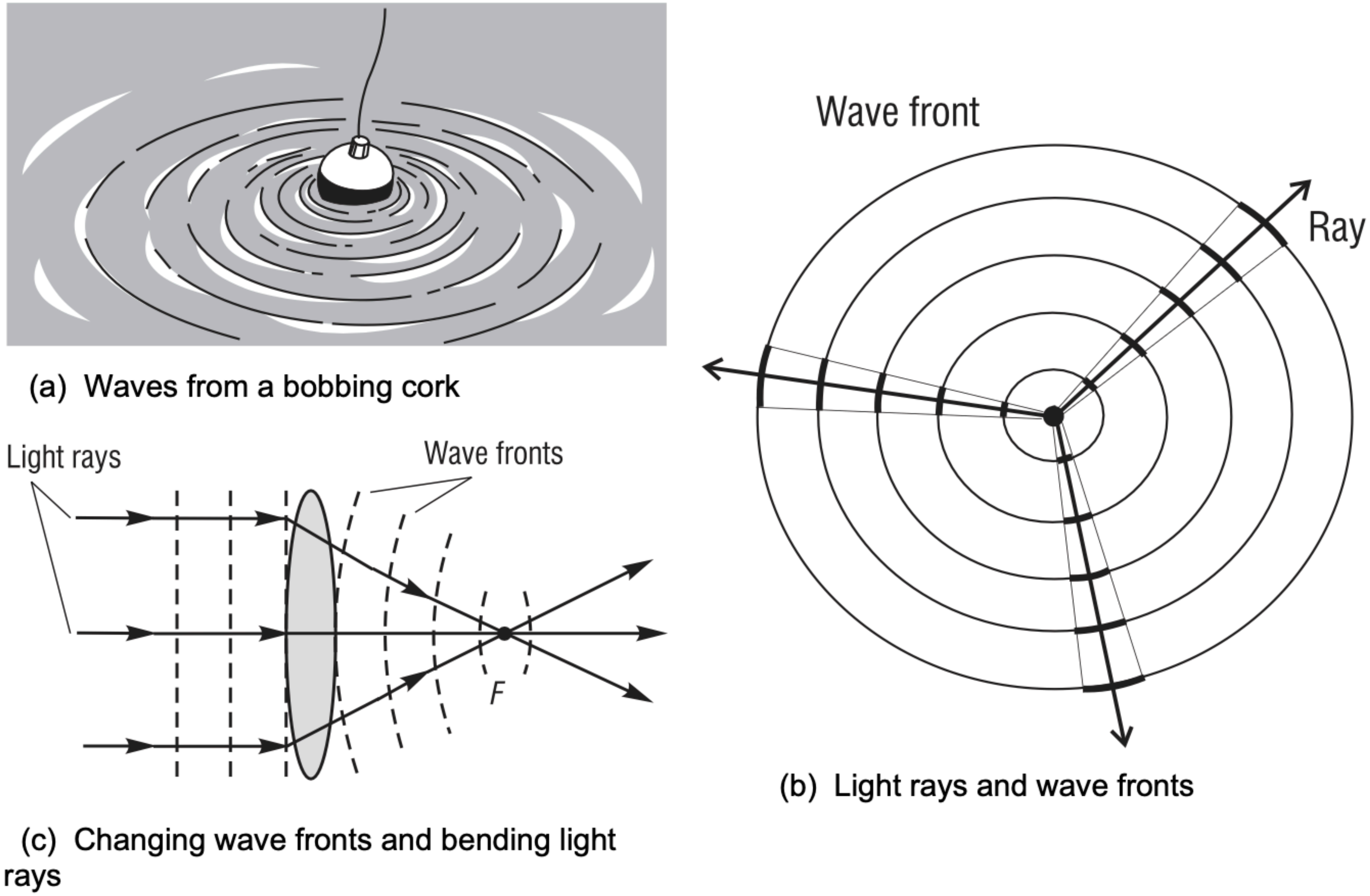
A light ray is nothing more than an imaginary line directed along the path that the light follows.

It is helpful to think of a light ray as a narrow pencil of light, very much like a narrow, well-defined laser beam.

A. Light rays and light waves

Before we look more closely at the use of light rays in geometrical optics, we need to say a brief word about light waves and the geometrical connection between light rays and light waves.

For most of us, wave motion is easily visualized in terms of water waves—such as those created on a quiet pond by a bobbing cork. See Figure 2a.



(a) Waves from a bobbing cork

(b) Light rays and wave fronts

(c) Changing wave fronts and bending light rays

Figure 2 Waves and rays

The successive high points (crests) and low points (troughs) occur as a train of circular waves moving radially outward from the bobbing cork.

Each of the circular waves represents a wave front.

A wave front is defined here as a locus of points that connect identical wave displacements—that is, identical positions above or below the normal surface of the quiet pond.

In Figure 2b, circular wave fronts are shown with radial lines drawn perpendicular to them along several directions.

Each of the rays describes the motion of a restricted part of the wave front along a particular direction.

Geometrically then, a ray is a line perpendicular to a series of successive wave fronts specifying the direction of energy flow in the wave.

Figure 2c shows plane wave fronts of light bent by a lens into circular (spherical in three dimensions) wave fronts that then converge onto a focal point F.

The same diagram shows the light rays corresponding to these wave fronts, bent by the lens to pass through the same focal point F.

Figure 2c shows clearly the connection between actual waves and the rays used to represent them.

In the study of geometrical optics, we find it acceptable to represent the interaction of light waves with plane and spherical surfaces—with mirrors and lenses—in terms of light rays.

With the useful geometric construct of a light ray we can illustrate propagation, reflection, and refraction of light in clear, uncomplicated drawings.

For example, in Figure 3a, the propagation of light from a "point source" is represented by equally spaced light rays emanating from the source.

Each ray indicates the geometrical path along which the light moves as it leaves the source. Figure 3b shows the reflection of several light rays at a curved mirror surface, and Figure 3c shows the refraction of a single light ray passing through a prism

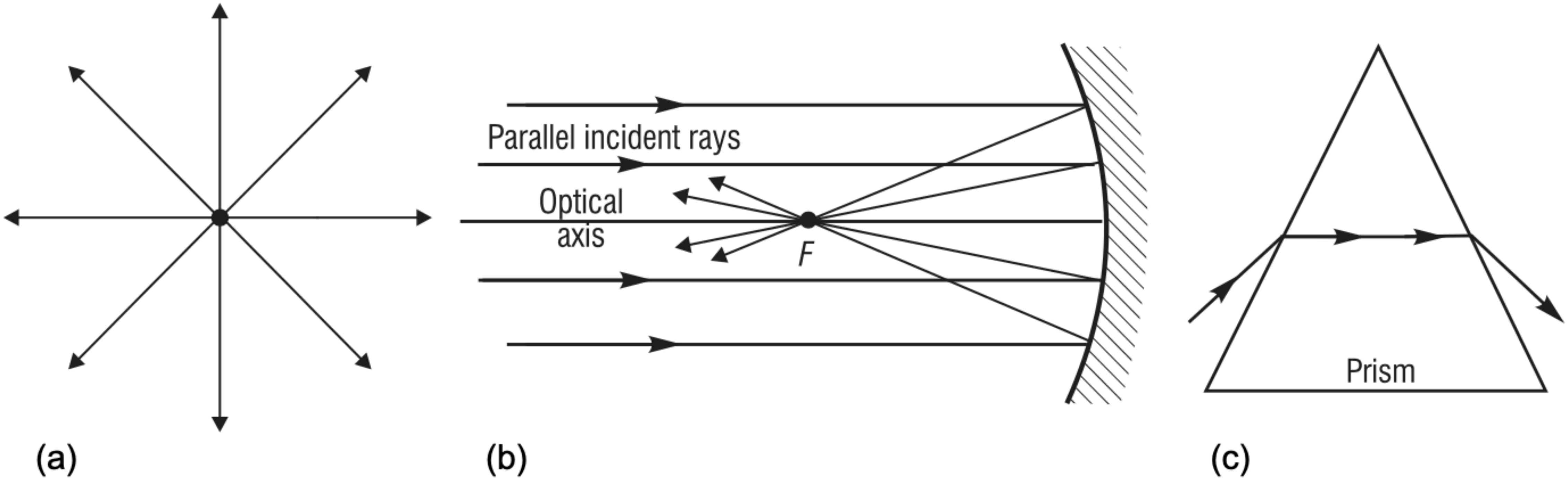


Figure 3 Typical light rays in (a) propagation, (b) reflection, and (c) refraction

B. Reflection of light from optical surfaces

When light is incident on an interface between two transparent optical media—such as between air and glass or between water and glass—four things can happen to the incident light.

- It can be partly or totally reflected at the interface.
- It can be scattered in random directions at the interface.
- It can be partly transmitted via refraction at the interface and enter the second medium.
- It can be partly absorbed in either medium.

In our study of geometrical optics we shall consider only smooth surfaces that give rise to specular (regular, geometric) reflections (Figure 4a) and ignore ragged, uneven surfaces that give rise to diffuse (irregular) reflections (Figure 4b).

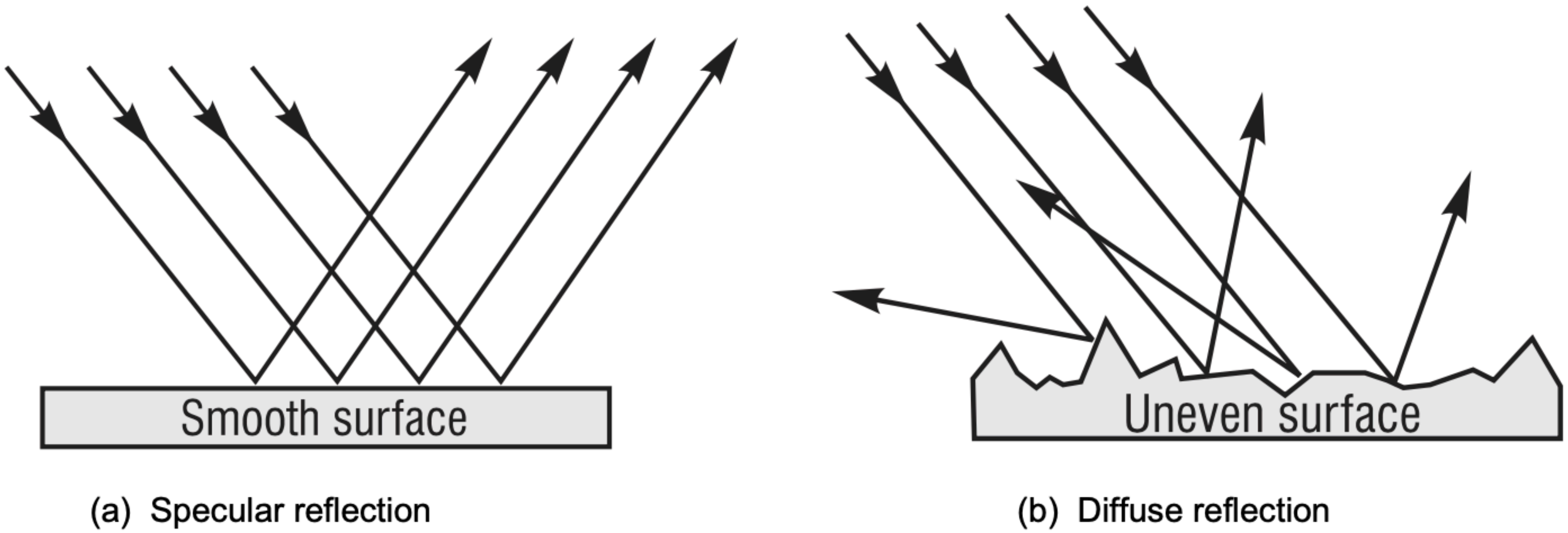


Figure 4 Specular and diffuse reflection

In addition, we shall ignore absorption of light energy along the path of travel, even though absorption is an important consideration when percentage of light transmitted from source to receiver is a factor of concern in optical systems.

1. The law of reflection: plane surface.

When light reflects from a plane surface as shown in Figure 5, the angle that the reflected ray makes with the normal (line perpendicular to the surface) at the point of incidence is always equal to the angle the incident ray makes with the same normal.

Note carefully that the incident ray, reflected ray, and normal always lie in the same plane.

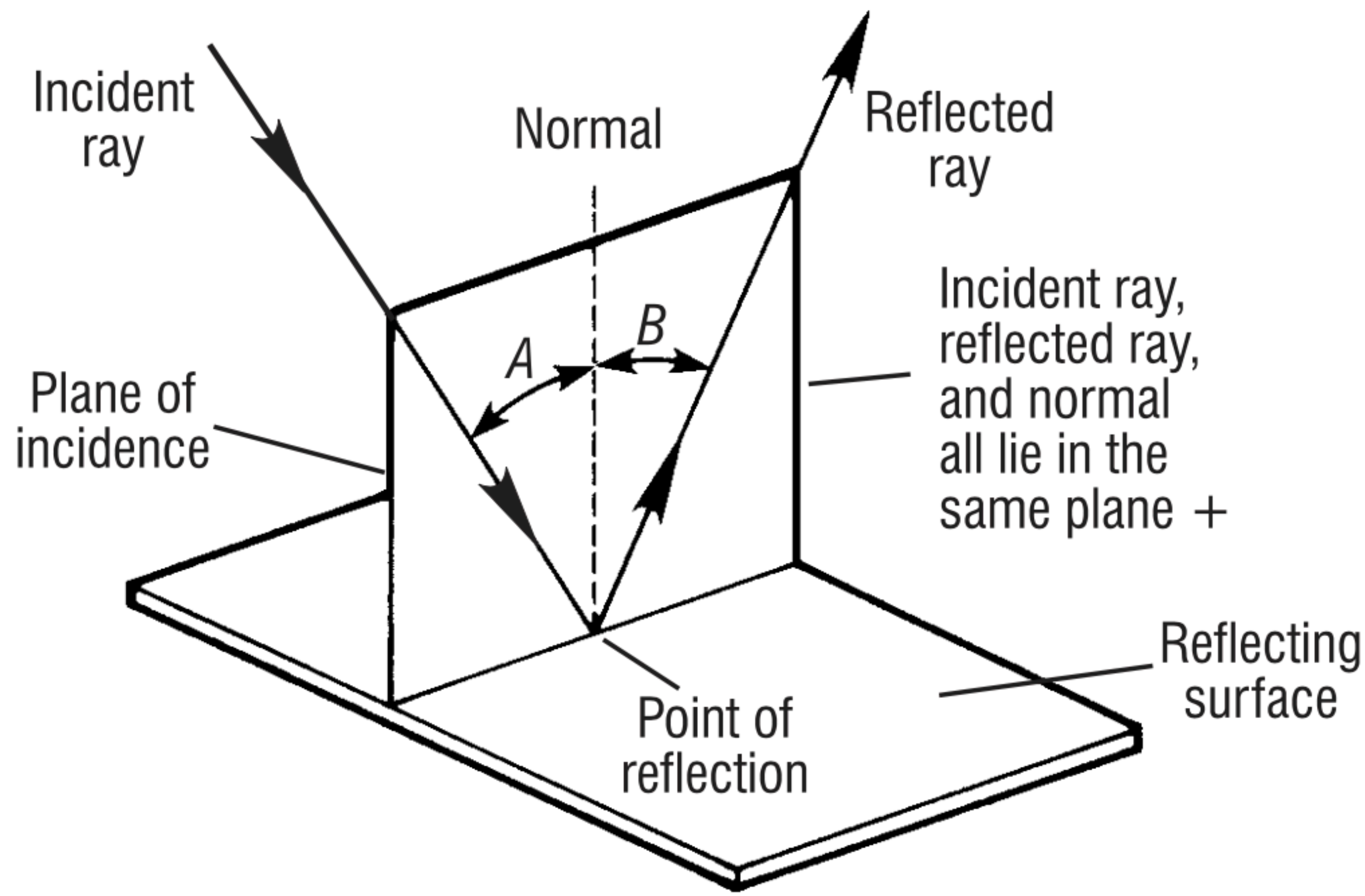


Figure 5 Law of reflection: Angle B equals angle A.

With the law of reflection in mind, we can see that, for the specular reflection shown earlier in Figure 4a, each of the incident, parallel rays reflects off the surface at the same angle, thereby remaining parallel in reflection as a group.

In Figure 4b, where the surface is made up of many small, randomly oriented plane surfaces, each ray reflects in a direction different from its neighbor, even though each ray does obey the law of reflection at its own small surface segment.

2. Reflection from a curved surface.

With spherical mirrors, reflection of light occurs at a curved surface.

The law of reflection holds, since at each point on the curved surface one can draw a surface tangent and erect a normal to a point P on the surface where the light is incident, as shown in Figure 6.

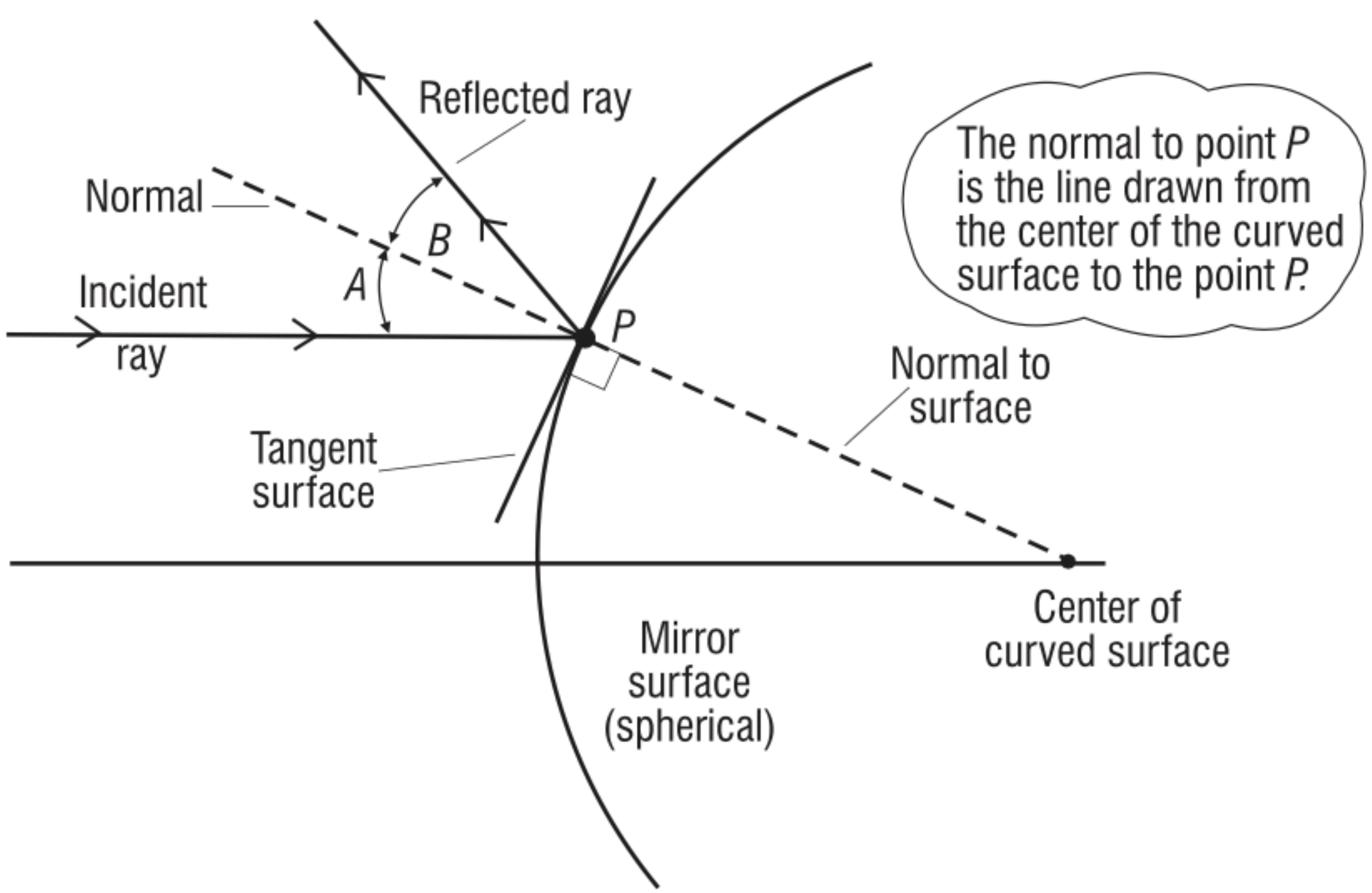


Figure 6 Reflection at a curved surface: Angle B equals angle A.

One then applies the law of reflection at point P just as was illustrated in Figure 3-5, with the incident and reflected rays making the same angles (A and B) with the normal to the surface at P.

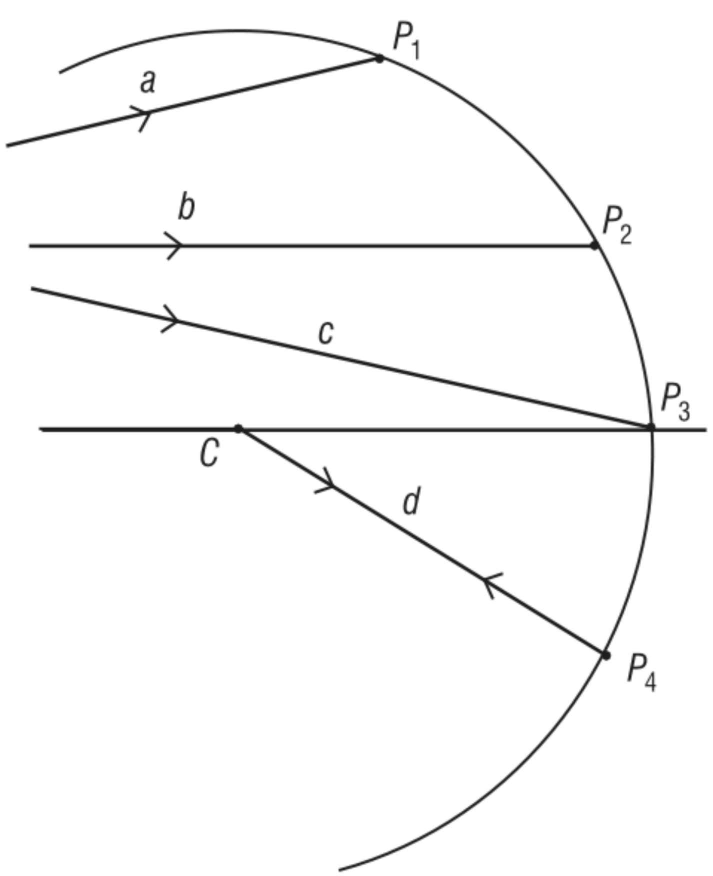
Note that successive surface tangents along the curved surface in Figure 3-6 are ordered (not random) sections of "plane mirrors" and serve—when smoothly connected—as a spherical surface mirror, capable of forming distinct images.

Since point P can be moved anywhere along the curved surface and a normal drawn there, we can always find the direction of the reflected ray by applying the law of reflection.

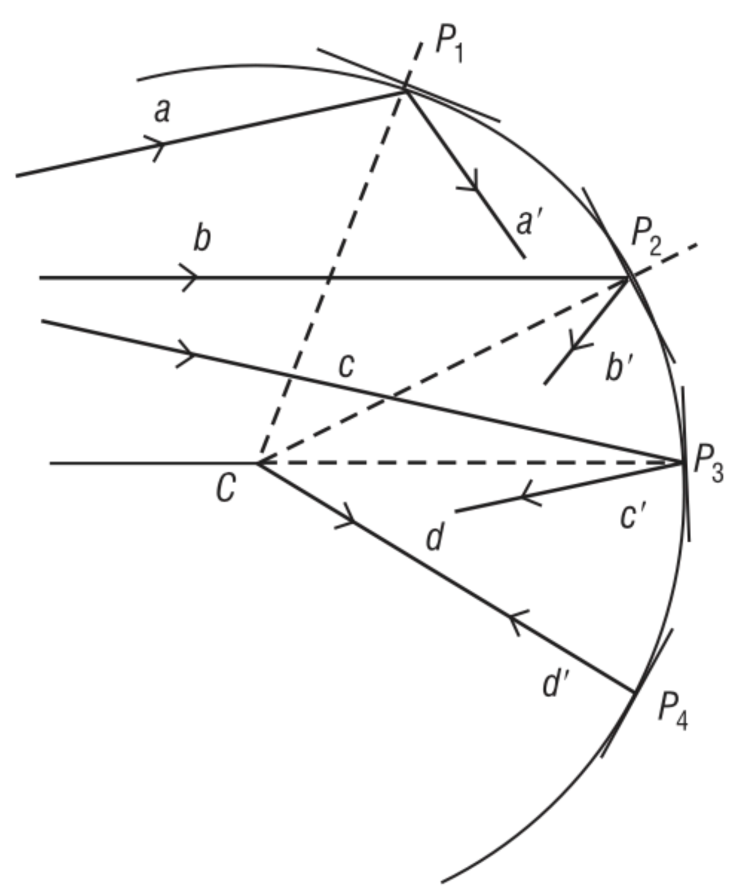
We shall apply this technique when studying the way mirrors reflect light to form images.

Example 1 - I will use examples in this part to illustrate the phenomena.

Using the law of reflection, complete the ray-trace diagram for the four rays (a, b, c, d) incident on the curved surface shown at the left below, given the center of the curved surface is at point C.



Beginning of ray trace



Completion of ray trace

Solution: Draw a normal (shown dashed) from point C to each of the points P_1, P_2, P_3, and P_3, as shown above in the drawing at the right. At each point, draw the appropriate reflected ray (a′,b′,c′,d′) so that it makes an angle with its normal equal to the angle made by the incident ray (a,b,c,d) at that point. Note that ray d reflects back along itself since it is incident along the line of the normal from C to point P_4.

C. Refraction of light from optical interfaces

When light is incident at an interface—the geometrical plane that separates one optical medium from another—it will be partly reflected and partly transmitted.

Figure 7 shows a three-dimensional view of light incident on a partially reflecting surface (interface), being reflected there (according to the law of reflection) and refracted into the second medium.

The bending of light rays at an interface between two optical media is called refraction.

Before we examine in detail the process of refraction, we need to describe optical media in terms of an index of refraction.

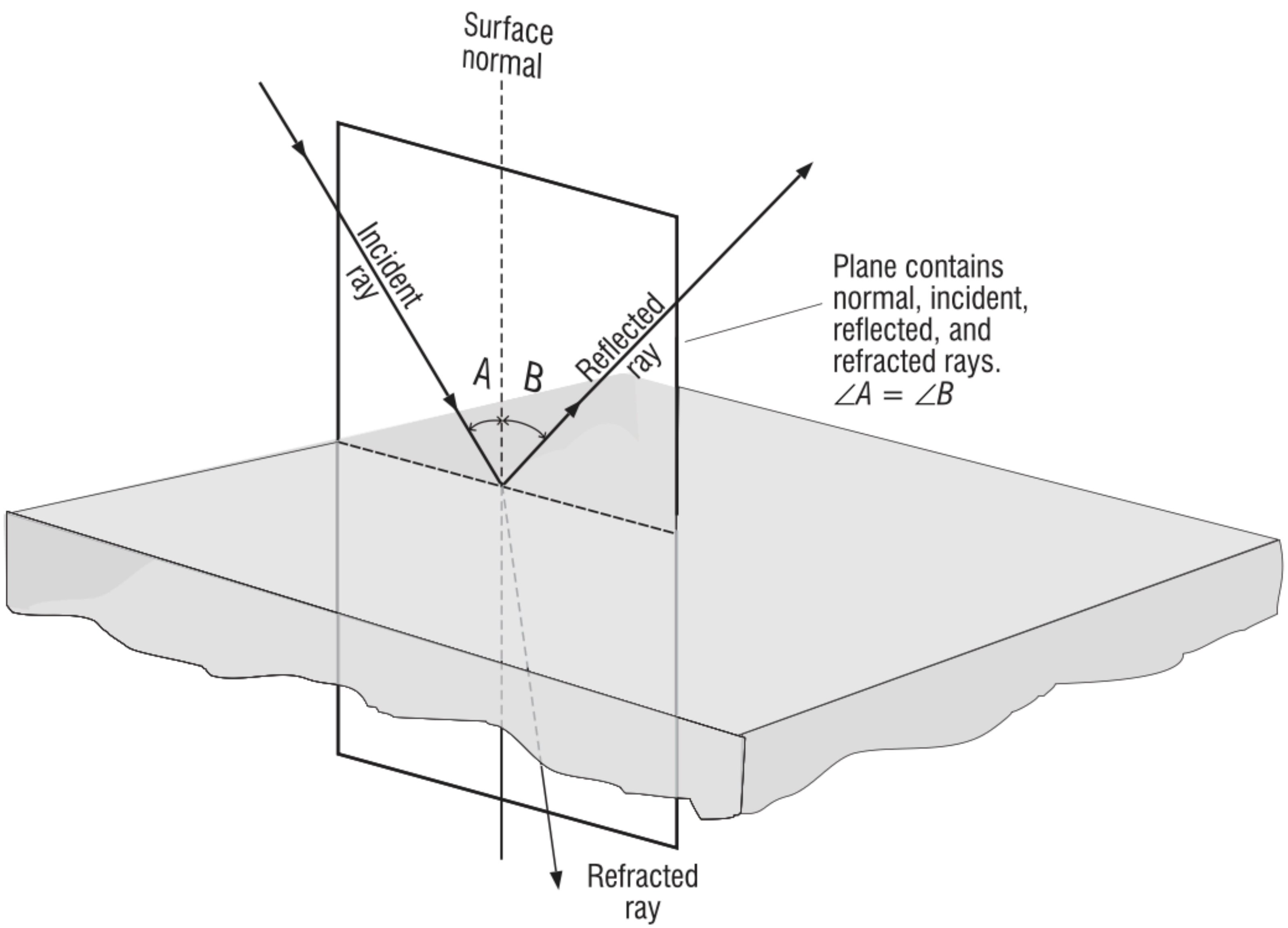


Figure 7 Reflection and refraction at an interface

1. Index of refraction.

The two transparent optical media that form an interface are distinguished from one another by a constant called the index of refraction, generally labeled with the symbol n.

The index of refraction for any transparent optical medium is defined as the ratio of the speed of light in a vacuum to the speed of light in the medium, as given in Equation 1.

$$\boxed{n = \frac{c}{\mathrm{v}}} \tag{1}$$

where

c = speed of light in free space (vacuum)
v = speed of light in the medium
n = index of refraction of the medium

The index of refraction for free space is exactly one.

For air and most gases it is very nearly one, so in most calculations it is taken to be 1.0.

For other materials it has values greater than one.

Table 3-1 lists indexes of refraction for common materials.

Table 1 Indexes of Refraction for Various Materials at 589 nm

Substance	*n*	Substance	*n*
Air	1.0003	Glass (flint)	1.66
Benzene	1.50	Glycerin	1.47
Carbon Disulfide	1.63	Polystyrene	1.49
Corn Syrup	2.21	Quartz (fused)	1.46
Diamond	2.42	Sodium Chloride	1.54
Ethyl Alcohol	1.36	Water	1.33
Gallium Arsenide (semiconductor)	3.40	Ice	1.31
Glass (crown)	1.52	Germanium	4.1
Zircon	1.92	Silicon	3.5

The greater the index of refraction of a medium, the lower the speed of light in that medium and the more light is bent in going from air into the medium.

Figure 8 shows two general cases, one for light passing from a medium of lower index to higher index, the other from higher index to lower index.

Note that in the first case (lower-to-higher) the light ray is bent toward the normal.

In the second case (higher-to-lower) the light ray is bent away from the normal.

It is helpful to memorize these effects since they often help one trace light through optical media in a generally correct manner.

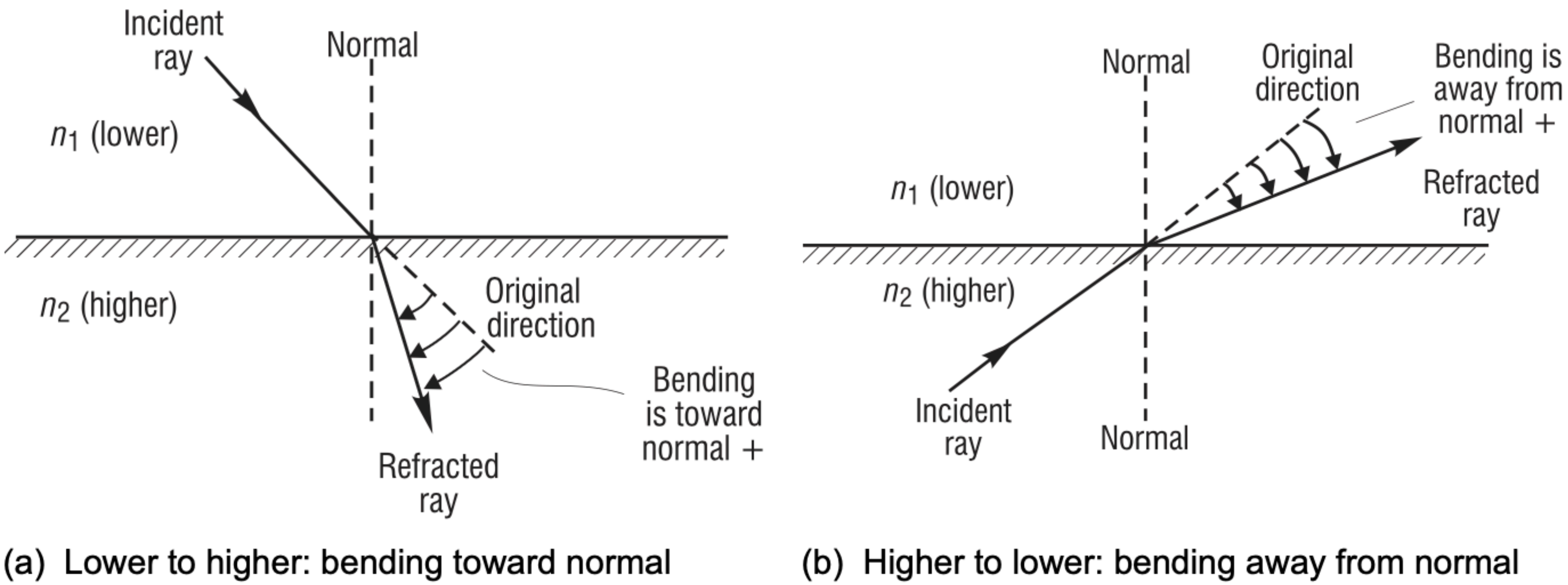


Figure 8 Refraction at an interface between media of refractive indexes n1 and n2

2. Snell's law.

Snell's law of refraction relates the sines of the angles of incidence and refraction at an interface between two optical media to the indexes of refraction of the two media.

The law is named after a Dutch astronomer, Willebrord Snell, who formulated the law in the 17th century.

Snell's law enables us to calculate the direction of the refracted ray if we know the refractive indexes of the two media and the direction of the incident ray.

The mathematical expression of Snell's law and an accompanying drawing are given in Figure 9.

Snell's Law

$$\frac{\sin\ i}{\sin\ r} = \frac{n_r}{n_i}$$, where

I is the angle of incidence

r is the angle of refraction

n_i is the index in the incident medium

n_r is the index in the refracting medium

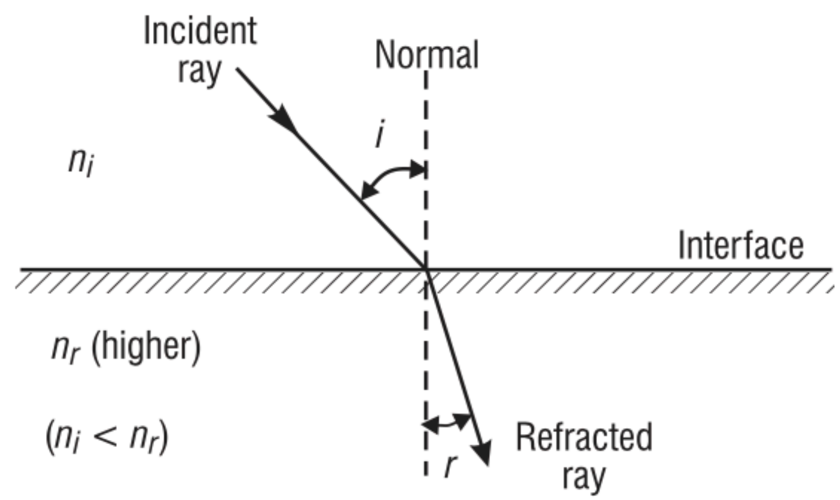


Figure 3-9 Snell's law: formula and geometry

Note carefully that both the angle of incidence (i) and refraction (r) are measured with respect to the surface normal.

Note also that the incident ray, normal, and refracted ray all lie in the same geometrical plane.

In practice Snell's law is often written simply as

$$\boxed{n_i \sin i = n_r \sin r} \qquad (2)$$

Now let's look at an example that make use of Snell's law

Example 2

In a handheld optical instrument used under water, light is incident from water onto the plane surface of flint glass at an angle of incidence of 45°.

(a) What is the angle of reflection of light off the flint glass?

(b) Does the refracted ray bend toward or away from the normal?

(c) What is the angle of refraction in the flint glass?

Solution:

(a) From the law of reflection, the reflected light must head off at an angle of 45° with the normal. (Note: the angle of reflection is not dependent on the refractive indexes of the two media.)

(b) From Table 3-1, the index of refraction is 1.33 for water and 1.63 for flint glass.

Thus, light is moving from a lower to a higher index of refraction and will bend toward the normal.

We know then that the angle of refraction r should be less than 45°.

(c) From Snell's law, Equation 2, we have:

$$n_i \sin i = n_r \sin r$$

where $n_i = 1.33$, $i = 45°$, and $n_i = 1.63$

Thus, $\sin r = \dfrac{1.33 \sin 45°}{1.63} = \dfrac{(1.33)(0.707)}{1.63} = 0.577$

So $r = \sin^{-1}(0.577) = 35.2°$

The angle of refraction is about 35°, clearly less than 45°, just as was predicted in part (b).

3. Critical angle and total internal reflection.

When light travels from a medium of higher index to one of lower index, we encounter some interesting results.

Refer to Figure 10, where we see four rays of light originating from point O in the higher-index medium, each incident on the interface at a different angle of incidence.

Ray 1 is incident on the interface at 90° (normal incidence) so there is no bending

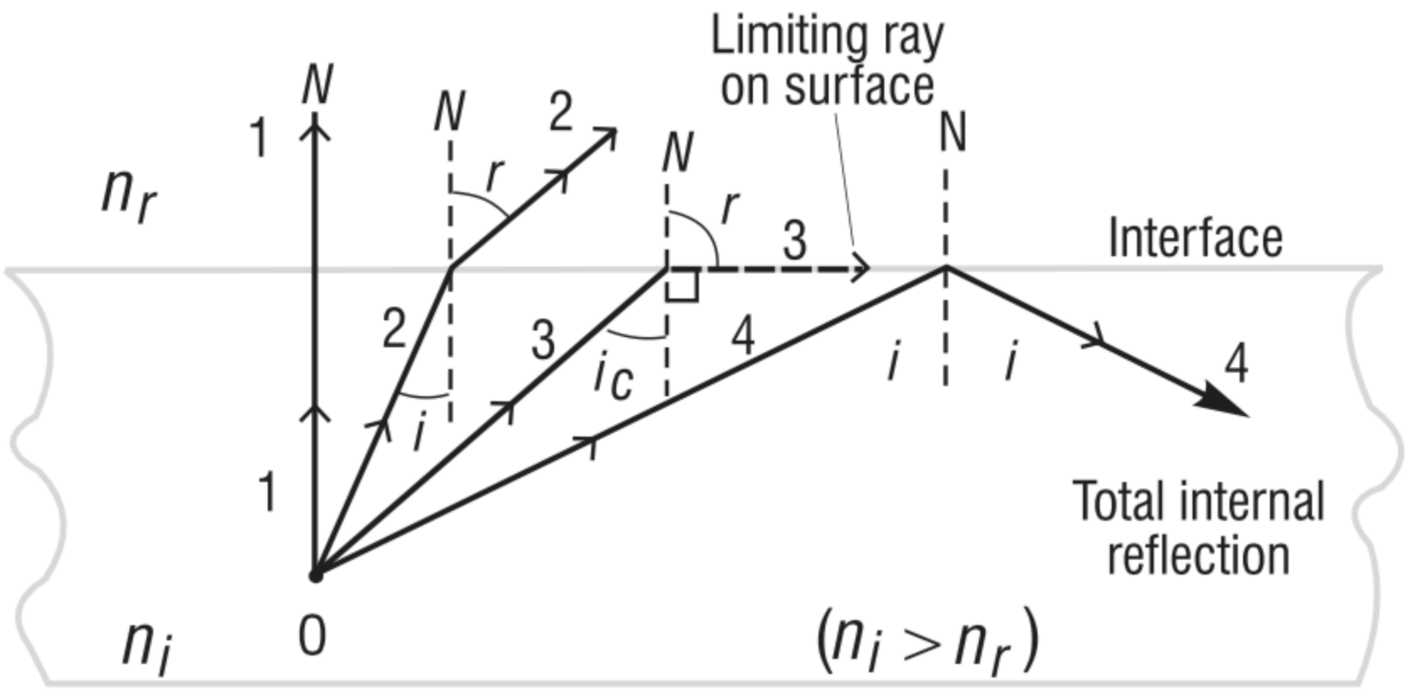


Figure 10 Critical angle and total internal reflection

The light in this direction simply speeds up in the second medium (why?) but continues along the same direction.

Ray 2 is incident at angle i and refracts (bends away from the normal) at angle r.

Ray 3 is incident at the critical angle i_c, large enough to cause the refracted ray bending away from the normal (N) to bend by 90°, thereby traveling along the interface between the two media. (This ray is trapped in the interface.)

Ray 4 is incident on the interface at an angle greater than the critical angle, and is totally reflected into the same medium from which it came.

Ray 4 obeys the law of reflection so that its angle of reflection is exactly equal to its angle of incidence.

We exploit the phenomenon of total internal reflection when designing light propagation in fibers by trapping the light in the fiber through successive internal reflections along the fiber.

We do this also when designing "retroreflecting" prisms.

Compared with ordinary reflection from mirrors, the sharpness and brightness of totally internally reflected light beams is enhanced considerably

The calculation of the critical angle of incidence for any two optical media—whenever light is incident from the medium of higher index—is accomplished with Snell's law.

Referring to Ray 3 in Figure10 and using Snell's law in Equation 2 appropriately, we have

$$n_i \sin i_c = n_r \sin 90°$$

where n_i is the index for the incident medium, i_c is the critical angle of incidence, n_r is the index for the medium of lower index, and r = 90° is the angle of refraction at the critical angle.

Then, since sin 90° = 1, we obtain for the critical angle,

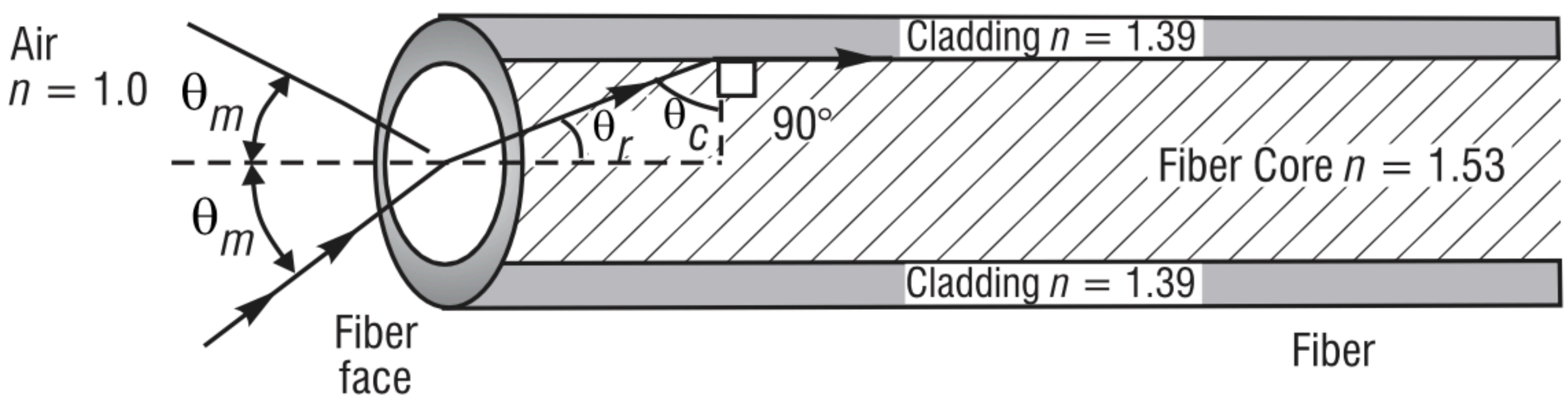
$$\boxed{i_c = \sin^{-1}\left(\frac{n_r}{n_i}\right)} \tag{3}$$

Let's use this result and Snell's law to determine the entrance cone for light rays incident on the face of a clad fiber if the light is to be trapped by total internal reflection at the core-cladding interface in the fiber.

Example 3
A step-index fiber 0.0025 inch in diameter has a core index of 1.53 and a cladding index of 1.39.

See drawing.

Such clad fibers are used frequently in applications involving communication, sensing, and imaging.



What is the maximum acceptance angle θ_m for a cone of light rays incident on the fiber face such that the refracted ray in the core of the fiber is incident on the cladding at the critical angle?

Solution: First find the critical angle θ_c in the core, at the core-cladding interface.

Then, from geometry, identify θ_r and use Snell's law to find θ_m.

(1) From Equation 3-3, at the core-cladding interface

$$\theta_c = \sin^{-1}\left(\frac{1.39}{1.53}\right) = 65.3°$$

(2) From right-triangle geometry, $\theta_r = 90 - 65.3 = 24.7°$

(3) From Snell's law, at the fiber face,

$$n_{air} \sin \theta_m = n_{core} \sin \theta_r$$

and $$\sin \theta_m = \left(\frac{n_{core}}{n_{air}}\right) \sin \theta_r = \left(\frac{1.53}{1.00}\right) \sin (24.7°)$$

from which $$\sin \theta_m = 0.639$$

and $$\theta_m = \sin^{-1} 0.639 = 39.7°$$

Thus, the maximum acceptance angle is 39.7° and the acceptance cone is twice that, or $2\theta_m = 79.4°$.

The acceptance cone indicates that any light ray incident on the fiber face within the acceptance angle will undergo total internal reflection at the core-cladding face and remain trapped in the fiber as it propagates along the fiber.

D. Refraction in prisms

Glass prisms are often used to bend light in a given direction as well as to bend it back again (retroreflection).

The process of refraction in prisms is understood easily with the use of light rays and Snell's law.

Look at Figure 11a.

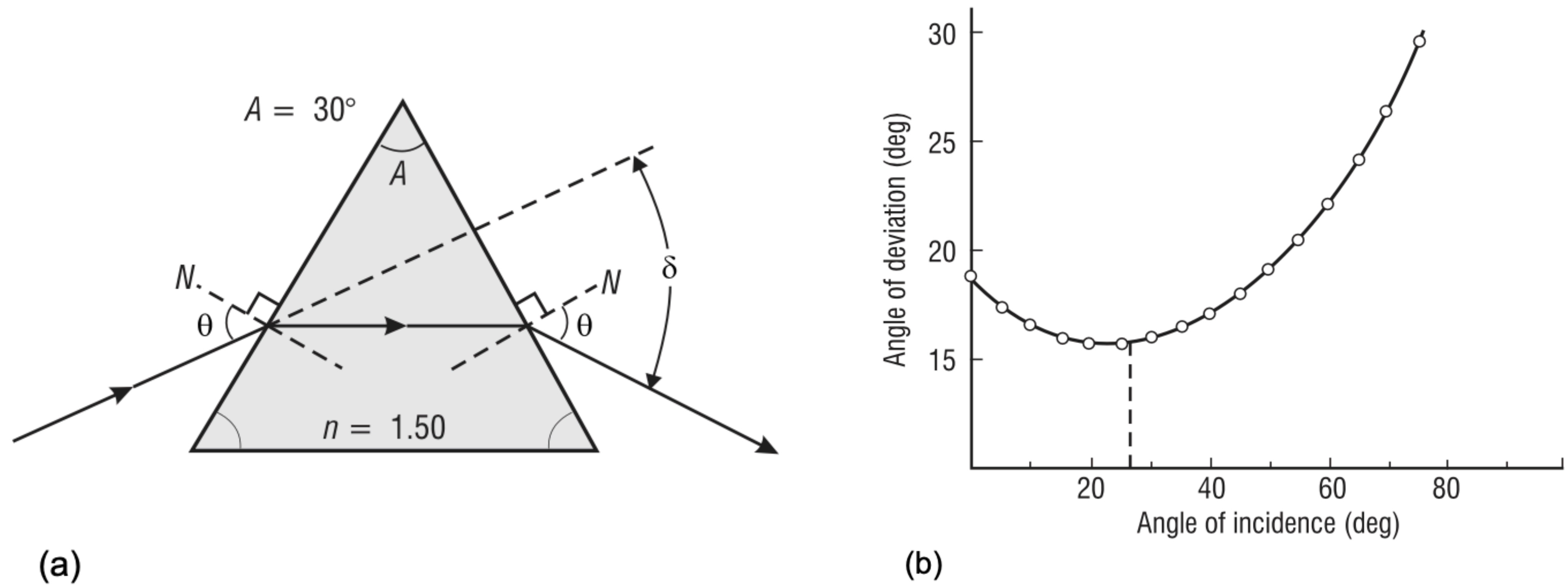


Figure 11 Refraction of light through a prism

When a light ray enters a prism at one face and exits,at another, the exiting ray is deviated from its original direction.

The prism shown is isosceles in cross section with apex angle A = 30° and refractive index n = 1.50.

The incident angle θ and the angle of deviation δ are shown on the diagram.

Figure 11b shows how the angle of deviation δ changes as the angle θ of the incident ray changes.

The specific curve shown is for the prism described in Figure 11a.

Note that δ goes through a minimum value, about 23° for this specific prism.

Each prism material has its own unique minimum angle of deviation.

1. Minimum angle of deviation.

It turns out that we can determine the refractive index of a transparent material by shaping it in the form of an isosceles prism and then measuring its minimum angle of deviation.

With reference to Figure 11a, the relationship between the refractive index n, the prism apex angle A, and the minimum angle of deviation δ_m is given by

$$\boxed{n = \frac{\sin\left(\dfrac{A+\delta_m}{2}\right)}{\sin\dfrac{A}{2}}} \tag{4}$$

where both A and δ_m are measured in degrees.

The derivation of Equation 4 is straightforward, but a bit tedious.

Details of the derivation — making use of Snell's law and geometric relations between angles at each refracting surface — can be found in most standard texts on geometrical optics.

Let's show how one can use Equation 4 in Example 4 to determine the index of refraction of an unknown glass shaped in the form of a prism.

Example 4

A glass of unknown index of refraction is shaped in the form of an isosceles prism with an apex angle of 25°.

In the laboratory, with the help of a laser beam and a prism table, the minimum angle of deviation for this prism is measured carefully to be 15.8°.

What is the refractive index of this glass material?

Solution: Given that $\delta_m = 15.8°$ and A = 25°, we use Equation 4 to calculate the refractive index.

$$n = \frac{\sin\left(\frac{A+\delta_m}{2}\right)}{\sin\left(\frac{A}{2}\right)} = \frac{\sin\left(\frac{25°+15.8°}{2}\right)}{\sin\left(\frac{25°}{2}\right)} = \frac{\sin(20.4°)}{\sin(12.5°)} = \frac{0.3486}{0.2164}$$

$$n = 1.61$$

2. Dispersion of light.

Table 3-1 lists indexes of refraction for various substances independent of the wavelength of the light.

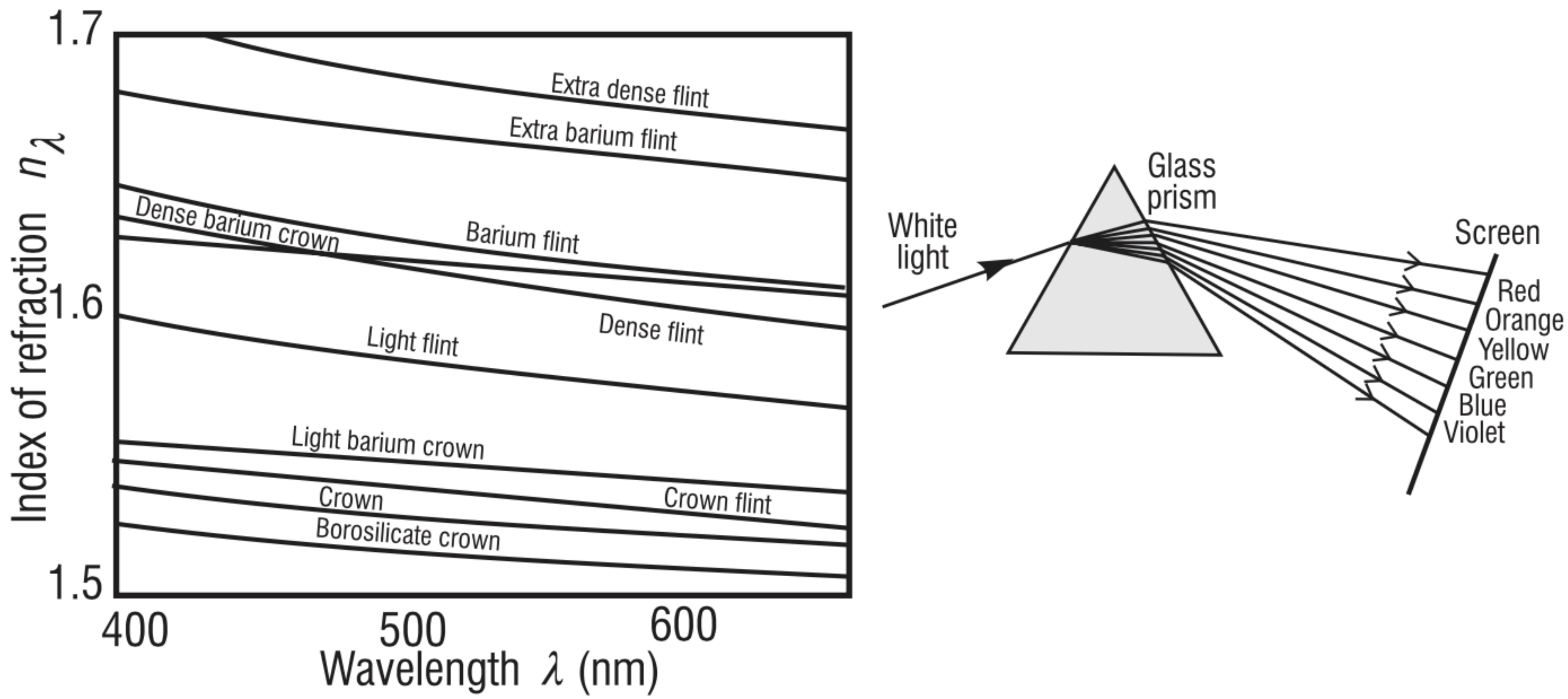
In fact, the refractive index is slightly wavelength dependent.

For example, the index of refraction for flint glass is about 1% higher for blue light than for red light.

The variation of refractive index n with wavelength λ is called dispersion.

Figure 12a shows a normal dispersion curve of n_λ versus λ for different types of optical glass.

Figure 12b shows the separation of the individual colors in white light—400 nm to 700 nm—after passing through a prism.



(a) Refraction by a prism (b) Optical glass dispersion curves

Figure 12 Typical dispersion curves and separation of white light after refraction by a prism

Note that n_λ decreases from short to long wavelengths, thus causing the red light to be less deviated than the blue light as it passes through a prism.

This type of dispersion that accounts for the colors seen in a rainbow, the "prism" there being the individual raindrops.

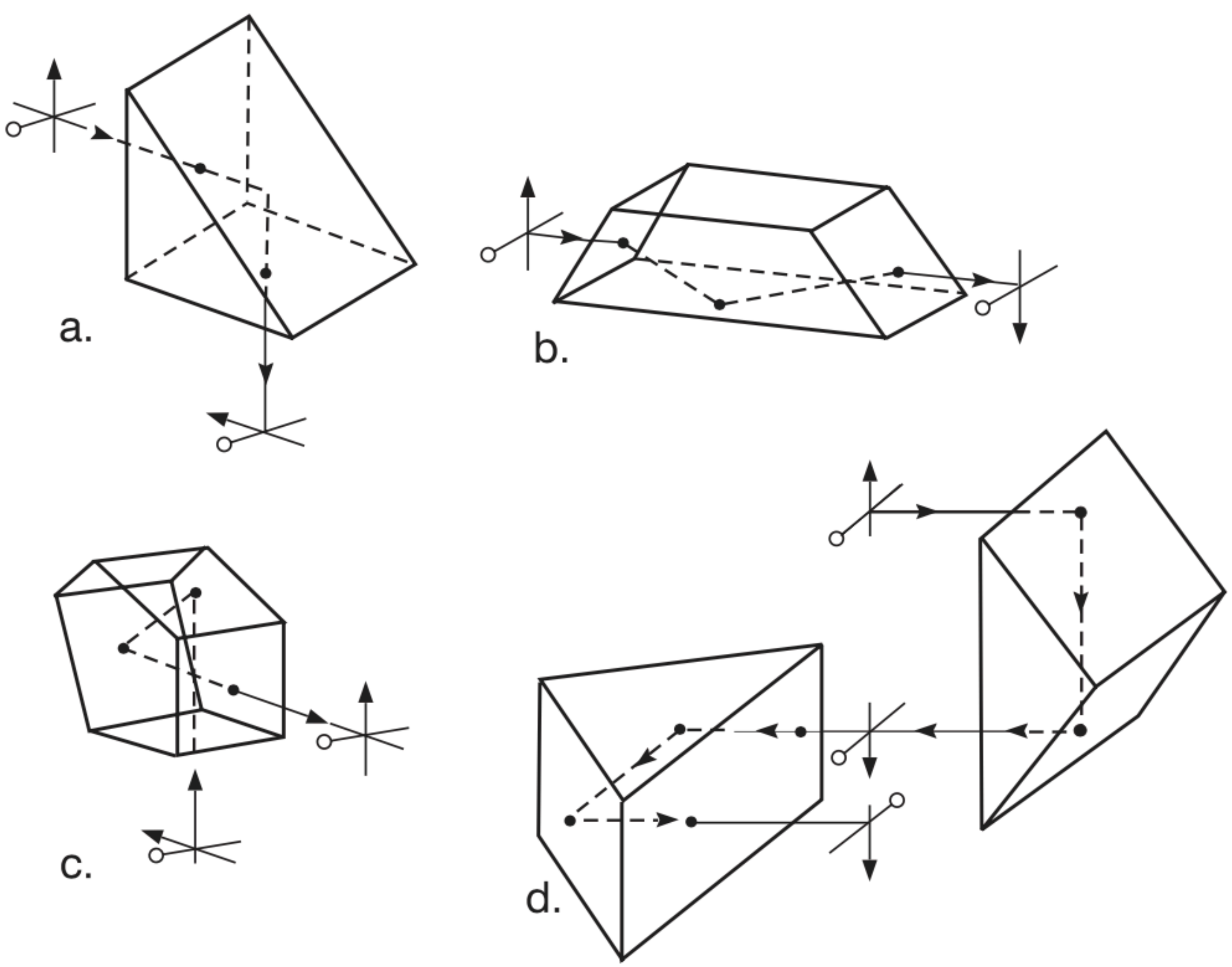
3. Special applications of prisms.

Prisms that depend on total internal reflection are commonly used in optical systems, both to change direction of light travel and to change the orientation of an image.

While mirrors can be used to achieve similar ends, the reflecting faces of a prism are easier to keep free of contamination and the process of total internal reflection is capable of higher reflectivity.

Some common prisms in use today are shown in Figure 13, with details of light redirection and image reorientation shown for each one.

If, for example, the Dove prism in Figure 13b is rotated about its long axis, the image will also be rotated.



(a) Right-angle prism
(b) Dove prism
(c) Penta prism
(d) Porro prism

Figure 13 Image manipulation with refracting prisms

The Porro prism, consisting of two right-angle prisms, is used in binoculars, for example, to produce erect final images and, at the same time, permit the distance between the object-viewing lenses to be greater than the normal eye-to-eye distance, thereby enhancing the stereoscopic effect produced by ordinary binocular vision

II. IMAGE FORMATION WITH MIRRORS

Mirrors, of course, are everywhere—in homes, auto headlamps, astronomical telescopes, and laser cavities, and many other places.

Plane and spherical mirrors are used to form three-dimensional images of three-dimensional objects.

If the size, orientation, and location of an object relative to a mirror are known, the law of reflection and ray tracing can be used to locate the image graphically.

Appropriate mathematical formulas can also be used to calculate the locations and sizes of the images formed by mirrors.

In this section we shall use both graphical ray tracing and formulas

A. Images formed with plane mirrors

Images with mirrors are formed when many nonparallel rays from a given point on a source are reflected from the mirror surface, converge, and form a corresponding image point.

When this happens, point by point for an extended object, an image of the object, point by point, is formed.

Image formation in a plane mirror is illustrated in several sketches shown in Figure 14.

In Figure 14a, point object S sends nonparallel rays toward a plane mirror, which reflects them as shown.

The law of reflection ensures that pairs of triangles like SNP and S′NP are equal, so that all reflected rays appear to originate at the image point S′ , which lies along the normal line SN, and at such depth that the image distance S′N equals the object distance SN.

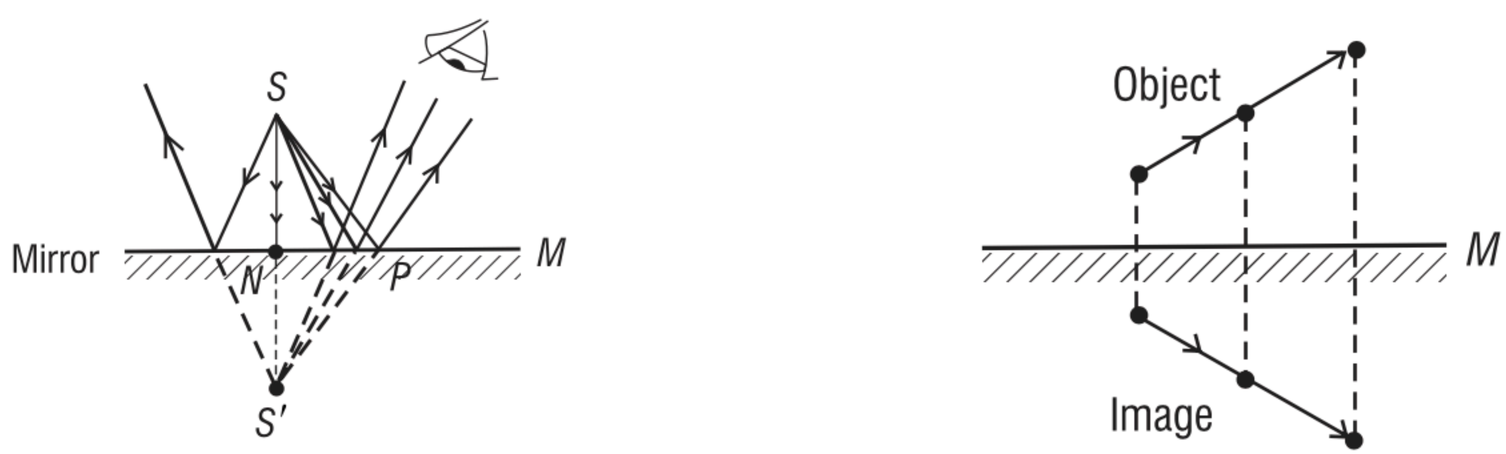
The eye sees a point image at S′ in exactly the same way it would see a real point object placed there.

Since the actual rays do not exist below the mirror surface, the image is said to be a virtual image.

The image S′ cannot be projected on a screen as in the case of a real image. An extended object, such as the arrow in Figure 14b, is imaged point by point by a plane mirror surface in similar fashion.

Each object point has its image point along its normal to the mirror surface and as far below the reflecting surface as the object point lies above the surface.

Note that image position does not depend on the position of the eye.

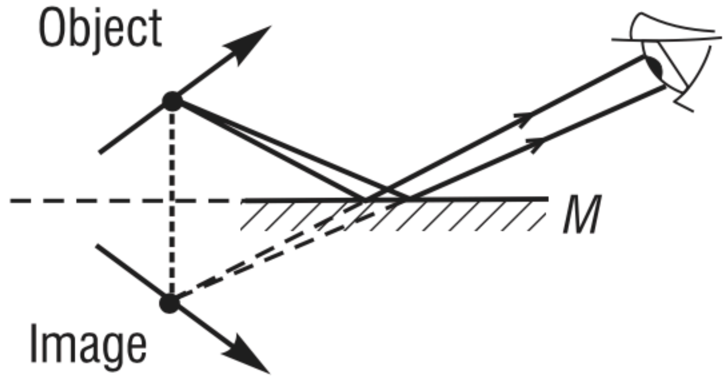


(a) Imaging a point surface (b) Imaging an extended object

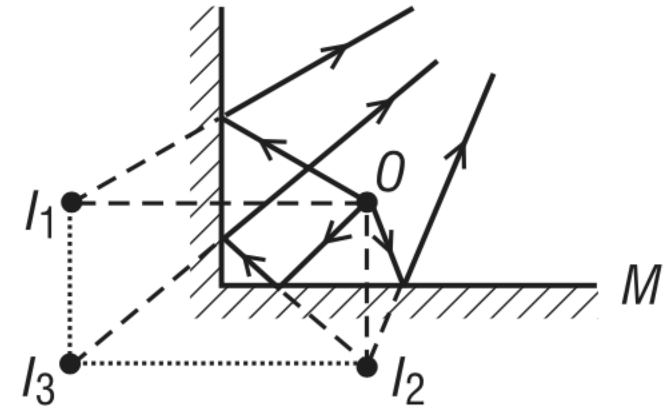
The construction in Figure 14b also makes clear that the image size is identical to the object size, giving a magnification of unity.

In addition, the transverse orientations of object and image are the same.

A right-handed object, however, appears left-handed in its image.



(c) Image is same size as object.



(d) Multiple images of point with inclined mirrors

In Figure 14c, where the mirror does not lie directly below the object, the mirror plane may be extended to determine the position of the image as seen by an eye positioned to receive reflected rays originating at the object.

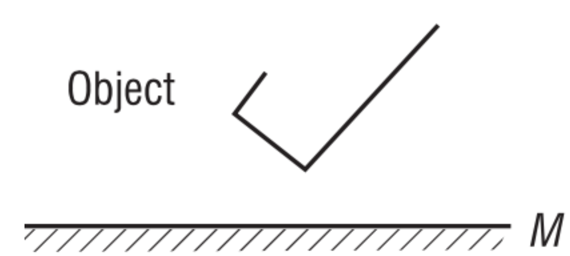
Figure 14d illustrates multiple images of a point object O formed by two perpendicular mirrors.

Each image, I_1 and I_2, results from a single reflection in one of the two mirrors, but a third image I_3 is also present, formed by sequential reflections from both mirrors.

All parts of Figure 14 and the related discussion above should be understood clearly because they are fundamental to the optics of images.

Example 5

Making use of the law of reflection and the conclusions drawn from Figure 14, draw the image of the letter L positioned above a plane mirror as shown below in (a).

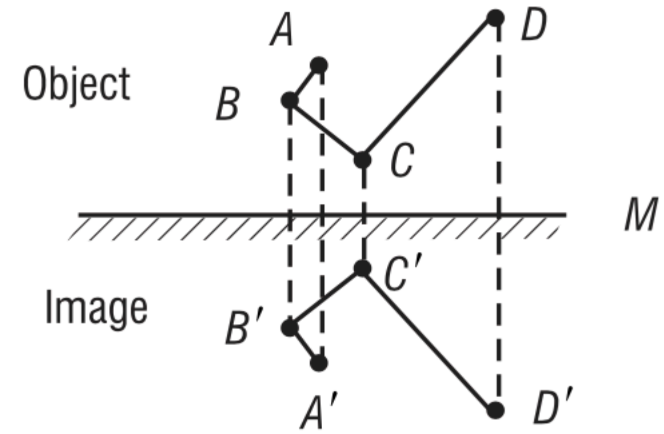


(a) Object

Solution: Make use of the fact that each point on the image is as far below the mirror—along a line perpendicular to the mirror—as the actual object point is above the mirror.

Indicate key points on the object and locate corresponding points on the image.

Sketch in the image as shown in (b).



(b) Image trace

B. Images formed with spherical mirrors

As we showed earlier in Figure 6, the law of reflection can be used to determine the direction along which any ray incident on a spherical mirror surface will be reflected.

Using the law of reflection, we can trace rays from any point on an object to the mirror, and from there on to the corresponding image point.

This is the method of graphical ray tracing.

1. Graphical ray-trace method.

To employ the method of ray tracing, we agree on the following:

- Light will be incident on a mirror surface *initially* from the left.
- The axis of symmetry normal to the mirror surface is its *optical axis*.
- The point where the optical axis meets the mirror surface is the *vertex*.

To locate an image we use two points common to each mirror surface, the center of curvature C and the focal point F.

They are shown in Figure 15, with the mirror vertex V, for both a concave and a convex spherical mirror.

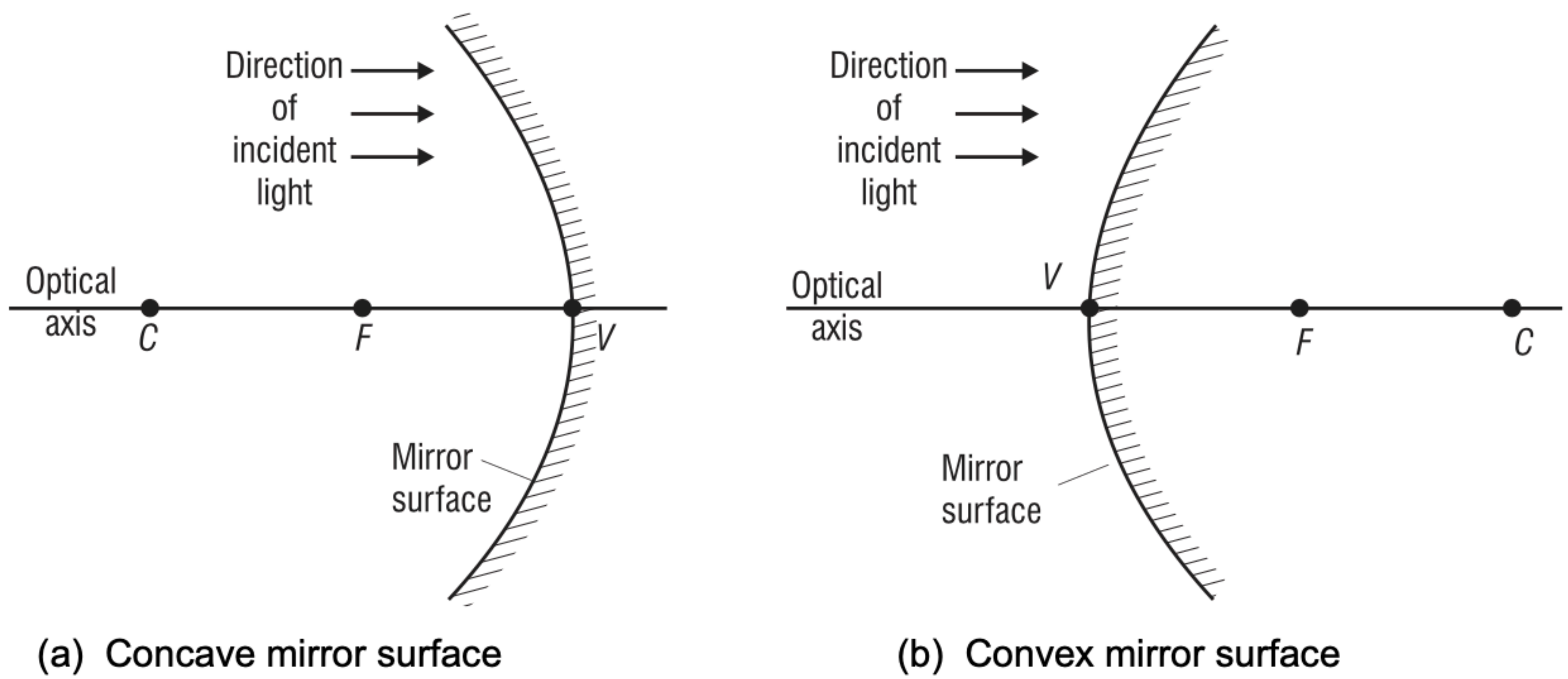


Figure 15 Defining points for concave and convex mirrors

The edges of concave mirrors always bend toward the oncoming light.

Such mirrors have their center of curvature C and focal point F located to the left of the vertex as seen in Figure 15a.

The edges of convex mirrors always bend away from the oncoming light, and their center of curvature C and focal point F are located to the right of the vertex as seen in Figure 15b.

The important connection between parallel rays and the focal points for mirror surfaces is shown in Figure 16 a, b.

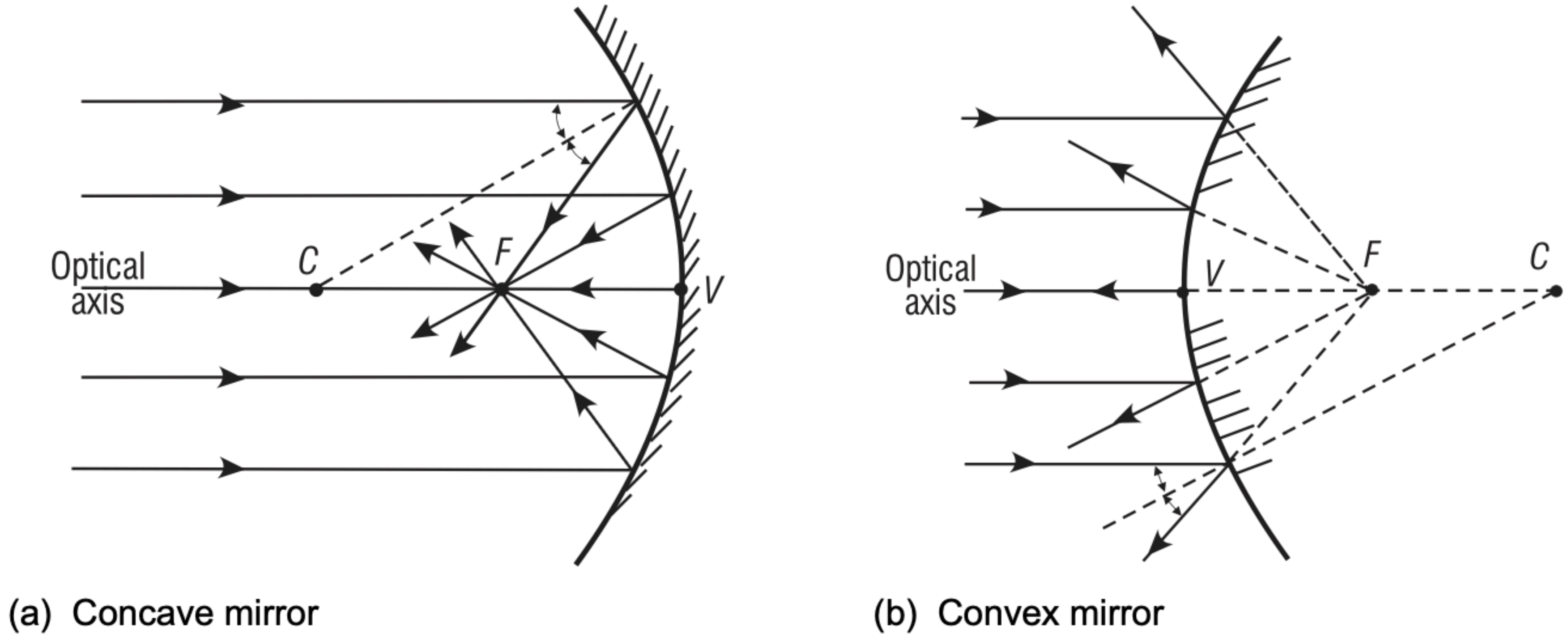


Figure 16 Parallel rays and focal points

Parallel rays are light rays coming from a very distant source (such as the sun) or from a collimated laser beam.

The law of reflection, applied at each point on the mirror surface where a ray is incident, requires that the ray be reflected so as to pass through a focal point F in front of the mirror (Figure 16a) or be reflected to appear to come from a focal point F behind the mirror (Figure 16b).

Notice that a line drawn from the center of curvature C to any point on the mirror is a normal line and thus bisects the angle between the incident and reflected rays.

As long as the transverse dimension of the mirror is not too large, simple geometry shows that the point F, for either mirror, is located at the midpoint between C and F, so that the distance FV is one-half the radius of curvature CV. The distance FV is called the focal length and is commonly labeled as f.

2. Key rays used in ray tracing.

Figure 17 shows three key rays—for each mirror—that are used to locate an image point corresponding to a given object point.

They are labeled 1, 2, and 3.

Any two, drawn from object point P, will locate the corresponding image point P′.

In most cases it is sufficient to locate one point, like P′, to be able to draw the entire image.

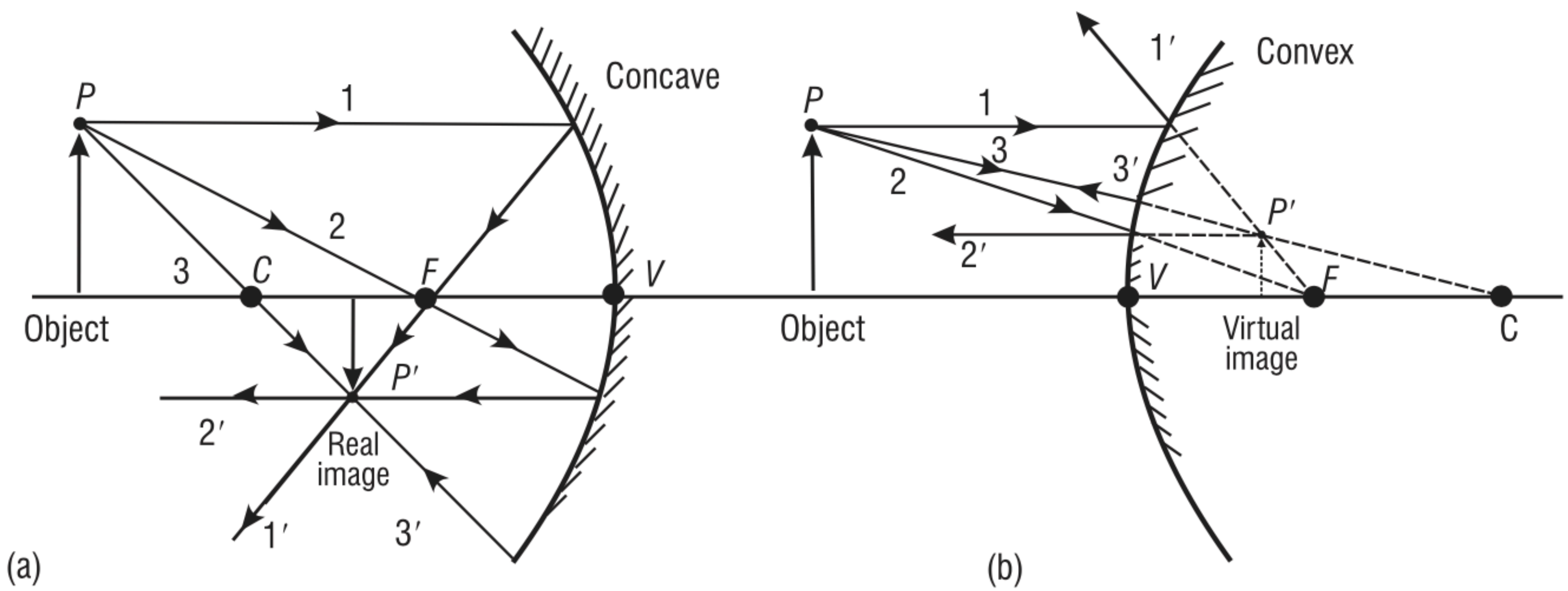
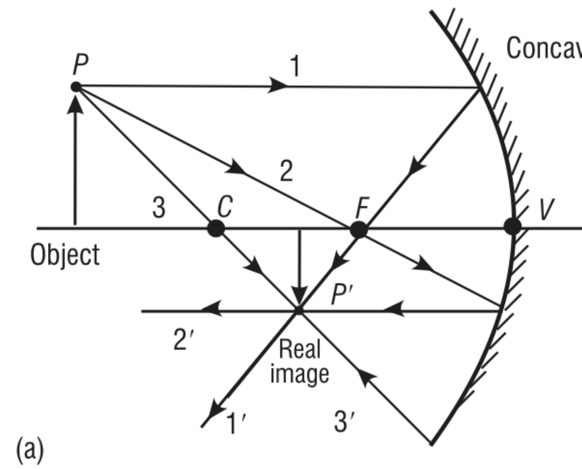


Figure 17 Key rays for graphical ray tracing with spherical mirrors

Note carefully, with reference to Figure 17a, b, the following facts:

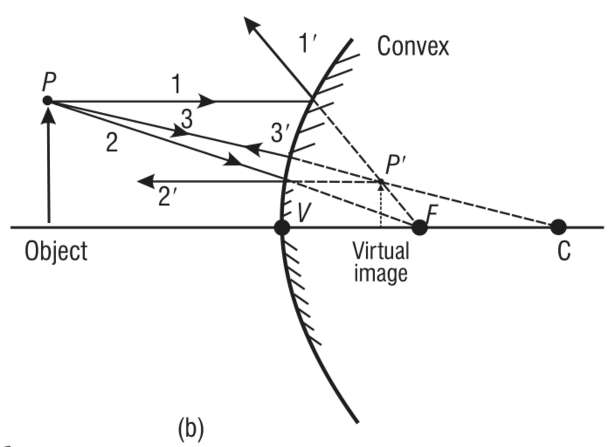
For a concave mirror:

- The ray from object point P parallel to the axis, such as ray 1, reflects from the mirror and *passes through the focal point F* (labeled ray 1′).
- The ray from P passing through the focal point F, such as ray 2, reflects from the mirror *as a ray parallel to the axis* (labeled ray 2′).
- The ray from P passing through the center of curvature C, such as ray 3, *reflects back along itself* (labeled ray 3′).
- Reflected rays 1′, 2′, and 3′ converge to locate point P' on the image. This image is a *real* image that can be formed on a screen located there.



For a convex mirror:

- The ray from object point P, parallel to the axis, such as ray 1, reflects from the mirror *as if to come from the focal point F behind* the mirror (labeled ray 1′).
- The ray from P, such as ray 2, headed toward the focal point F behind the mirror, reflects from the mirror in a direction parallel to the optical axis (labeled ray 2′).
- The ray from P, such as ray 3, headed toward the center of curvature C behind the mirror, reflects back along itself (labeled ray 3′).
- Rays 1′, 2′, and 3′ diverge after reflection. A person looking toward the mirror intercepts the diverging rays and sees them appearing to come from their common intersection point P', behind the mirror. The image is *virtual* since it cannot be formed on a screen placed there.



Example 6

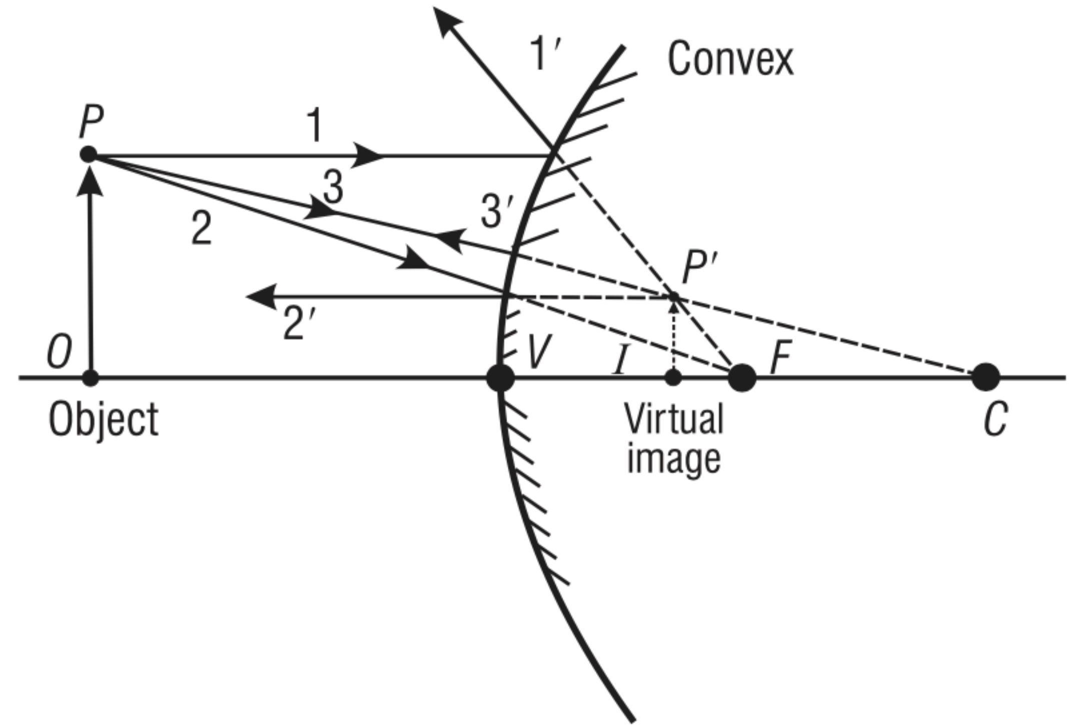
The passenger-side mirror on an automobile is a convex mirror.

It provides the driver with a wide field of view, but significantly reduced images.

Assume that object OP is part of an automobile trailing the driver's car.

See diagram below.

Use three key rays to locate the reduced, virtual image of the trailing auto.



Solution: Using key rays 1, 2, and 3 incident on the mirror from point P on object OP, in conjunction with points C and F, draw the appropriate reflected rays, as show below, to locate P′ on image IP′
.

The three reflected rays 1′, 2′, and 3′ diverge after reflection.

They appear to come from a common point P′ behind the mirror.

This locates virtual image IP′, reduced in size, about one-third as large as object OP.

As a result, drivers are always cautioned that images seen in the passenger-side mirror are actually NEARER than they appear to be.

C. Mirror formulas for image location

In place of the graphical ray-tracing methods described above, we can use formulas to calculate the image location.

We shall derive below a "mirror formula" and then use the formula to determine image location.

The derivation is typical of those found in geometrical optics, and is instructive in its combined use of algebra, geometry, and trigonometry.

1. Derivation of the mirror formula.

The drawing we need to carry out the derivation is shown in Figure 18.

The important quantities are the object distance p, the image distance q, and the radius of curvature r.

Both p and q are measured relative to the mirror vertex, as shown, and the sign on r will indicate whether the mirror is concave or convex.

All other quantities in Figure 18 are used in the derivation but will not show up in the final "mirror formula"

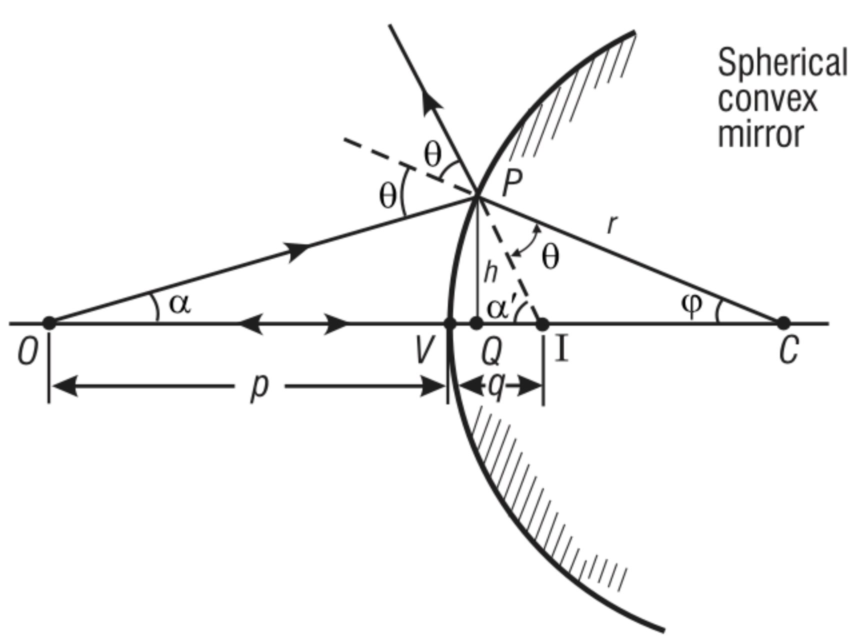
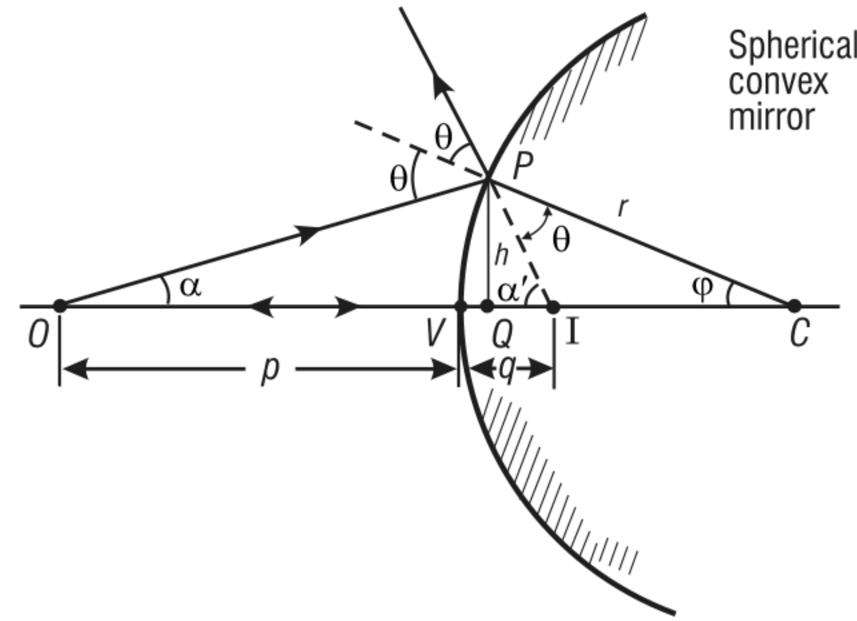


Figure 18 Basic drawing for deriving the mirror formula

The mirror shown in Figure 18 is convex with center of curvature C on the right.



Two rays of light originating at object point O are drawn, one normal to the convex surface at its vertex V and the other an arbitrary ray incident at P.

The first ray reflects back along itself; the second reflects at P as if incident on a plane tangent at P, according to the law of reflection.

Relative to each other, the two reflected rays diverge as they leave the mirror.

The intersection of the two rays (extended backward) determines the image point I corresponding to object point O.

The image is virtual and located behind the mirror surface.

Object and image distances measured from the vertex V are shown as p and q, respectively.

A perpendicular of height h is drawn from P to the axis at Q.

We seek a relationship between p and q that depends on only the radius of curvature r of the mirror.

As we shall see, such a relation is possible only to a first-order approximation of the sines and cosines of angles such as α and ϕ made by the object and image rays at various points on the spherical surface.

This means that, in place of expansions of $\sin\phi$ and $\cos\phi$ in series as shown here,

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots$$

we consider the first terms only and write

$$\sin\varphi \cong \varphi \quad \text{and} \quad \cos\varphi \cong 1, \quad \text{so that} \quad \tan\varphi = \frac{\sin\varphi}{\cos\varphi} \cong \varphi$$

These relations are accurate to 1% or less if the angle ϕ is 10° or smaller.

This approximation leads to first-order (or Gaussian) optics, after Karl Friedrich Gauss, who in 1841 developed the foundations of this subject.

Returning now to the problem at hand—that of relating p, q, and r—notice that two angular relationships may be obtained from Figure 18, because the exterior angle of a triangle equals the sum of its interior angles.

Thus,

$$\theta = \alpha + \varphi \text{ in } \Delta OPC \quad \text{and} \quad 2\theta = \alpha + \alpha' \quad \text{in } \Delta OPI$$

which combine to give

$$\alpha - \alpha' = 2\varphi$$

Using the small-angle approximation, the angles α, α', and ϕ above can be replaced by their tangents, yielding

$$\frac{h}{p} - \frac{h}{q} = -2\frac{h}{r}$$

Note that we have neglected the axial distance VQ, small when ϕ is small.

Cancellation of h produces the desired relationship,

$$\frac{1}{p} - \frac{1}{q} = -\frac{2}{r} \tag{5}$$

If the spherical surface is chosen to be concave instead, the center of curvature will be to the left.

For certain positions of the object point O, it is then possible to find a real image point, also to the left of the mirror.

In these cases, the resulting geometric relationship analogous to Equation 3-5 consists of the same terms, but with different algebraic signs, depending on the sign convention employed.

We can choose a sign convention that leads to a single equation, the mirror equation, valid for both types of mirrors.

It is Equation 6.

$$\boxed{\frac{1}{p} + \frac{1}{q} = -\frac{2}{r}} \tag{6}$$

2. Sign convention.

The sign convention to be used in conjunction with Equation 6 and Figure 18 is as follows.

- Object and image distances *p* and *q* are both *positive* when located to the *left* of the vertex and both *negative* when located to the *right*.
- The radius of curvature *r* is *positive* when the center of curvature *C* is to the *left* of the vertex (concave mirror surface) and *negative* when *C* is to the *right* (convex mirror surface).
- Vertical dimensions are positive above the optical axis and negative below.

In the application of these rules, light is assumed to be directed initially, as we mentioned earlier, from left to right.

According to this sign convention, positive object and image distances correspond to real objects and images, and negative object and image distances correspond to virtual objects and images.

Virtual objects occur only with a sequence of two or more reflecting or refracting elements.

3. Magnification of a mirror image.

Figure 19 shows a drawing from which the magnification—ratio of image height h_i to object height h_o—can be determined.

Since angles θ_i, θ_r, and α are equal, it follows that triangles VOP and VIP′ are similar.

Thus, the sides of the two triangles are proportional and one can write

$$\frac{h_i}{h_o} = \frac{q}{p}$$

This gives at once the magnification m to be

$$m \equiv \frac{h_i}{h_o} = \frac{q}{p}$$

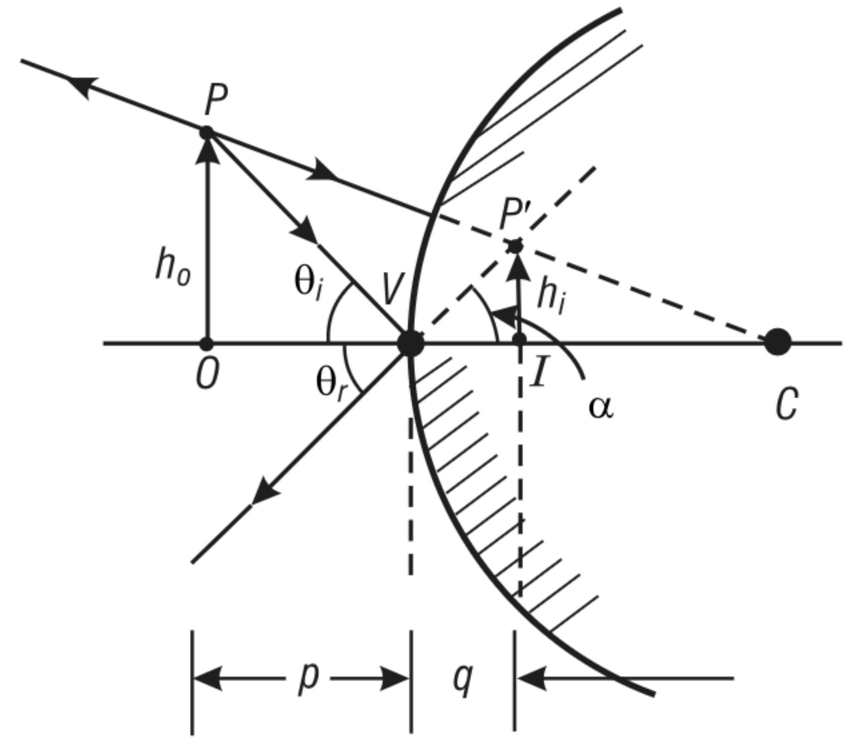


Figure 19 Construction for derivation of mirror magnification formula

When the sign convention is taken into account, one has, for the general case, a single equation, Equation 7, valid for both convex and concave mirrors.

$$\boxed{m = -\frac{q}{p}} \tag{7}$$

If, after calculation, the value of m is positive, the image is erect.

If the value is negative, the image is inverted.

Let us now use the mirror formulas in Equations 6 and 7, and the sign convention, to locate an image and determine its size.

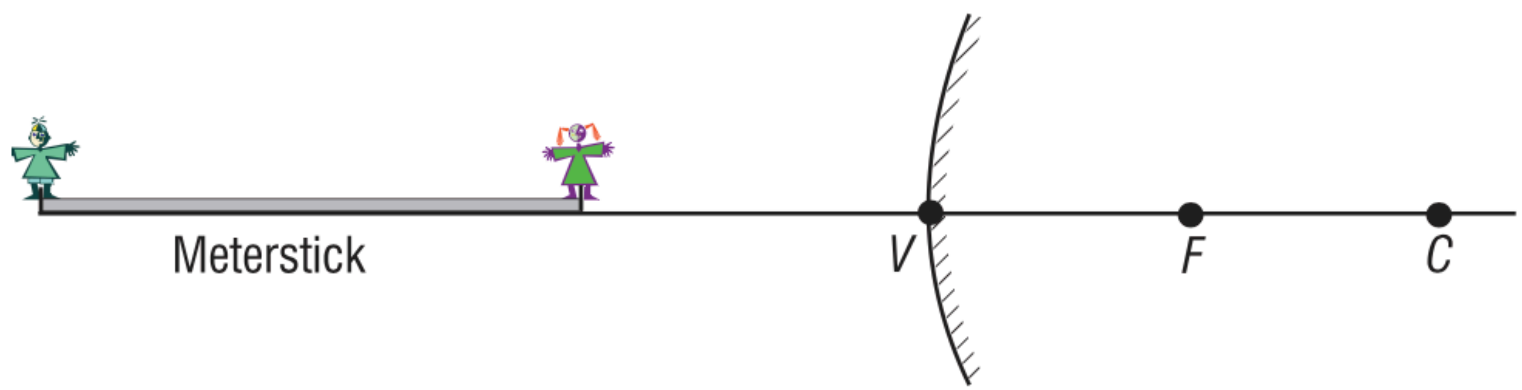
Example 7

A meterstick lies along the optical axis of a convex mirror of focal length 40 cm, with its near end 60 cm from the mirror surface.

Five-centimeter toy figures stand erect on both the near and far ends of the meterstick.

(a) How long is the virtual image of the meterstick?

(b) How tall are the toy figures in the image, and are they erect or inverted?



Solution: Use the mirror equation $\frac{1}{p} + \frac{1}{q} = -\frac{2}{r}$ twice, once for the near end and once for the far end of the meterstick.

Use the magnification equation $m = -\frac{q}{p}$ for each toy figure.

(a) **Near end:** Sign convention gives $p = +60$ cm, $r = 2f = -(2 \times 40) = -80$ cm

$\therefore \quad \frac{1}{60} + \frac{1}{q_n} = -\frac{2}{80}$, so $q_n = -24$ cm

Negative sign indicates image is *virtual*, 24 cm to the right of V.

Far end: $p = +160$ cm, $r = -80$ cm

$$\frac{1}{160} + \frac{1}{q_f} = -\frac{1}{40}, \text{ so } q_f = -32 \text{ cm}$$

Far-end image is *virtual*, 32 cm to the right of V.

∴ Meterstick image is 32 cm – 24 cm = 8 cm long.

(b) **Near-end toy figure:**

$$m_n = \frac{-q}{p} = \frac{-(-24)}{60} = +0.4 \text{ (Image is erect since m is positive.)}$$

The toy figure is 5 cm × 0.4 = 2 cm tall, at near end of the meterstick image.

Far-end toy figure:

$$m_f = \frac{-q}{p} = \frac{-(-32)}{160} = +0.2 \quad \text{(Image is erect since m is positive.)}$$

The toy figure is 5 cm × 0.2 = 1 cm tall, at far end of the meterstick image.

III. IMAGE FORMATION WITH LENSES

Lenses are at the heart of many optical devices, not the least of which are cameras, microscopes, binoculars, and telescopes.

Just as the law of reflection determines the imaging properties of mirrors, so Snell's law of refraction determines the imaging properties of lenses.

Lenses are essentially light-controlling elements, used primarily for image formation with visible light, but also for ultraviolet and infrared light.

In this section we shall look first at the types and properties of lenses, then use graphical ray-tracing techniques to locate images, and finally use mathematical formulas to locate the size, orientation, and position of images in simple lens systems.

A. Function of a lens

A lens is made up of a transparent refracting medium, generally of some type of glass, with spherically shaped surfaces on the front and back.

A ray incident on the lens refracts at the front surface (according to Snell's law) propagates through the lens, and refracts again at the rear surface.

Figure 20 shows a rather thick lens refracting rays from an object OP to form an image O′P′.

The ray-tracing techniques and lens formulas we shall use here are based again on Gaussian optics, just as they were for mirrors.

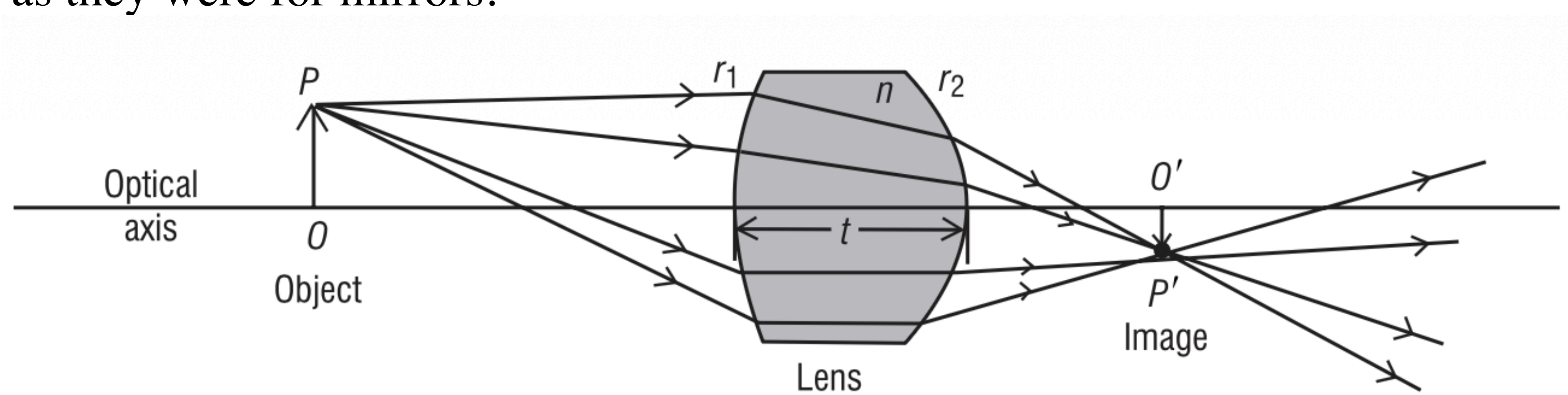


Figure 20 Refraction of light rays by a lens

As we have seen, *Gaussian optics*—sometimes called *paraxial optics*—arises from the basic approximations $\sin\phi \cong \phi$, $\tan\phi \cong \phi$, and $\cos\phi \cong 1$.

These approximations greatly simplify ray tracing and lens formulas, but they do restrict the angles the light rays make with the optical axis to rather small values of 20° or less.

B. Types of lenses

If the axial thickness of a lens is small compared with the radii of curvature of its surfaces, it can be treated as a *thin* lens.

Ray-tracing techniques and lens formulas are relatively simple for thin lenses.

If the thickness of a lens is not negligible compared with the radii of curvature of its faces, it must be treated as a *thick* lens.

Ray-tracing techniques and lens-imaging formulas are more complicated for thick lenses, where computer programs are often developed to trace the rays through the lenses or make surface-by-surface calculations.

In this basic introduction of geometrical optics, we shall deal with only thin lenses.

1. Converging and diverging thin lenses.

In Figure 21, we show the shapes of several common "thin" lenses.

Even though a "thickness" is shown, the use of thin lenses assumes that the rays simply refract at the front and rear faces without a translation through the lens medium.

The first three lenses are thicker in the middle than at the edges and are described as converging or positive lenses

They are converging because they cause parallel rays passing through them to bend toward one another.

Such lenses give rise to positive focal lengths.

The last three lenses are thinner in the middle than at the edges and are described as diverging or negative lenses.

In contrast with converging lenses, they cause parallel rays passing through them to spread as they leave the lens.

These lenses give rise to negative focal lengths.

In Figure 21, names associated with the different shapes are noted.

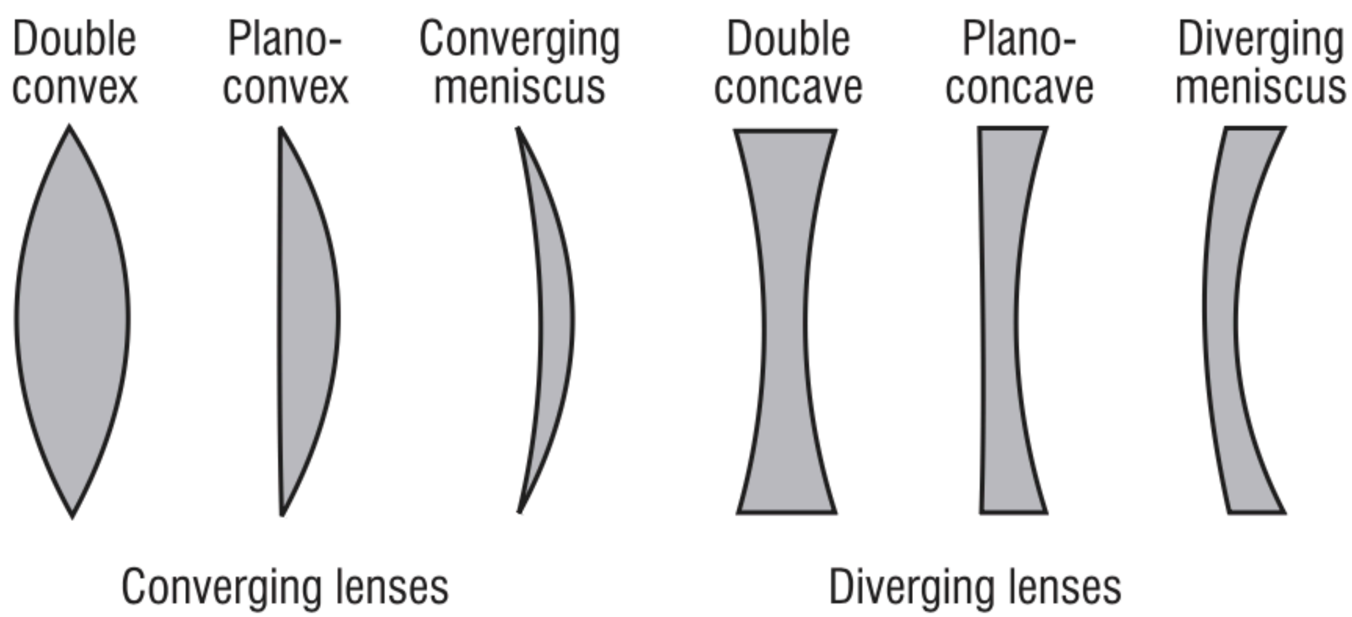


Figure 21 Shapes of common thin lenses

2. Focal points of thin lenses.

Just as for mirrors, the *focal points* of lenses are defined in terms of their effect on parallel light rays and plane wave fronts.

Figure 22 shows parallel light rays and their associated plane wave fronts incident on a *positive* lens (Figure 22a) and a *negative* lens (Figure 22b).

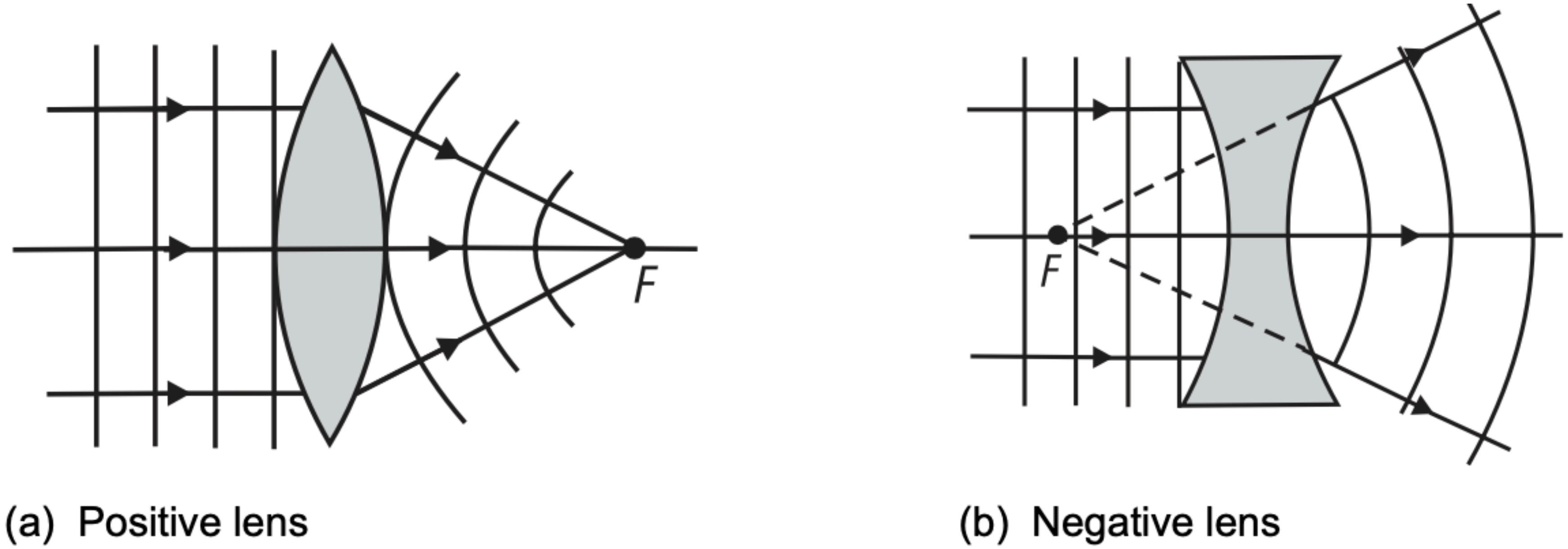


Figure 22 Focal points for positive and negative lenses

For the positive lens, refraction of the light brings it to focal point *F* (real image) to the right of the lens.

For the negative lens, refraction of the light causes it to diverge as if it is coming from focal point *F* (virtual image) located to the left of the lens.

Note how the plane wave fronts are changed to converging spherical wave fronts by the positive lens and to diverging spherical wave fronts by the negative lens.

This occurs because light travels more slowly in the lens medium than in the surrounding air, so the thicker parts of the lens retard the light more than do the thinner parts.

Recall that, for mirrors, there is but a *single* focal point for each mirror surface since light remains always on the same side of the mirror.

For thin lenses, there are *two* focal points, symmetrically located on each side of the lens, since light can approach from either side of the lens.

The sketches in Figure 23 indicate the role that the two focal points play, for positive lenses (Figure 23a) and negative lenses (Figure 23b).

Study these figures carefully.

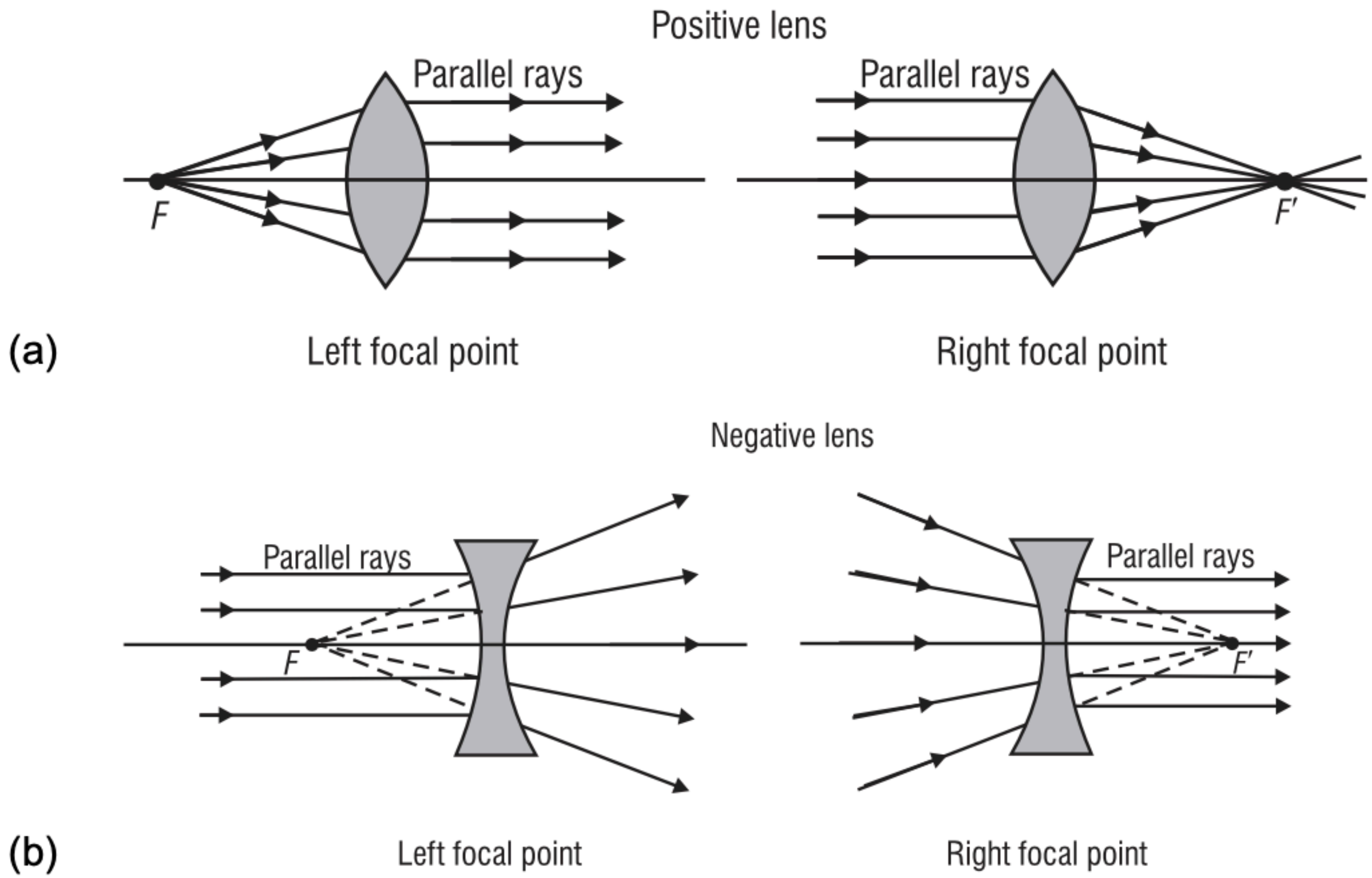


Figure 23 Relationship of light rays to right and left focal points in thin lenses

3. *f*-number and numerical aperture of a lens.

The size of a lens determines its light-gathering power and, consequently, the brightness of the image it forms.

Two commonly used indicators of this special characteristic of a lens are called the *f-number* and the *numerical* aperture.

The *f-number*, also referred to as the *relative aperture* and the *f/stop*, is defined simply as the ratio of the focal length *f* of the lens, to its diameter *D*, as given in Equation 3-8.

$$\boxed{f\text{-number} = \frac{f}{D}} \qquad (8)$$

For example, a lens of focal length 4 cm *stopped down* to an aperture of 0.5 cm has an *f*-number of 4/0.5 = 8.

Photographers usually refer to this situation as a lens with an *f/stop* of *f*/8.

Before the advent of fully automated cameras ("point and shoot"), a photographers would routinely select an aperture size for a given camera lens—(thereby setting the *f*/stop), a shutter speed, and a proper focus to achieve both the desired image brightness and sharpness.

Table 2 lists the usual choices of *f*/stops (*f*-numbers) available on cameras and the corresponding image irradiance or "brightness"—in watts per square meter.

Table 2. Relative Image Irradiance (Brightness) as a Function of *f* /stop Setting

f /stop or *f*-number	Relative Image Irradiance in watts/m²
1	E_0
1.4	$E_0/2$
2	$E_0/4$
2.8	$E_0/8$
4	$E_0/16$
5.6	$E_0/32$
8	$E_0/64$
11	$E_0/128$
16	$E_0/256$
22	$E_0/512$

The listing gives the irradiance E_0 as the value for an *f*/stop of 1 and shows how the image irradiance decreases as the lens is "stopped down," that is, as the adjustable aperture size behind the camera lens is made smaller.

From Equation 8, it should be clear that, for a given camera lens of focal length *f*, the *f*/stop or *f*-number increases as *D* decreases, that is, as the aperture size decreases.

Clearly then, increasing the *f*-number of a lens decreases its light-gathering power.

Since the total exposure in joules/m^2 on the film is the product of the irradiance in joules/(m^2-s) and the exposure time (shutter speed) in seconds, a desirable film exposure can be obtained in a variety of ways.

Accordingly, if a particular film—whose speed is described by an ASA number—is perfectly exposed by light from a particular scene with a shutter speed of 1/50 second and an *f*/stop of *f*/8 (irradiance equals *Eo*/64 from Table 2), it will also be perfectly exposed by any other combination that gives the same total exposure.

For example, by choosing a shutter speed of 1/100 second and an *f*/stop of *f*/5.6, the exposure time is cut in half while the irradiance ($E_0/32$) is doubled, thereby leaving *no net change* in the film exposure (J/m^2).

The *numerical aperture* is another important design parameter for a lens, related directly to how much light the lens gathers.

If the focal length of a design lens increases and its diameter decreases, the solid angle (cone) of useful light rays from object to image for such a lens decreases.

For example, the concept of a numerical aperture finds immediate application in the design of the objective lens (the lens next to the specimen under observation) for a *microscope*, as we show below.

Light-gathering capability is crucial for microscopes.

Figure 24 depicts the light-gathering power of a lens relative to a point *O* on a specimen covered by a glass slide.

Lens *L* is the objective lens of a microscope focused on the specimen.

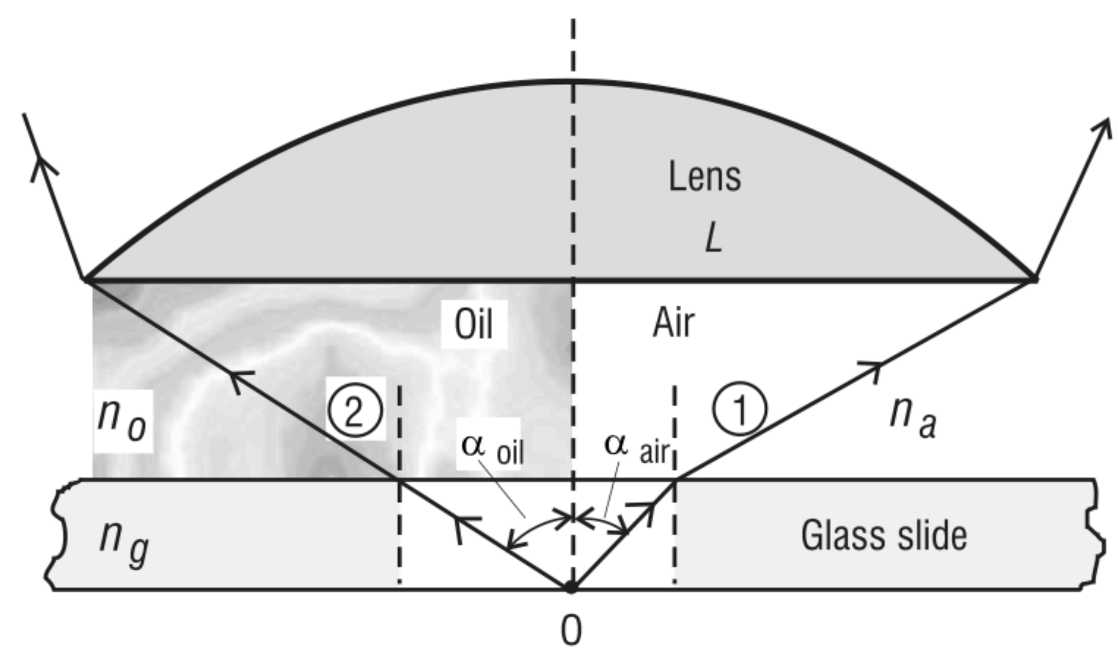
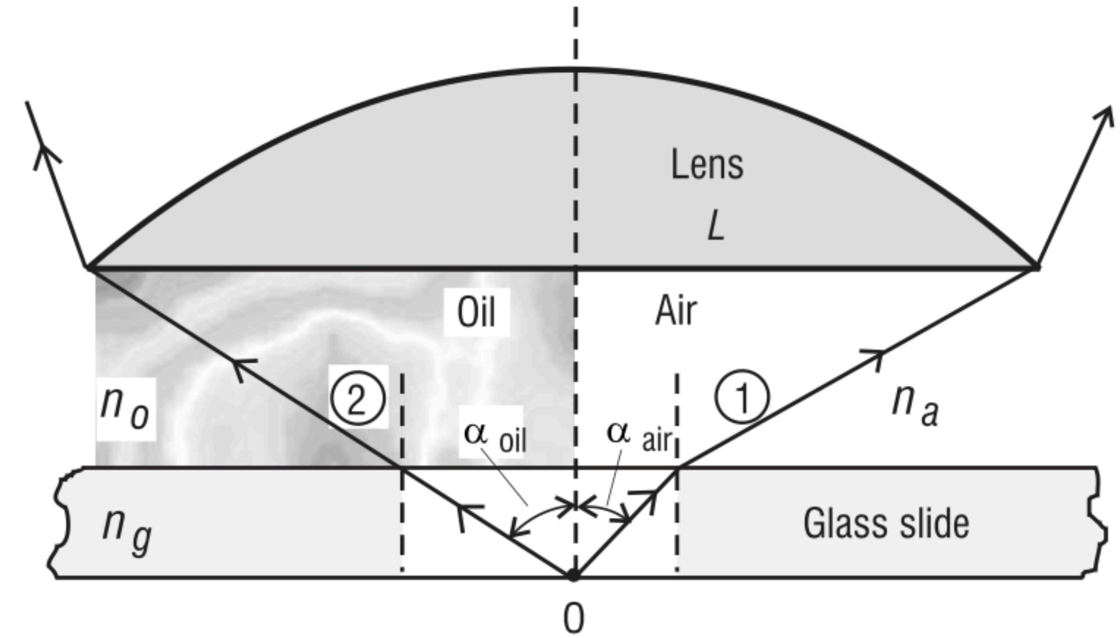


Figure 3-24 Light-gathering power of oil-immersion and air-immersion lens, showing that α_{oil} is greater than α_{air}

On the right side of the symmetry axis of the lens, the light-gathering power of the lens—*with* air between the cover slide and the lens—is depicted in terms of half-angle α_{air}.

On the left side, by contrast, the increased light-gathering power of the lens—*with oil situated between the cover slide and the lens*—is shown in terms of the *larger* half-angle α_{oil}.



The oil is chosen so as to have an index of refraction (n_0) very near that of the cover slide (n_g) so that little or no refraction occurs for limiting ray 2 at the glass-oil interface.

Consequently the half-angle α_{oil} is greater than the half-angle α_{air}.

As Figure 24 shows, ray 1 suffers refraction at the glass-air interface, thereby restricting the cone of rays accepted by the lens to the smaller half-angle α_{air}.

The *numerical aperture* of a lens is defined so as to exhibit the difference in solid angles (cones) of light accepted, for example, by an "oil-immersion" arrangement versus an air-immersion setup.

The definition of numerical aperture (*N.A.*) is given in Equation 9 as

$$\boxed{N.A. = n \sin \alpha} \tag{9}$$

where n is the index of refraction of the intervening medium between object and lens and α is the half-angle defined by the limiting ray (α_{air} or α_{oil} in Figure 24).

The "light-gathering" power of the microscope's objective lens is thus increased by increasing the refractive index of the intervening medium.

In addition, the numerical aperture is closely related to the acceptance angle discussed in Example 3 for both graded-index and step-index optical fibers.

Since the rays entering the fiber face are in air, the numerical aperture $N.A.$ is equal simply to $N.A. = \sin \alpha$.

It is shown in most basic books on optics that image brightness is dependent on values of the *f*-number or numerical aperture, in accordance with the following proportionalities:

$$\text{image brightness} \propto \frac{1}{(f\text{-number})^2}$$

$$\text{image brightness} \propto (N.A.)^2$$

In summary, one can *increase* the light-gathering power of a lens and the brightness of the image formed by a lens by *decreasing* the *f*-number of the lens (increasing lens diameter) or by *increasing* the numerical aperture of the lens (increasing the refraction index and thus making possible a larger acceptance angle).

C. Image location by ray tracing

To locate the image of an object formed by a thin lens, we make use of three key points for the lens and associate each of them with a defining ray.

The three points are the left focal point F, the right focal point F', and the lens vertex (center) V.

In Figure 25 the three rays are shown locating an image point P' corresponding to a given object point P, for both a positive and a negative lens.

The object is labeled OP and the corresponding image IP'.

The defining rays are labeled to show clearly their connection to the points F, F', and V.

In practice, of course, only two of the three rays are needed to locate the desired image point.

Note also that the location of image point P' is generally sufficient to sketch in the rest of the image IP', to correspond with the given object OP.

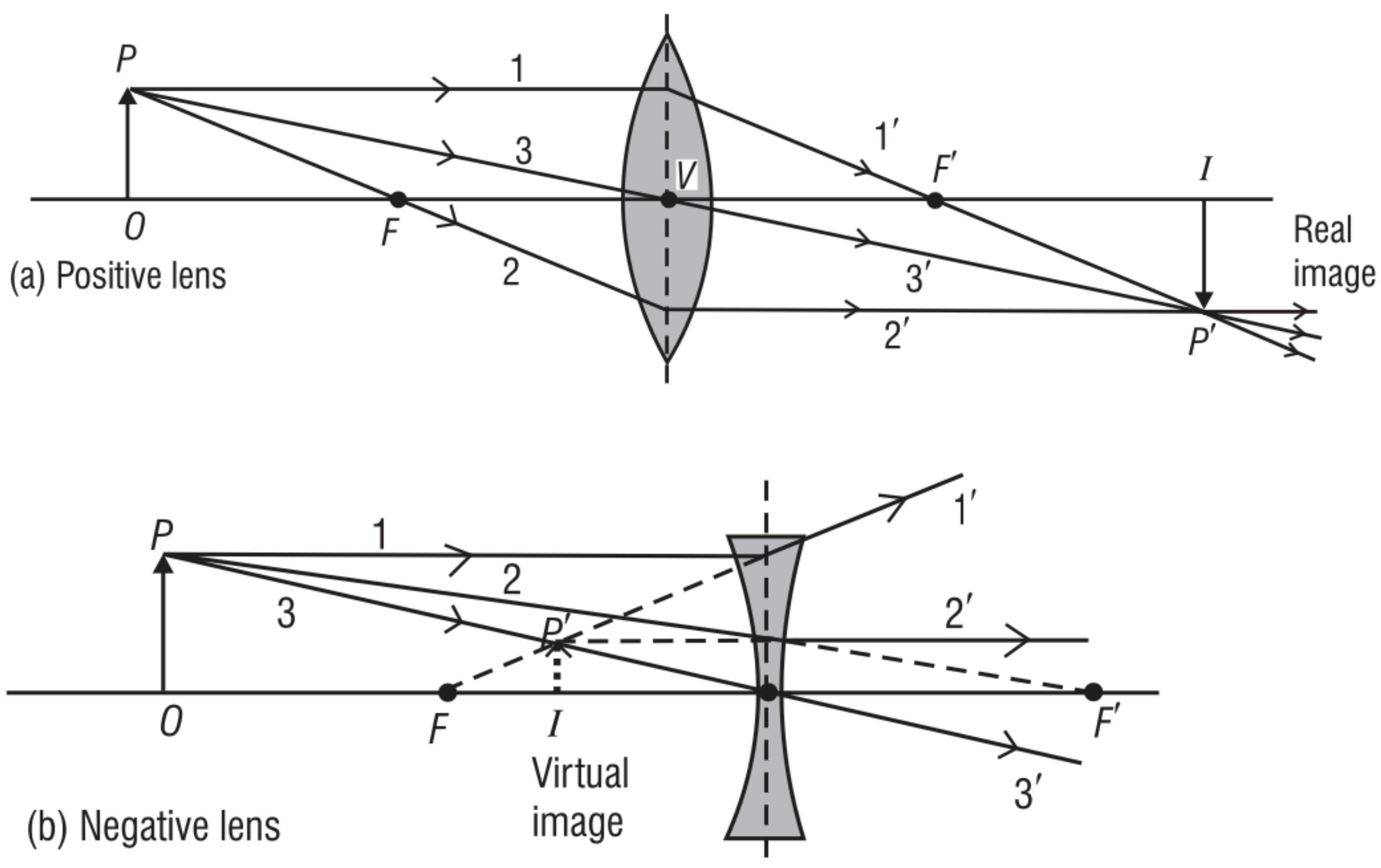


Figure 25 Ray diagrams for image formation by positive and negative lenses

The behavior of rays 1 and 2—connected with the left and right focal points for both the positive and negative lenses—should be apparent from another look at Figure 23.

The behavior of ray 3—going straight through the lens at its center V—is a consequence of assuming that the lens has zero thickness.

Note, in fact, that, for both Figures 23 and 25, all the bending is assumed to take place at the dashed vertical line that splits the drawn lenses in half.

Also, it should be clear in Figure 25 that the positive lens forms a real image while the negative lens forms a virtual image.

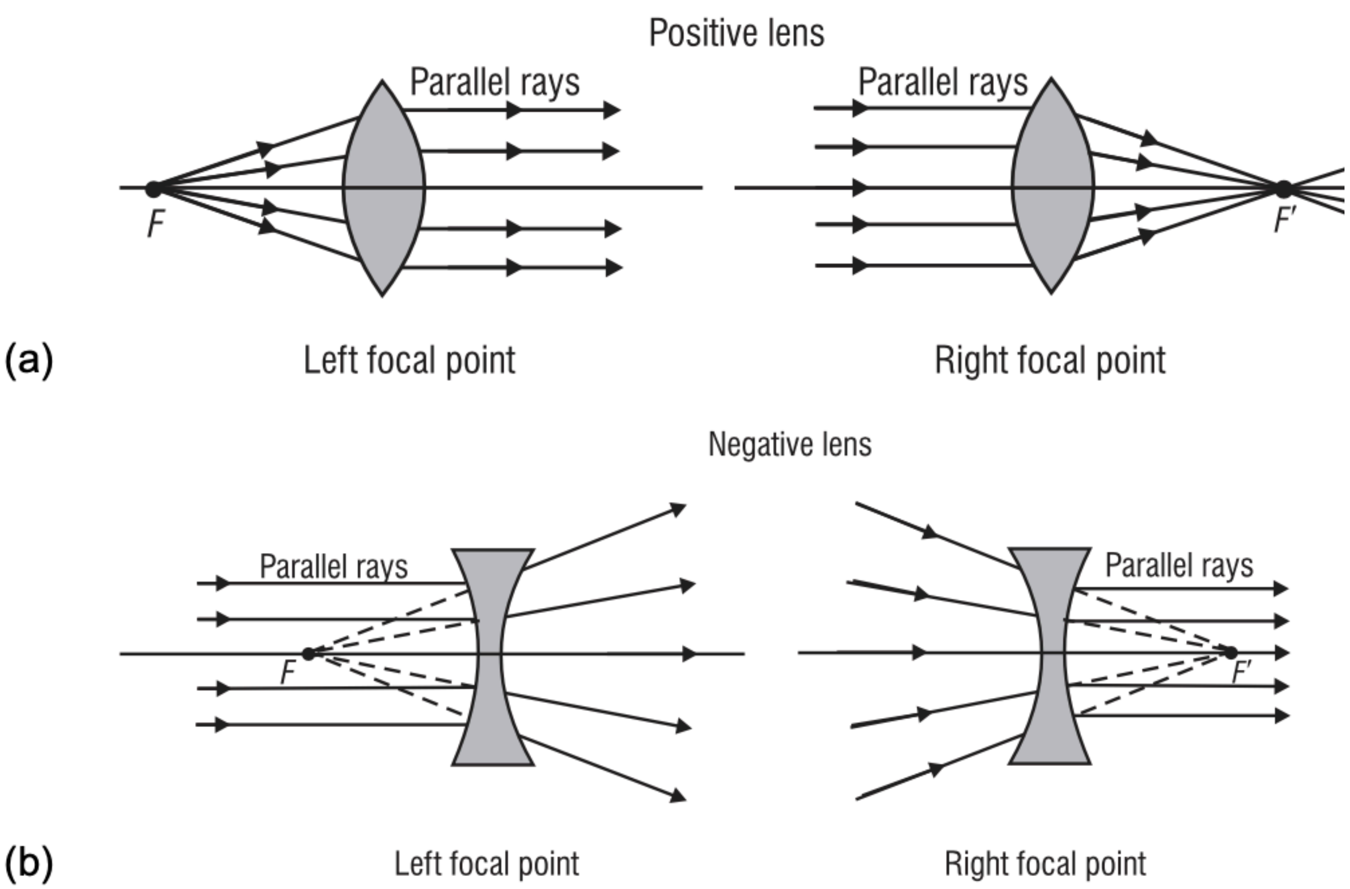


Figure 23

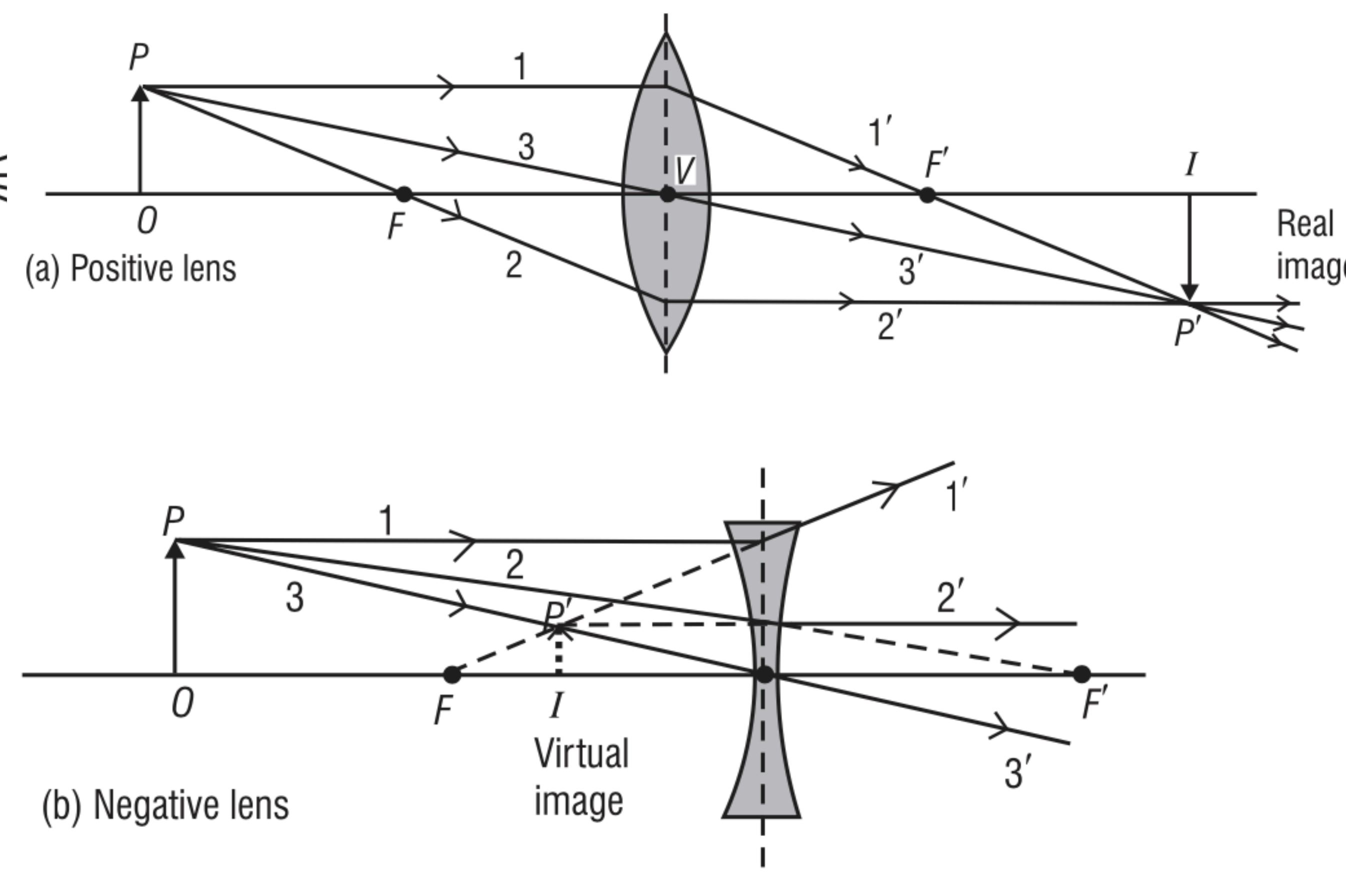


Figure 25

One can apply the principles of ray tracing illustrated in Figure 25 to a train of thin lenses.

Figure 26 shows a ray trace through an "optical system" made up of a positive and a negative lens.

For accuracy in drawing, a common practice used here is to show the positive lens as a vertical line with normal arrowheads and the negative lens as a vertical line with inverted arrowheads, and to show all ray bending at these lines.

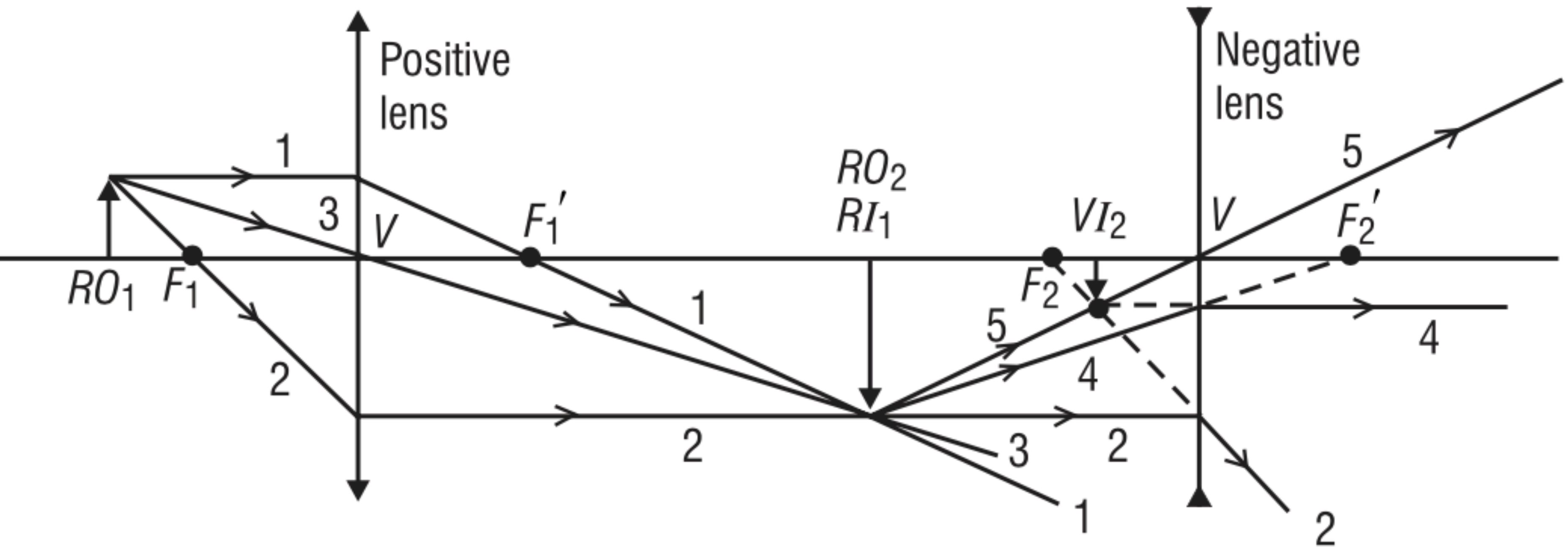


Figure 3-26 Ray diagram for image formation through two lenses

Note that the primary object is labeled RO_1 (real object 1) and its image formed by the positive lens is labeled RI_1 (real image 1).

The image RI_1 then serves as a real object (RO_2) for the negative lens, leading finally to a virtual image VI_2.

Test your understanding of ray tracing through thin lenses by accounting for each numbered ray drawn in the figure.

What happens to rays 1 and 3 relative to the negative lens?

Why are rays 4 and 5 introduced? Is this a "fair" practice?

D. Lens formulas for thin lenses

As with mirrors, convenient formulas can be used to locate the image mathematically.

The derivation of such formulas—as was carried out for spherical mirrors in the previous section—can be found in most texts on geometrical optics.

The derivation essentially traces an arbitrary ray *geometrically* and *mathematically* from an object point through the two surfaces of a thin lens to the corresponding image point.

Snell's law is applied for the ray at each spherical refracting surface.

The details of the derivation involve the geometry of triangles and the approximations mentioned earlier —$\sin \phi \cong \phi$, $\tan \phi \cong \phi$, and $\cos \phi \cong 1$—to simplify the final results.

Figure 27 shows the essential elements that show up in the final equations, relating object distance p to image distance q, for a lens of focal length f with radii of curvature r_1 and r_2 and refractive index n_g.

For generality, the lens is shown situated in an arbitrary medium of refractive index n.

If the medium is air, then, of course, $n = 1$.

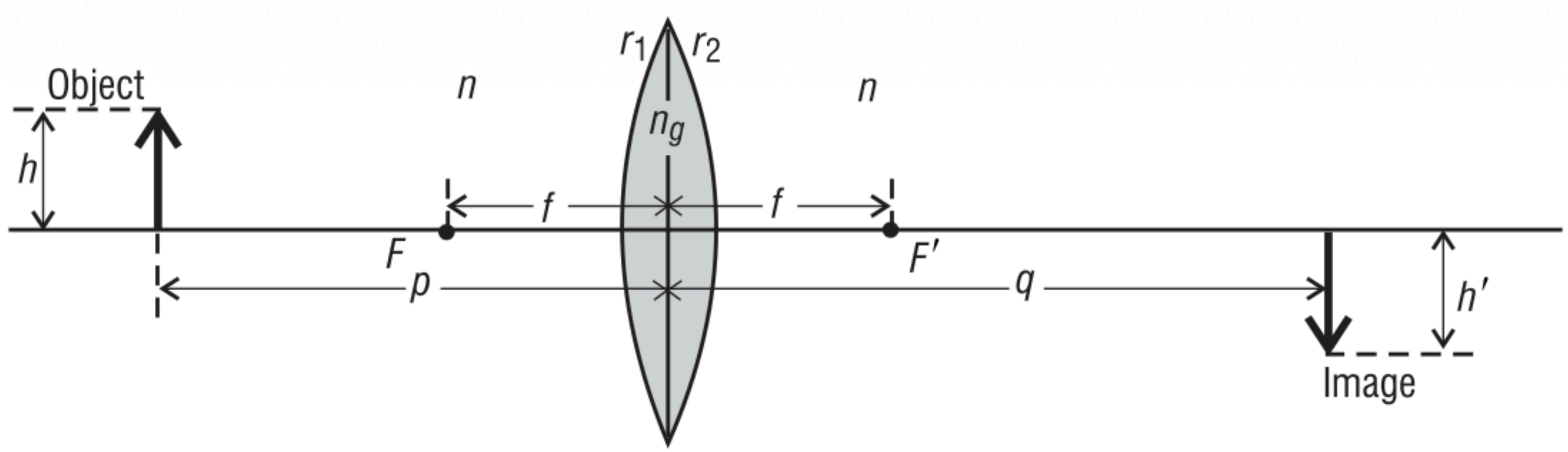


Figure 27 Defining quantities for image formation with a thin lens

1. Equations for thin lens calculations.

The *thin lens equation* is given by Equation 10.

$$\boxed{\frac{1}{p}+\frac{1}{q}=\frac{1}{f}} \tag{10}$$

where p is the object distance (from object to lens vertex V)

q is the image distance (from image to lens vertex V)

and f is the focal length (from either focal point F or F' to the lens vertex V)

For a lens of refractive index n_g situated in a medium of refractive index n, the relationship between the parameters n, n_g, r_1, r_2 and the focal length f is given by the *lensmaker's equation* in Equation 11.

$$\boxed{\frac{1}{f}=\left(\frac{n_g-n}{n}\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \tag{11}$$

where n is the index of refraction of the surrounding medium

n_g is the index of refraction of the lens materials

r_1 is the radius of curvature of the front face of the lens

r_2 is the radius of curvature of the rear face of the lens

The magnification m produced by a thin lens is given in Equation 12.

$$\boxed{m = \frac{h_i}{h_o} = -\frac{q}{p}} \tag{12}$$

where

m is the magnification (ratio of image size to object size)

h_i is the transverse size of the image

h_o is the transverse size of the object

p and q are object and image distance respectively

2. Sign convention for thin lens formulas.

Just as for mirrors, we must agree on a sign convention to be used in the application of Equations 10, 11, and 12.

It is:

- Light travels initially from left to right toward the lens.
- Object distance p is *positive* for *real* objects located to the *left* of the lens and *negative* for *virtual* objects located to the *right* of the lens.
- Image distance q is *positive* for *real* images formed to the *right* of the lens and *negative* for *virtual* images formed to the *left* of the lens.
- The focal length f is *positive* for a *converging* lens, *negative* for a *diverging* lens.
- The radius of curvature r is *positive* for a *convex* surface, *negative* for a *concave* surface.
- Transverse distances (h_o and h_i) are *positive above* the optical axis, *negative below*.

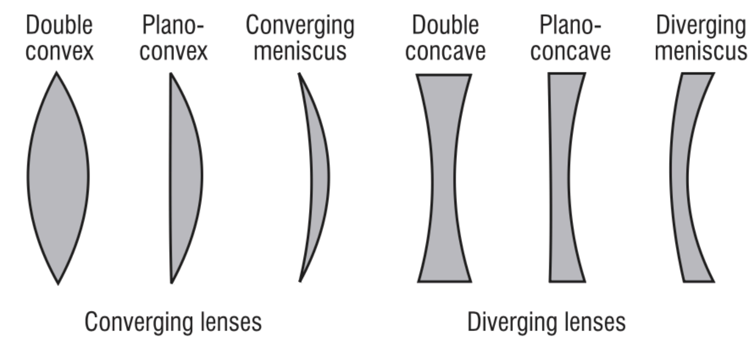
Now let's apply Equations 10, 11, and 12 in several examples, where the use of the sign convention is illustrated and where the size, orientation, and location of a final image are determined.

Example 8

A double-convex thin lens such as that shown in Figure 21 can be used as a simple "magnifier."

It has a front surface with a radius of curvature of 20 cm and a rear surface with a radius of curvature of 15 cm.

The lens material has a refractive index of 1.52.



Answer the following questions to learn more about this simple magnifying lens.

(a) What is its focal length in air?

(b) What is its focal length in water ($n = 1.33$)?

(c) Does it matter which lens face is turned toward the light?

(d) How far would you hold an index card from this lens to form a sharp image of the sun on the card?

Solution:

(a) Use the lensmaker's equation. With the sign convention given, we have $n_g = 1.52$, $n = 1.00$, $r_1 = +20$ cm, and $r_2 = -15$ cm. Then

$$\frac{1}{f} = \left(\frac{n_g - n}{n}\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \left(\frac{1.52-1}{1}\right)\left(\frac{1}{20} - \frac{1}{-15}\right) = 0.0607$$

So $f = +16.5$ cm (a converging lens, so the sign is positive, as it should be)

(b) $$\frac{1}{f} = \left(\frac{1.52-1.33}{1.33}\right)\left(\frac{1}{20} - \frac{1}{-15}\right) = 0.0167$$

$f = 60$ cm (converging but less so than in air)

(c) No, the magnifying lens behaves the same, having the same focal length, no matter which surface faces the light. You can prove this by reversing the lens and repeating the calculation with Equation 3-11. Results are the same. But **note carefully**, reversing a *thick* lens changes its effect on the light passing through it. The two orientations are *not* equivalent.

(d) Since the sun is very far away, its light is collimated (parallel rays) as it strikes the lens and will come to a focus at the lens focal point. Thus, one should hold the lens about 16.5 cm from the index card to form a sharp image on the card.

the the the the the

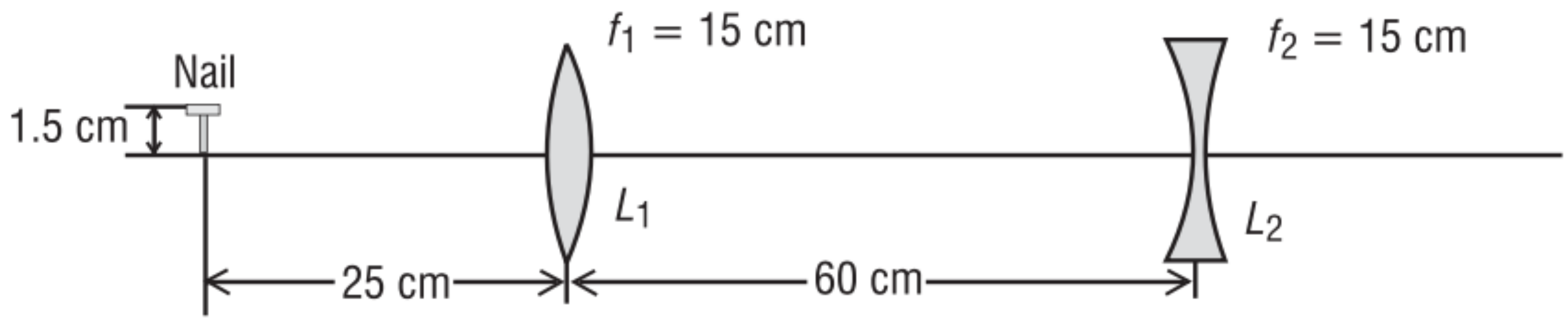
Example 9

A two-lens system is made up of aconverging lens followed by a diverging lens, each of focal length 15 cm.

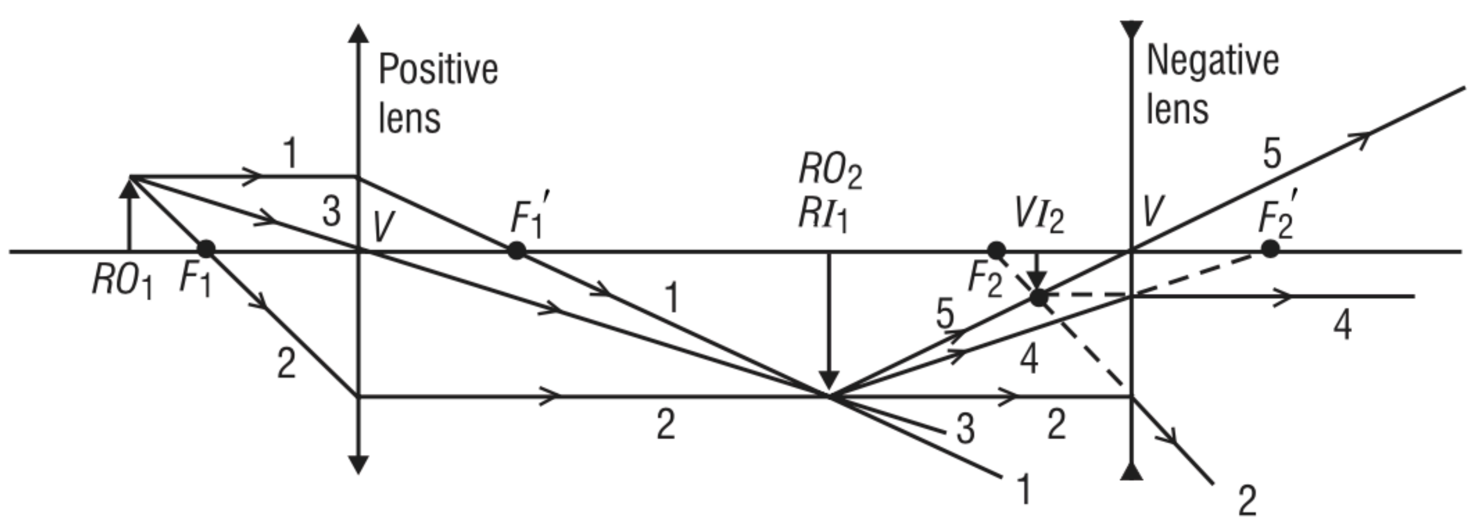
The system is used to form an image of a short nail, 1.5 cm high, standing erect, 25 cm from the first lens.

The two lenses are separated by a distance of 60 cm.

See accompanying diagram.



Refer to Figure 26 for a ray-trace diagram of what’s going on in this problem.



Locate the final image, determine its size, and state whether it is real or virtual, erect or inverted.

Solution: We apply the thin lens equations to each lens in turn, making use of the correct sign convention at each step.

Lens L_1: $\dfrac{1}{p_1}+\dfrac{1}{q_1}=\dfrac{1}{f_1}$ *or* $\dfrac{1}{25}+\dfrac{1}{q_1}=\dfrac{1}{15}$ (f_1 is + since lens L_1 is converging.)

$q_1 = +37.5$ cm (Since the sign is positive, the image is real and located 37.5 cm to the right of lens L_1.

Lens L_2: $\dfrac{1}{p_2}+\dfrac{1}{q_2}=\dfrac{1}{f_2}$ where $p_2 = (60 - 37.5) = 22.5$ cm

Since the first image, a distance q_1 from L_1, serves as the object for the lens L_2, this object is to the left of lens L_2, and thus its distance p_2 is positive. The focal length for L_2 is negative since it is a diverging lens. So, the thin lens equation becomes

$$\frac{1}{22.5}+\frac{1}{q_2}=\frac{1}{-15}, \text{ giving } q_2 = -9\text{cm}$$

Since q_2 is negative, it locates a *virtual* image, 9 cm to the left of lens L_2. See Figure 26.

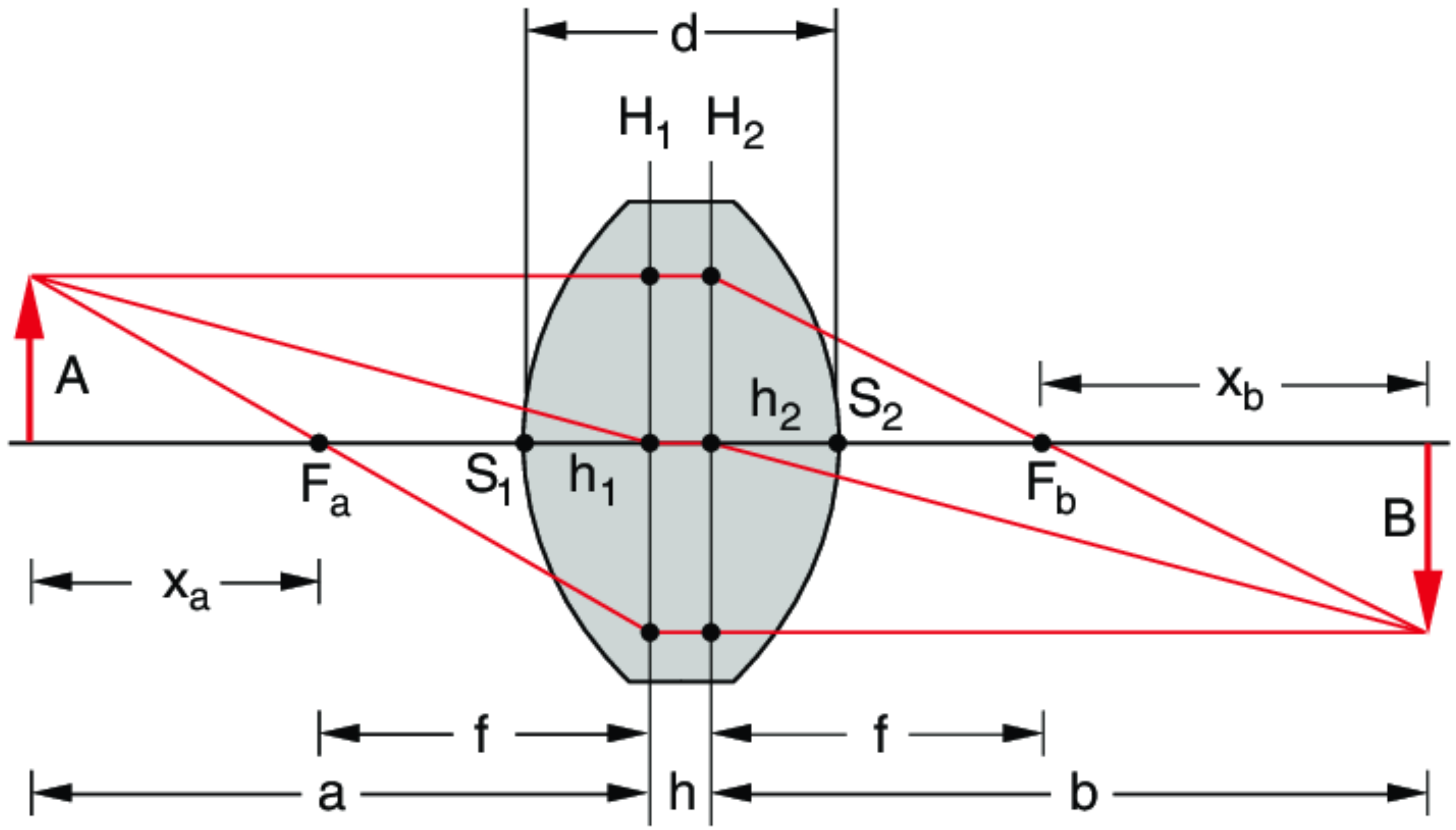
The overall magnification for the two-lens system is given by the combined magnification of the lenses. Then

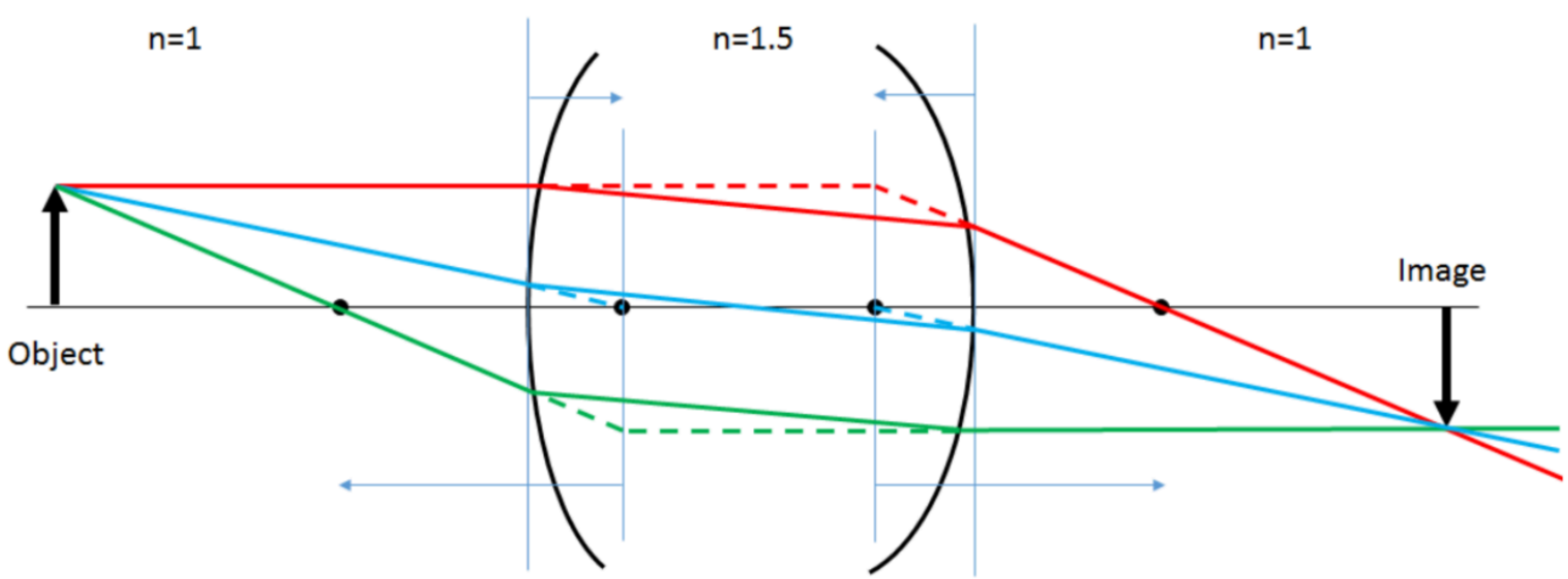
$$m_{sys} = m_1 \times m_2 = \left(-\frac{q_1}{p_1}\right)\left(-\frac{q_2}{p_2}\right) = \left(-\frac{37.5}{25}\right)\left(-\frac{-9}{22.5}\right) = -0.6$$

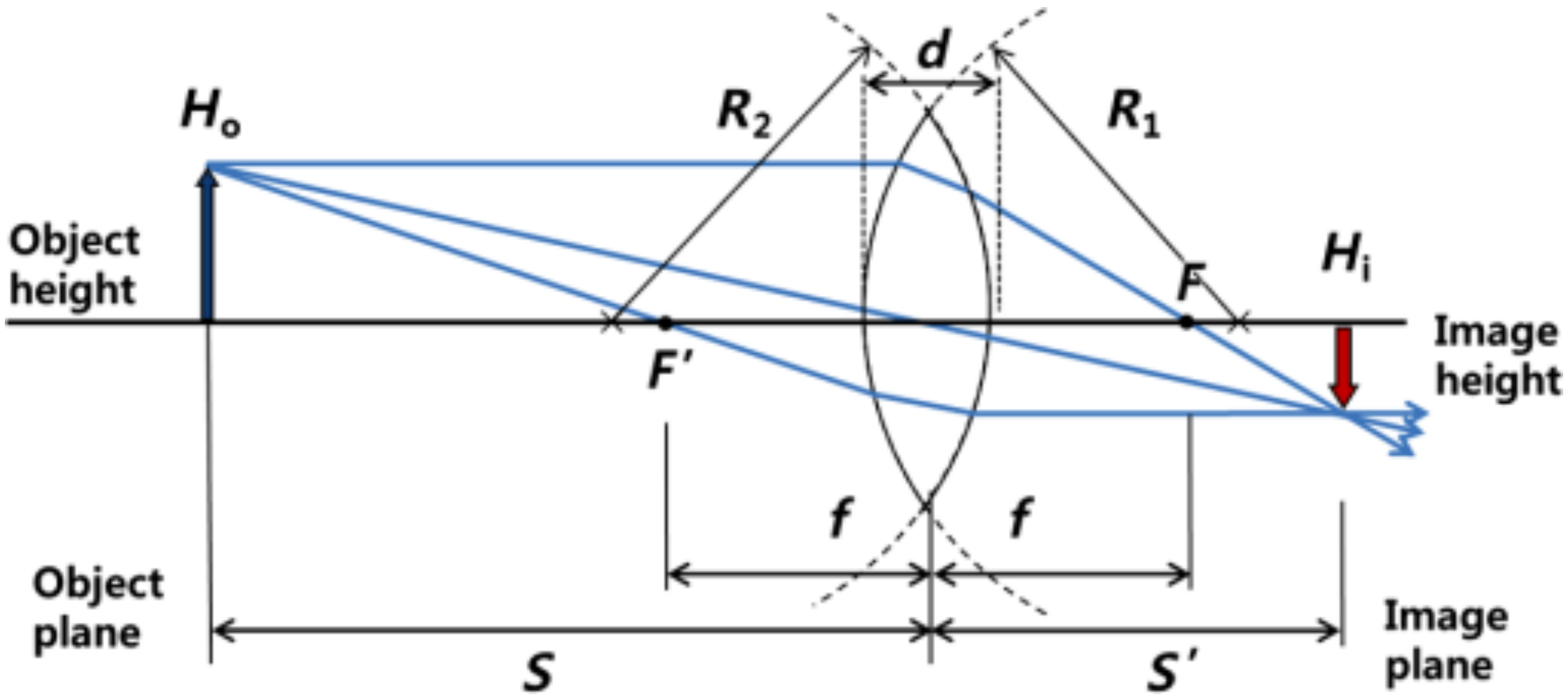
Thus, the final image is inverted (since overall magnification is negative) and is of final size $(0.6 \times 1.5 \text{ cm}) = 0.9$ cm.

Addendum

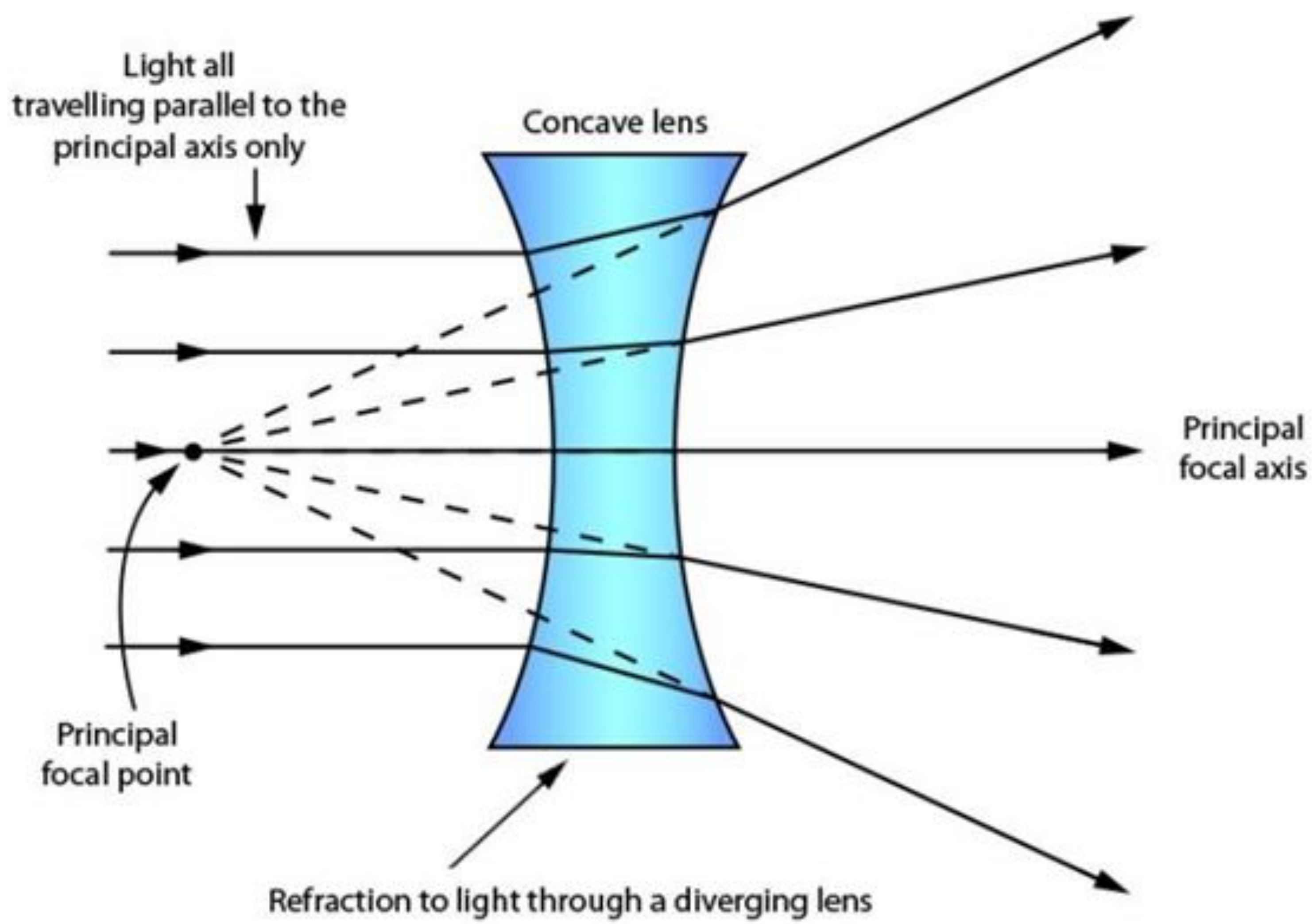
Construction of the ray path in a thick lens - just present some examples - you should be able to figure out the method.







d
R_2
R_1
H_o
Object height
F
H_i
F'
Image height
f
f
Object plane
S
S'
Image plane



Light all travelling parallel to the principal axis only
Concave lens
Principal focal axis
Principal focal point
Refraction to light through a diverging lens

Having now covered the basics of geometrical optics, the rest of the class will delve more deeply and more mathematically in the subject of optics.

Section 1

Matrices and Matrix Operations

The idea of a **matrix** arises in this way.

Suppose we have a pair of linear equations

$$U = Ax + By$$
$$V = Cx + Dy$$

where A,B,C and D are known constants and x and y are variables.

We can write this pair of equations as:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where [..] is called a **matrix**.

$$\begin{bmatrix} U \\ V \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

are 2x1 matrices or **column matrices** or **column vectors**.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a 2x2 matrix.

We can have n x n matrices, n x 1 column vectors and 1 x n **row vectors**.

We can then write

$$C_2 = \begin{bmatrix} U \\ V \end{bmatrix} \qquad S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad C_1 = \begin{bmatrix} x \\ y \end{bmatrix}$$

so that we can write the **matrix equation** as

$$C_2 = SC_1$$

Equation (2) clearly defines the operation of **matrix multiplication**.

We also have the equations

$$L = PU + QV$$

$$M = RU + TV$$

or using matrices

$$C_3 = \begin{bmatrix} L \\ M \end{bmatrix} = \begin{bmatrix} P & Q \\ R & T \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = K \begin{bmatrix} U \\ V \end{bmatrix}$$

or

$$C_3 = KC_2$$

where

$$\mathrm{K} = \begin{bmatrix} P & Q \\ R & T \end{bmatrix}$$

Continuing the algebra…..

$$L = P(Ax + By) + Q(Cx + Dy)$$

$$M + R(Ax + By) + T(Cx + Dy)$$

or

$$\begin{bmatrix} L \\ M \end{bmatrix} = \begin{bmatrix} PA + QC & PB + QD \\ RA + TC & RB + TD \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$C_3 = FC_1 = (KS)C_1$$

where

$$F = KS = \begin{bmatrix} PA + QC & PB + QD \\ RA + TC & RB + TD \end{bmatrix}$$

Thus, we obtain the general rule for matrix multiplication

$$\begin{bmatrix} P & Q \\ R & T \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} PA + QC & PB + QD \\ RA + TC & RB + TD \end{bmatrix}$$

Generalizing to arbitrary size we have

$$(C_2)_i = \sum_k S_{ik}(C_1)_k \qquad (C_3)_i = \sum_k K_{ik}(C_2)_k$$

$$(C_3)_i = \sum_j \left(\sum_k K_{ik} S_{kj} \right)(C_1)_j = \sum_j F_{ij}(C_1)_j$$

$$F_{ij} = \sum_k K_{ik} S_{kj} = K_{ik} S_{kj}$$

Einstein summation convention

Matrix Addition/Subtraction

$$F_{ij} = K_{ij} \pm S_{ij}$$

Special Matrices

Null matrix: $A_{ij} = 0 \, , \ all \ \ i,j$

Unit or identity matrix I : $A_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Diagonal matrix: $A_{ij} = 0 \quad i \neq j$

Matrix Transpose: $\left(A^T\right)_{ij} = A_{ji} \qquad \left(AB\right)^T = B^T A^T$

Determinants (we are interested in 2x2 matrices only)

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad , \quad \det(P) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC$$

$$\det(AB) = \det(A)\det(B)$$

Matrix Inversion (interested in 2x2 matrices only)

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad , \quad P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

where

$$PP^{-1} = I$$

Note that det(P) = 1 (unimodular) for matrices of interest to us.

Matrix Diagonalization - Suppose we have the relation

$$M = F\Lambda F^{-1}$$

where Λ is a diagonal matrix.

This implies that

$$M^N = F\Lambda F^{-1}F\Lambda F^{-1} \cdots\cdots F\Lambda F^{-1} = F\Lambda^N F^{-1}$$

Thus, once the diagonalizing transformation F has been found, the N^{th} power of the original matrix is found by taking the diagonal matrix to the N^{th} power, which is easy since all we have to do is replace each diagonal element by its N^{th} power.

The diagonal elements of the matrix Λ are the eigenvalues of M and the columns of the diagonalizing matrix are its eigenvectors.

This is shown below:

$$
\begin{aligned}
&C_r = column\ vector\ with\ 1\ in\ r^{th}\ place \\
&F_r = FC_r = r^{th} column\ of\ F \\
&MF_r = (F\Lambda F^{-1})(FC_r) = F\Lambda C_r = \lambda_r FC_r = \lambda_r F_r \\
&\lambda_r =\ r^{th}\ diagonal\ element\ of\ \Lambda
\end{aligned}
$$

Eigenvalues and Eigenvectors for 2x2 Unimodular Matrices - In general, eigenvalues are determined using the **characteristic** equation

$$\det(\lambda I - M) = 0$$

For a 2x2 unimodular matrix, we have

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\det(M) = 1 = AD - BC$$

$$\det(\lambda I - M) = 0 = \det\begin{bmatrix} \lambda - A & -B \\ -C & \lambda - D \end{bmatrix}$$

$$= (\lambda - A)(\lambda - D) - BC$$

$$= (\lambda - A)(\lambda - D) + (1 - AD)$$

or

$$\lambda^2 - (A + D)\lambda + 1 = 0 = \lambda^2 - (Tr(M))\lambda + \det(M)$$

`Tr(...) = sum of diagonal elements`

and we find

$$\lambda_{1,2} = \frac{1}{2}\left[(A + D) \pm \sqrt{(A + D)^2 - 4}\right]$$

so that we have two eigenvalues satisfying

$$\lambda_1 + \lambda_2 = A + D = Tr(M) \quad , \quad \lambda_1\lambda_2 = 1 = \det(M)$$

If Tr(M) = A+D lies between 2 and -2 in value, then the two eigenvalues can be written conveniently in terms of an angle θ, chosen so that it lies between 0 and π and such that A+D = 2cosθ.

We then find (a general property of unitary matrices)

$$\lambda_1 = \cos\theta + i\sin\theta = \exp(i\theta)$$
$$\lambda_2 = \cos\theta - i\sin\theta = \exp(-i\theta)$$

If Tr(M) = A+D is greater than 2 or less than -2, we can choose a positive quantity t such that A+D = 2cosh(t) (or -2cosh(-t) if A+D is negative).

The eigenvalues then take the form

$$\lambda_1 = \exp(t)\ \ (or\ \ -\exp(t)\ \ if\ \ (A+D)\ \ is\ \ negative)$$
$$\lambda_2 = \exp(-t)\ \ (or\ \ -\exp(-t)\ \ if\ \ (A+D)\ \ is\ \ negative)$$

We determine the diagonalizing matrix F as follows.

$$M = F\Lambda F^{-1} \Rightarrow MF = F\Lambda$$

$$MF = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = F\Lambda$$

where AD-BC=1 since det(M)=1.

On multiplying we get

$$\begin{bmatrix} AF_{11} + BF_{21} & AF_{12} + BF_{22} \\ CF_{11} + DF_{21} & CF_{12} + DF_{22} \end{bmatrix} = \begin{bmatrix} F_{11}\lambda_1 & F_{12}\lambda_2 \\ F_{21}\lambda_1 & F_{22}\lambda_2 \end{bmatrix}$$

which gives

$$\frac{F_{11}}{F_{21}} = \frac{\lambda_1 - D}{C} = \frac{B}{\lambda_1 - A} \quad , \quad \frac{F_{12}}{F_{22}} = \frac{\lambda_2 - D}{C} = \frac{B}{\lambda_2 - A}$$

twhere the eigenvectors are

$$F_1 = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} \quad , \quad F_2 = \begin{bmatrix} F_{12} \\ F_{22} \end{bmatrix}$$

One possible solution has the form:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 - D & \lambda_2 - D \\ C & C \end{bmatrix}$$

We will use all these ideas at some point.

Matrix methods in Paraxial(Gaussian) Optics

Let us discuss how matrices can be used to describe the geometric formation of images by a centered lens system - a succession of spherical refracting surfaces all centered on the same optical axis.

The results are valid within two approximations.

First, the basic assumption of all geometric optics is that the wavelength of light is small and the propagation of light can be explained in terms of individual rays instead of wavefronts.

As can be shown by means of Huygens' construction(proved later), if light waves are allowed to travel without encountering any obstacles, they are propagated along a direction which is normal to the wavefronts.

The concept of a geometric ray is a idealization of this wave-normal.

Each ray obeys Fermat's principle of least time(proved later) - if we consider the neighborhood of any short section of the ray path, the path which the ray chooses between a given entry point and a given exit point is that which minimizes the time taken(proof later).

The second approximation is that we only consider paraxial rays - those that remain close to the axis and almost parallel to it so that we can use the first-order approximations for sines or tangents of any angles.

$$\sin\theta \approx \tan\theta \approx \theta$$

Ray-Transfer Matrices - Now consider the propagation of a paraxial ray through a system of centered lenses.

We use a Cartesian coordinate system where the z-axis (horizontal) represents the optical axis.

The y-axis is orthogonal to the optical axis.

All rays will lie in the yz-plane and close to the z-axis.

The trajectory of a ray as it passes through the various refracting surfaces of the system will consist of a series of straight lines.

We specify a ray by the height y of a point on the ray and the angle (anticlockwise from z-axis is positive direction) the ray makes with the z-axis as shown in figure 1.

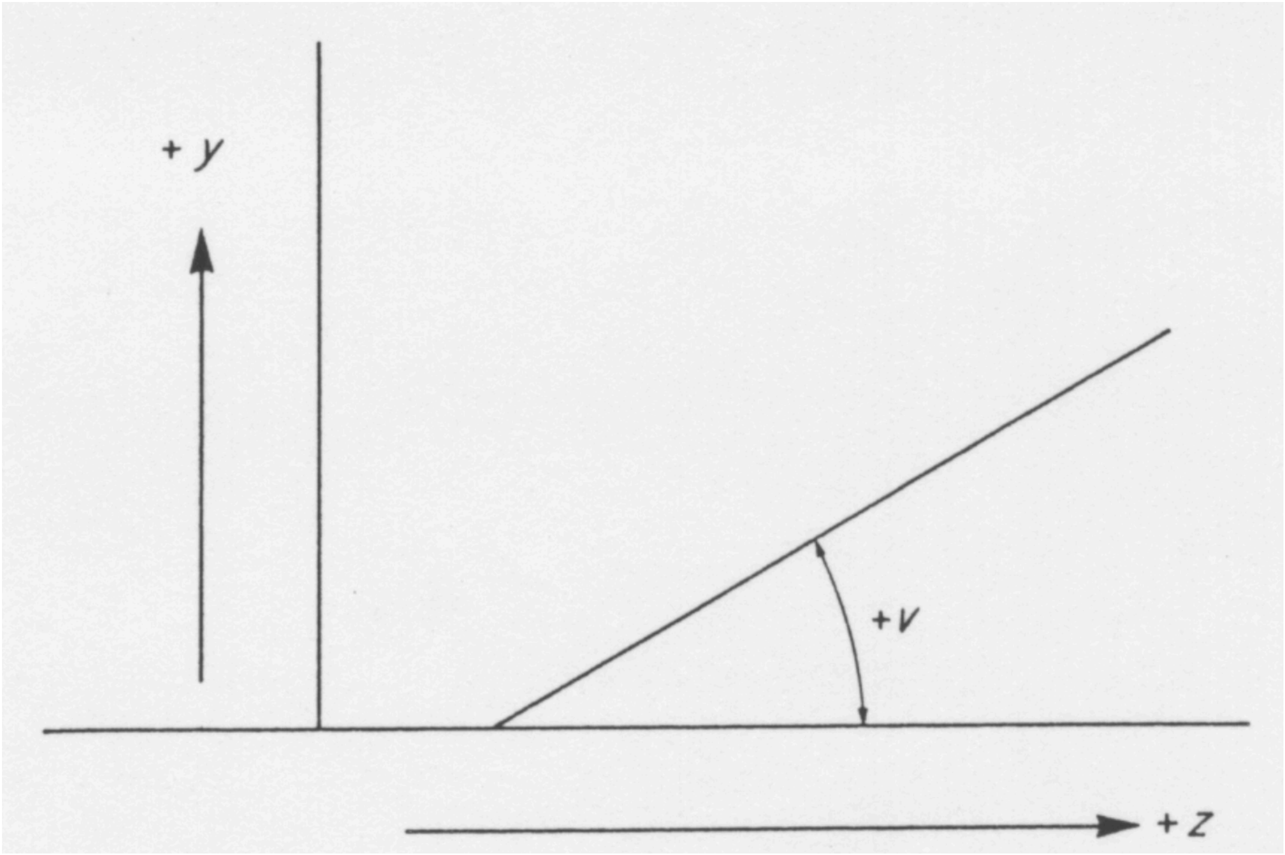


Figure 1

Later calculations are made more convenient by replacing the ray angle v with the corresponding **optical direction-cosine** V = nv (or strictly, V = nsin(v)) where n is the refractive index of the medium in which the ray is traveling.

The optical direction-cosine has the property that, by Snell's law(will derive later),

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \Rightarrow n_1\theta_1 = n_2\theta_2 \Rightarrow V_1 = V_2$$

it will remain unchanged as it crosses a plane boundary between two different media.

It will also make all matrices involved unimodular (unitary with determinant = 1), which will be very useful.

As a ray passes through a refracting lens system, there are only two basic types of process that we need to consider in order to determine its progress:

(a) A **translation** during which the ray simply moves in a straight line to the next refracting surface.

We need to know the distance translated (thickness of material) and the corresponding refractive index n.

(b) **Refraction** at the boundary surface between two regions of different refractive index.

To determine how much bending the ray undergoes, we need to know the radius of curvature of the refracting surface and the two values of refractive index.

We now investigate the effect that each of these two basic elements has on the y-value and the V-value of a ray passing between two so-called reference planes (one on either side of the element), namely RP_1 and RP_2 (ses Figure 2), orthogonal to the optical axis(z-axis).

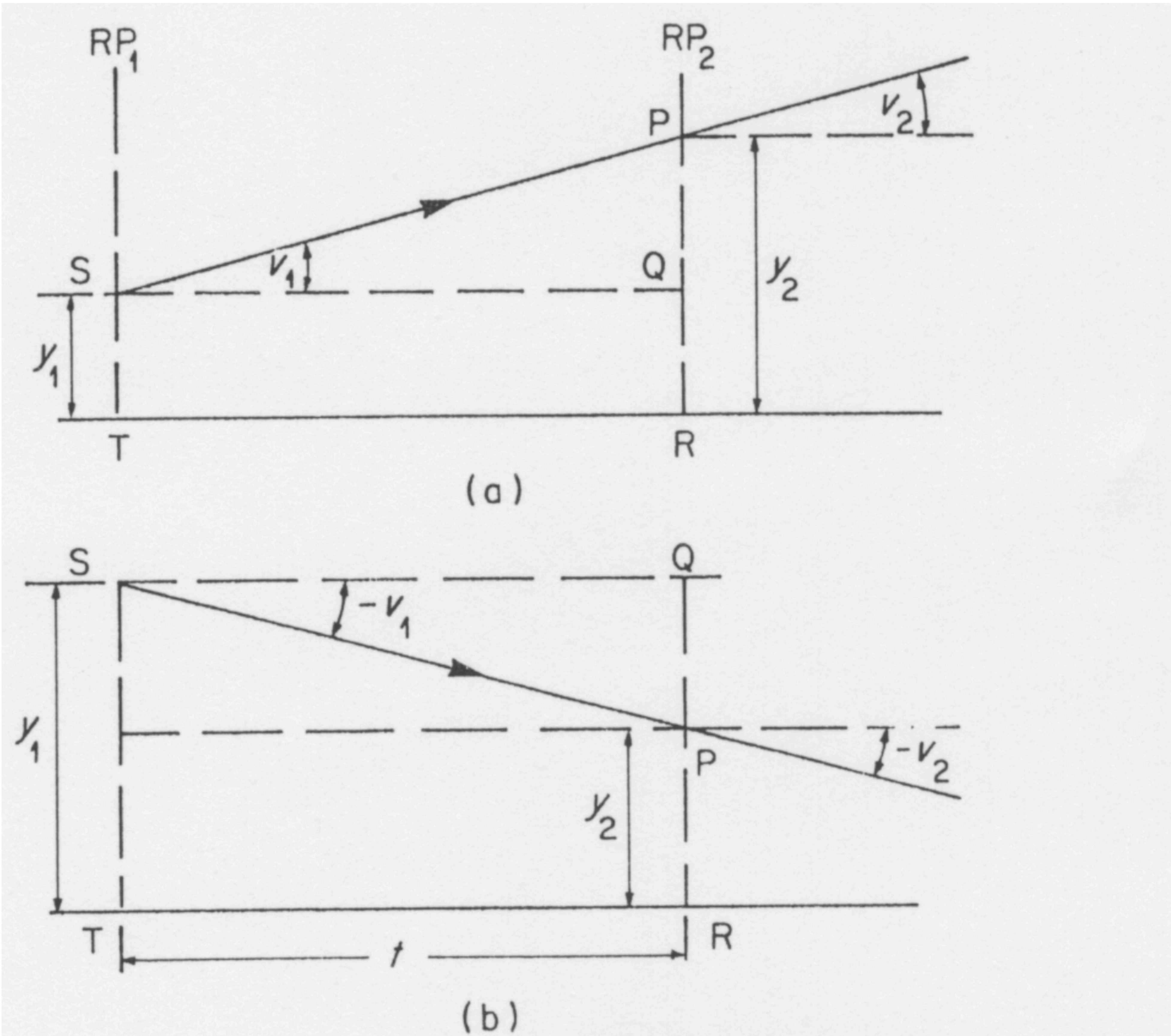


Figure 2

The ray first passes through RP_1 with value y_1 and V_1, then through the optical element and then through RP_2, with values y_2 and V_2.

We want equations expressing y_2 and V_2 in terms of y_1 and V_1 and the properties of the optical element.

We will find that for both types of elements (translation and refraction) the equations are linear and thus their effect can be written using 2x2 matrices as

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

where the determinant of the transformation matrix(ray-transfer matrix) is equal to 1, i.e., AD-BC=1.

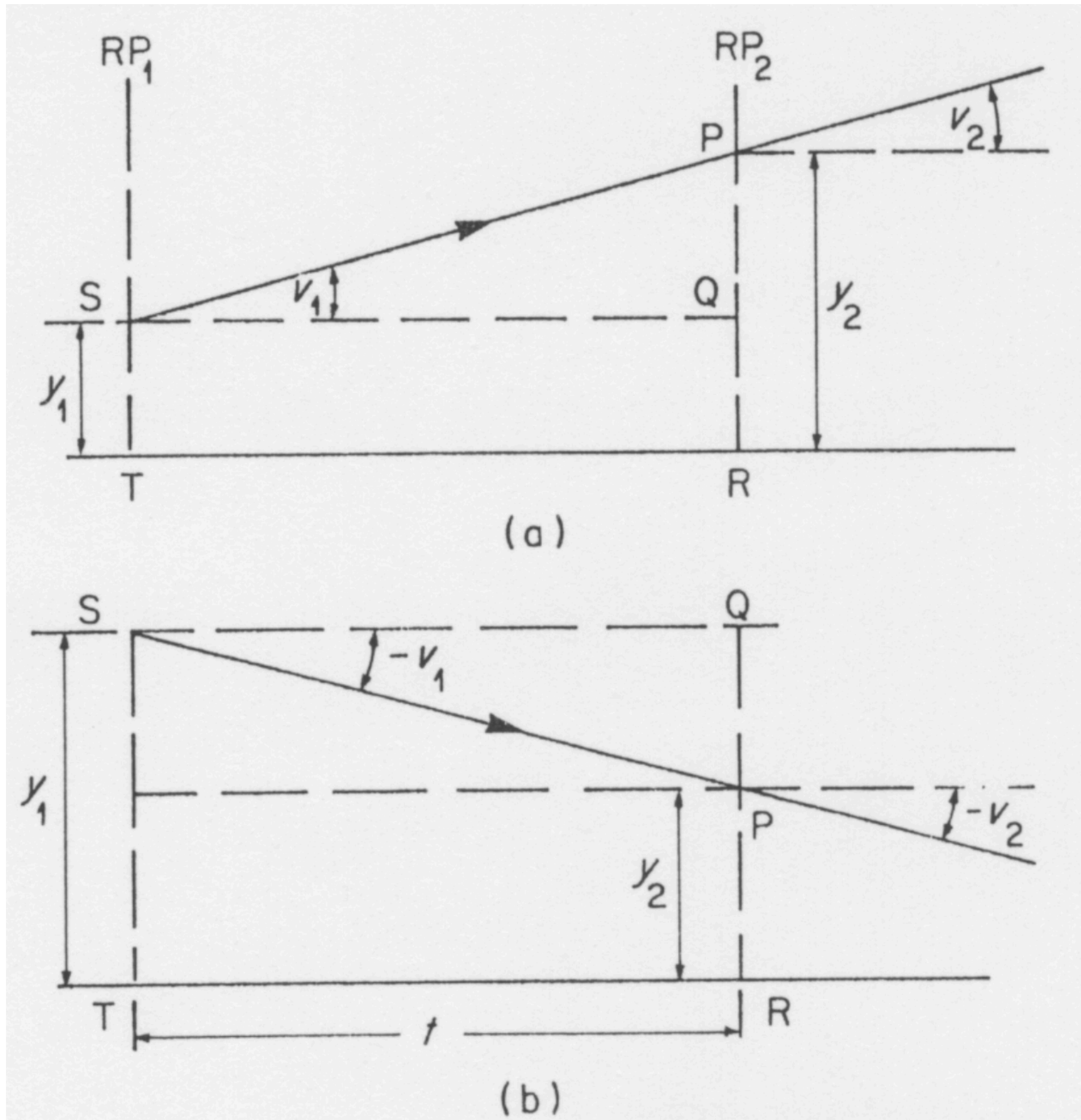
Alternatively, if we want to trace a ray backwards, the matrix equation can be inverted to give

$$\begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} y_2 \\ V_2 \end{bmatrix}$$

The ray-transfer matrix representing a complex optical element is the matrix product of all the ray-transfer matrices of which it is composed.

The Translation Matrix

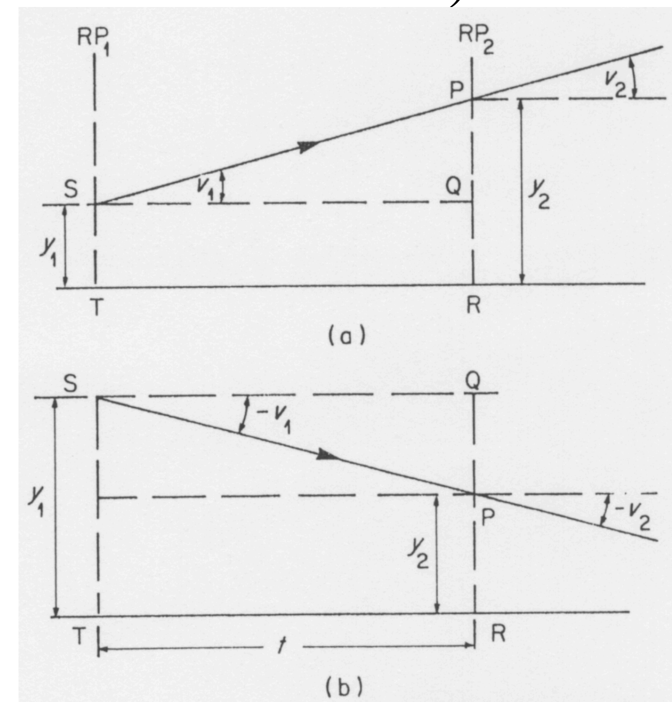
The figures 2(a) and 2(b) below show two examples of a ray which travels a distance t to the right between the reference planes.



Clearly the angle of the ray will remain unchanged during the translation, but not the distance y from the optical axis.

Figure 2(a) illustrates the case where both y- and V-values remain positive and Figure 2(b) illustrates the case where the V-value is negative.

In both figures the angles v are greatly exaggerated(generally less than 0.1 rad or 6°)



Referring to Figure 2(a)

$$y_2 = RP = RQ + QP$$
$$= TS + SQ\tan(PSQ)$$
$$= y_1 + t\tan(v_1)$$
$$= y_1 + tv_1$$

Referring to Figure 2(b)

$$y_2 = RP = RQ - QP$$
$$= TS - SQ\tan(PSQ)$$
$$= y_1 - t\tan(-v_1)$$
$$= y_1 + tv_1$$

If n is the refractive index of the medium between RP_1 and RP_2, we can write

$$y_2 = y_1 + tv_1 = y_1 + \left(\frac{t}{n}\right)nv_1 = (1)y_1 + TV_1$$

where

$$T = \frac{t}{n} = \text{reduced thickness}$$

It is clear from the diagrams that $v_1 = v_2$ so that we can write

$$V_2 = nv_2 = nv_1 = (0)y_1 + (1)V_1$$

This pair of equations can be written using matrices as

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

Thus, the matrix representing a translation to the right through a reduced distance T is

$$\Im = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Clearly $\det(\Im) = 1$.

Compound Layers and Plane-Parallel Plates

If we imagine that we divide the distance t into two segments t_1 and t_2, both with index of refraction n, we can write two successive translation matrices

$$\Im_1 = \begin{bmatrix} 1 & T_1 \\ 0 & 1 \end{bmatrix} \quad , \quad \Im_2 = \begin{bmatrix} 1 & T_2 \\ 0 & 1 \end{bmatrix}$$

$$T_1 = \frac{t_1}{n} \quad , \quad T_2 = \frac{t_2}{n}$$

and we note that

$$\Im_1\Im_2 = \Im_2\Im_1 = \begin{bmatrix} 1 & T_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & T_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T_1 + T_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \Im$$

According to our rules above we should write $\mathfrak{I} = \mathfrak{I}_1\mathfrak{R}\mathfrak{I}_2$ where $\mathfrak{R}$ is the refraction matrix at the boundary between the two segments.

Clearly, $\mathfrak{R} = I$ = identity matrix in this case(same medium on both sides of boundary).

We will prove this shortly.

A similar situation applies if we divide into many segments.

We have

$$\mathfrak{I} = \mathfrak{I}_1\mathfrak{R}\mathfrak{I}_2\mathfrak{R}\mathfrak{I}_3\ldots\ldots\ldots\ldots\mathfrak{R}\mathfrak{I}_N = \mathfrak{I}_1\mathfrak{I}_2\mathfrak{I}_3\ldots\ldots\ldots\ldots\mathfrak{I}_N = \begin{bmatrix} 1 & \sum_{i=1}^{N} T_i \\ 0 & 1 \end{bmatrix}$$

Clearly, the order of the segments in this case is irrelevant.

When we say that a plane-parallel glass plate of refractive index n and thickness t has a reduced-thickness t/n we are using very appropriate language.

If we are looking at an object on the far side of the plate the light actually takes longer to reach us then if the plate were absent, but the object certainly looks closer.

Replacing a layer of air by the same thickness of glass makes the world seem closer to the observer by a distance

$$\frac{t}{1} - \frac{t}{n} = \frac{t(n-1)}{n}$$

which is about 1/3 the thickness of the plate.

For an object submerged in water, the factor (n-1)/n is only about 1/4, but the depth t may be very large - even a bear fishing with his paw needs to know about reduced thickness!

The Refraction Matrix $\Re$

We now consider the action of a curved surface separating two regions of refractive index n_1 and n_2.

The radius of curvature of the surface will be taken to be positive if the center of curvature lies to the right of the surface.

This situation is shown in Figure 3 below, which shows a surface of positive curvature with the refractive index n_2 on the right of the surface greater than that on the left (n_1).

The ray also shows that we have positive y- and V-values on both sides of the surface.

The angles are again greatly exaggerated.

As a consequence, RP_1 appears well separated from RP_2.

But for paraxial rays, the separation between these two planes will be r(1-cosα) and is very small since α (like v_1 and v_2) is very small.

We therefore have $y_1 = y_2$.

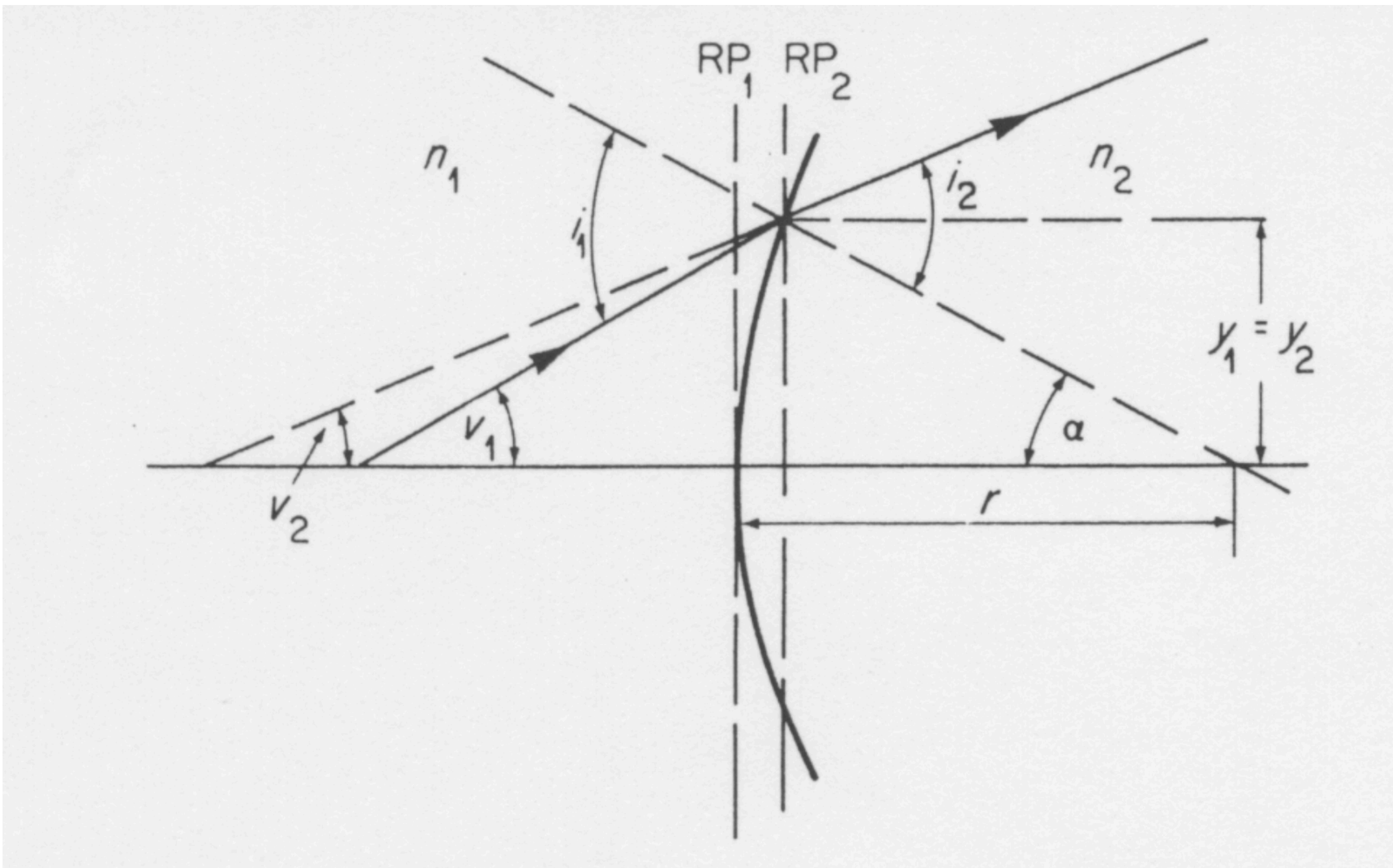


Figure 3

Applying Snell's law in the diagram, we have

$$n_1 \sin i_1 = n_2 \sin i_2 \rightarrow n_1 i_1 = n_2 i_2$$

in the paraxial approximation.

But simple geometry says

$$i_1 = v_1 + \alpha = v_1 + \frac{y_1}{r} \quad and \quad i_2 = v_2 + \alpha = v_2 + \frac{y_2}{r}$$

Hence,

$$n_1\left(v_1 + \frac{y_1}{r}\right) = n_2\left(v_2 + \frac{y_1}{r}\right)$$

or

$$V_1 + \frac{n_1 y_1}{r} = V_2 + \frac{n_2 y_1}{r}$$

Thus, rearranging the equations in matrix form, we get

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{n_2 - n_1}{r} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

The quantity $(n_2-n_1)/r$ is the refractive power of the surface.

We thus have the refraction matrix

$$\Re = \begin{bmatrix} 1 & 0 \\ -\dfrac{n_2 - n_1}{r} & 1 \end{bmatrix}$$

Even though this was derived for a special case (all quantities positive), it turns out to be general and works in all other cases.

Two special cases are:

$$n_2 = n_1 \text{ which says } \mathfrak{R} = I$$

and $r \to \infty$ (a plane surface) which also says that $\mathfrak{R} = I$

We used this result earlier when discussing plane-parallel plates.

Thin Lens Approximation

When several refracting surfaces are so close together that the intervening gaps are very small, the translation matrices for the gaps reduce to identity matrices.

Thus we have the situation where

$$\mathfrak{I}_1\mathfrak{R}_1\mathfrak{I}_2\mathfrak{R}_2\mathfrak{I}_3\mathfrak{R}_3\mathfrak{I}_4 \to \mathfrak{I}_1\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3\mathfrak{I}_4$$

that is, the transfer matrix for the system is just the matrix product of refraction matrices.

If the i_{th} refracting surface has a curvature r_i and refractive indices n_i and n_{i+1}, we are able to represent its refracting power by

$$P_i = \frac{n_{i+1} - n_i}{r_i}$$

The thin lens system is represented by

$$\mathfrak{I}_1\mathfrak{R}_1\mathfrak{I}_2\mathfrak{R}_2\mathfrak{I}_3 \rightarrow \mathfrak{I}_1\mathfrak{R}_1\mathfrak{R}_2\mathfrak{I}_4$$

so we are interested in the matrix product $\mathfrak{R}_1\mathfrak{R}_2$ where

$$\mathfrak{R}_1 = \begin{bmatrix} 1 & 0 \\ -\dfrac{n-1}{r_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix} \quad , \quad \mathfrak{R}_2 = \begin{bmatrix} 1 & 0 \\ -\dfrac{1-n}{r_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix}$$

$$\mathfrak{R}_1\mathfrak{R}_2 = \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(P_1 + P_2) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\dfrac{n-1}{r_1} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{1-n}{r_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

The refracting power P is measured in diopters, the focal length f and radii of curvature r_1 and r_2 in meters.

With these simple matrices we can do all of geometrical optics

In general, if calculations are to be made with a series of thin lenses whose focal lengths are given, it is convenient to replace each focal length f_i by its corresponding refracting power or focal power $P_i = 1/f_i$.

The refraction matrix for the i_{th} lens is then

$$\Re_i = \begin{bmatrix} 1 & 0 \\ -P_i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_i} & 1 \end{bmatrix}$$

Clearly, it is simple to collapse a succession of $\Re$ -matrices or a succession of $\Im$ - matrices and be careful about the order in which they arise.

For any refraction-translation product, matrix multiplication is **non-commutative**.

For example, we have

$$\mathfrak{R}\mathfrak{I} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ -P & 1-PT \end{bmatrix}$$

$$\mathfrak{I}\mathfrak{R} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} = \begin{bmatrix} 1-PT & T \\ -P & 1 \end{bmatrix}$$

The Ray-Transfer Matrix for a System

We consider the propagation of a paraxial ray through an optical system containing n refracting surfaces separated by (n-1) gaps.

We choose the first "input" reference plane RP_1 at distance d_a to the left of the first refraction surface. RP_2 and RP_3 will then lie immediately to the left and to the right of the first refracting surface; RP_4 and RP_5 will lie on either side of the second surface, and so on until we reach RP_{2n} and RP_{2n+1} on either side of the n_{th} surface.

Our final "output" reference plane RP_{2n+2} will then be taken to lie a distance d_b to the right of this last refracting surface.

We now want to find an overall ray-transfer matrix M that will enable us to directly convert an input ray vector

$$\begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

into an output ray vector

$$\begin{bmatrix} y_{2n+2} \\ V_{2n+2} \end{bmatrix}$$ (see Figure 4 below)

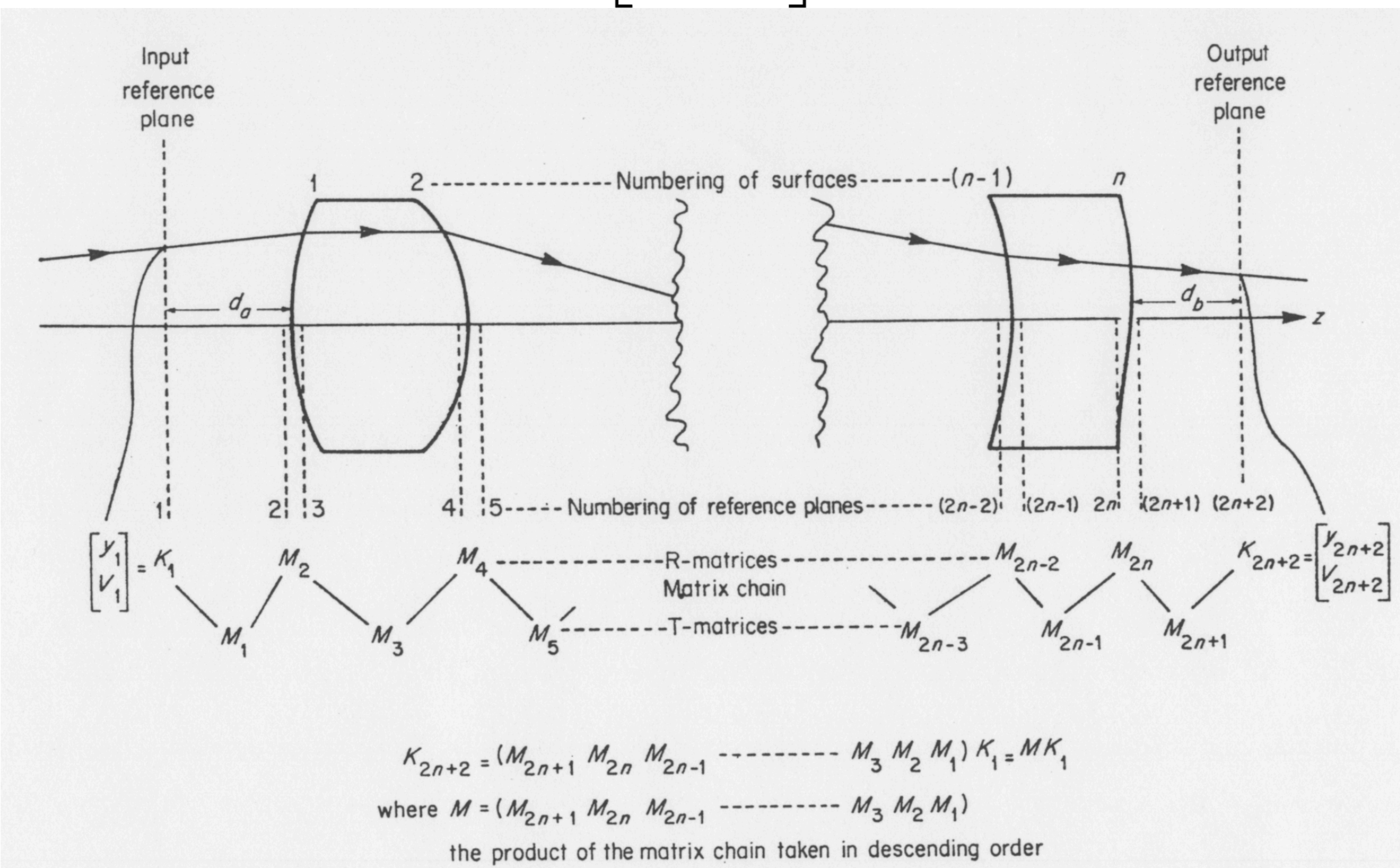


Figure 4

The next step is to write down translation or refraction matrices that represent each of the elements between the various reference planes.

Working from left to right across the diagram, we number these matrices $M_1 M_2 M_3 M_4 ....M_{2n+1}$, the rule being that the number assigned to each matrix is the same as for the reference plane on its left.

If we use the symbol K_r to denote the ray vector

$$\begin{bmatrix} y_r \\ V_r \end{bmatrix}$$

for a ray traversing the r_{th} reference plane, then for transfer of ray data from RP_r to RP_{r+1} we have the basic recurrence relation $K_{r+1} = M_r K_r$ and similarly $K_r = M_{r-1} K_{r-1}$ etc.

Calculation of Output Ray Produced by a Given Input Ray

By repeatedly using this recurrence relation, and also the associative property of matrix multiplication, we now find

$$\begin{aligned} K_{2n+2} &= M_{2n+1} K_{2n+1} = M_{2n+1}\left(M_{2n} K_{2n}\right) = \left(M_{2n+1} M_{2n}\right)\left(M_{2n-1} K_{2n-1}\right) \\ &= \left(M_{2n+1} M_{2n} M_{2n-1} M_{2n-2} ....M_3 M_2 M_1\right) K_1 \end{aligned}$$

Hence $K_{2n+2} = MK_1$ where M represents the complete matrix product taken in descending order $M = \left(M_{2n+1} M_{2n} M_{2n-1} M_{2n-2} ....M_3 M_2 M_1\right).$

Calculation of Input Ray Required to produce Given Output Ray

If we want to proceed from first principles, then we can invert each of the individual matrices and use the inverted recurrence relation

$$K_r = M_r^{-1} K_{r+1}$$

We then get

$$K_1 = M_1^{-1} K_2 = M_1^{-1} M_2^{-1} K_3 = \left(M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1} \ldots\ldots\ldots M_{2n+1}^{-1} \right) K_{2n+2}$$

Derivation of Properties of a System from its Matrix

Let us suppose that we have numerically evaluated the matrix M for a complete optical system, and that the equation $K_2 = MK_1$ when written out fully reads

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

where AD-BC=1.

In order to understand better the significance of these four quantities A, B, C, and D, let us consider what happens if one of them vanishes.

(a) If $D = 0$, the equation for V_2 reads $V_2 = (C)y_1+(0)V_1 = Cy_1$.

This means that all raysentering the input plane from the same point (characterized by y_1) emerge at the output plane making the same angle $V_2 = Cy_1$ with the axis, no matter what angle V_1 they entered the system.

It follows that the input plane RP_1 must be the **first focal plane** of the system as shown in Figure 5 below.

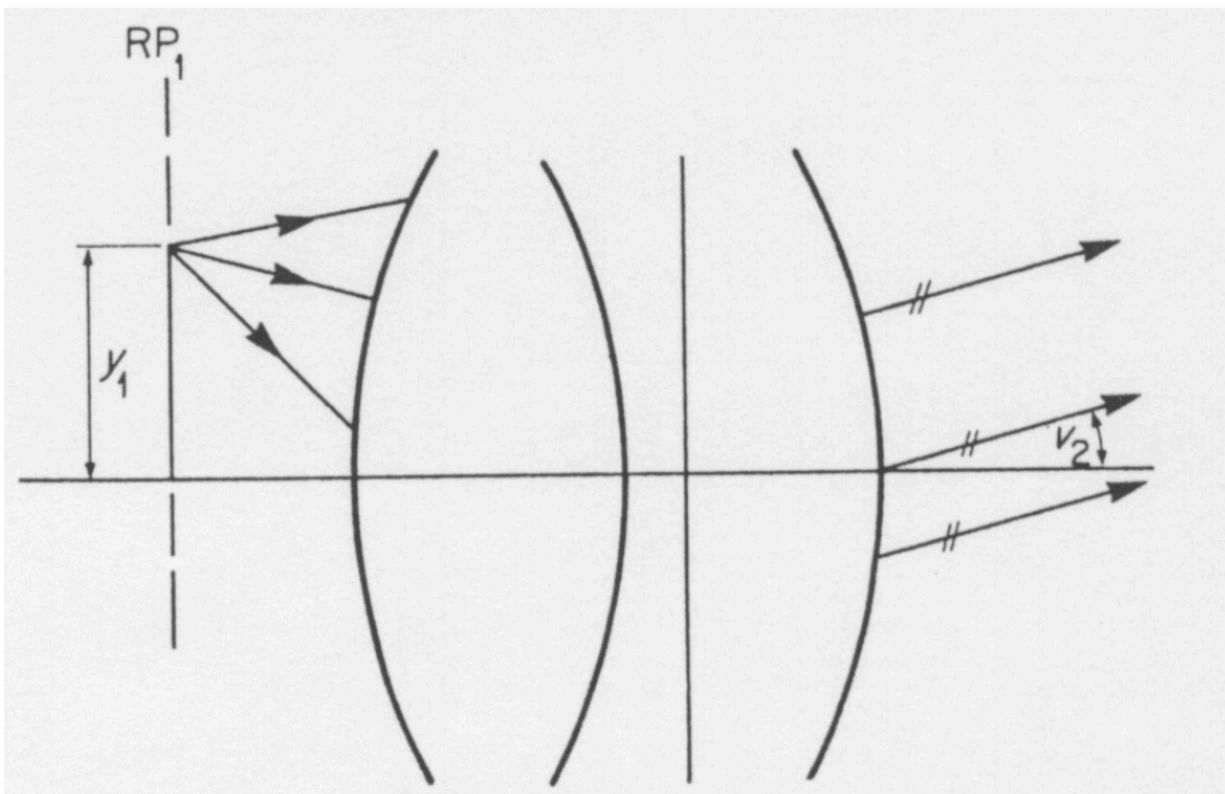


Figure 5

(b) If $B = 0$, the equation for y_2 reads $y_2 = (A)y_1+(0)V_1 = Ay_1$.

This means O (characterized by y_1) in RP_1 will pass through the same point I (characterized by y_2) in RP_2.

Thus O and I are object and image points, so that **RP_1 and RP_2 are now conjugate planes**.

In addition, $A = y_2/y_1$ gives the **magnification** produced by the system in this case as shown in Figure 6 below.

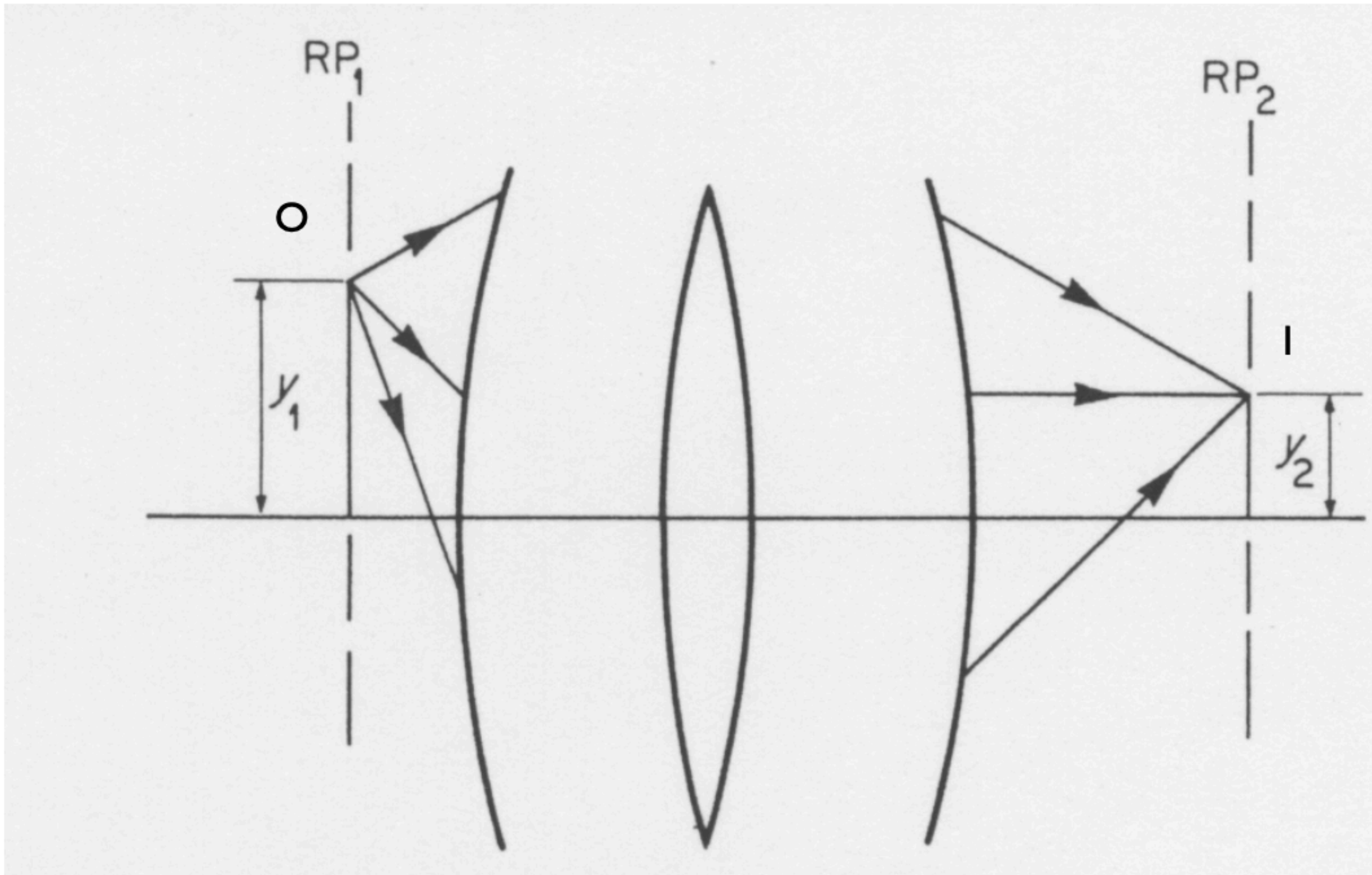


Figure 6

(c) If $C = 0$, then $V_2 = DV_1$.

This means that all rays which enter the system parallel to one another (V_1 gives their direction on entry) will emerge parallel to one another in a new direction V_2.

A lens system like this, which transforms a parallel beam into a parallel beam in a different direction, is called an **afocal** or **telescopic** system

In this case, $n_1D/n_2 = v_2/v_1$ is the angular magnification produced by the system as shown in Figure 7.

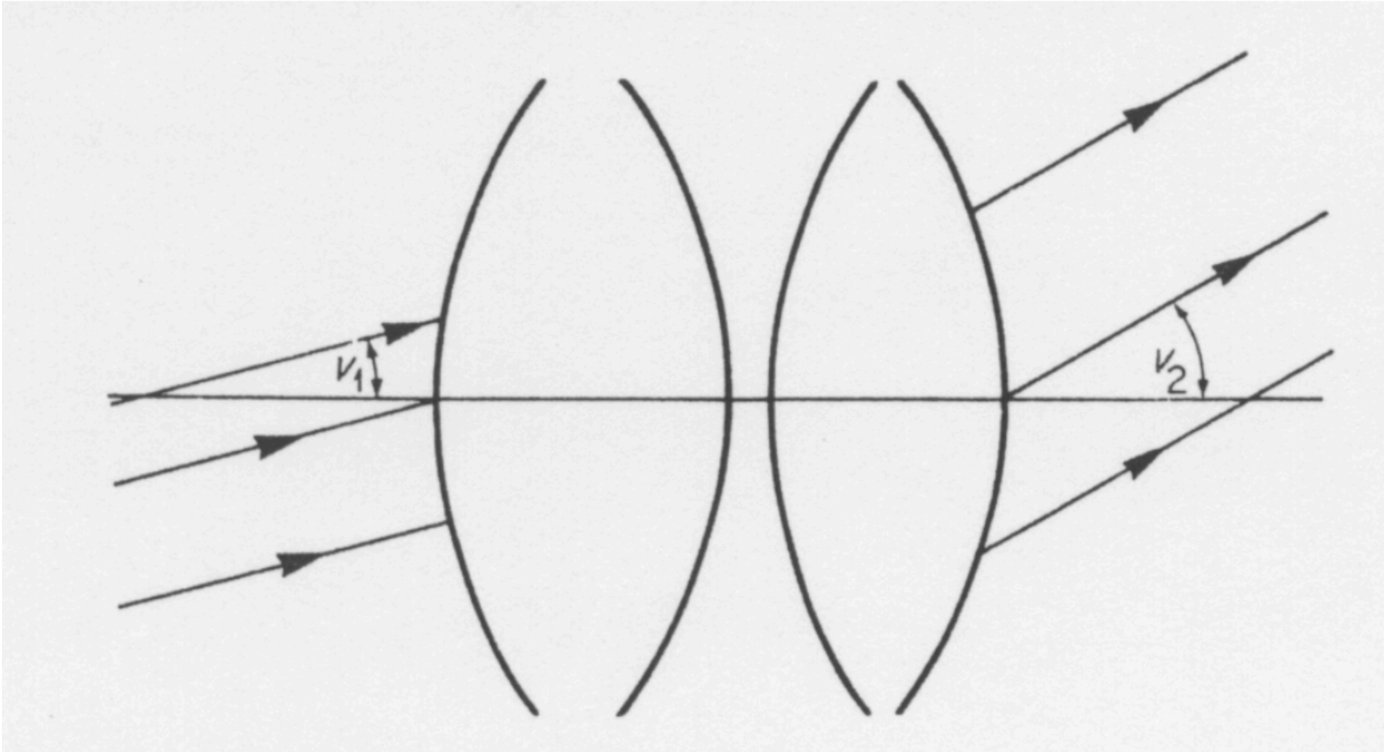


Figure 7

(d) If A = 0, the equation for y_2 reads $y_2 = BV_1$.

This means that all rays entering the system at the same angle (characterized by V_1) will pass through the same point (characterized by y_2) in the output plane RP_2.

Thus, the system brings bundles of parallel rays to focus at points in RP_2, that is, RP_2 is the **second focal plane** of the system as shown in Figure 8 below.

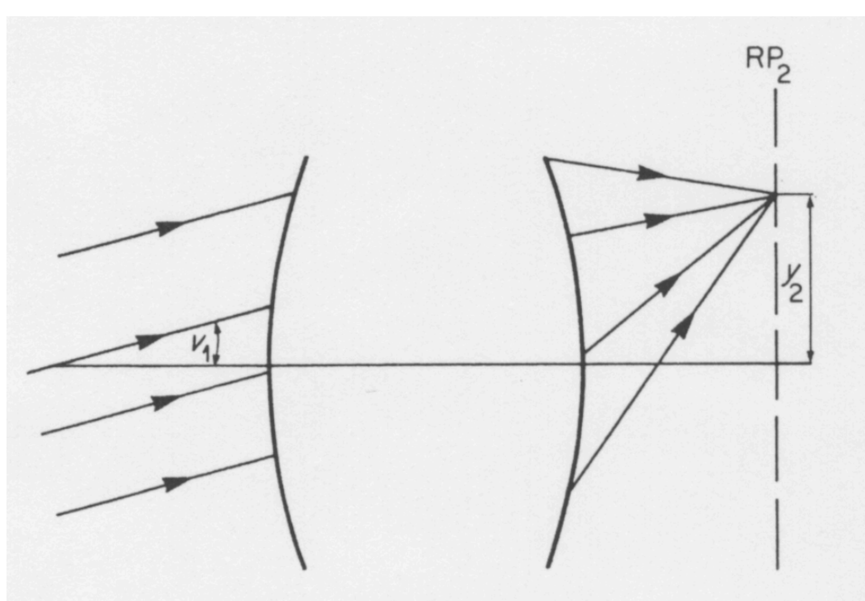


Figure 8

(e) It is worth remembering that, if either A or D in a ray-transfer matrix vanishes , the equation AD-BC=1 then requires that BC=-1.

Likewise, if either B or C vanishes, A must be the reciprocal of D.

The fact that B vanishes when the object-image relationship holds between the first and second reference planes, and that A or 1/D is then the transverse magnification, leads to an experimental method for finding the matrix elements of an optical system without dismantling it or measuring its individual components.

We will show this method after doing some examples.

In general, unless the V-value of a ray is zero it will cut the axis somewhere.

With respect to a given reference plane in which the ray height is y and the ray angle v is V/n, the z-position where it cuts the axis will be (-y/v) = (-ny/V).

If y and V are both positive or both negative, the point will be to the left of the reference plane as shown in Figure 9 below.

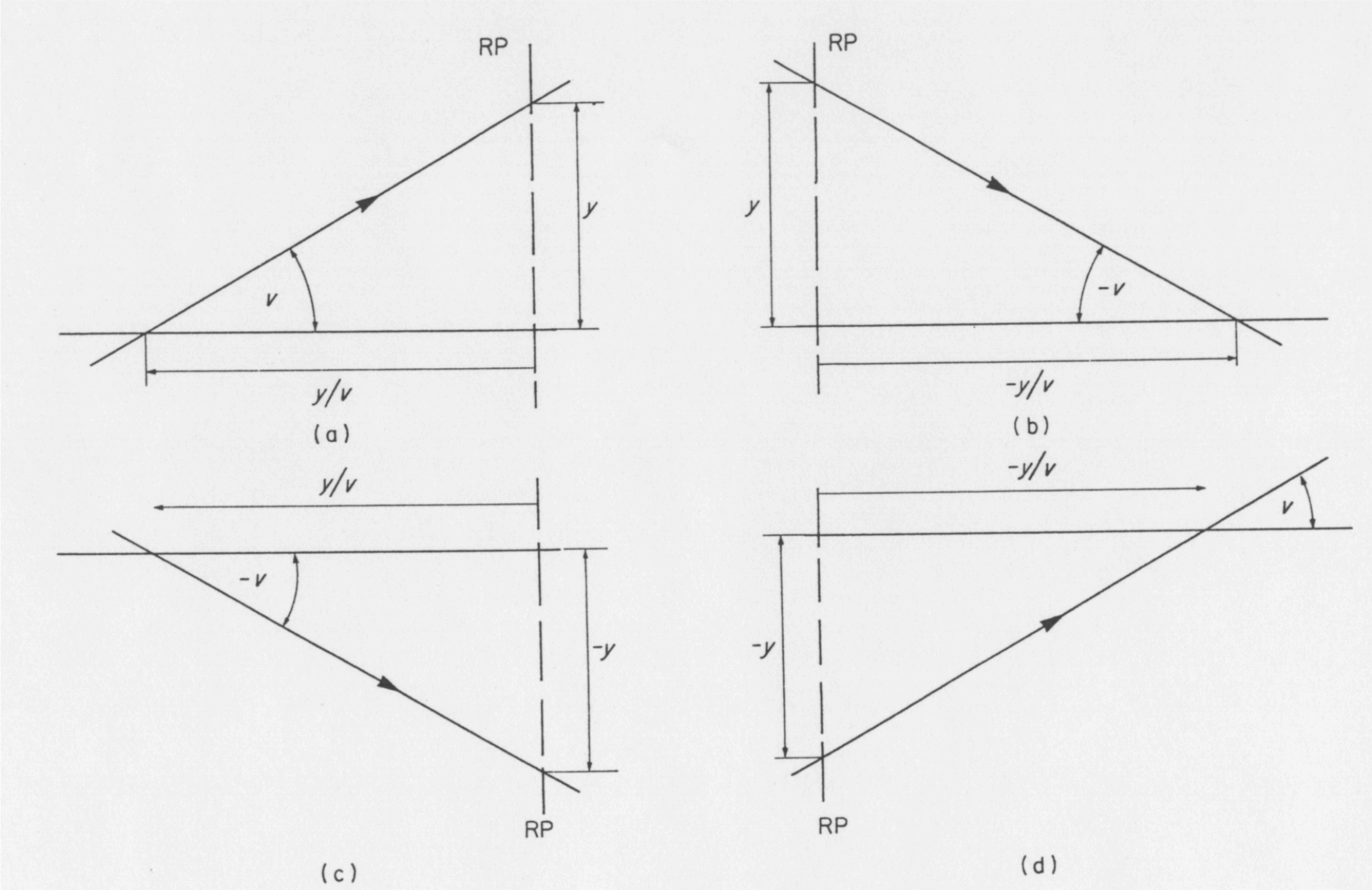


Figure 9

Example 1

The left hand of a long plastic rod of refractive index 1.56 is ground and polished to a convex (outward) spherical surface of radius 2.8 cm.

An object 2 cm tall is located in the air and on the axis at a distance of 15 cm from the vertex.

See Figure 10 below.

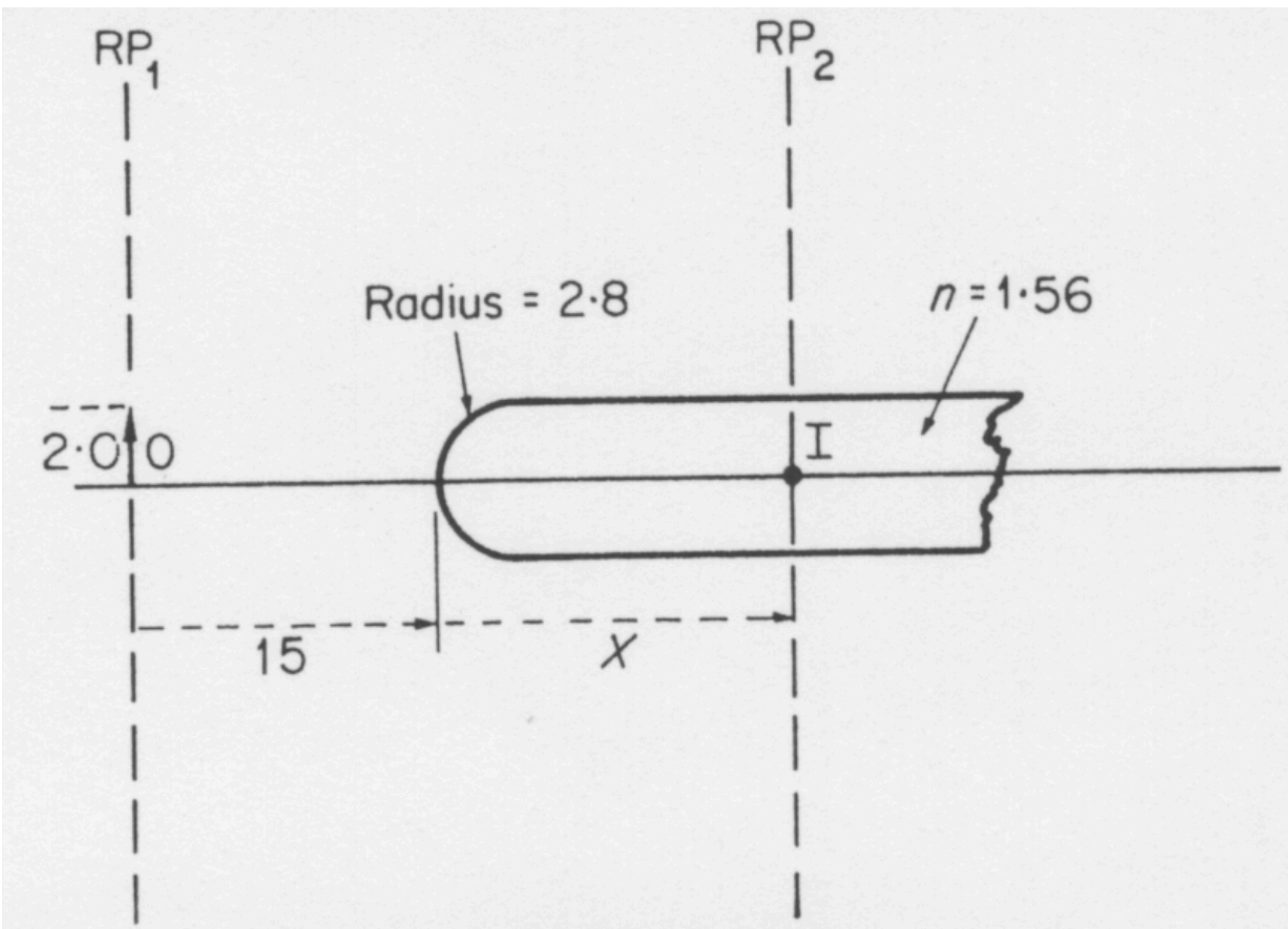


Figure 10

Find the position and size of the image inside the rod.

Solution: Using cm as the unit of length for both $\Re$- and $\Im$-matrices.

We proceed as follows.

The $\mathfrak{R}$-matrix of the surface is

$$\mathfrak{R} = \begin{bmatrix} 1 & 0 \\ -\dfrac{n_2 - n_1}{r} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1.56 - 1}{2.8} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix}$$

If the image is at X cm to the right of the curved surface, the matrix chain leading from this image back to the object is

$$M = \mathfrak{I}_{\substack{\text{from image} \\ \text{to surface}}} \mathfrak{R}_{\text{surface}} \mathfrak{I}_{\substack{\text{from surface} \\ \text{to object}}}$$

$$\mathrm{M} = \begin{bmatrix} 1 & X / n_{plastic} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 15 / n_{air} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & X / 1.56 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 15 / 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - X / 7.8 & 15 - X / 0.78 \\ -0.2 & -2 \end{bmatrix}$$

An easy check shows that det(M) = AD-BC = 1.

For the object-image relationship to hold, the top right element, B, must vanish:

$$15 - X / 0.78 = 0 \rightarrow X = 11.7 cm$$

The image is thus 11.7 cm inside the rod.

The lateral magnification is given by either A or 1/D and is -0.5.

Therefore, the image is (0.5 x 2) = 1 cm tall, and is inverted.

Example 2

A parallel beam of light enters a clear plastic spherical bead whose diameter is 2 cm and refractive index is 1.4 as shown in Figure 11 below:

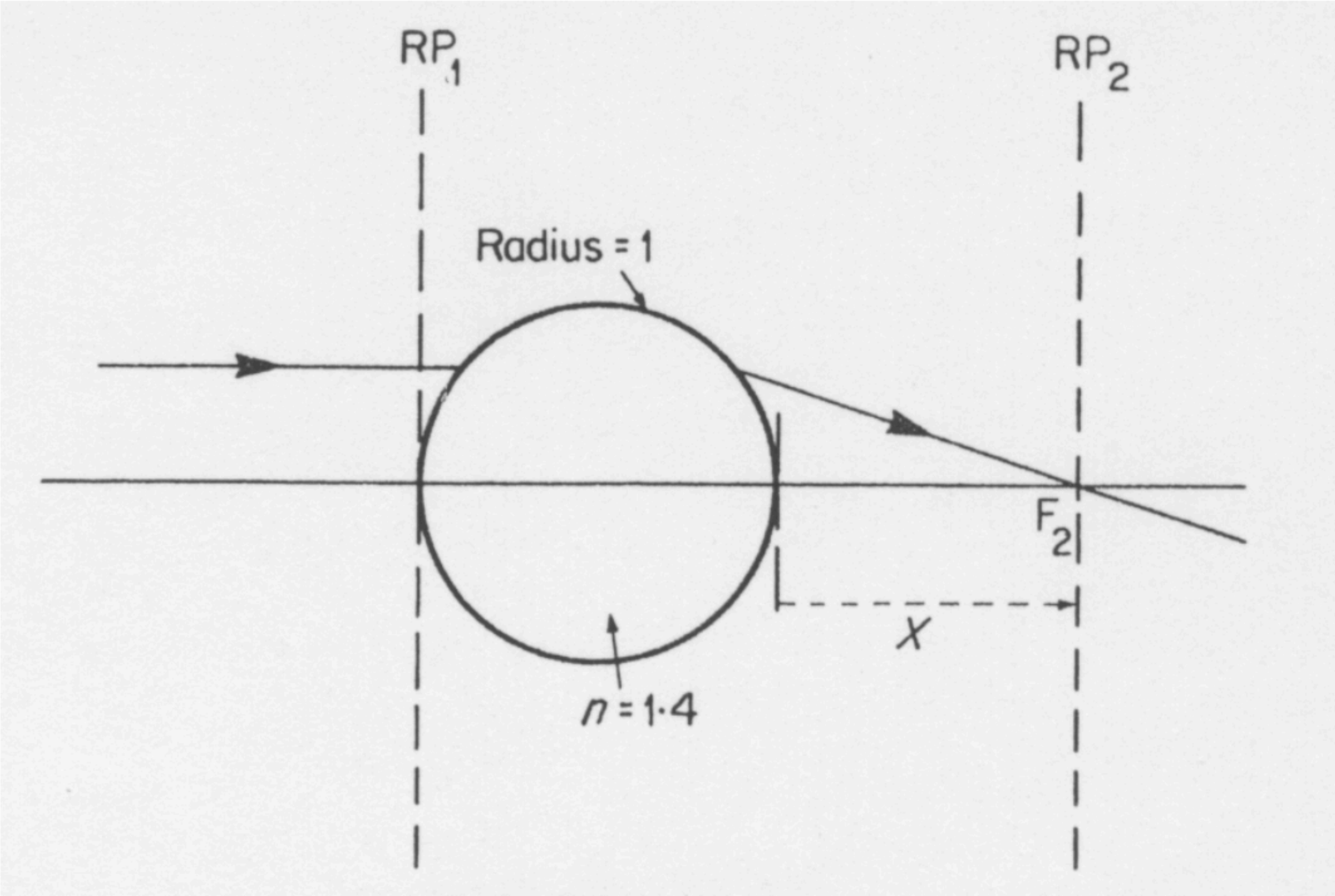


Figure 11

At what point beyond the bead is the light brought to a focus?

Solution: If we choose an output reference plane RP2 X cm to the right of the bead, the matrix chain is

$$M = \underset{\text{air gap}}{\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}} \underset{\text{right face}}{\begin{bmatrix} 1 & 0 \\ -\dfrac{(1-1.4)}{-1} & 1 \end{bmatrix}} \underset{\substack{\text{between} \\ \text{faces}}}{\begin{bmatrix} 1 & \dfrac{2}{1.4} \\ 0 & 1 \end{bmatrix}} \underset{\text{left face}}{\begin{bmatrix} 1 & 0 \\ -\dfrac{(1.4-1)}{1} & 1 \end{bmatrix}}$$

$$M = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.429 & 1.429 \\ -0.571 & 0.429 \end{bmatrix} = \begin{bmatrix} 0.429 - 0.571X & 1.429 + 0.429X \\ -0.571 & 0.429 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The condition for parallel light to come to a focus in RP2 is that element A vanishes.

$$0.429 - 0.571X = 0 \rightarrow X = 0.75$$

Alternatively, this result could be derived directly from the matrix for the bead

$$\begin{bmatrix} 0.429 & 1.429 \\ -0.571 & 0.429 \end{bmatrix}$$

For, if $V_1 = 0$,

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Ay_1 \\ Cy_1 \end{bmatrix}$$

Therefore, the distance at which the $\begin{bmatrix} y_2 \\ V_2 \end{bmatrix}$ ray cuts the axis is $-ny_2 / V_2 = -A / C$ *to the right of the bead = 0.429/0.571 = 0.75.*

This calculation is only valid for central paraxial rays of the beam - the outer region will show an effect called spherical aberration.

Example 3

A slide 2 inches tall is located 10.5 feet from a screen as shown in Figure 12 below.

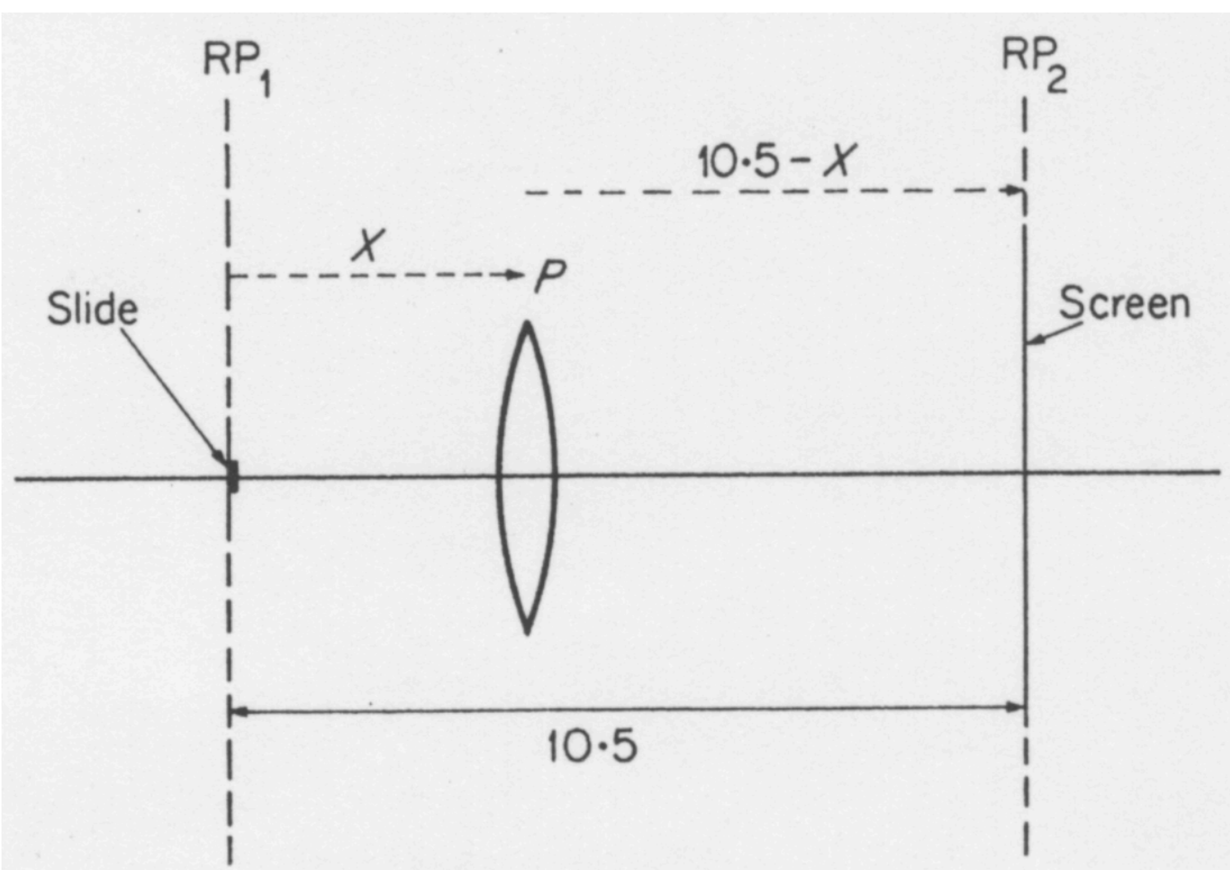


Figure 12

What is the focal length of the lens which will project an image measuring 40 inches on the screen, and where must it be placed?

Solution: Although the object and image sizes are given in inches, it is more convenient to use feet for calculating the matrix elements.

The lens will be taken to be a thin positive lens whose power P is measured in reciprocal foot units.

Let it be placed X feet from the slide and (10.5-X) feet from the screen.

The matrix chain from screen to slide is then

$$M = \underbrace{\begin{bmatrix} 1 & 10.5 - X \\ 0 & 1 \end{bmatrix}}_{\text{screen to lens}} \underbrace{\begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}}_{\text{lens}} \underbrace{\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}}_{\text{lens to slide}}$$

$$= \begin{bmatrix} 1 & 10.5 - X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & X \\ -P & 1 - PX \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 - 10.5P + PX & X + (10.5 - X)(1 - PX) \\ -P & 1 - PX \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The real image produced from a real object by a single lens is always inverted, so the magnification specified must be -40/2 = -20.

In the above matrix, therefore, A = 1/D = -20 and B = 0, to secure the object-image relationship.

Since D = 1-PX = -0.05, the equation for B becomes

$$B = X - 0.05(10.5 - X) = 0 \rightarrow X = 0.5$$

The equation for D now becomes

$$1.05P = -0.05 \rightarrow P = +2.1$$

The focal length of the lens is therefore 1/2.1 feet = 5.7 inches.

Thus, the lens needs to have a positive focal length of 5.7 inches and it should be placed 6 inches from the slide.

Example 4

An object is located at a distance U to the left of a thin lens and its image is formed a distance V to the left of the same lens as shown in Figure 13 below:

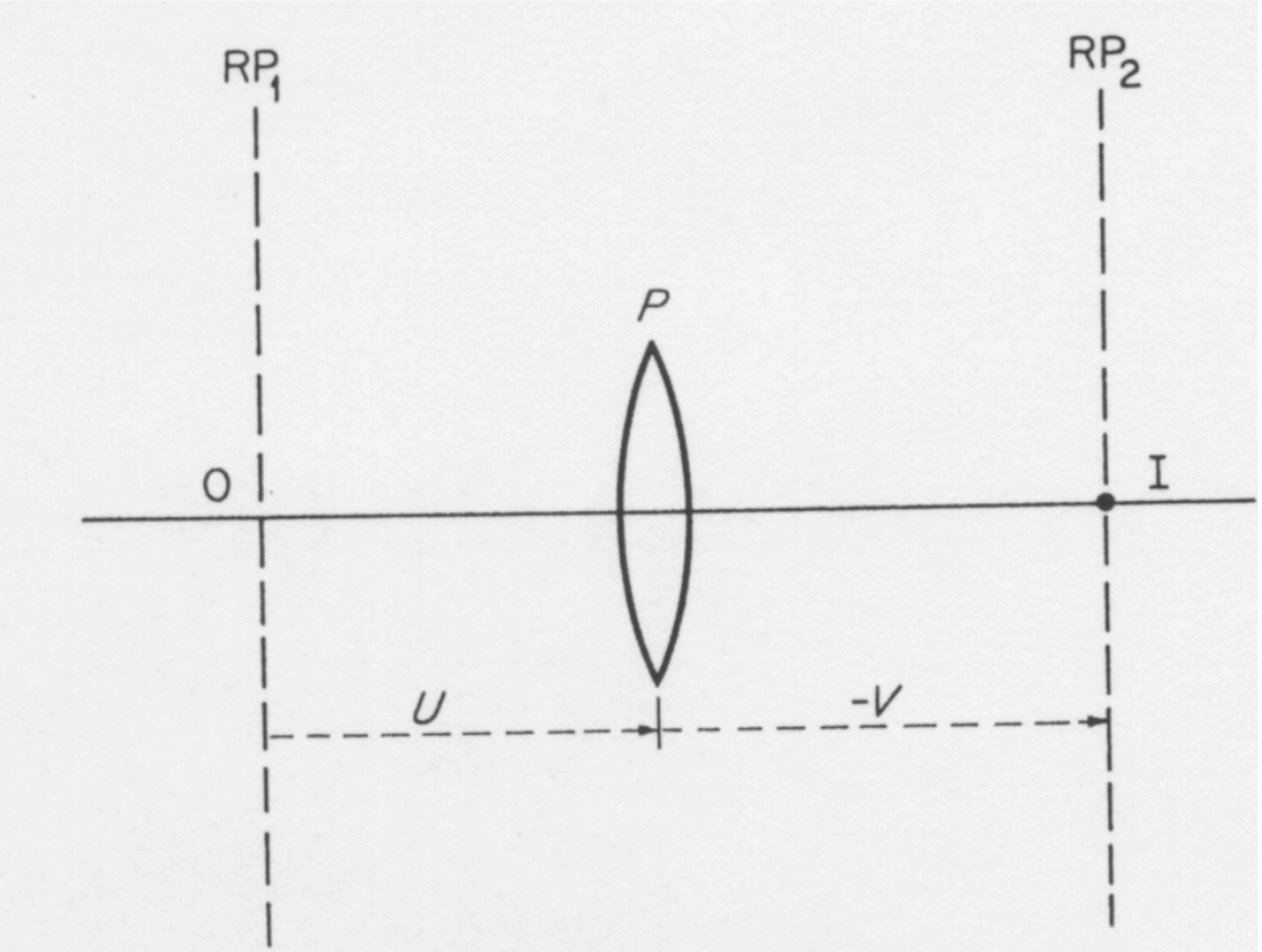


Figure 13

If now the object is displaced axially a small distance dU to the left, find an expression for the corresponding displacement dV of the image; dU/dV is called the longitudinal magnification.

Show that it is the square of the lateral magnification.

Note that if the image formed is a real one, V will have a negative value; but, whether the thin lens is positive or negative, there will be some object distance for which the image is virtual, in which case V is positive.

Solution: The matrix chain from image back to object is

$$M = \underset{\substack{\text{image to}\\ \text{lens with}\\ \text{V negative}\\ \text{as measured}\\ \text{in - z-direction}}}{\begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix}} \underset{\text{lens}}{\begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}} \underset{\substack{\text{lens to}\\ \text{object}}}{\begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & U \\ -P & 1-PU \end{bmatrix} = \begin{bmatrix} 1+PV & U-V+PUV \\ -P & 1-PU \end{bmatrix}$$

For the object-image relationship, the top right-hand element U-V-PUV must vanish.

Hence

$$V = \frac{U}{1-PU}$$

and differentiating

$$\frac{dV}{dU} = \frac{(1-PU)(1)+UP}{(1-PU)^2} = \frac{1}{(1-PU)^2}$$

As the bottom right-hand element indicates, one expression for the lateral magnification is 1/(1-PU), whose square agrees with the expression for dV/dU.

Other expressions that can be found for the lateral magnification are (1+PV) and V/U.

We note that in obtaining this solution, we have determined a basic relationship between P, U, and V for a thin lens.

Since -V+PUV = 0, we can divide by UV and rearrange to obtain the familiar lens formula

$$\frac{1}{U}-\frac{1}{V}=P=\frac{1}{f}$$

where f is the focal length of the lens.

Example 5

Prove that the distance between the real object and real image formed by a thin positive lens cannot be less than four times the focal length. See Figure 14 below.

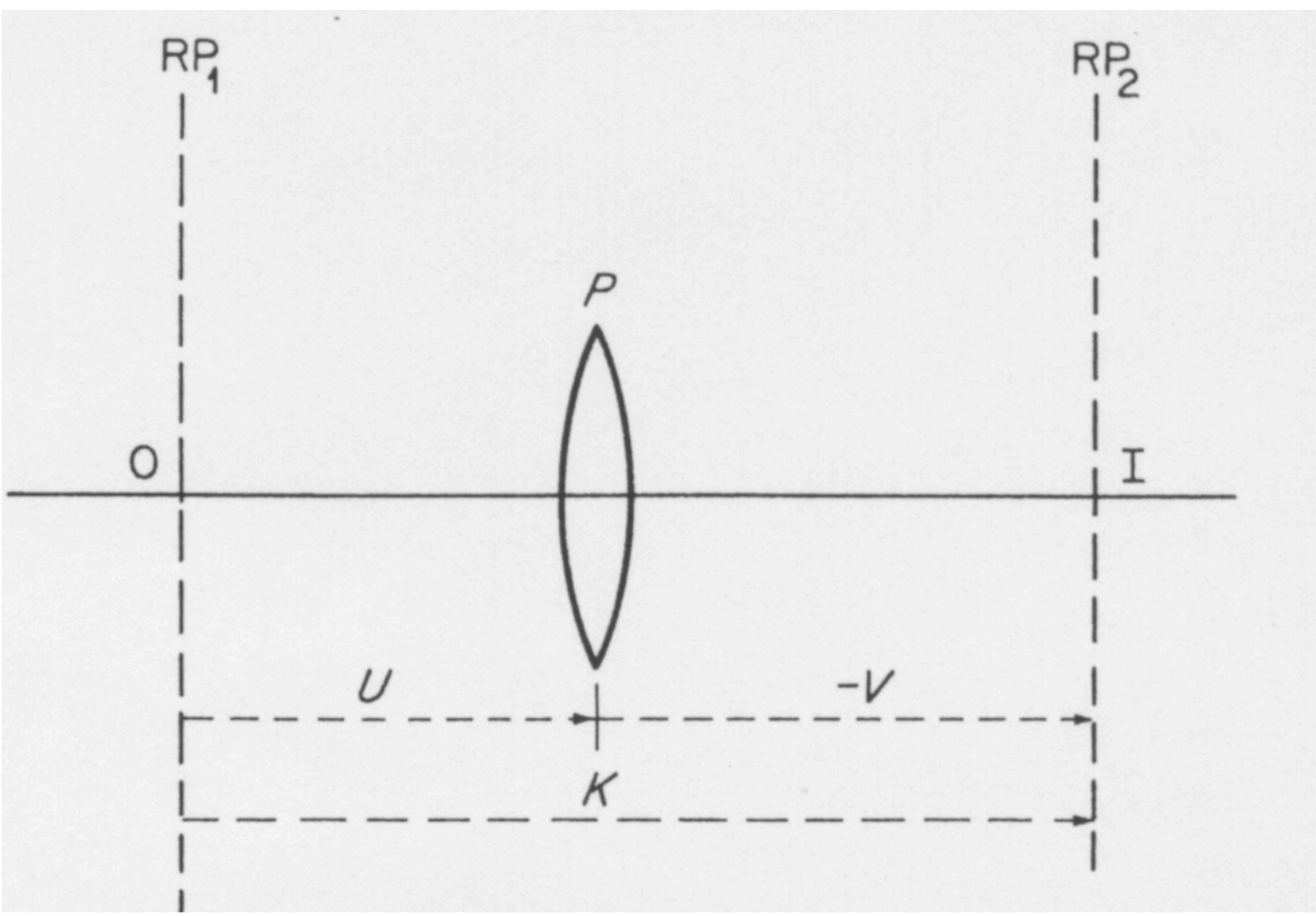


Figure 14

Solution: As in Example 4, the top right-hand element in the image-to-object matrix is U-V+PUV, and this must vanish.

Furthermore, since U and V are both measured to the left of the lens, U must be positive, but V must be negative for a real image to be formed from a real object.

If the object-image distance is K, then K = U-V and V = U-K. Substituting for V, therefore, we have

$$U-(U-K)+PU(U-K)=0 \rightarrow K=\frac{U}{PU-1}+U$$

Therefore,

$$\frac{dK}{dU}=\frac{(PU-1)-UP}{(PU-1)^2}+1=1-\frac{1}{(PU-1)^2}$$

For dK/dU to vanish, we must have |PU-1| = 1, hence PU = 2 or PU = 0 (the trivial case).

Since both P and U are assumed to be positive, the case PU = 2 needs to be considered further by considering the second derivative.

$$\frac{d^2K}{dU^2}=\frac{d}{dU}\left(-\frac{1}{(PU-1)^2}\right)=\frac{2P}{(PU-1)^3}$$

Now for the case PU = 2, the denominator is unity and the numerator is positive; so d^2K/dU^2 is positive and the value of K must be passing through a minimum. If we now substitute U = 2/P in the expression for K, we find

$$K_{\min} = \frac{2/P}{1} + 2/P = 4/P = 4f$$

where f = 1/P is the focal length of the lens.

Alternatively, if we had used +V instead of -V

$$U + V - PUV = 0 \quad , \quad K = U + V$$

$$U + (K - U) - PU(K - U) = 0 = K - PUK + PU^2$$

$$\Rightarrow K = -\frac{PU^2}{1 - PU} = \frac{PU^2}{PU - 1} = \frac{U}{PU - 1} + U$$

which is the same result as before.

Experimental Determination of Matrix Elements of Optical System

We assume initially that the system that we are dealing with is a positive lens system located in air.

The first step is to choose two conveniently accessible planes near the first and last surfaces of the system - these will be the input reference plane RP_1 and the output reference plane RP_2.

We want to determine elements A, B, C, and D such that

$$\begin{bmatrix} y_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ V_1 \end{bmatrix}$$

The procedure that we will follow is to locate an object of known size at various distances to the left of RP_1 and then to measure the size and position of each of the resultant real images.

In the general case, let R represent the displacement in the +z-direction from the object to RP_1, and let S represent the displacement, again in the +z-direction, from RP_2 to the real image.

We will operate so that R and S are positive.

The ray-transfer matrix chain from the plane containing the image back to the plane containing the object is then

$$M = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & AR+B \\ C & CR+D \end{bmatrix}$$

$$= \begin{bmatrix} A+SC & AR+B+S(CR+D) \\ C & CR+D \end{bmatrix}$$

For this matrix we know that its determinant is unity and that its upper right-hand element vanishes because of the object-image relationship.

The upper left-hand element A+SC represents the transverse magnification and the lower right-hand element CR+D represents its reciprocal, which we denote by α.

The quantities that we can measure experimentally are the distances R and S and the ratio

$$\frac{height\ of\ object}{height\ of\ image} = \alpha = CR + D$$

Note that if the image is inverted, as it often is in practice, then α is negative.

If α is measured for several values of the object distances R and the results plotted as a graph of α versus R, this graph will be a straight line whose slope is C and whose intercept on the α-axis is D.

Thus, C and D are determined.

From the vanishing of the upper right-hand element, we have AR+B = -S(CR+D) = -S α = β, where β is known.

Once again, if β is plotted versus R, the slope of the graph will be A and the intercept on the β-axis will be B.

Once all four elements have been determined, a check should be made that the determinant AD-BC is approximately unity.

Locating the Cardinal Points of a System

We assume that, either by calculation or by experimental method we have determined, the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

which links the chosen output plane RP_2 to a chosen input plane RP_1.

We now want to locate, with respect to these reference planes, the two focal points, the principal planes of unit transverse magnification and the nodal planes of unit angular magnification.

We assume that n_1 and n_2 are the refractive indices to the left and to the right of the system.

(a) Consider first, as in Figure 15a below, a ray which enters the system parallel to the axis at a height y_1.

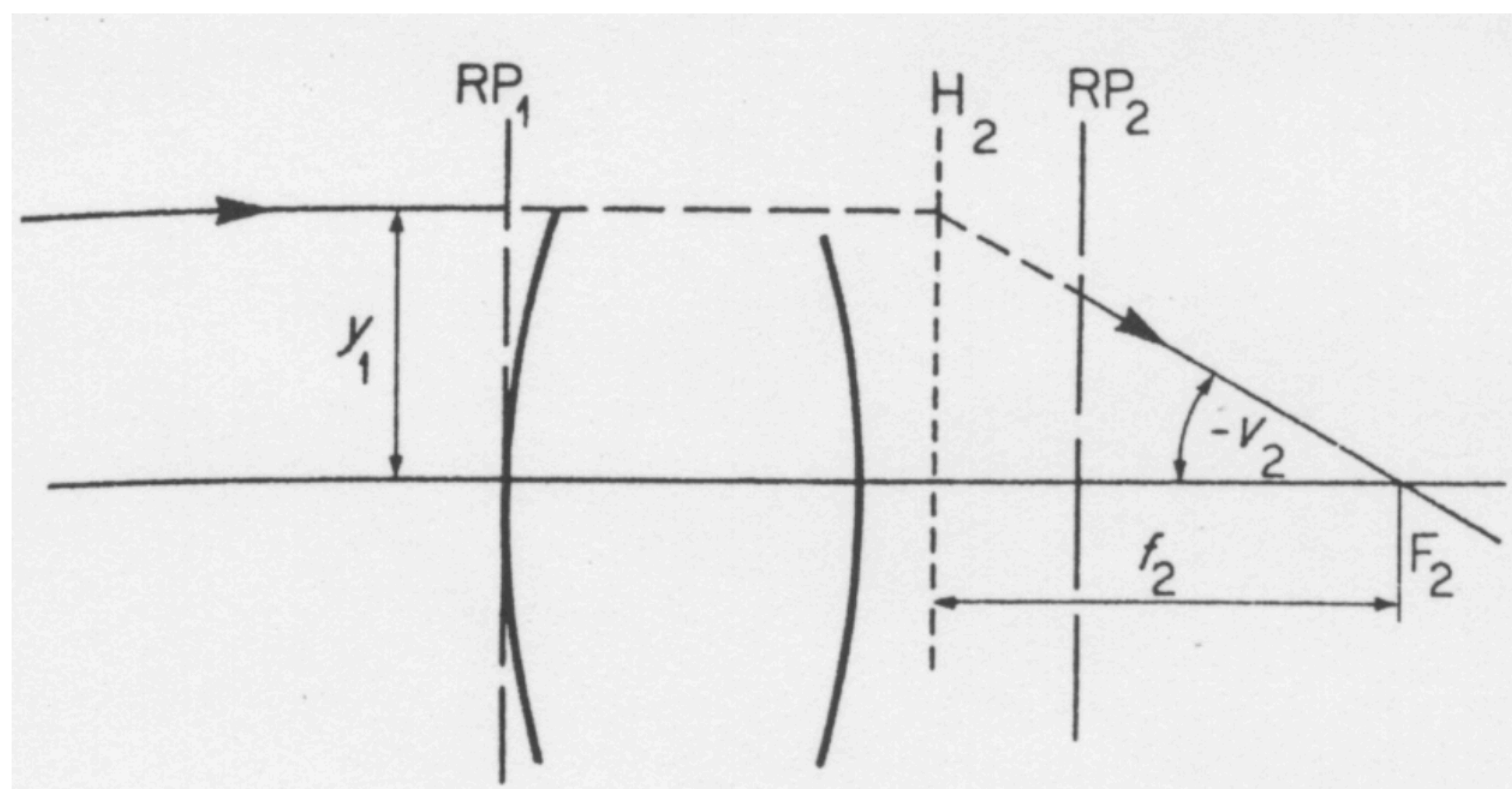


Figure 15a

The second principal plane H_2 is defined as shown in the figure.

The parallel ray arrives with the same height at the second principal plane H_2 and is there effectively bent in such a way that it emerges through the second focal point F_2.

For this ray, v_1 and therefore V_1 both vanish, so that at RP_2 the ray-transfer matrix give
$y_2 = Ay_1$ and $v_2 = V_2/n_2 = Cy_1/n_2$.

$$\begin{pmatrix} y_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$$

$$y_2 = Ay_1$$

$$V_2 = Cy_1 = n_2 v_2$$

$$\Rightarrow v_2 = \frac{Cy_1}{n_2}$$

If t_2 is the displacement measured in the +z-direction from RP_2 to F_2, then we must have $t_2 = -y_2/v_2 = -n_2A/C$.

This locates the second focus.

When the ray that we have been considering emerged from the second principal plane H_2, its y-coordinate was y_1.

Therefore, if we define the second focal length f_2 as the displacement from H_2 to F_2, its value must be $f_2 = -y_1/v_2$.

Therefore, $f_2 = -n_2y_1/Cy_1 = -n_2/C$, and the second focal length is determined.

The displacement from RP_2 to the second principal plane is therefore

$$s_2 = t_2 - f_2 = (-f_2) - (-t_2) = n_2(1-A)/C.$$

(b) Consider now the ray which enters the system at angle v_1 after passing through the first focus F_1 as shown in Figure 15b below.

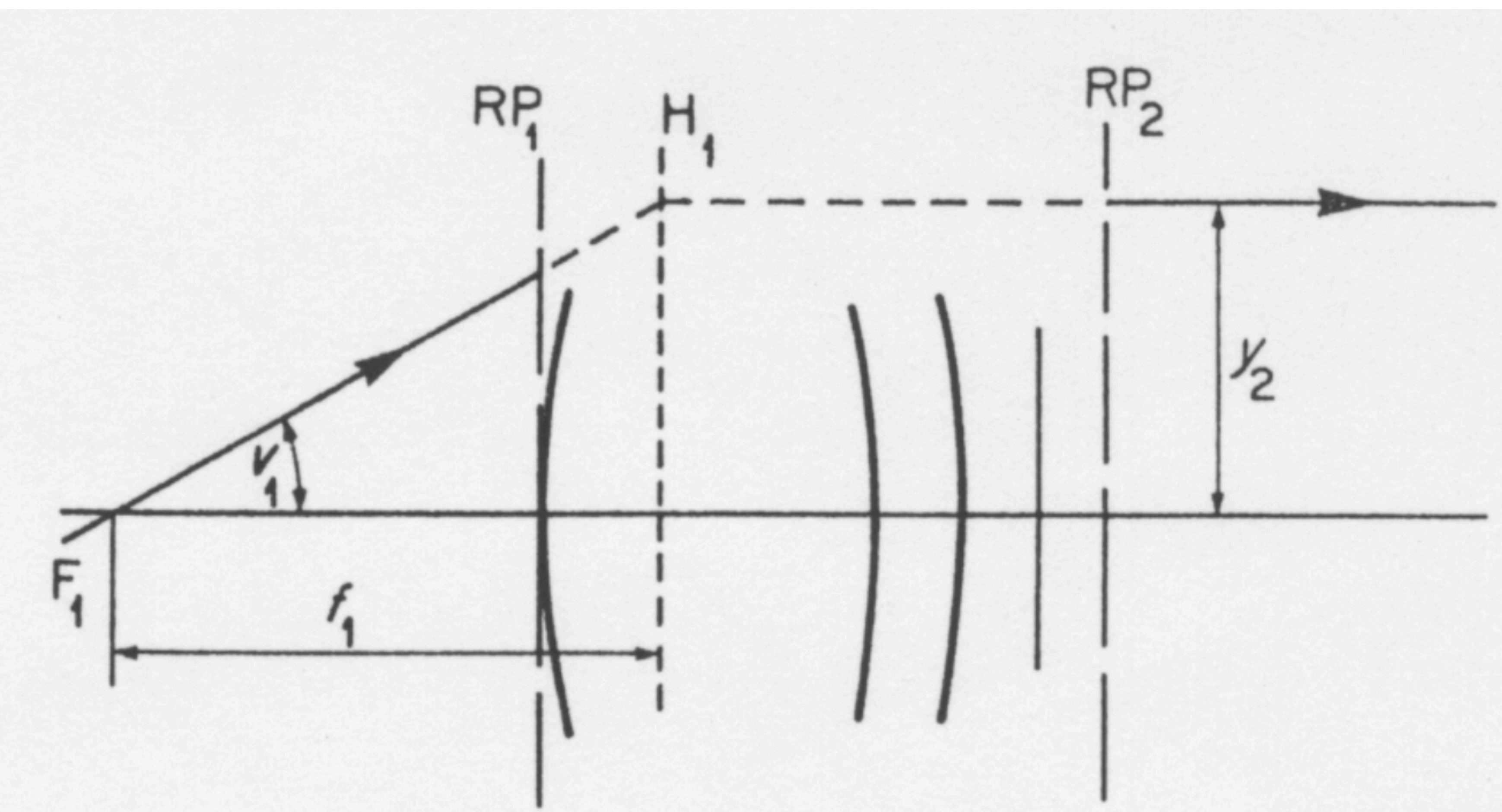


Figure 15b

It is effectively bent parallel to the axis at the first principal plane H_1 and continues through RP_2 with v_2, and therefore V_2, equal to zero.

We can therefore write $V_2 = Cy_1 + Dn_1v_1 = 0$ and $y_1 = -Dn_1v_1/C$.

From the diagram, the displacement t_1 of F_1 from RP_1 is given by $t_1 = -y_1/v_1 = n_1D/C$.

This locates the first focus.

Still considering the same ray, we recall that at the first principal plane its y-coordinate must be $y_2 = Ay_1 + Bn_1v_1$.

Therefore, if the first focal length f_1 is defined as the displacement from F_1 to H_1, we must have $f_1 = y_2/v_1 = -DAn_1/C + Bn_1 = -n_1(AD-BC)/C = -n_1/C$ for the first focal length since $\det(M) = AD-BC = 1$.

The displacement s_1 of the first principal plane H_1 from RP_1 is then

$$s_1 = t_1 + f_1 = -(-t_1) + f_1 = n_1(D-1)/C.$$

(c) Finally, we want to determine the two nodal points L_1 and L_2 such that any ray entering the system directed towards L_1 appears on emergence as a ray coming from L_2 and making the same angle $v_2 = v_1$ with the axis as shown in Figure 15c below.

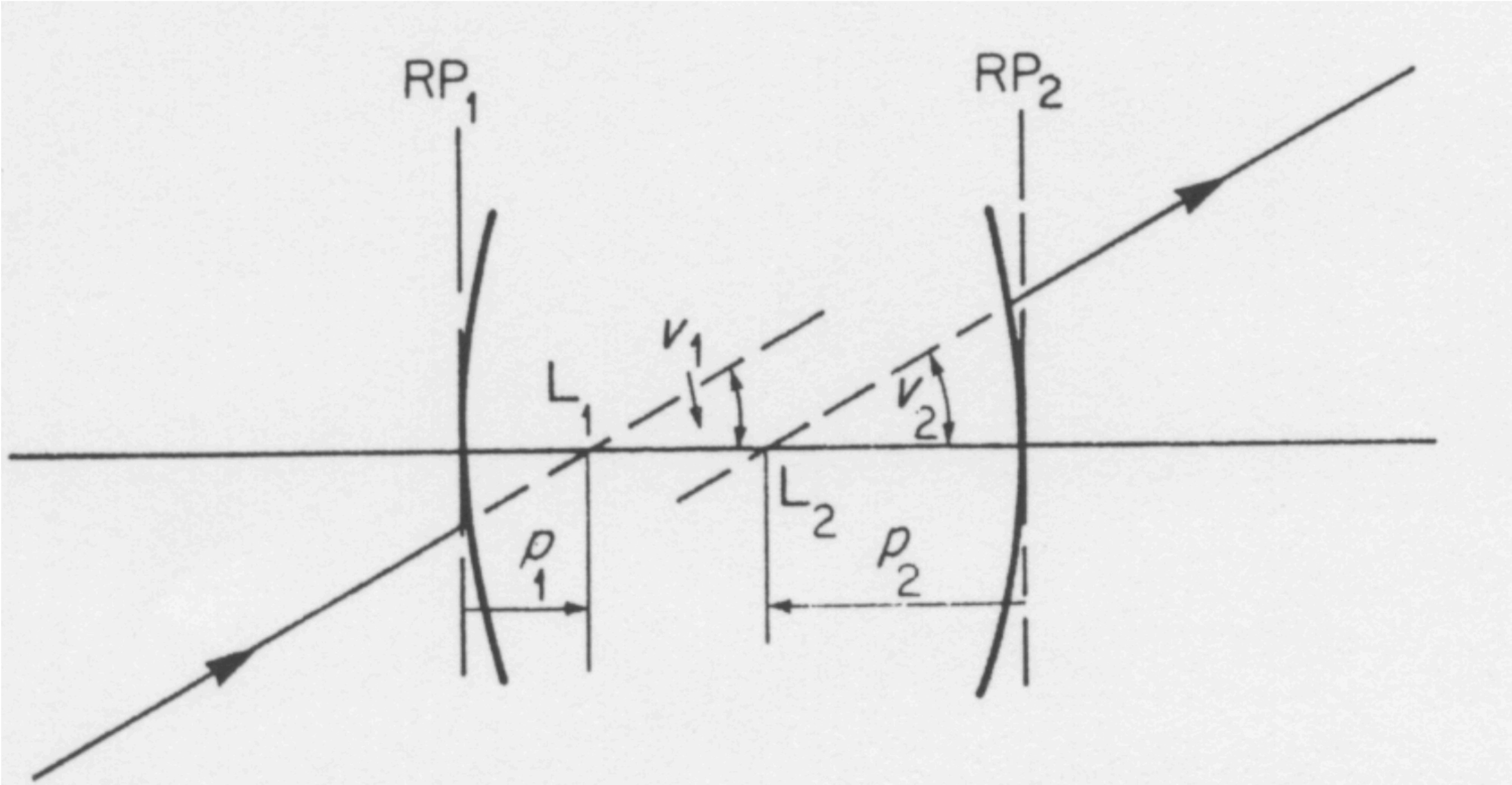


Figure 15c

Let the displacements of L_1 and L_2 from RP_1 and RP_2 respectively be denoted by p_1 and p_2 respectively.

The matrix chain linking the second nodal point L_2 back to the first nodal point L_1 will then be:

$$\underset{L_2\ to\ RP_2}{\begin{bmatrix} 1 & p_2/n_2 \\ 0 & 1 \end{bmatrix}}\underset{RP_2\ to\ RP_1}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}\underset{RP_1\ to\ L_1}{\begin{bmatrix} 1 & -p_1/n_1 \\ 0 & 1 \end{bmatrix}}$$

$$=\begin{bmatrix} 1 & p_2/n_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & -(Ap_1/n_1)+B \\ C & -(Cp_1/n_1)+D \end{bmatrix} = \begin{bmatrix} A+(Cp_2/n_2) & -(Ap_1/n_1)+B+p_2\left[-(Cp_1/n_1)+D\right]/n_2 \\ C & -(Cp_1/n_1)+D \end{bmatrix} = \Phi$$

Note that reduced displacements have been used and that $-p_1$ rather than p_1 is used for the $\mathfrak{I}$-matrix from RP_1 to L_1.

We denote the matrix just obtained by ϕ and use y_0, v_0 to denote the coordinates of a ray entering the L_1 plane and y_3, v_3 to denote the coordinates of a ray leaving the L_2 plane.

We then have

$$y_3 = \phi_{11}y_0 + \phi_{12}V_0 = \phi_{11}y_0 + \phi_{12}(n_1v_0)$$

and

$$n_2v_3 = V_3 = \phi_{21}y_0 + \phi_{22}V_0 = \phi_{21}y_0 + \phi_{22}(n_1v_0)$$

But if L_1 and L_2 are nodal points and $y_0 = 0$, then whatever value of v_0 we must have $y_3 = 0$ and $v_3 = v_0$.

This will be true only if $\phi_{12} = 0$ and $\phi_{22}(n_1/n_2) = 1$, that is, the matrix is an object-image matrix with transverse (linear) magnification $1/\phi_{22} = (n_1/n_2)$.

From the equation $\phi_{22} = (n_2/n_1)$ we obtain $p_1 = (Dn_1-n_2)/C$.

Substituting for p_1 in the equation $\phi_{12} = 0$, we obtain eventually $p_2 = (n_1-An_2)/C$.

We summarize these results in the table below:

System parameter described	Measured From	To	Function of matrix elements	Special case $n_1 = n_2 = 1$
1st focal point	RP_1	F_1	$n_1 D/C$	D/C
1st focal length	F_1	H_1	$-n_1/C$	-1/C
1st principal point	RP_1	H_1	$n_1(D-1)/C$	(D-1)/C
1st nodal point	RP_1	L_1	$(Dn_1-n_2)/C$	(D-1)/C
2nd focal point	RP_2	F_2	$-n_2 A/C$	-A/C
2nd focal length	H_2	F_2	$-n_2/C$	-1/C
2nd principal point	RP_2	H_2	$n_2(1-A)/C$	(1-A)/C
2nd nodal point	RP_2	L_2	$(n_1-An_2)/C$	(1-A)/C

The right-hand column of this table shows that, for the most frequently encountered case of an optical system located in air, the nodal points coincide with the principal points; this is because the conditions for unit angular and unit linear magnification are then the same.

Secondly, both focal lengths are given by -1/C.

When the refractive index differs from unity, it is the reduced focal length that is given by -1/C.

To complete this treatment of paraxial refracting systems, let us see what happens to an

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

matrix if we transform it so that, instead of converting ray data from RP1 to RP2, it operates (a) between two principal planes or (b) between the two focal planes.

(a) For operation between two principle planes, the new matrix will be

$$\begin{bmatrix} 1 & \frac{1-A}{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & \frac{1-D}{C} \\ 0 & 1 \end{bmatrix}$$

H_2 *to* RP_2 RP_2 *to* RP_1 RP_1 *to* H_1 (note sign reversal)

$$= \begin{bmatrix} 1 & \frac{1-A}{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & -A\left(\frac{1-D}{C}\right) + B \\ C & -C\left(\frac{1-D}{C}\right) + D \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1-A}{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & \frac{A-1}{C} \\ C & 1 \end{bmatrix} \quad \text{since AD-BC = 1}$$

$$= \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}$$

Between these two planes, we have a refraction matrix which is the same as for a thin lens of power $P = -C = 1/f$.

As we should expect, there is an object-image relationships between RP_1 and RP_2, with transverse magnification unity.

(b) For the operation between two focal planes, the new matrix will be

$$\begin{bmatrix} 1 & -\dfrac{A}{C} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & -\dfrac{D}{C} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -A/C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & -AD/C+B \\ C & -CD/C+D \end{bmatrix}$$

F_2 *to* RP_2 RP_2 *to* RP_1 RP_1 *to* F_1 (note sign reversal)

$$= \begin{bmatrix} 1 & -A/C \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & -1/C \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix}$$

This matrix expresses the well-known result that the ray height in the second focal plane depends only on the ray angle in the first, while the ray angle in the second focal plane depends only on the ray height in the first.

Furthermore, if we use y_1/V_1 to represent the focal distance z_1 of an object point (measured to the left from F_1) and y_2/V_2 to represent the focal distance z_2 of a corresponding image point, we find immediately that

$$z_1 z_2 = (y_1/V_1)(y_2/V_2) = -f^2 \quad \text{(Newton's equation)}$$

Again, as we should expect, the vanishing of the principal elements of the matrix indicates that RP_1 and RP_2 are located at the focal planes.

Given that the equivalent focal length is unchanged by this transformation, the element C remains the same and the upper right-hand element has to equal -1/C for the matrix to remain unimodular.

Example 6

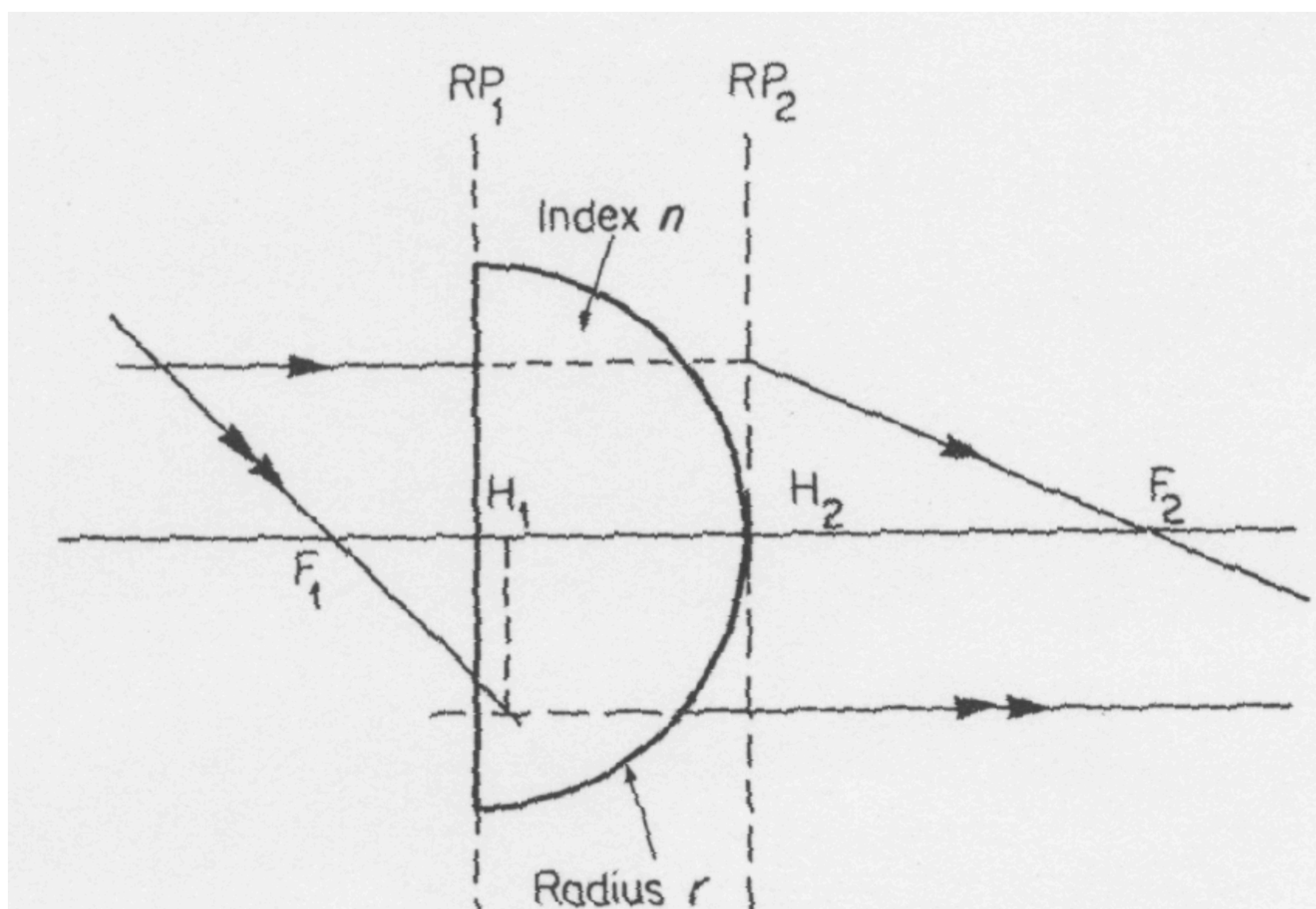
A glass hemisphere of radius r and refractive index n is used as a lens, rays through it being limited to those nearly coinciding with the axis.

Show that, if the plane surface faces to the left, the second principal point coincides with the intersection of the convex surface with the axis (its vertex).

Show also that the first principal point is inside the lens at a distance r/n from the plane surface and that the focal length of the lens is equal to r/(n-1).

See Figure 16.

Figure 16



Solution: If we locate the reference planes at the two surfaces of the hemisphere, the matrix chain is

$$M = \underbrace{\begin{bmatrix} 1 & 0 \\ -\dfrac{1-n}{-r} & 1 \end{bmatrix}}_{\substack{\text{refraction}\\ \text{at curved}\\ \text{surface}\\ RP_2}} \underbrace{\begin{bmatrix} 1 & r/n \\ 0 & 1 \end{bmatrix}}_{\substack{\text{reduced}\\ \text{thickness}\\ \text{of lens}}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{trivial}\\ \text{case of}\\ \text{refraction by}\\ \text{plane surface}\\ \text{at } RP_1}} = \begin{bmatrix} 1 & r/n \\ -\dfrac{n-1}{r} & 1/n \end{bmatrix}$$

The required results can be found in the right-hand column of the earlier table.

The distance from RP_2 to the second principal point is (1-A)/C, which vanishes.

The distance from RP_1 to the first principal point

$$(D-1)/C = (1/n - 1) \times r/(1-n) = r/n$$

Since n > 1, this distance is less than the thickness r of the hemispherical lens.

Finally, the focal length measured in air in either direction is -1/C = r/(n-1)

Example 7

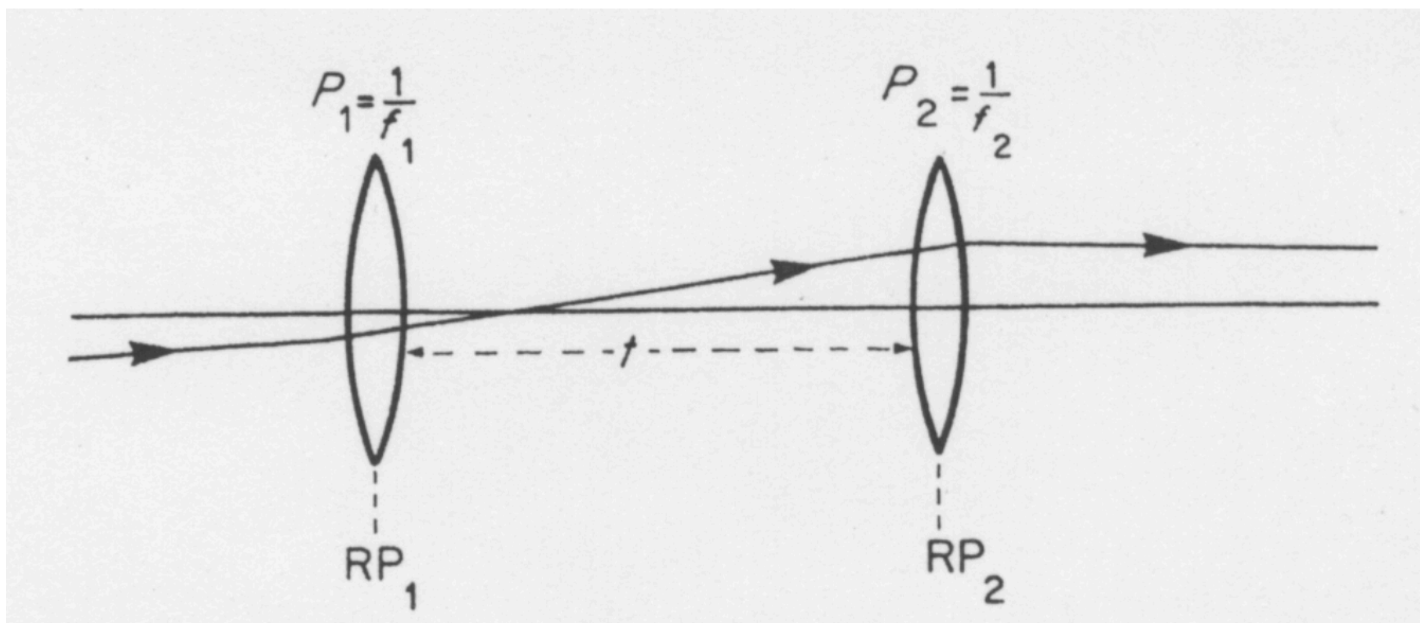
Two positive thin lenses of powers P1 and P2 are used with a lens separation t to form an eyepiece for a reflecting telescope.

Show that the equivalent focal length of the eyepiece is

$$\frac{1}{P_1 + P_2 - P_1 P_2 t}$$

See Figure 17.

Figure 17



Solution: We choose the two lenses as the reference planes and obtain the system matrix

$$\begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - P_1 t & t \\ -P_2 - P_1 + P_1 P_2 t & 1 - P_2 t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

From the lower left-hand element, the equivalent focal length is evidently given by

$$1/f = P_1 + P_2 - P_1 P_2 t$$

Section 2 - Fermat's Principle of Least Time(this will show how powerful mathematics is)

Many problems in Newtonian mechanics are more easily analyzed by means of alternative statements of the laws, including Lagrange's equation and Hamilton's prinicple.

In order to derive these new techniques, we must consider some general principles of the techniques of the **calculus of variations**.

Our primary interest(considering we are studying optics) at this point is in determining the path that gives extremum solutions, for example, the shortest distance or time between two points.

A well-known example of the use of the theory of variations is **Fermat's principle** that light travels on the path that takes the least amount of time.

Statement of the Problem

The basic problem of the calculus of variations is to determine the function y(x) such that the integral

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx \qquad (01)$$

is an extremum (i.e., either a maximum or a minimum).

In equation (01), $y'(x) = dy/dx$, and the semicolon in f separates the independent variable x from the dependent variable y(x) and its derivative $y'(x)$.

The functional f (f depends on the functional form of the dependent variable y(x)) is considered as given, and the limits of integration are fixed.

The function y(x) is then to be varied until an extreme value of J is found.

By this we mean that if a function y = y(x) gives the integral J a minimum value, then any **neighboring function**, no matter how close to y(x), must make J increase.

The definition of a neighboring function may be made as follows.

We give all possible functions y a parametric representation $y = y(\alpha,x)$ such that, for $\alpha = 0$, $y = y(0,x) = y(x)$ is the function that yields an extremum for J.

We can then write

$$y(\alpha,x) = y(0,x) + \alpha\eta(x) \tag{02}$$

where $\eta(x)$ is some function of x that has a continuous first derivative and that vanishes at x_1 and x_2, because the varied function $y(\alpha,x)$ must be identical with y(x) at the endpoints of the path:
$\eta(x_1) = \eta(x_2) = 0$.

The situation is depicted schematically in Figure 01.

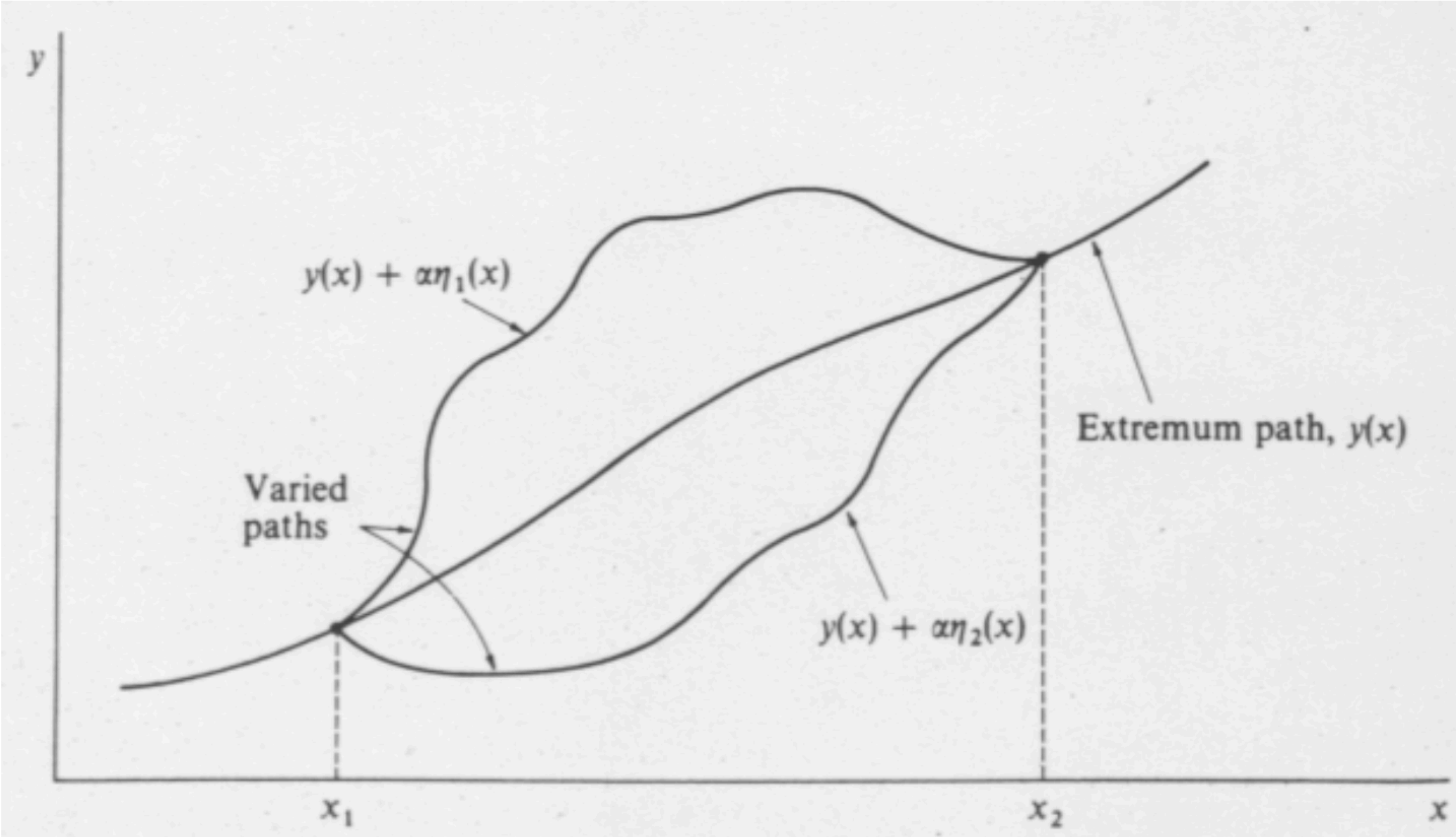


Figure 01

If functions of the type given by equation (02) are considered, the integral J becomes a function of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha,x), y'(\alpha,x); x\} dx \tag{03}$$

The condition that the integral have a **stationary value** (i.e., that an extremum results) is that J be independent of α in first order along the path giving the extremum (α = 0), or, equivalently, that

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{04}$$

for all functions . This is only a necessary condition; it is not sufficient.

Example: Consider the function f = (dy/dx)2, where y(x) = x.

Add to y(x) the function η(x)=sin(x), and find J(α) between the limits of x = 0 and x = 2π.

Show that the stationary value of J(α) occurs for α = 0.

Solution: We can construct the neighboring varied paths by adding to y(x) = x the sinusoidal variation function αsin(x) so that $y(\alpha, x) = x + \alpha \sin x$.

These paths are illustrated in Figure 02 for α = 0 and for two different non-vanishing values of α.

Clearly, the function η(x)=sin(x) obeys the endpoint conditions, that is, $\eta(0)=0=\eta(2\pi)$.

To determine f(y, y′;x), we first determine

$$\frac{dy(\alpha,x)}{dx} = 1+\alpha\cos x$$

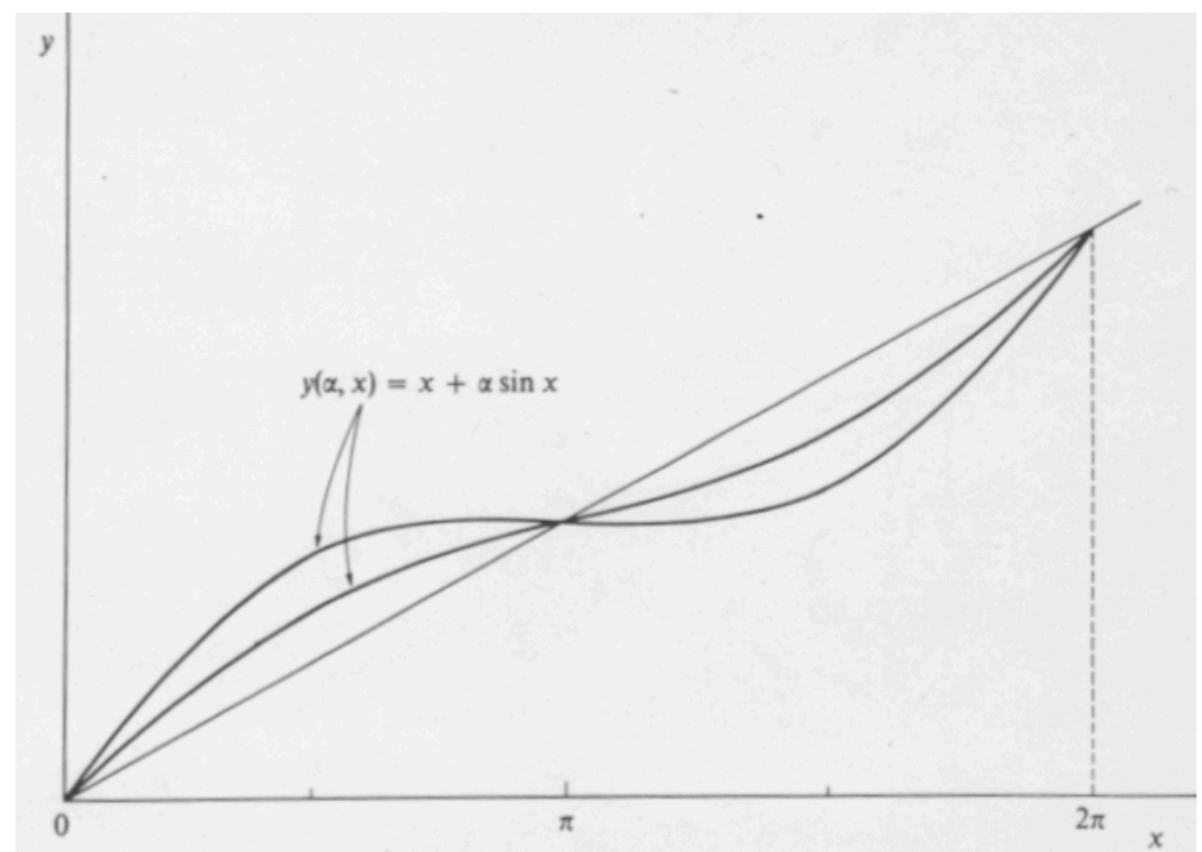


Figure 02

then

$$f = \left(\frac{dy(\alpha,x)}{dx}\right)^2 = 1 + 2\alpha\cos x + \alpha^2\cos^2 x$$

Equation 03 now becomes

$$J(\alpha) = \int_0^{2\pi} \left(1 + 2\alpha\cos x + \alpha^2\cos^2 x\right)dx = 2\pi + \alpha^2\pi$$

Thus, we see the value of J(α) is always greater than J(0), no matter what value (positive or negative) we choose for α.

The condition of equation 04 is also satisfied.

Euler's Equation

To determine the result of the condition expressed by equation (04), we perform the indicated differentiation in equation (03):

$$\frac{\partial J}{\partial\alpha} = \frac{\partial}{\partial\alpha}\int_{x_1}^{x_2} f\{y, y'; x\}dx \tag{05}$$

Because the limits of integration are fixed, the differential operation affects only the integrand. Hence,

$$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial\alpha}\right)dx \tag{06}$$

From equation (02), we have

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad , \quad \frac{\partial y'}{\partial \alpha} = \frac{d\eta(x)}{dx} \tag{07}$$

Equation (06) becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx} \right) dx \tag{08}$$

The second term in the integrand can be integrated by parts:

$$\int u\, dv = uv - \int v\, du \tag{09}$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx} dx = \frac{\partial f}{\partial y'}\eta(x)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right)\eta(x) dx \tag{10}$$

The integrated term vanishes because $\eta(x_1) = \eta(x_2) = 0$.

Therefore, equation (06) becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right)\eta(x) \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right)\eta(x) dx \tag{11}$$

The integral in equation(11) now appears to be independent of α.

But the functions y and y' with respect to which the derivatives of f are taken are still functions of α.

Because $(\partial J / \partial \alpha)_{\alpha=0}$ must vanish for the extremum value and because $\eta(x)$ is an arbitrary function (subject to the conditions already stated), the integrand of equation (11) must itself vanish for $\alpha = 0$:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad (12)$$

where now y and y′ are the original functions, independent of α.

This result is know as **Euler's equation**, which is a necessary condition for J to have an extremum value.

When applied to mechanical systems, this is known as the **Euler-Lagrange equation**.

Example: We can use the calculus of variations to solve a classic problem in the history of physics: the **brachistochrone**.

Consider a particle moving in a constant force field starting at rest from some point (x_1,y_1) to some lower point (x_2,y_2).

Find the path that allows the particle to accomplish the transit in the least possible time.

Solution: The coordinate system may be chosen so that the point (x_1,y_1) is at the origin.

Further, let the force field be directed along the positive x-axis as in (Figure 03).

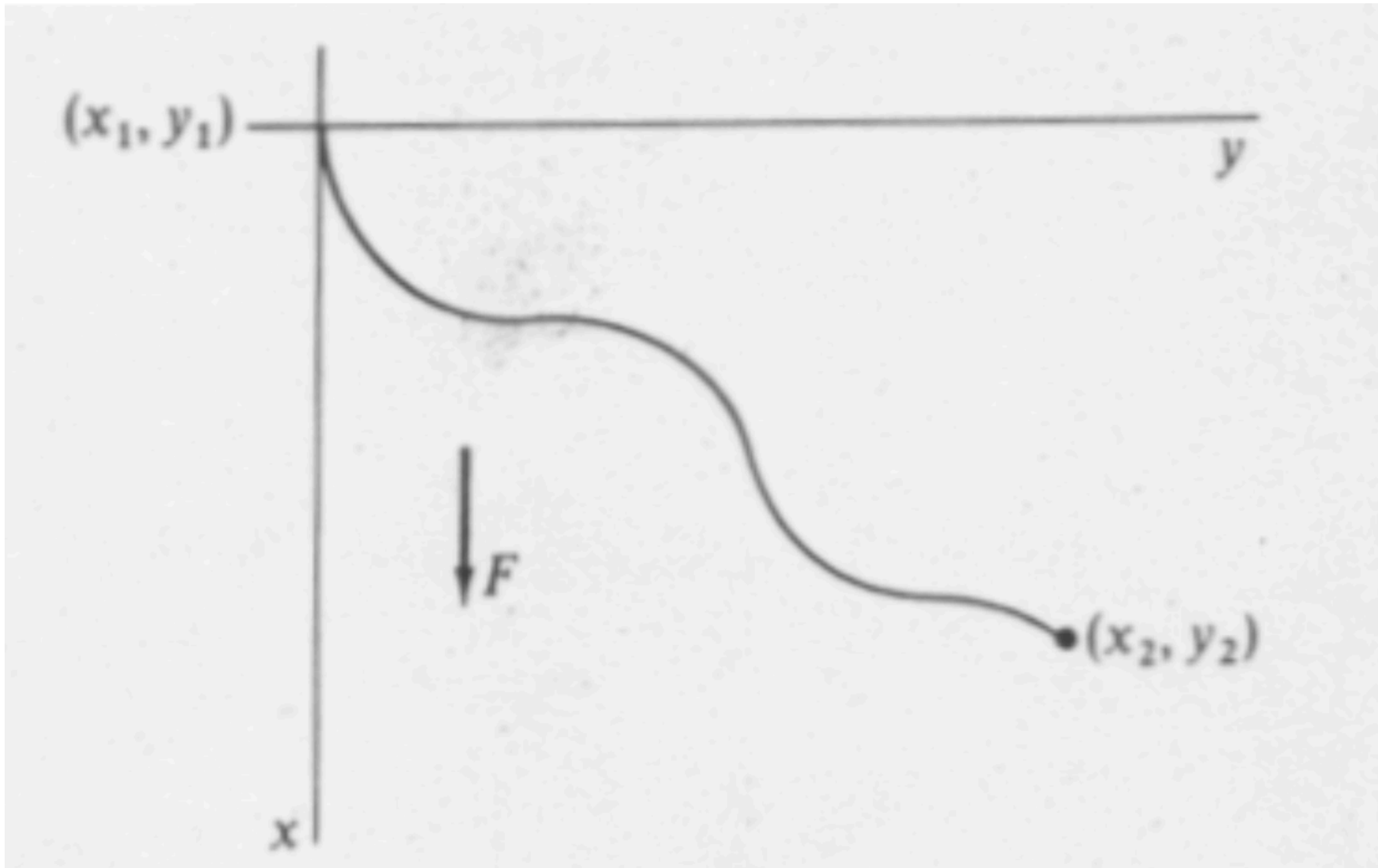


Figure 03

Because the force on the particle is constant - and if we ignore the possibility of friction - the field is conservative, and the total energy of the particle is T + U=constant.

If we measure the potential from the point x = 0 (i.e., U(x = 0) = 0), then, because the particle starts from rest T + U = 0, the kinetic energy is $T = mv^2/2$ and the potential energy is U = -Fx = -mgx, where g is the acceleration imparted by the force.

Thus,

$$v = \sqrt{2gx} \tag{13}$$

The time required for the particle to make the transit from the origin to (x_2,y_2) is

$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\left(dx^2 + dy^2\right)^{1/2}}{(2gx)^{1/2}} = \int_{x_1=0}^{x_2} \left(\frac{1+y'^2}{2gx}\right)^{1/2} dx \tag{14}$$

The time of transit is the quantity for which a minimum is desired.

Because the constant $(2g)^{-1/2}$ does not affect the final equation, the functional f may be identified as

$$f = \left(\frac{1+y'^2}{x}\right)^{1/2} \tag{15}$$

and because $\partial f / \partial y = 0$, the Euler equation (equation (12)) becomes

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{16}$$

or

$$\frac{\partial f}{\partial y'} = \text{constant} = (2a)^{-1/2} \tag{17}$$

where a is a new constant.

thePerforming the differentiation $\partial f / \partial y'$ on equation (15) and squaring the result, we have

$$\frac{y'^2}{x\left(1+y'^2\right)}=\frac{1}{2a} \tag{18}$$

This may be put in the form

$$y=\int\frac{xdx}{(2ax-x^2)^{1/2}} \tag{19}$$

We now make the following change of variable:

$$x=a(1-\cos\theta) \quad , \quad dx=a\sin\theta d\theta \tag{20}$$

The integral in equation (19) then becomes

$$y=\int a(1-\cos\theta)d\theta \tag{21}$$

and the most general solution is

$$y=a(\theta-\sin\theta)+\text{constant} \tag{22}$$

Now the parametric equations for a cycloid passing through the origin are

$$x=a(1-\cos\theta) \quad , \quad y=a(\theta-\sin\theta) \tag{23}$$

which is just the solution found, with the constant of integration set equal to zero to conform with the requirement that (0,0) is the starting point of the motion.

The path is then as shown in Figure 04, and the constant a must be adjusted to allow the cycloid to pass through the specified point (x_2,y_2).

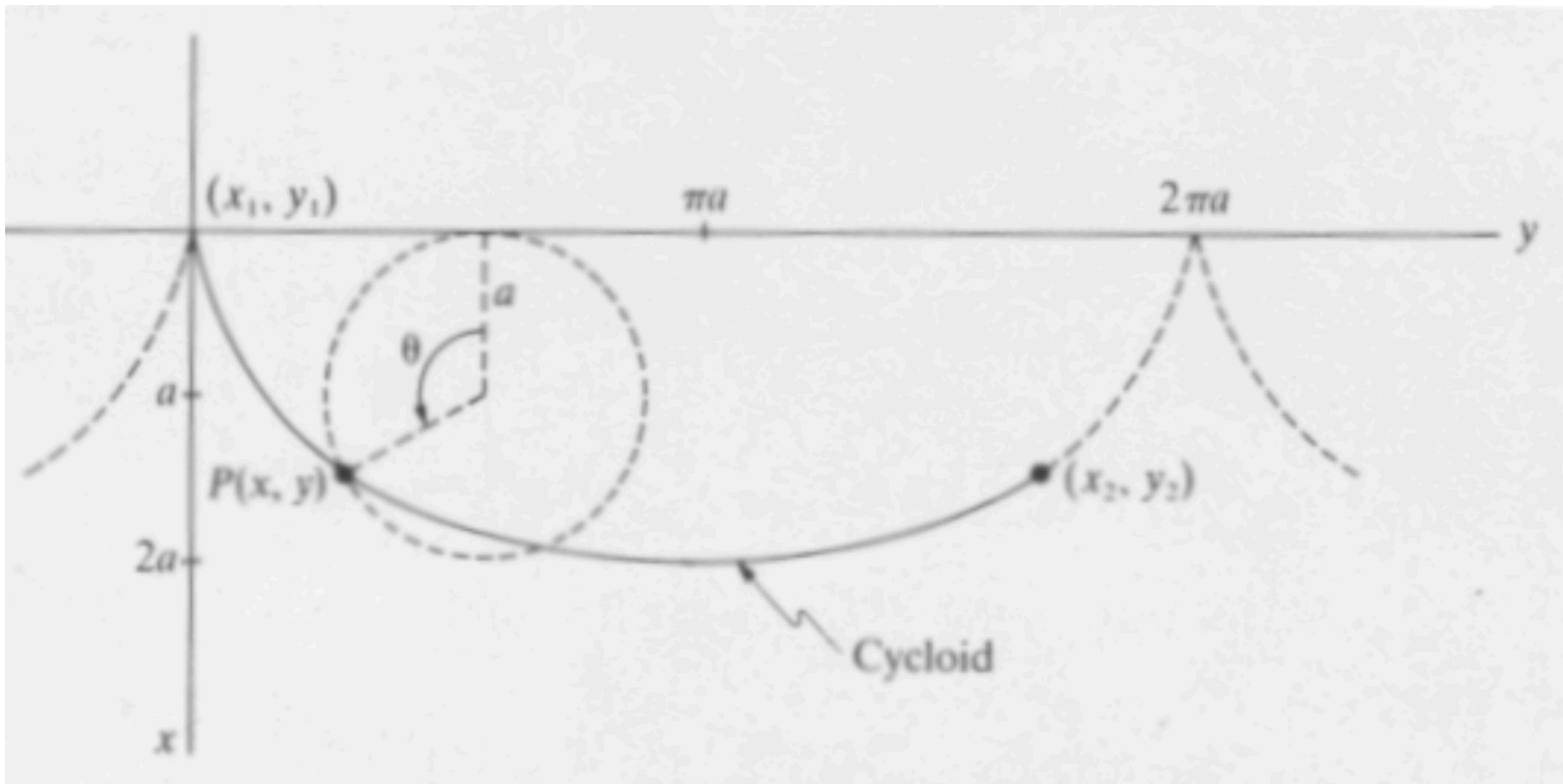


Figure 04

Solving the problem of the brachistochrone does indeed yield a path the particle traverses in **minimum** time.

The procedures of variational calculus are designed only to produced an extremum - either a minimum or a maximum.

It is almost always the case in dynamics that we desire(and find) a minimum for the problem.

A second equation may be derived from Euler's equation that is convenient for functions that do not explicitly depend on x, i.e., $\partial f/\partial x = 0$.

We first note that for any function f(y, y′;x) the derivative is a sum of terms

$$
\begin{aligned}
\frac{df}{dx} = \frac{d}{dx} f(y,y';x) &= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} \\
&= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}
\end{aligned}
\tag{24}
$$

Also

$$
\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} \right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{25}
$$

or, substituting from equation (24) y″∂f/∂y',

$$
\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} \right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{26}
$$

The last two terms in equation (26) may be written as

$$
y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right) \tag{27}
$$

We can use this so-called "second form" of the Euler equation in cases in which f does not depend explicitly on x, and $\partial f / \partial x = 0$.

Then

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} \quad \left(\text{for } \frac{\partial f}{\partial x} = 0 \right) \tag{28}$$

Example: A **geodesic** is a line that represents the shortest path between two points when the path is restricted to a particular surface.

Find the geodesic on a sphere.

Solution: The element of length on the surface of a sphere of radius R is given by

$$ds = R\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{29}$$

The distance between points 1 and 2 is therefore

$$s = R\int_1^2 \left[\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right]^{1/2} d\phi \tag{30}$$

and, if s is to be a minimum, f is identified as

$$f = \left[\theta'^2 + \sin^2\theta\right]^{1/2} \tag{31}$$

where $\theta' = d\theta / d\phi$.

Because $\partial f/\partial\phi = 0$ where equation (equation 28), which yields

$$\left[\theta'^2+\sin^2\theta\right]^{1/2}-\theta'\frac{\partial}{\partial\theta'}\left[\theta'^2+\sin^2\theta\right]^{1/2} = \text{constant} = a \tag{32}$$

Differentiating and multiplying through by f, we have

$$\sin^2\theta = a\left[\theta'^2+\sin^2\theta\right]^{1/2} \tag{33}$$

This may be solved for $d\phi/d\theta = \theta'^{-1}$ with the result

$$\frac{d\phi}{d\theta} = \frac{a\csc^2\theta}{\left(1-a^2\csc^2\theta\right)^{1/2}} \tag{34}$$

Solving for ϕ , we obtain

$$\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right)+\alpha \tag{35}$$

where α is the constant of integration and $\beta^2 = (1-a^2)/a^2$.

Rewriting equation 35 produces

$$\cot\theta = \beta\sin(\phi-\alpha) \tag{36}$$

To interpret this result, we convert the equation to rectangular coordinates by multiplying through by $R\sin\theta$ to obtain, on expanding $\sin(\phi\text{-}\alpha)$,

$$(\beta\cos\alpha)R\sin\theta\sin\phi-(\beta\sin\alpha)R\sin\theta\cos\phi=R\cos\theta \tag{37}$$

Because α and β are constants, we may write them as

$$\beta\cos\alpha=A \quad , \quad \beta\sin\alpha=B$$

Then equation 37 becomes

$$A(R\sin\theta\sin\phi)-B(R\sin\theta\cos\phi)=(R\cos\theta)$$

The quantities in the parentheses are just the expressions for y, x, and z, respectively, in spherical coordinates, therefore equation (39) may be written as

$$Ay-Bx=z$$

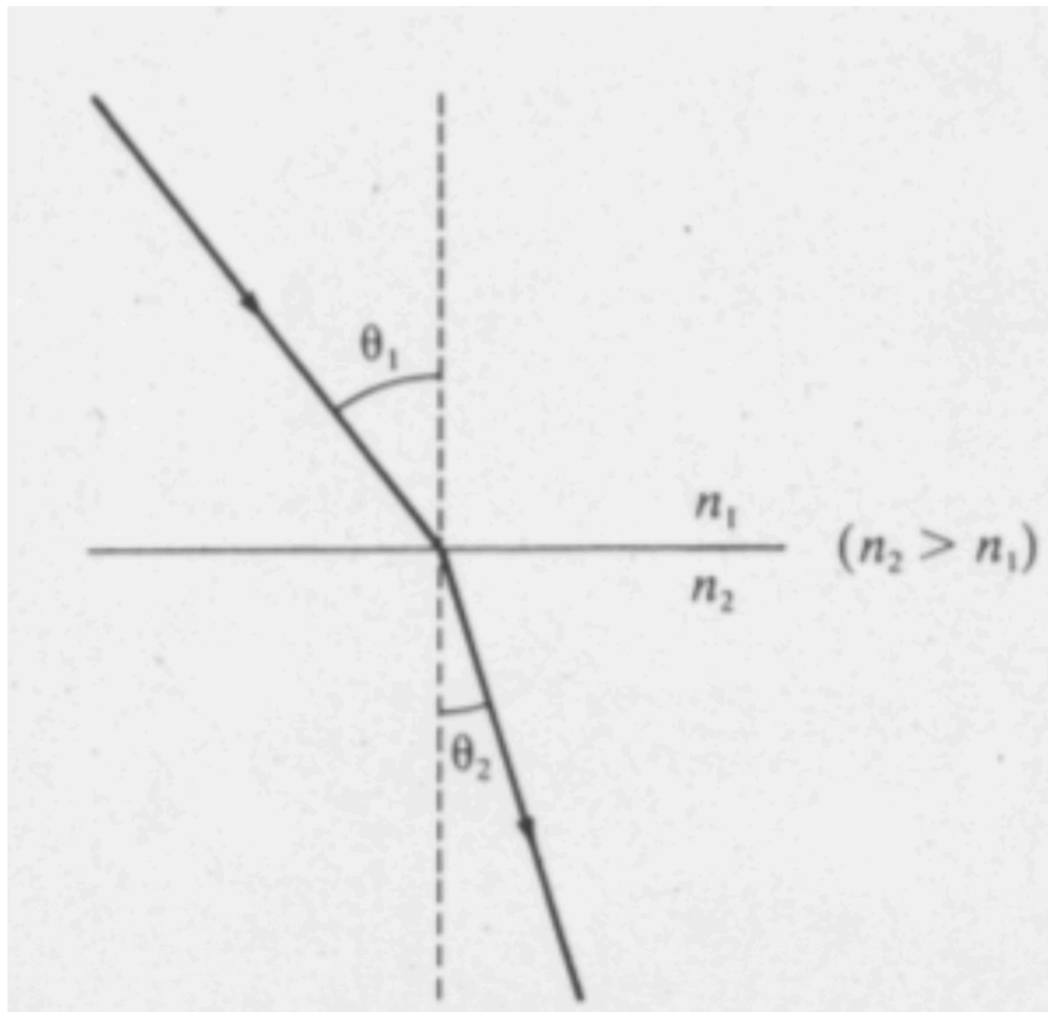
which is the equation of a plane passing through the center of the sphere.

Hence the geodesic on a sphere is the path that the plane forms at the intersection with the surface of the sphere = a great circle.

Note that the great circle is the maximum as well as the minimum "straight line" distance between two points on the surface of a sphere.

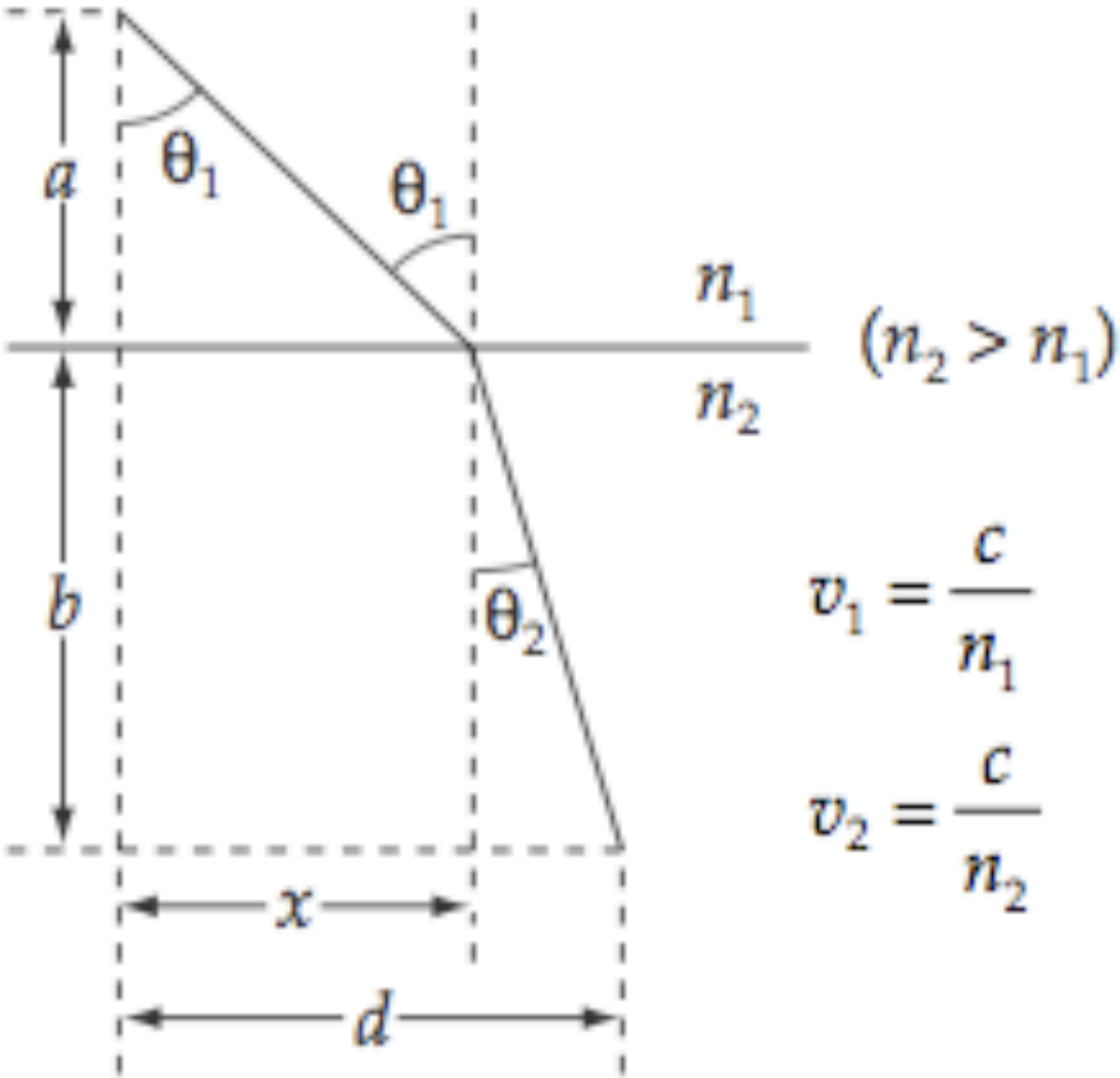
Snell's Law

Finally, we consider light passing from one medium with index of refract n_1 into another medium with index of refraction n_2 (see figure).



We now use Fermat's principle to minimize time and derive the law of refraction.

Consider the expanded diagram



The time to travel the path shown is

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{1+y'^2}}{v} dx$$

Although we have v = v(y), we only have dv/dy$\neq$ 0 when y = 0.

The Euler equation tells us

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

or

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 = \frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2}}\right]$$

Now we use v = c/n and y′ = -tan(θ) so that we have

$$\frac{y'}{v\sqrt{1+y'^2}} = \frac{-n\tan\theta}{c\sqrt{1+\tan^2\theta}} = \text{constant}$$

$$\frac{-n\tan\theta\cos\theta}{c} = \text{constant} \Rightarrow n\sin\theta = \text{constant}$$

which is Snell's law of refraction.

Alternatively, we can write

$$t = t_{upper} + t_{lower} = \frac{\ell_{upper}}{v_{upper}} + \frac{\ell_{lower}}{v_{lower}} = \frac{\sqrt{a^2+x^2}}{c/n_1} + \frac{\sqrt{b^2+(d-x)^2}}{c/n_2}$$

We now minimize this time as function of x to determine the point on the boundary between the media which minimizes the travel time.

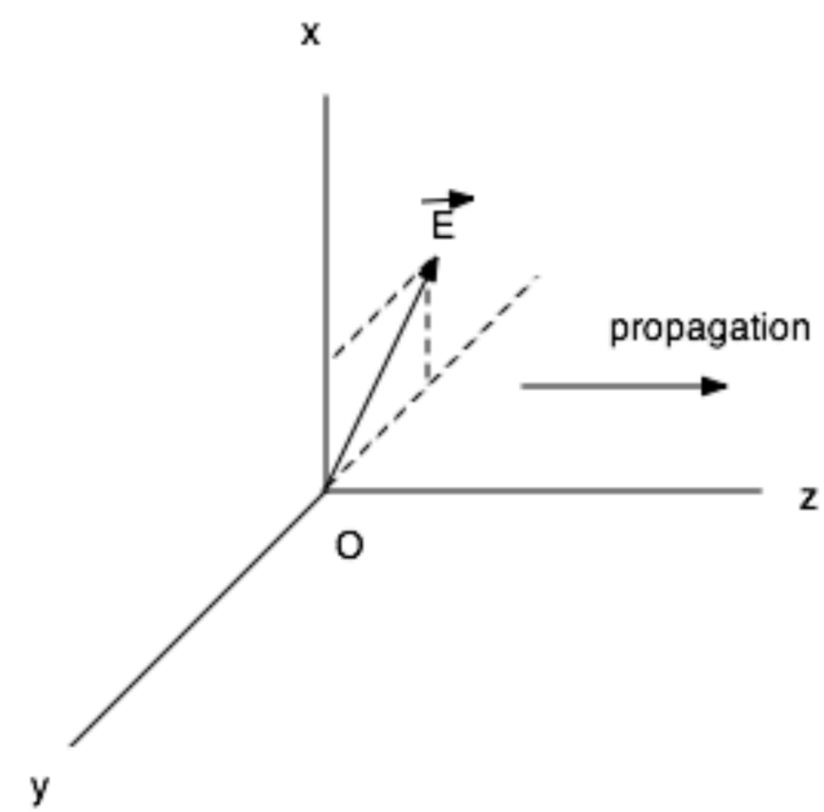
$$\frac{dt}{dx} = 0 = \frac{n_1 x}{c\sqrt{a^2 + x^2}} - \frac{n_2(d-x)}{c\sqrt{b^2 + (d-x)^2}}$$

$$\frac{n_1 x}{\sqrt{a^2 + x^2}} = \frac{n_2(d-x)}{\sqrt{b^2 + (d-x)^2}} \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Section 3 - Matrices in Polarization Optics

Polarized Light - Its Production and Analysis

For all electromagnetic radiation, the oscillating components of the electric and magnetic fields are directed at right angles to each other and to the direction of propagation.



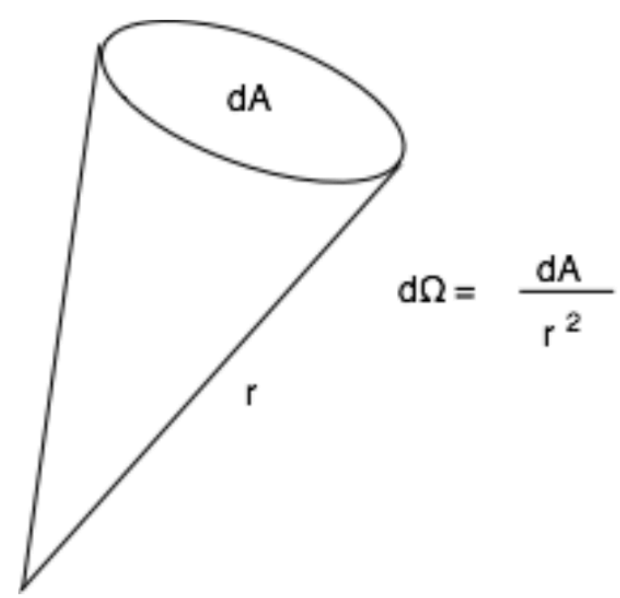
We assume a right-handed set of axes, with the Oz-axis pointing along the direction of propagation.

In dealing with polarization, we can ignore the magnetic field and concentrate on how the electric field vector is oriented with respect to the transverse xy-plane.

Most ordinary sources of light, such as the sun or an incandescent light bulb, produces light that is incoherent and unpolarized; light of this sort is a chaotic jumble of almost innumerable independent disturbances, each with their own direction of travel, its own optical frequency and its own state of polarization.

Just how innumerable are these "states" or "modes" of the radiation field?

As far as transverse modes are concerned, for a source of area A radiating into a **solid** angle Ω



the number of distinguishable directions of travel is given by $A\Omega/\lambda^2$ - for a source of 1 cm^2 area radiating into a hemisphere, this will be well over 10^8.

Secondly, if we consider a source which is observed over a period of one second and ask how many optical frequencies, in theory at least, can be distinguished, we find approximately 10^{14}.

By contrast with these numbers, if we consider the polarization of an electromagnetic wave field, we find the number of independent orthogonal states is just **two**.

Unpolarized light is sometimes said to be a random mixture of "all sorts" of polarization; it would be better to say that, whenever we try to analyze it , in terms of a chosen orthogonal pair of states, we can find no evidence for preference of one state over the other.

If we could make our observations fast enough, we would find that the "instantaneous" state of polarization is passing very rapidly through all possible combinations of the two states we have chosen, in a statistically random way.

In most cases, the rate at which these changes occur is over 10^{12} per second, so that what we observe is a smoothed average.

Unpolarized light is easy to produce but difficult to describe, whether mathematically or in terms of a model.

At the opposite extreme is the completely coherent light that is generated in a single mode laser.

This light is as simple as one could specify.

We assume that we have a plane wave of angular frequency ω which is traveling with velocity c in the direction Oz.

Since we know that the vibrations of the electric field are transverse, they can be specified in terms of an x-component E_x, of peak amplitude H, and a y-component E_y, of peak amplitude K.

We thus have

$$E_x = H\cos[\omega(t - z/c) + \phi_x] = \mathrm{Re}\left(H\exp\left(i[\omega(t - z/c) + \phi_x]\right)\right)$$

$$E_y = V\cos[\omega(t - z/c) + \phi_y] = \mathrm{Re}\left(V\exp\left(i[\omega(t - z/c) + \phi_y]\right)\right)$$

These functional forms satisfy the wave equation with wave propagation velocity = c, k = $2\pi/\lambda = \omega/c$ and wave propagation direction along the z-axis (Poynting vector direction).

If we use Δ to represent the phase difference ($\phi_y - \phi_x$), and if the symbols $\hat{i}$ and $\hat{j}$ denote unit vectors directed along the axes Ox and Oy, then these two equations can be combined in the space vector form:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} \exp\left[i[\omega(t - z/c) + \phi_x]\right]$$

There is no dependence on x or y since a plane wave of indefinite lateral extent is assumed.

A column vector or ket such as $\begin{bmatrix} He^{i\phi_x} \\ Ke^{i\phi_y} \end{bmatrix}$ or $\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix}$ is usually referred to as a Jones vector.

The Jones vector provides a complete description of the state of polarization of any light beam that is fully polarized.

For the coherent plane wave described above we see that, if either H or K vanishes, the transverse vibrations must be vertically or horizontally polarized.

If the phase difference Δ vanishes, the polarization is linear and if H = K, while Δ = π/2 we have light that is circularly polarized.

For the general case, the light is elliptically polarized.

Some examples are:

$$H=1, K=0, \Delta=0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow linear - polarization(x-0°)$$

$$H=0, K=1, \Delta=0 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow linear - polarization(y-90°)$$

$$H=\frac{1}{\sqrt{2}}, K=\frac{1}{\sqrt{2}}, \Delta=0 \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow linear - polarization(45°)$$

$$H=\frac{1}{\sqrt{2}}, K=\frac{\pm 1}{\sqrt{2}}, \Delta=90° \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \Rightarrow \begin{bmatrix} right \\ left \end{bmatrix} - circular - polarization$$

If these geometrical descriptions of the transverse vibrations are difficult to visualize, it is helpful to think of them in terms of a model.

Imagine that, in a given xy-plane, we are following the motion of a very small charged particle, which is simultaneously attached to the origin by a weak spring and pulled away from it by the oscillating electric field vector.

If we choose the time-origin so that $\phi_x = 0$, then the instantaneous x- and y-coordinates of the test charge, its displacement from the origin, will vary with time according to the equations

$$x = H\cos(\omega t) \quad , \quad y = K\cos(\omega t + \Delta)$$

It follows that, whenever changes through a range of 2π (which happens at least 10^{14} times every second), the test charge executes one cyclic traversal of a single Lissajous figure as shown in Figure 1 below.

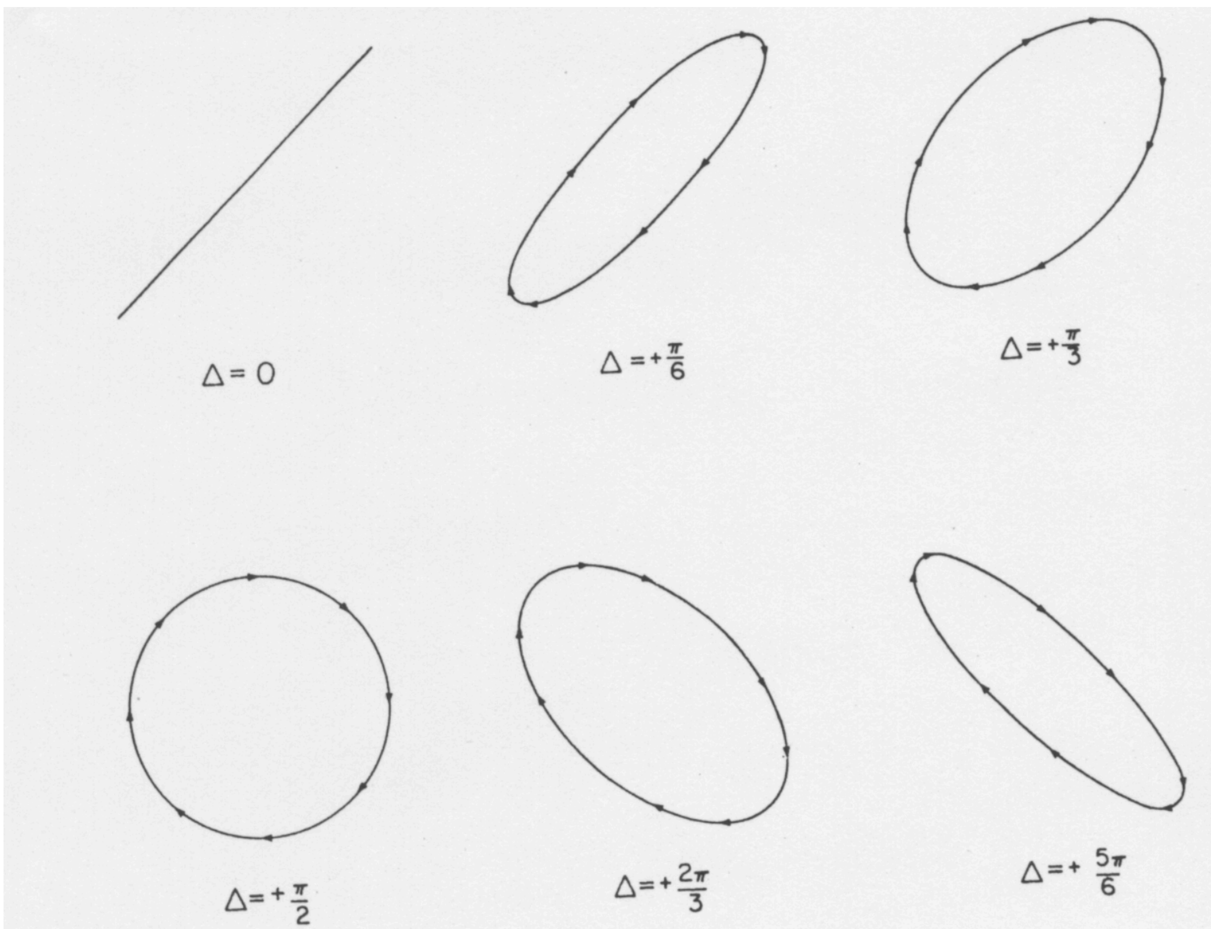


Figure 1

Each of these patterns represents the locus of the tip of the oscillating electric field vector.

In order to analyze their geometry, we must eliminate the time parameter ωt from the above pair of equations.

Plane-polarized light

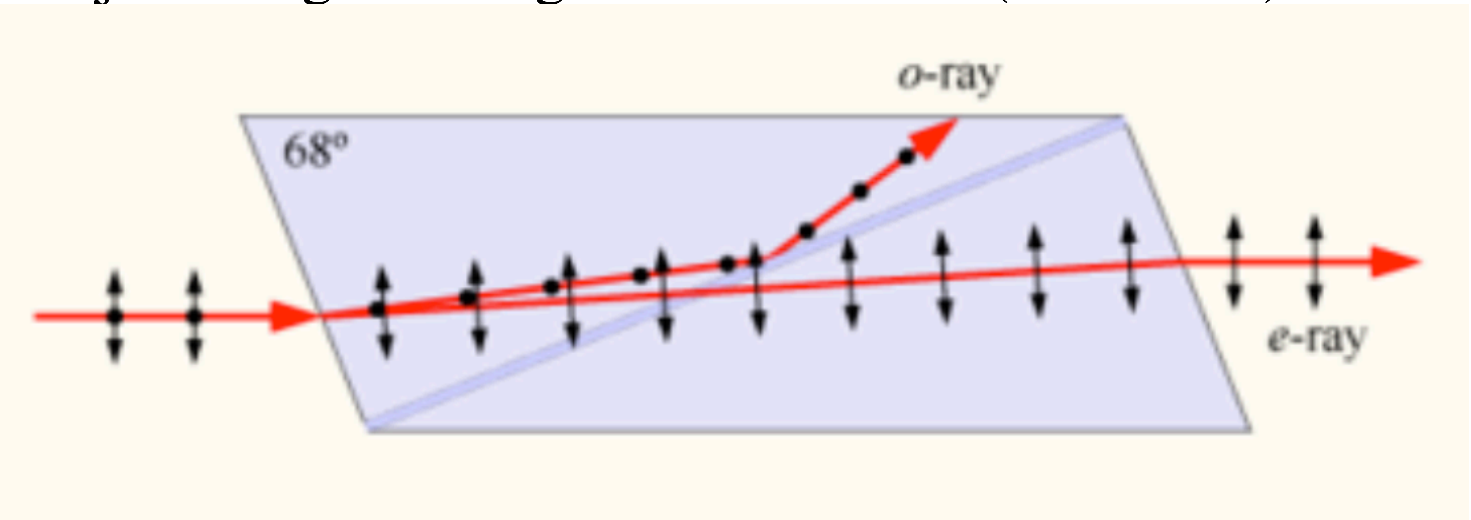
Plane-polarized light is obtained by passing unpolarized light through a polarizing filter.

One of the most efficient types of polarizing filters is based on the double-refraction properties of a uniaxial crystal such as calcite.

A **Nicol prism** is a type of polarizer, an optical device used to generate a beam of polarized light.

It was the first type of polarizing prism to be invented, in 1828 by William Nicol (1770-1851) of Edinburgh.

It consists of a rhombohedral crystal of calcite (Iceland spar) that has been cut at a 68° angle, split diagonally, and then joined again using Canada balsam (see below).



Unpolarized light enters one end of the crystal and is split into two polarized rays by birefringence.

One of these rays (the **ordinary** or o-ray) experiences a refractive index of $n_o = 1.658$ and at the balsam layer (refractive index $n = 1.55$) undergoes total internal reflection at the interface, and is reflected to the side of the prism.

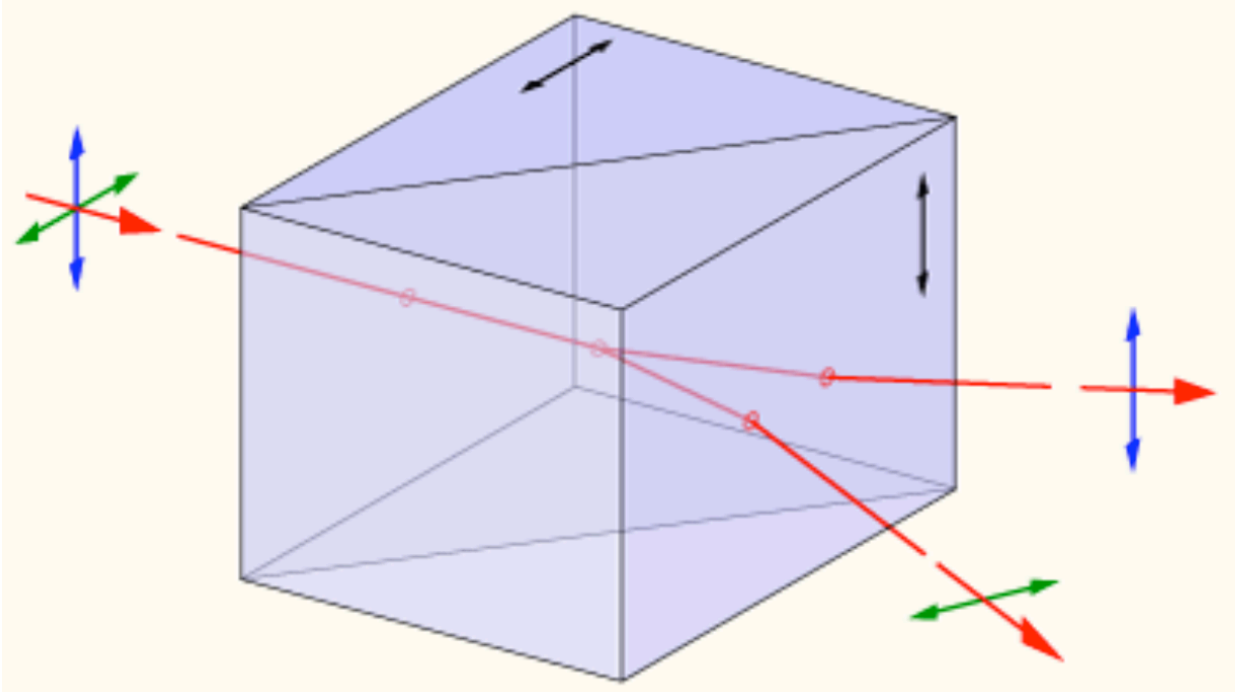
The other ray (the **extraordinary** or e-ray) experiences a lower refractive index (n_e = 1.486), is not reflected at the interface, and leaves through the second half of the prism as plane polarized light.

A **Wollaston prism** is an optical device, invented by William Hyde Wollaston, that manipulates polarized light.

It separates randomly polarized or unpolarized light into two orthogonal, linearly polarized outgoing beams.

The Wollaston prism consists of two orthogonal calcite prisms, cemented together on their base (typically with Canada balsam) to form two right triangle prisms with perpendicular optic axes.

Outgoing light beams diverge from the prism, giving two polarized rays, with the angle of divergence determined by the prisms' wedge angle and the wavelength of the light (see below).



Commercial prisms are available with divergence angles from 15° to about 45°

Also, remarkably efficient, over a wide range of wavelengths and directions, are the various polarizing sheets of **Polaroid** material.

Imagine that we have converted unpolarized light into plane-polarized light by passing it through a sheet polarizer.

The vibrations of the electric field vector now lie entirely in one direction in the transverse xy-plane(the plane of polarization).

Suppose that the direction of polarization makes an angle θ with the horizontal x-axis.

The equations describing the transverse electric field components are then of the form:

$$E_x = A\cos\theta\cos(\omega t+\phi) \quad , \quad E_y = A\sin\theta\cos(\omega t+\phi)$$

Now consider the hypothetical small test charge once again.

Under the influence of this synchronized pair of field vectors, its displacement will be given by

$$x = A\cos\theta\cos(\omega t+\phi) \quad , \quad y = A\sin\theta\cos(\omega t+\phi)$$

Multiplying the first equation by $\sin\theta$ and the second by $\cos\theta$ we find that the locus of the transverse displacement is the equation - a straight line motion in the plane of polarization.

Elliptically Polarized Light

A convenient method for producing elliptically polarized light is to send a plane-polarized beam through a "phase plate', that is, a slice of uniaxial crystal.

Such a slice (as we will see later) introduces a phase difference between the components of the field vectors parallel and perpendicular to a special direction in the crystal, which is known as the **optic(fast) axis**.

The field vector parallel to the optic(fast) axis is called the **extraordinary** component (e-ray) and that perpendicular to the optic(fast) axis is called the **ordinary** component (o-ray).

For most phase plates, the refractive index for the e-ray is smaller than for the o-ray.

We suppose that the phase plate has its optic(fast) axis parallel to the x-axis and that its thickness is such that it advances the o-ray by angle Δ radians with respect to the e-ray.

The field vectors coming out of the phase plate are then given by

$$x = A\cos\theta\cos(\omega t) \quad , \quad y = A\sin\theta\cos(\omega t + \Delta)$$

If we eliminate ωt between these two equations, we get an equation linking the x- and y-components of the field vector in the resultant beam:

$$\frac{x^2}{A^2\cos^2\theta} - \frac{2xy\cos\Delta}{A^2\cos\theta\sin\theta} + \frac{y^2}{A^2\sin^2\theta} = \sin^2\Delta$$

or

$$\frac{x^2}{H^2} - \frac{2xy\cos\Delta}{HK} + \frac{y^2}{K^2} = \sin^2\Delta$$

where $H = A\cos\theta$ and $K = A\sin\theta$.

Remember, H is the component of the original field vector parallel to the x-axis and K is the component of the original field vector parallel to the y-axis.

Clearly, we have $H^2 + K^2 = A^2 = I$ = intensity (proportional to energy flow).

Some special cases are:

If $\Delta = 0$ or no phase plate, then $\cos\Delta = 1$, $\sin\Delta = 0$ and the equation becomes

$$\frac{x^2}{H^2} - \frac{2xy}{HK} + \frac{y^2}{K^2} = 0 \rightarrow \left(\frac{x}{H} - \frac{y}{K}\right)^2 = 0 \rightarrow \frac{x}{H} = \frac{y}{K}$$

This just describes the plane-polarized condition of the original beam.

If $\Delta = \pi$ or a half-wave plate $\cos\Delta = -1$, $\sin\Delta = 0$, we have

$$\frac{x}{H} = -\frac{y}{K}$$

This means that the ratio of the displacement parallel to the two axes is the same as before, but with reversed sign, so that when x is at its maximum positive, y is at its maximum negative, and vice versa.

We still have plane-polarized light, but its direction is now an angle θ on the other side of the optic(fast) axis, that is, the half-wave plate has turned the direction of polarization through and angle of 2θ.

If $\Delta = \pi/2$, or a quarter-wave plate, so that $\cos\Delta = 0$, $\sin\Delta = 1$ so that linking x and y becomes

$$\frac{x^2}{H^2} + \frac{y^2}{K^2} = 1$$

This is the well-known equation of an ellipse lying with its major and minor axes parallel to the x- and y-axes.

The semi-major axis parallel to the x-axis is H and that parallel to the y-axis is K.

If $\theta = 45°$, then H and K are equal and the equation becomes

$$x^2 + y^2 = \frac{A^2}{2}$$

and we say that the light is **circularly polarized**.

If a beam of fully polarized light is viewed through a Polaroid sheet (polarizer), and the polarizer is gradually rotated through 360°, it is quite easy to distinguish experimentally between light that is linearly, elliptically or circularly polarized.

For linear states, there will be two orientations at which the intensity of the light will be zero.

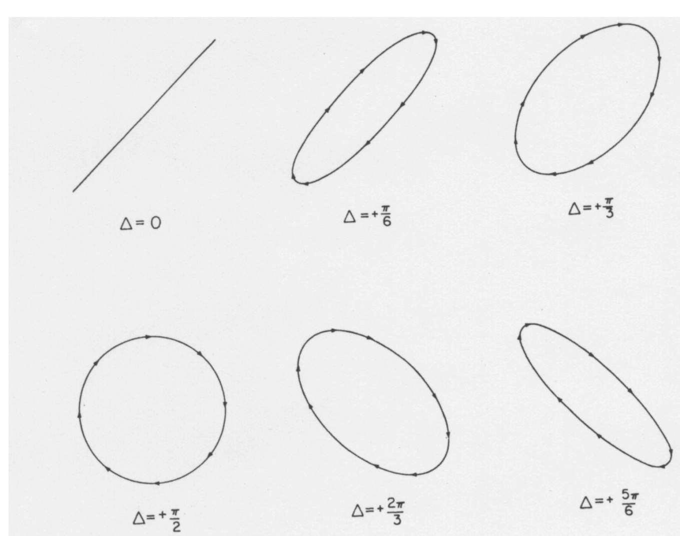
For elliptical states, there will be two maxima and two minima of intensity, but the minima will not be zero.

For circularly polarized light, the intensity will remain constant.

Any state of polarization which is elliptical or circular is called either "right-handed" or "left-handed".

For a right-handed state, the phase angle Δ by which the y-component leads the x-component must lie between 0 and $\pi/2$.

In that case, the tip of the electric field vector travels around its elliptical path in a clockwise direction as seen from an observer looking from the +z-direction towards the light source(see Figure 1).



Use of Jones Calculus for Transforming a Ket or Column Vector

In the Jones calculus, optical devices correspond to 2x2 matrices with complex elements.

There are no redundant elements and every Jones matrix that can be written corresponds to a device which, in principle at least, can be physically realized.

The Jones matrices representing devices act on the optical column vectors (kets) described earlier.

The elements of these vectors represent the amplitudes and phases of the components of the transverse electric field.

The electric field components in the beam leaving the devices are linear functions of the electric field components in the input beam and the matrix which connects the components in the beam leaving the devices with those entering the devices gives a complete(as far as polarization is concerned) characterization of the devices.

We saw earlier the representation of the transverse electric fields corresponding to any fully polarized beam can be represented by a column vector.

For a plane wave which is being propagated in the direction +z, the electric field can be regarded as the real part of the complex vector

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} H \exp\left[i \left\{ \omega \left(t - \frac{z}{c} \right) + \phi_x \right\} \right] \\ K \exp\left[i \left\{ \omega \left(t - \frac{z}{c} \right) + \phi_y \right\} \right] \end{bmatrix}$$

In most cases the time-dependence and the z-dependence of the beam are transferred outside the column as a scalar multiplier, so that we have

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} H\exp[i\phi_x] \\ K\exp[i\phi_y] \end{bmatrix} \exp\left[i\left\{ \omega\left(t - \frac{z}{c} \right) \right\} \right]$$

A useful rule for calculating the intensity of the beam is to premultiply the ket by its hermitian conjugate, an operation known as finding the bra(c)ket product.

We obtain

$$I = \begin{bmatrix} E_x^* & E_y^* \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} H\exp[-i\phi_x] & K\exp[-i\phi_y] \end{bmatrix} \begin{bmatrix} H\exp[i\phi_x] \\ K\exp[i\phi_y] \end{bmatrix}$$

$$= H^2 + K^2$$

Often the absolute amplitude is not required and can be transferred outside the column into the scalar multiplier.

For a beam of unit intensity, we must have $H^2 + K^2 = 1$ so we can write $H = \cos\theta$ and $K = \sin\theta$ for some θ. In terms of this "normalized" column vector, the expression for the beam becomes

$$\begin{bmatrix} \cos\theta \\ \sin\theta e^{i\Delta} \end{bmatrix} A\exp\left[i\left\{ \omega\left(t - \frac{z}{c} \right) \right\} \right]$$

This expression provides all that is needed to represent a completely coherent plane-monochromatic wave disturbance.

It is an experimental fact that at optical frequencies the response of the medium to an electromagnetic wave field is almost perfectly linear.

We can therefore predict that, if

$$E_1 = \begin{bmatrix} H_1 e^{i\phi_1} \\ K_1 e^{i\psi_1} \end{bmatrix}$$

represents the ket of the beam entering some kind of polarizing device, the ket representing the beam leaving the device can be represented as

$$E_2 = \begin{bmatrix} H_2 e^{i\phi_2} \\ K_2 e^{i\psi_2} \end{bmatrix}$$

and

$$H_2 e^{i\phi_2} = J_{11} H_1 e^{i\phi_1} + J_{12} K_1 e^{i\psi_1}$$
$$K_2 e^{i\psi_2} = J_{21} H_1 e^{i\phi_1} + J_{22} K_1 e^{i\psi_1}$$

or in matrix form

$$\begin{bmatrix} H_2 e^{i\phi_2} \\ K_2 e^{i\psi_2} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} H_1 e^{i\phi_1} \\ K_1 e^{i\psi_1} \end{bmatrix}$$

In general, the four elements of the 2x2 matrix are complex and depend only on the device.

It will be noticed that in the equations above, we have, for the sake of generality, used two separate phase angles ϕ and Ψ in each of the kets.

As far as the state of polarization is concerned, it is only the phase difference $(\phi - \psi) = \Delta$ which has any physical significance.

But it is not always possible to ensure, even for an input for which the phase angle ϕ_1 is zero, that the corresponding output phase angle ϕ_2 for the x-component is also zero.

Once and output ket has been calculated, it is very easy to multiply the whole ket by any desired phase factor.

Starting with the above equation, we can now develop the Jones calculus.

The matrix consisting of the four J_{ij}'s is called the **Jones matrix** J of the device, so that the matrix equation can be written $E_2 = JE_1$.

Suppose now we have two devices, whose Jones matrices are J_a and J_b.

We pass a beam of light through the two devices in series.

Let the ket of the original beam of light be E_1, that of the beam between the devices E_2 and that after the second device E_3.

Then, we have $E_2 = J_a E_1$ and $E_3 = J_b E_2$.

Substituting the first of these equations into the second, we get

$$E_3 = J_b\left(J_a E_1\right) = \left(J_b J_a\right) E_1$$

using the associativity property of matrix multiplication.

Thus, we see that the effect of a number of devices in series on a beam of light can be obtained by multiplying their Jones matrices together.

Derivation of Jones Matrices - For simplicity we will work with equation of the form

$$H_2 = J_{11} H_1 + J_{12} K_1 e^{i\Delta_1}$$
$$K_2 = J_{21} H_1 + J_{22} K_1 e^{i\Delta_1}$$

which is sufficient for all examples we will consider.

We note the connection using the form

$$A\begin{bmatrix} \cos\theta \\ \sin\theta e^{i\Delta} \end{bmatrix} = \begin{bmatrix} H \\ K e^{i\Delta} \end{bmatrix}$$

The Polarizer (such as a sheet of Polaroid)

(a) First, we consider an ideal linear polarizer, whose pass-plane is horizontal, that is, parallel to the x-direction.

This allows only electric field vectors parallel to the x-direction to pass through the device, so that in the general equations describing the behavior of an ideal device H_2 is equal to H_1 and K_2 is zero, no matter what the values of H_1 and K_1 may be.

Thus, the equations become

$$H_2 = H_1 = J_{11}H_1 + J_{12}K_1e^{i\Delta_1}$$
$$K_2 = 0 = J_{21}H_1 + J_{22}K_1e^{i\Delta_1}$$

for all values of H_1, K_1, and Δ_1.

Set $K_1=0$.

The equations become $H_1 = J_{11}H_1$ or $J_{11} = 1$ and $0 = J_{21}H_1$ for all H1, that is, $J_{21} = 0$.

Set $H_1=0$.

The equations become $0 = J_{12}K_1\exp(i\Delta_1)$ and $0 = J_{22}K_1\exp(i\Delta_1)$ for all K_1 and Δ_1, that is, J_{12} and J_{22} are both zero.

The Jones matrix for this ideal x-polarizer is thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_x = P_{\theta=0}$$

In practice, of course, a sheet of Polaroid introduces some attenuation, even for the preferred direction of polarization and its optical thickness will be sufficient to introduce at least several hundred wavelengths of retardation.

Thus, for some purposes, for example, in an interferometer calculation, the above matrix needs to be multiplied by a complex scalar, which represents the complex amplitude transmittance of the polarizer.

The definition of the Jones matrix is arbitrary up to an over all phase factor(intensity only).

Thus, a general representation of the Jones matrix for this ideal polarizer is

$$\begin{bmatrix} e^{i\delta} & 0 \\ 0 & 0 \end{bmatrix} = P_x = P_{\theta=0}$$

In a similar manner we can show that, for a polarizer with its pass-plane vertical, that is, parallel to the y-axis, the Jones matrix can be represented by

$$\begin{bmatrix} 0 & 0 \\ 0 & e^{i\alpha} \end{bmatrix} = P_y = P_{\theta=\pi/2}$$

(b) We now consider the more general case of a polarizer with its pass-plane making and angle θ with the x-direction as shown in Figure 2 below.

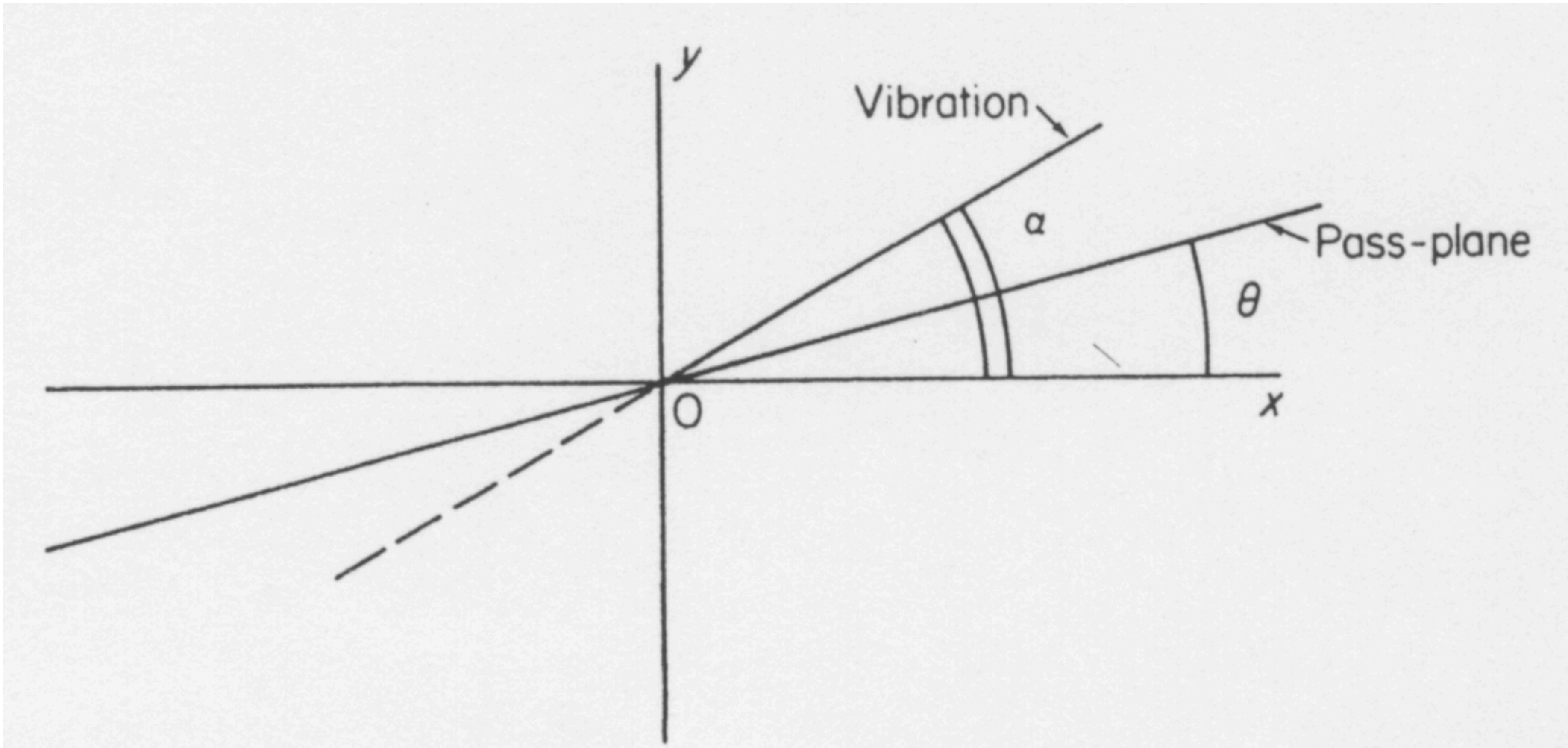


Figure 2

Suppose we have an plane-polarized electric field incident such that its polarization direction makes an angle α with the x-axis.

Suppose the amplitude of this incident beam is A.

Then, by definition $X_1 = A\cos\alpha$ and $Y_1 = A\sin\alpha$.

Only the component of the electric field along the direction of the pass-plane of the Polaroid will pass through the device.

The amplitude of this component is

$$U = A\cos(\alpha - \theta) = A\cos\alpha\cos\theta + A\sin\alpha\sin\theta$$
$$= X_1\cos\theta + Y_1\sin\theta$$

The component of the emerging vector parallel to the x-axis is

$$Y_2 = U\sin\theta = X_1\sin\theta\cos\theta + Y_1\sin^2\theta$$

Writing the last two equation in matrix form we have

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

so the Jones matrix for this device is

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = P_\theta$$

Jones Matrix of the General Linear Retarder

We now consider a crystal plate with its optic(fast) axis at an angle α to the x-axis and we suppose that this plate retards the phase of the o-ray (orthogonal to the optic(fast) axis) by δ relative to the field component along the optic(fast) axis, that is, to the e-ray.

This device makes no difference to the ket of a plane-polarized vibration parallel to the optic(fast) axis.

Suppose the amplitude of this field is A, then, as in the discussion of the Polaroid, we know that the components are $X_1 = A\cos\alpha$ and $Y_1 = A\sin\alpha$.

Since this field component is unaffected by the device, the emerging components are the same as the entering ones, that is, $X_2 = A\cos\alpha$ and $Y_2 = A\sin\alpha$.

Substituting in the general equation for the behavior of the device, we find the equations

$$A\cos\alpha = J_{11}A\cos\alpha + J_{12}A\sin\alpha$$
$$A\sin\alpha = J_{21}A\cos\alpha + J_{22}A\sin\alpha$$

which give

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1-J_{11}}{J_{12}} = \frac{J_{21}}{1-J_{22}}$$

as long as $\cos\alpha$ does not vanish.

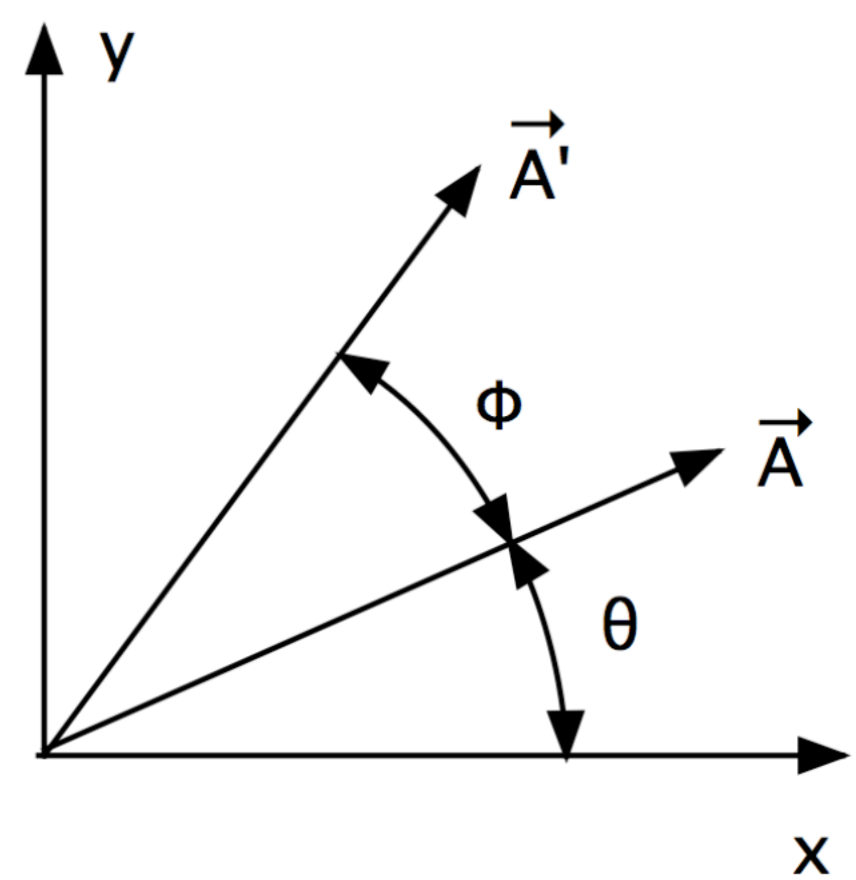
We now consider the action of the device on a beam of plane-polarized light with it direction of polarization parallel to the x-axis and of amplitude A.

Digression to rotation matrices:

Vector Rotation

Consider a vector

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 = A\cos\theta\hat{e}_1 + A\sin\theta\hat{e}_2$$



Now rotate the vector through an angle Φ as shown:

We then have

$$\vec{A}' = A'_1\,\hat{e}_1 + A'_2\,\hat{e}_2 = A\cos(\theta+\phi)\hat{e}_1 + A\sin(\theta+\phi)\hat{e}_2$$

Therefore we find that the components transform as

$$A'_1 = A_1\cos\phi - A_2\sin\phi$$
$$A'_2 = A_1\sin\phi + A_2\cos\phi$$

Using matrix notation we have

$$[A'] = [R(\phi)][A]$$

$$\begin{bmatrix} A'_1 \\ A'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

The matrix [$R(\phi)$] is called the rotation matrix and is defined by (for a rotation about the z-axis or 3-axis) as

$$[R(\phi)] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

In summation notation we have

$$A'_i = R_{ij}A_j \rightarrow \vec{A}' = R_{ij}A_j\hat{e}_i$$

The rotation matrix is an orthogonal matrix since

$$[R(\phi)]^+[R(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I] = \text{identity matrix}$$

This says that the length of the vector does not change under the rotation transformation,i.e.,

$$\begin{aligned}\vec{A}'\cdot\vec{A}' &= \left(R_{ij}A_j\hat{e}_i\right)\cdot\left(R_{mn}A_n\hat{e}_m\right) = A_jA_nR_{ij}R_{mn}\hat{e}_i\cdot\hat{e}_m = A_jA_nR_{ij}R_{mn}\delta_{im} \\ &= A_jA_nR_{ij}R_{in} = A_jA_n([R^+])_{ji}R_{in} = A_jA_n\left([R^+R]\right)_{jn} = A_jA_n\delta_{jn} = A_jA_j \\ &= \vec{A}\cdot\vec{A}\end{aligned}$$

Continuing: For this plane-polarized beam, the ket is

$$E_1 = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

We now need to consider the components of this field orthogonal and parallel to the optic(fast) axis.

To find these components we rotate the axes:

$$\begin{bmatrix} \text{component along optic axis} \\ \text{component perpendicular to optic axis} \end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} A\cos\alpha \\ -A\sin\alpha \end{bmatrix}$$

The component perpendicular to the optic(fast) axis is now retarded by a phase angle δ, that is to say V_1 is multiplied by $e^{-i\delta}$.

The matrix of components parallel and orthogonal to the optic(fast) axis thus become

$$\begin{bmatrix} U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} A\cos\alpha \\ -Ae^{-i\delta}\sin\alpha \end{bmatrix}$$

We now need to return to the original axes. We thus rotate from the UV axes back to the XY axes.

The final form of the ket is thus

$$E_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} A\cos\alpha \\ -Ae^{-i\delta}\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} A\cos^2\alpha + Ae^{-i\delta}\sin^2\alpha \\ A\cos\alpha\sin\alpha - Ae^{-i\delta}\cos\alpha\sin\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} A(\cos^2\alpha + e^{-i\delta}\sin^2\alpha) \\ A\cos\alpha\sin\alpha(1 - e^{-i\delta}) \end{bmatrix}$$

Now we must have

$$E_2 = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} J_{11}A \\ J_{21}A \end{bmatrix}$$

Thus,

$$J_{11} = \cos^2\alpha + e^{-i\delta}\sin^2\alpha$$

$$J_{21} = \cos\alpha\sin\alpha(1 - e^{-i\delta})$$

Using the two equations we derived earlier for $\tan\alpha$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1-J_{11}}{J_{12}} = \frac{J_{21}}{1-J_{22}}$$

we find

$$J_{22} = \sin^2\alpha + e^{-i\delta}\cos^2\alpha$$

$$J_{12} = \cos\alpha\sin\alpha(1-e^{-i\delta})$$

so that the Jones matrix of the general phase plate in general orientation α is

$$\begin{bmatrix} \cos^2\alpha + e^{-i\delta}\sin^2\alpha & \cos\alpha\sin\alpha(1-e^{-i\delta}) \\ \cos\alpha\sin\alpha(1-e^{-i\delta}) & \sin^2\alpha + e^{-i\delta}\cos^2\alpha \end{bmatrix}$$

The Jones matrix for a simple rotator, that is, a device which twists the polarization direction of a beam of plane-polarized light through angle θ counterclockwise (turpentine) is

$$R(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In general, if J represents the Jones matrix already calculated for a particular device, the new matrix for the same device rotated through an angle θ will be given by the triple matrix product

$$R(-\theta)JR(\theta)$$

As an example of this rule, one can verify that the general retarder at angle θ is equal to the product

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\delta} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

which agree with our earlier result.

The table below gives the Jones matrices for ideal linear polarizers, linear retarders, rotation axes and circular retarders.

The angle θ specifies how the pass-plane of a polarizer or the optic(fast) axis of a linear retarder is oriented with respect to the x-axis.

Type of device	$\theta = 0$	$\theta = \pm\pi/4$	$\theta = \pi/2$	$\theta =$ any value
Ideal linear polarizer at angle θ	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_1^2 & C_1S_1 \\ C_1S_1 & S_1^2 \end{bmatrix}$
Quarter-wave linear retarder with optic axis at angle θ	$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1-i & \pm(1+i) \\ \pm(1+i) & 1-i \end{bmatrix}$	$\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_1^2 - iS_1^2 & C_1S_1(1+i) \\ C_1S_1(1+i) & -iC_1^2 + S_1^2 \end{bmatrix}$

$C_1 = \cos\theta$,
$S_1 = \sin\theta$

The matrix for $\theta = \pm\,\pi/4$ can be multiplied by $e^{i\pi\theta}$ to give

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}$$

Type of device	$\theta=0$	$\theta=\pm\pi/4$	$\theta=\pi/2$	$\theta=$ any value
Half-wave linear retarder with optic axis at angle θ	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_2 & S_2 \\ S_2 & -C_2 \end{bmatrix}$
Linear retarder with retardation δ and with optic axis at angle θ	$\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\delta} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} e^{-i\delta}+1 & \pm\left(1-e^{-i\delta}\right) \\ \pm\left(1-e^{-i\delta}\right) & e^{-i\delta}+1 \end{bmatrix}$ *or* $e^{-i\delta/2}\begin{bmatrix} \cos\frac{\delta}{2} & \pm i\sin\frac{\delta}{2} \\ \pm i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{bmatrix}$	$\begin{bmatrix} e^{-i\delta} & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_1^2+S_1^2e^{-i\delta} & C_1S_1(1-e^{-i\delta}) \\ C_1S_1(1-e^{-i\delta}) & C_1^2e^{-i\delta}+S_1^2 \end{bmatrix}$

$C_2 = \cos 2\theta$, $S_2 = \sin 2\theta$

Experimental Determination of Elements of Jones Matrix and Light Column Vector

We now describe methods for determining the ket vector of any beam of polarized light, whether plane or elliptically polarized, and method by which the Jones matrix of any device can be found experimentally.

These methods involve passing beams of light through the device and through various Polaroids and phase plates, and then measuring the intensity of each beam that emerges.

First, we consider determination of the ket vector of a beam of light.

Suppose the ket vector of the beam is

$$\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix}$$

so that the intensity is

$$\begin{bmatrix} H & Ke^{-i\Delta} \end{bmatrix}\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = H^2 + K^2 = 1$$

that is, we assume unit intensity.

First, we pass the beam through a polaroid with its pass-plane horizontal, that is, parallel to the x-axis.

The ket vector of the emerging beam is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = \begin{bmatrix} H \\ 0 \end{bmatrix}$$

Therefore, the intensity $I_1 = H^2$, so that $H = \sqrt{I_1}$, where I_1 is the measured intensity.

We now pass the original beam through a polaroid with its pass-plane vertical (parallel to the y-axis), so that the ket vector becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = \begin{bmatrix} 0 \\ Ke^{i\Delta} \end{bmatrix}$$

Therefore, the intensity $I_2 = K^2$, so that $K = \sqrt{I_2}$, where I_2 is the measured intensity.

We now pass the beam through a polaroid with its pass-plane at an angle 45° with the x-axis in the first and third quadrants for which the Jones matrix is

$$\begin{bmatrix} \cos^2 45 & \cos 45 \sin 45 \\ \cos 45 \sin 45 & \sin^2 45 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1/2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The emerging ket vector is

$$1/2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = 1/2\begin{bmatrix} H - Ke^{i\Delta} \\ -H + Ke^{i\Delta} \end{bmatrix}$$

The intensity is

$$I_4 = \frac{1}{2}\left(H^2 - 2HK\cos\Delta + K^2\right)$$

Therefore

$$I_3 - I_4 = 2HK\cos\Delta$$

The values of H and K are already known, so this equation determines $\cos\Delta$, but Δ can still be positive or negative.

There is an uncertainty of sign, so we now determine $\sin\Delta$ to resolve this uncertainty.

We pass the original beam through a quarter-wave plate whose optic(fast) axis is horizontal.

Using the fact that $\delta = 90°$ $\theta = 0°$ for a quarter-wave plate and that find that the Jones matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

and the ket vector of the beam becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix} = \begin{bmatrix} H \\ -iKe^{i\Delta} \end{bmatrix}$$

We put the beam from the quarter-wave plate through a polaroid inclined at 45° to the axis.

The ket vector becomes

$$1/2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}H\\ -iKe^{i\Delta}\end{bmatrix}=1/2\begin{bmatrix}H-iKe^{i\Delta}\\ H-iKe^{i\Delta}\end{bmatrix}$$

The intensity is

$$I_6=\frac{1}{2}\left(H^2+2HK\sin\Delta+K^2\right)$$

Therefore

$$I_6-I_5=2HK\sin\Delta$$

This determines $\sin\Delta$. Knowing $\sin\Delta$ and $\cos\Delta$, we obtain the value of Δ.

We now know all variables of the ket vector of the original beam.

We now describe the method by which the Jones matrix of any device can be determined from intensity measurements.

Suppose the Jones matrix of the device is

$$J=\begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}=\begin{bmatrix}X_{11}+iY_{11} & X_{12}+iY_{12}\\ X_{21}+iY_{21} & X_{22}+iY_{22}\end{bmatrix}=\begin{bmatrix}\mathrm{R}_{11}\mathrm{e}^{i\theta_{11}} & \mathrm{R}_{12}\mathrm{e}^{i\theta_{12}}\\ \mathrm{R}_{21}\mathrm{e}^{i\theta_{21}} & \mathrm{R}_{22}\mathrm{e}^{i\theta_{22}}\end{bmatrix}$$

A. We pass into the device a beam of unit intensity plane-polarized parallel to the x-axis so the incident ket vector is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

On emerging from the device the ket vector will be

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix}$$

A.1. The beam is now passed through a polaroid with its pass-plane horizontal and the ket vector becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} J_{11} \\ 0 \end{bmatrix}$$

The intensity is

$$I_2 = J_{11}^* J_{11} = X_{11}^2 + Y_{11}^2 = R_{11}^2$$

A.2. The beam from the device is now put through a polaroid with its pass-plane vertical.The ket vector becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ J_{21} \end{bmatrix}$$

The intensity is

$$I_3 = R_{21}^2$$

B. A beam of unit intensity polarized parallel to the y-axis and having a ket vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is now passed into the device.

The emerging ket vector is

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} J_{12} \\ J_{22} \end{bmatrix}$$

B.1. The beam emerging from the device is now passed through a polaroid with its pass-plane horizontal. The emerging ket vector is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J_{12} \\ J_{22} \end{bmatrix} = \begin{bmatrix} J_{12} \\ 0 \end{bmatrix}$$

The intensity is

$$I_4 = R_{12}^2$$

B.2. The beam emerging from the device is now passed through a polaroid with its pass-plane vertical. The emerging ket vector is

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J_{12} \\ J_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ J_{22} \end{bmatrix}$$

The intensity is

$$I_5 = R_{22}^2$$

We have now determined the magnitude of all four Jones matrix elements.

It remains only to find the angles (phase) in these matrix elements.

C. We now pass into the device a beam of right-handed circularly polarized light of unit intensity so that

$$H = K \quad , \quad H^2 + K^2 = 1$$

Thus,

$$H = K = \frac{1}{\sqrt{2}}$$

The ket vector of the incident beam is therefore $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$

On emerging from the device the ket vector of the beam is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}J_{11}+iJ_{12}\\ J_{21}+iJ_{22}\end{bmatrix}$$

C.1. The beam is now passed through a polaroid with its pass-plane horizontal so the ket vector becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}J_{11}+iJ_{12}\\ J_{21}+iJ_{22}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}J_{11}+iJ_{12}\\ 0\end{bmatrix} = \begin{bmatrix}(X_{11}-Y_{12})+i(Y_{11}+X_{12})\\ 0\end{bmatrix}$$

The intensity is thus,

$$I_6 = \frac{1}{2}\left[\left(X_{11} - Y_{12}\right)^2 + \left(Y_{11} + X_{12}\right)^2\right]$$

Thus,

$$\frac{2I_6 - I_2 - I_4}{\sqrt{I_2 I_4}} = 2\sin\left(\theta_{11} - \theta_{12}\right)$$

C.2. The beam from the device is now passed through a polaroid with its pass-plane vertical.

The ket vector becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} J_{11} + iJ_{12} \\ J_{21} + iJ_{22} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ J_{21} + iJ_{22} \end{bmatrix}$$

so that the intensity is

$$I_7 = \frac{1}{2}\left[\left(X_{21} - Y_{22}\right)^2 + \left(Y_{21} + X_{22}\right)^2\right]$$

and thus

$$\frac{2I_7 - I_3 - I_5}{\sqrt{I_3 I_5}} = 2\sin\left(\theta_{21} - \theta_{22}\right)$$

We now know the sines of the angles $(\theta_{11} - \theta_{12})$ and $(\theta_{21} - \theta_{22})$ but this still leaves doubt as to the value of the angles because $\sin(\pi - \theta) = \sin\theta$.

To determine the angles completely we need to know the cosines of the angles as well.

D. We pass into the device a beam of light of unit intensity plane polarized at 45° to the x-axis.

For this beam

$$H = K = \frac{1}{\sqrt{2}}$$

and $\Delta = 0$, so that $e^{i\Delta} = 1$.

For the original beam, the ket vector is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that after the device the ket vector is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} J_{11} + J_{12} \\ J_{21} + J_{22} \end{bmatrix}$$

D.1. The beam is now passed through a polaroid with its pass-plane horizontal so the ket vector becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}J_{11}+J_{12}\\ J_{21}+J_{22}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}J_{11}+J_{12}\\ 0\end{bmatrix}$$

and the intensity is

$$I_9=\frac{1}{2}\left[\left(X_{21}+X_{22}\right)^2+\left(Y_{21}+X_{22}\right)^2\right]$$

and thus

$$\frac{2I_9-I_3-I_5}{\sqrt{I_3 I_5}}=2\cos\left(\theta_{22}-\theta_{21}\right)$$

We now completely know the angles $(\theta_{11}-\theta_{12})$ and $(\theta_{21}-\theta_{22})$.

To determine the individual angles we put a beam that is plane-polarized horizontally into the device so that as earlier the ket emerging from the device is

$$\begin{bmatrix}J_{11}\\ J_{21}\end{bmatrix}$$

We pass this beam through a polaroid with is pass-plane inclined at 45° to the axis.

The ket vector becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} J_{11} + J_{21} \\ J_{11} + J_{21} \end{bmatrix}$$

We now put the beam through another polaroid with its axis horizontal.

The ket vector becomes

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} J_{11} + J_{12} \\ J_{21} + J_{22} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} J_{11} + J_{12} \\ 0 \end{bmatrix}$$

As in D.1 or D.2, the intensity becomes

$$I_{10} = \frac{1}{4}\left[\left(X_{11} + X_{21}\right)^2 + \left(Y_{11} + X_{21}\right)^2\right]$$

so that

$$\frac{4I_{10} - I_2 - I_3}{\sqrt{I_2 I_3}} = 2\cos\left(\theta_{11} - \theta_{21}\right)$$

We now put the same beam from the device

$$\begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix}$$

through a quarter-wave plate with its optic(fast) axis vertical.

The ket vector becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} J_{11} \\ iJ_{21} \end{bmatrix}$$

We now pass the beam through a polaroid with its axis at 45°.

The ket vector becomes

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} J_{11} \\ iJ_{21} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} J_{11} + iJ_{21} \\ J_{11} + iJ_{21} \end{bmatrix}$$

We now put the beam through another polaroid with its axis horizontal.

The ket vector becomes

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} J_{11} + iJ_{21} \\ J_{11} + iJ_{21} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} J_{11} + iJ_{21} \\ 0 \end{bmatrix}$$

As in C.1 or C.2, the intensity is now

$$I_{11} = \frac{1}{4}\left[\left(X_{11} - X_{21}\right)^2 + \left(Y_{11} + X_{21}\right)^2\right]$$

so that

$$\frac{4I_{11} - I_2 - I_3}{\sqrt{I_2 I_3}} = 2\sin\left(\theta_{11} - \theta_{21}\right)$$

The angle $(\theta_{11} - \theta_{12})$ is thus determined and so the differences between θ_{11} and all the other three angles are now known.

We can regard one of the θ's as arbitrary, that is, we can arbitrarily set θ_{11} to zero.

This means that the angles in the other three Jones elements are now known and we have completely determined the Jones matrix.

Example 1

A source emits plane-polarized light of unit intensity which falls on an ideal linear polarizer.

Prove that the intensity of the light coming through the polaroid is $\cos^2\theta$ where θ is the angle of the polaroid measured from a position of maximum transmission.

Polaroid is set to extinguish the transmitted beam and a second polaroid is then placed between the source and the first polaroid.

Show that some light can now pass through both polaroids and prove that its intensity is proportional to $\sin^2\phi$, where ϕ where is the angle of the second polaroid measured from the position of extinction.

Solution: Suppose the incident beam of light is plane-polarized horizontal so that its ket vector is

$$E_1 = \begin{bmatrix} H \\ 0 \end{bmatrix} \quad , \quad H = 1 = \text{unit intensity}$$

The Jones matrix of the polaroid with its pass-plane at angle θ with the x-direction is

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

so that the ket vector of the beam emerging from the polaroid is

$$E_2 = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} H \\ 0 \end{bmatrix} = \begin{bmatrix} H\cos^2\theta \\ H\cos\theta\sin\theta \end{bmatrix}$$

The intensity of the emergent beam is given by

$$\begin{aligned} E_2^\dagger E_2 &= H^2\cos^4\theta + H^2\cos^2\theta\sin^2\theta \\ &= H^2\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right) = H^2\cos^2\theta \end{aligned}$$

Suppose we now pass the beam through another polaroid with its pass-plane vertical, so that its Jones matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The ket vector emerging from this polaroid will therefore be

$$E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H\cos^2\theta \\ H\cos\theta\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ H\cos\theta\sin\theta \end{bmatrix}$$

The intensity is

$$E_3^\dagger E_3 = H^2\cos^2\theta\sin^2\theta = \frac{H^2\sin^2 2\theta}{4}$$

There are thus four orientations of the intermediate polaroid which give extinction.

Example 2

Elliptically polarized light is passed through a quarter-wave plate and then through a polaroid.

Extinction occurs when the optic(fast) axis of the plate and the pass-plane of the polaroid are inclined to the horizontal at angles 30° and 60° respectively, the angles being measured in the same direction.

Find the orientation of the ellipse.

Solution: If we substitute the given angles into the standard formulae for the Jones matrix of the polaroid and of the quarter-wave plate, we get the matrix of the polaroid followed by the quarter-wave plate

$$\begin{bmatrix} \cos^2 60 & \cos 60\sin 60 \\ \cos 60\sin 60 & \sin^2 60 \end{bmatrix} \begin{bmatrix} \cos^2 30 - i\sin^2 30 & \cos 30\sin 30(1+i) \\ \cos 30\sin 30(1+i) & -i\cos^2 30 + \sin^2 30 \end{bmatrix}$$

$$\underset{\text{polaroid}}{\frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}}\underset{\text{quarter-wave plate}}{\frac{1}{4}\begin{bmatrix} 3-i & \sqrt{3}(1+i) \\ \sqrt{3}(1+i) & 1-3i \end{bmatrix}}$$

$$=\frac{1}{8}\begin{bmatrix} 3+i & \sqrt{3}(1-i) \\ \sqrt{3}(3+i) & 3-3i \end{bmatrix}$$

Suppose the ket vector of the entering beam is

$$\begin{bmatrix} H \\ Ke^{i\Delta} \end{bmatrix}$$

If we assign unit intensity to this beam, then in the usual notation

$$H=\cos\theta \quad , \quad K=\sin\theta$$

so that the incident ket is

$$\begin{bmatrix} \cos\theta \\ \sin\theta e^{i\Delta} \end{bmatrix}$$

Since extinction is produced, the ket vector of the emerging beam from the pair of devices must have both of its elements equal to zero so that

$$\begin{bmatrix} 3+i & \sqrt{3}(1-i) \\ \sqrt{3}(3+i) & 3-3i \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta e^{i\Delta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now suppose we have a 2x2 square matrix multiplying a column matrix to produce azero column matrix thus:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This says that AX+BY=0 and CX+DY=0.

Given that neither X nor Y vanishes for an elliptically polarized beam, we know at once from these equations that

$$\frac{X}{Y} = -\frac{B}{A} = -\frac{D}{C}$$

We have and so that $B = \sqrt{3}(1-i)$ and $A = 3+i$ so that

$$\frac{X}{Y} = \frac{B}{A} = \frac{\sqrt{3}(1-i)}{3+i} = \frac{\sqrt{3}(1-i)(1+i)}{(3+i)(1+i)} = \frac{\sqrt{3}}{1+2i}$$

Therefore

$$\frac{\cos\theta}{\sin\theta e^{i\Delta}} = \frac{X}{Y} = -\frac{B}{A} = -\frac{\sqrt{3}}{1+2i}$$

and inverting

$$\frac{\sin\theta\cos\Delta + i\sin\theta\sin\Delta}{\cos\theta} = \frac{-2i-1}{\sqrt{3}}$$

Equating real imaginary parts we get

$$\tan\theta\cos\Delta = -\frac{1}{\sqrt{3}} \quad , \quad \tan\theta\sin\Delta = -\frac{2}{\sqrt{3}}$$

Dividing the second equation by the first now gives

$$\tan\Delta = 2 \rightarrow \Delta = 63°26' \rightarrow \cos\Delta = 0.447$$

The equation between the real parts now gives

$$\tan\theta = -\frac{1}{\cos\Delta\sqrt{3}} = -\frac{1}{0.447x1.732} = -1.292 \rightarrow \theta = -52°16'$$

Example 3

A beam of right-handed circularly polarized light is passed normally through (a) a quarter-wave plate and (b) an eighth-wave plate.

Both plates are to be regarded as having their optic(fast) axes vertical.

Describe the state of polarization of the light as it emerges from each plate.

Solution: The circularly polarized light beam has its components H and K parallel to the two axes equal.

For a right-handed state the phase difference Δ between them is 90° so the normalized ket vector is

$$\sqrt{\frac{1}{2}}\begin{bmatrix}1\\(1)e^{i\pi/2}\end{bmatrix}=\sqrt{\frac{1}{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

The Jones matrix of the quarter-wave plate with its optic(fast) axis vertical can be written

$$\begin{bmatrix}-i & 0\\0 & 1\end{bmatrix}(i)=\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}$$

This corresponds to H = K = 1 and $\Delta=\pi$.

The emerging beam is linearly polarized at an angle of -45° to the x-axis and lying in the second and fourth quadrants.

For the eighth-wave plate the phase shift produced between the ordinary and extraordinary waves is 45°, so that $\cos\delta$ and $\sin\delta$ are both equal to $1/\sqrt{2}$.

In this case $\theta = 90°$ and the Jones matrix can be written in the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{bmatrix}\left(e^{i\pi/4}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}$$

so that the emerging ket vector is

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix}\sqrt{\frac{1}{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \sqrt{\frac{1}{2}}\begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}}(i-1) \end{bmatrix}$$

From the ratios of the real and imaginary parts of the second component we see that
$\tan\Delta = -1 \rightarrow \Delta = 135°$, $\sin\Delta = -1/\sqrt{2} = \cos\Delta$

Inserting these values into the matrix, we see that H = K = 1, so that using the standard form for the equation of the ellipse

$$\frac{x^2}{H^2} - \frac{2xy}{HK}\cos\Delta + \frac{y^2}{K^2} = \sin^2\Delta$$

we get $x^2 - \sqrt{2}xy + y^2 = 1$ because Δ lies between 0° and 180°, the elliptical state is right-handed.

Example 4

A wave described by: $x = A\cos(\omega t + \pi/4)$ and $y = A\cos(\omega t)$ is incident on a polaroid which is rotated in its own plane until the transmitted intensity is a maximum.

(a) In what direction does the pass-plane of the polaroid now lie?

(b) Compute the ratio of the transmitted intensities observed with the polaroid so oriented and oriented with its pass-plane in the y-direction.

Solution: We first set up the ket vector of the original beam.

This is

$$A\begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix} \quad \text{or} \quad A\begin{bmatrix} e^{i\pi/4} \\ 1 \end{bmatrix}\left(e^{-i\pi/4}\right) = A\begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix}$$

After passing through the polarizer at an angle θ , the ket vector becomes

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} A\begin{bmatrix} 1 \\ e^{-i\pi/4} \end{bmatrix} = A\begin{bmatrix} \cos^2\theta + \cos\theta\sin\theta e^{-i\pi/4} \\ \cos\theta\sin\theta + \sin^2\theta e^{-i\pi/4} \end{bmatrix}$$

The intensity is given by

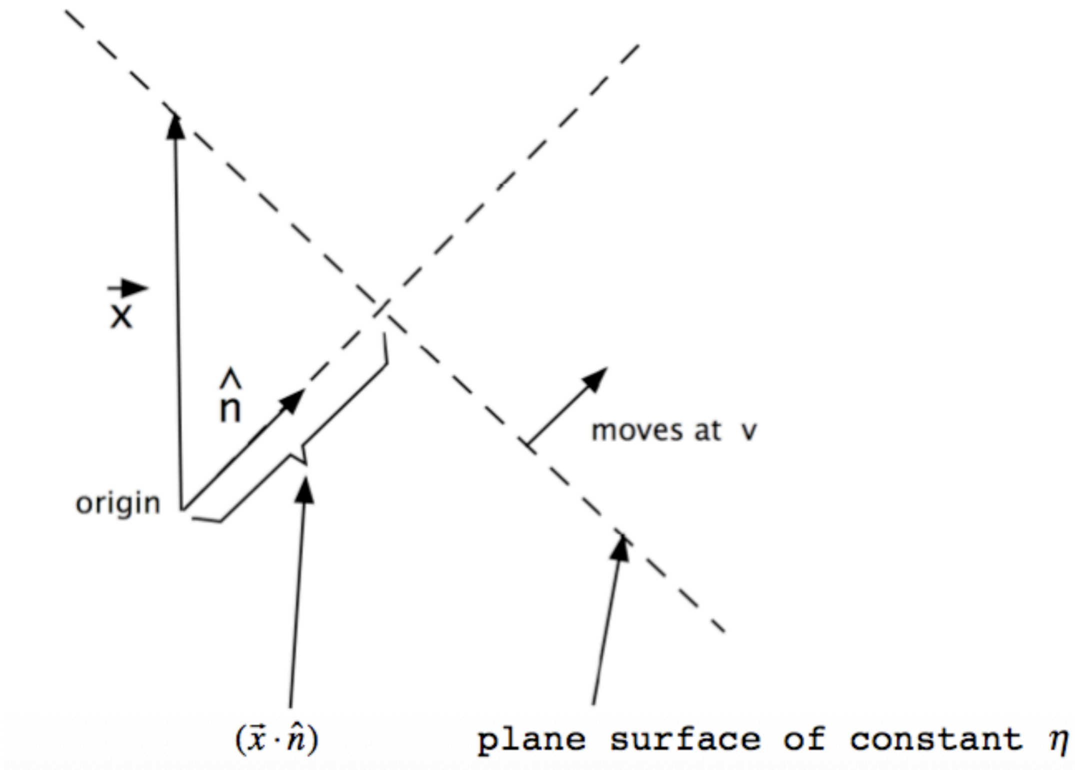
$$I = A^2\left(1 + \frac{1}{\sqrt{2}}\sin 2\theta\right)$$

We now take a short excursion into some real physics calculations.

Just sit back and observe what a full physics major does all the time.

Part 4 - Waves and Diffraction:

Let $\vec{x} = (x, y, z), |\vec{x}| = r$ locate a point in space with respect to some origin (position vector) (see diagram).



Let $\hat{n}$ = a unit vector ($|\hat{n}| = 1$) pointing in a **fixed** direction.

$\hat{n}$ is dimensionless (it has no units).

Let $\eta(\vec{x}, t)$ be some physical quantity defined at each point $\vec{x}$ at some time t.

We refer to $\eta(\vec{x}, t)$ as a "disturbance".

In this discussion, we will generalize slightly and let $\eta(\vec{x}, t)$ take on complex values.

For example, $\eta(\vec{x}, t)$ can refer to one of the Cartesian components of the electric field $\vec{E}(\vec{x}, t)$ or the magnetic field $\vec{B}(\vec{x}, t)$.

Definition: $\eta(\vec{x},t)$ is a plane wave if it can be written in the form

$$\eta(\vec{x},t) = F(\vec{x}\cdot\hat{n} - vt)$$

where F is some arbitrary function of one variable, where $\hat{n}$ is some fixed unit vector and υ = a real constant, $\upsilon > 0$.

Clearly, υ has units of

$$\frac{\vec{x}\cdot\hat{n}}{t} \rightarrow \text{units of velocity}$$

Consider the points in space for which $\vec{x}\cdot\hat{n} - vt$ = constant (η remains constant at these points).

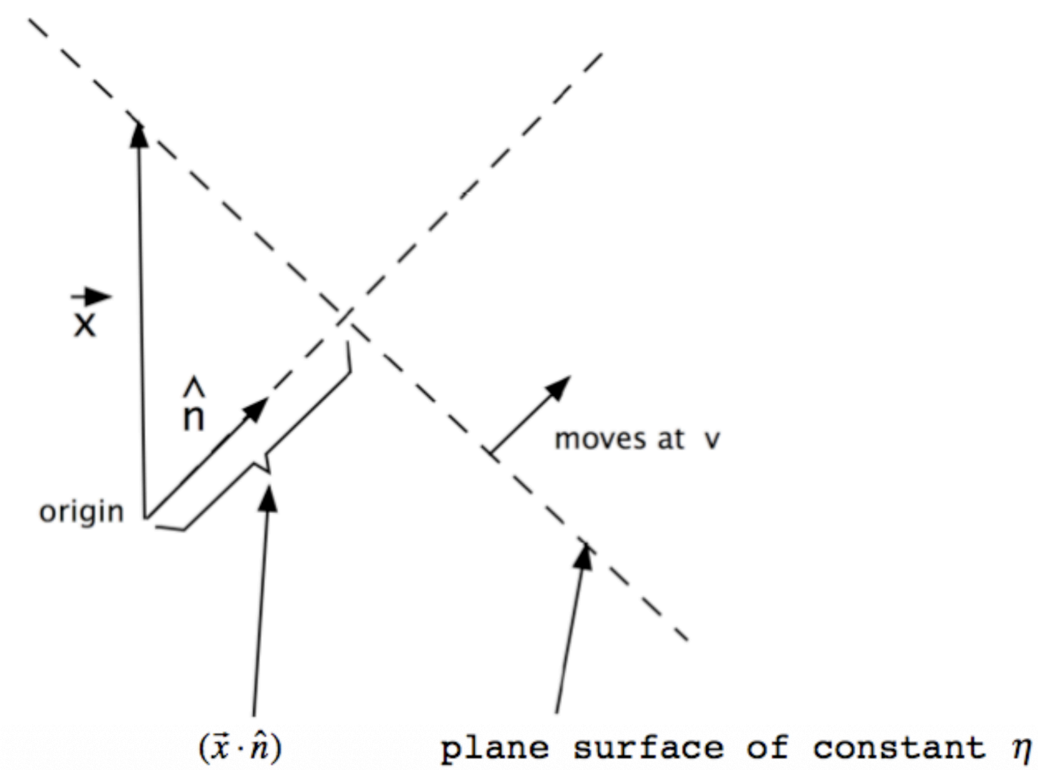
Therefore

$$\vec{x}\cdot\hat{n} = \text{constant} + vt = \texttt{ projection of } \vec{x} \texttt{ onto the } \hat{n} \texttt{ direction}$$

This projection moves with constant speed υ.

All points on the plane perpendicular to $\hat{n}$ have the same value of $\vec{x}\cdot\hat{n}$.

These relationships are shown in the figure



Thus, the surfaces on which $\eta(\vec{x},t)$ has a fixed value are plane surfaces perpendicular to $\hat{n}$ which move at speed v in the $\hat{n}$ direction (hence the name plane wave).

```
v = phase velocity of the plane wave, i.e.,
    the velocity of the planes of constant phase
n̂ = direction of propagation
```

That is what a plane wave is!!!

Now

$$\frac{\partial\eta}{\partial t} = F'\frac{\partial(\vec{x}\cdot\hat{n} - vt)}{\partial t} = -vF'$$

Similarly

$$\frac{\partial\eta}{\partial x} = F'\frac{\partial(\vec{x}\cdot\hat{n} - vt)}{\partial x} = F'\frac{\partial(xn_x + yn_y + zn_z - vt)}{\partial x} = n_x F'$$

$$\frac{\partial^2\eta}{\partial x^2} = n_x F''\frac{\partial(\vec{x}\cdot\hat{n} - vt)}{\partial x} = n_x^2 F''$$

and

$$\frac{\partial^2\eta}{\partial y^2} = n_y^2 F'' \quad , \quad \frac{\partial^2\eta}{\partial z^2} = n_z^2 F''$$

Therefore, the

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+\frac{\partial^2\eta}{\partial z^2}=(n_x^2+n_y^2+n_z^2)F''=(\hat{n}\cdot\hat{n})F''=F''=\frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\eta=\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\eta=0$$

which is the **classical wave equation**.

It is obeyed by **any** plane wave of the form

$$\eta(\vec{x},t)=F(\vec{x}\cdot\hat{n}-vt)$$

where υ = phase velocity of the plane wave.

Note that:

(1) The classical wave equation is **linear**, that is, if $\eta_1(\vec{x},t)$ and $\eta_2(\vec{x},t)$ are separately solutions of the wave equation, then any linear superposition (combination)

$$a_1\eta_1(\vec{x},t)+a_2\eta_2(\vec{x},t)$$

where a_1 and a_2 are constant, is also a solution.

(2) If $\eta(\vec{x},t)$ is a complex solution of the wave equation, then $\mathrm{Real}(\eta(\vec{x},t))$ and $\mathrm{Imag}(\eta(\vec{x},t))$ separately satisfy the wave equation (because $1/v^2$ is real).

(3) The classical wave equation has solutions other than plane wave solutions, for example, the linear superposition of 2 plane waves traveling in different directions

$$\eta(\vec{x},t) = F_1(\vec{x}\cdot\hat{n}_1 - vt) + F_2(\vec{x}\cdot\hat{n}_2 - vt)$$

is a solution.

Definition: $\eta(\vec{x},t)$ is a spherical wave if it can be written in the form

$$\eta(\vec{x},t) = \begin{cases} \frac{1}{r}G(r-vt) \rightarrow \text{spherical outgoing wave} \\ \frac{1}{r}G(r+vt) \rightarrow \text{spherical incoming wave} \end{cases}$$

where, $r = |\vec{x}|$, G is some arbitrary function on one variable and where υ is a real constant ($\upsilon > 0$) and υ has dimensions of velocity.

Now consider the points in space for which $r \pm vt$ = constant (G remains constant at these points).

We consider $r + vt$ and $r - vt$ as two separate cases.

Therefore, $r = \text{constant} \pm vt$ a sphere (centered at r = 0 that moves (outward or inward) with speed v.

Thus, the constant phase surfaces on which ($r \cdot \eta(\vec{x},t)$) has a fixed value are spheres (with center at the origin which move (**outward** or **inward**) with speed v.

v = phase velocity of the spherical wave.

The 1/r factor was used in the definition of $\eta(\vec{x},t)$ so that $\eta(\vec{x},t)$ obeys the classical wave equation for $r \neq 0$.

We prove this as follows.

First consider a function f(r).

We need to calculate $\nabla^2 f(r)$ where $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$.

We have

$$\frac{\partial r}{\partial x}=\frac{x}{r} \quad , \quad \frac{\partial r}{\partial y}=\frac{y}{r} \quad , \quad \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\frac{\partial f}{\partial x}=\frac{df}{dr}\frac{\partial r}{\partial x}=\frac{df}{dr}\frac{x}{r} \quad , \quad \frac{\partial f}{\partial y}=\frac{df}{dr}\frac{\partial r}{\partial y}=\frac{df}{dr}\frac{y}{r} \quad , \quad \frac{\partial f}{\partial z}=\frac{df}{dr}\frac{\partial r}{\partial z}=\frac{df}{dr}\frac{z}{r}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{df}{dr}\frac{x}{r}\right)=\frac{df}{dr}\frac{1}{r}+x\frac{df}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{\partial}{\partial x}\left(\frac{df}{dr}\right)\frac{x}{r}$$

$$=\frac{df}{dr}\frac{1}{r}+x\frac{df}{dr}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}+\frac{x}{r}\frac{\partial}{\partial r}\left(\frac{df}{dr}\right)\frac{\partial r}{\partial x}$$

$$=\frac{df}{dr}\frac{1}{r}+x\frac{df}{dr}\left(-\frac{1}{r^2}\right)\frac{x}{r}+\frac{x}{r}\frac{\partial}{\partial r}\left(\frac{d^2 f}{dr^2}\right)\frac{x}{r}$$

and similarly for $\partial^2 f/\partial y^2$ and $\partial^2 f/\partial z^2$.

We then obtain

$$\nabla^2 f = \frac{d^2 f}{dr^2}\left(\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}\right)+\frac{df}{dr}\left(\frac{3}{r}-\frac{x^2}{r^3}-\frac{y^2}{r^3}-\frac{z^2}{r^3}\right)=\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}$$

But

$$\frac{1}{r}\frac{d^2}{dr^2}(rf(r))=\frac{1}{r}\frac{d}{dr}\left(f+r\frac{df}{dr}\right)=\frac{1}{r}\left(r\frac{d^2 f}{dr^2}+2\frac{df}{dr}\right)$$

Therefore,

$$\nabla^2 f=\frac{1}{r}\frac{d^2}{dr^2}(rf(r))$$

for $r \neq 0$ since things are not defined for $r = 0$.

Now for
$$\eta(\vec{x},t)=\frac{1}{r}G(r\mp vt)$$

$$\frac{\partial\eta}{\partial t}=\frac{1}{r}G'(\mp v)\to\frac{\partial^2\eta}{\partial t^2}=\frac{1}{r}G''(\mp v)^2=v^2\frac{1}{r}G''$$

$$\nabla^2\eta=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\eta(r))=\frac{1}{r}\frac{\partial^2}{\partial r^2}(G(r\mp vt))=\frac{1}{r}G''$$

$$\nabla^2\eta-\frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2}=0$$

so that spherical waves also obey the classical wave equation (for $r \neq 0$).

The $1/r$ factor is required physically of course to account for the $1/r^2$ decrease in the intensity of a spherical wave with distance.

Definition: A function $\eta(\vec{x},t)$ has a definite frequency if it is of the form

$$\eta(\vec{x},t) = f_+(\vec{x})e^{i\omega t} + f_-(\vec{x})e^{-i\omega t}$$

where $f_\pm(\vec{x})$ are arbitrary functions of $\vec{x}$ and where ω is a real number such that $\omega \geq 0$.

ω = the angular frequency.

Now

$$e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$$

so that this $\eta(\vec{x},t)$ is a linear superposition of $\cos\omega t$ and $\sin\omega t$.

In addition, at a point $\vec{x}$ since $e^{\pm 2\pi i} = +1$, this function $\eta(\vec{x},t)$ repeats itself in time after a time interval T (period of $\eta(\vec{x},t)$) where $\omega T = 2\pi$ or

$$T = \frac{2\pi}{\omega}$$

independent of $\vec{x}$.

The frequency of $\eta(\vec{x},t)$ is then

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Plane Wave of Definite Frequency: we have

$$\eta(\vec{x},t) = f_+(\vec{x})e^{i\omega t} + f_-(\vec{x})e^{-i\omega t} = F(\vec{x}\cdot\hat{n} - vt)$$

Therefore, we can write

$$f_\pm(\vec{x})e^{\pm i\omega t} = f_\pm(\vec{x})e^{\pm i\frac{\omega}{v}vt} = A_\pm e^{\pm i\frac{\omega}{v}(vt - \vec{x}\cdot\hat{n})} = A_\pm e^{\pm i(\omega t - \frac{\omega}{v}\vec{x}\cdot\hat{n})}$$

where $A_\pm$ are constants so that it fits the standard functional form.

If we let $\omega/v = k$, $\vec{k} = k\hat{n}$ = propagation vector so that $\vec{k}$ is parallel to $\hat{n}$ = direction of wave propagation, then

$$\eta(\vec{x},t) = A_+ e^{+i(\omega t - \vec{k}\cdot\vec{x})} + A_- e^{-i(\omega t - \vec{k}\cdot\vec{x})}$$

Now let $\hat{n} = \hat{z}$ (propagation in the z-direction) for definiteness,

Therefore

$$\eta(\vec{x},t) = A_+ e^{+i(\omega t - kz)} + A_- e^{-i(\omega t - kz)}$$

At a given time, $\eta(\vec{x},t)$ repeats itself in z after a distance λ (wavelength of $\eta(\vec{x},t)$) where

$k\lambda = 2\pi$ or

$$k = \frac{2\pi}{\lambda} \quad , \quad \lambda \textit{ independent of } t$$

Now

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = \frac{2\pi f}{v} \rightarrow \lambda f = v$$

or

```
wavelength x frequency = phase velocity
```

Note: $e^{\pm i(\omega t - kz)}$ has **phase** $(\omega t - kz)$.

This phase is constant when

$$\omega t - kz = \text{constant}$$

or when

$$z = -\frac{\text{constant}}{k} + \frac{\omega}{k}t = -\frac{\text{constant}}{k} + vt$$

Therefore, the planes of constant phase move with velocity $\omega/k = v$ in the +z direction; hence the name phase velocity.

Spherical Waves of Definite Frequency are given by

$$\eta(\vec{x},t) = f_+(\vec{x})e^{i\omega t} + f_-(\vec{x})e^{-i\omega t} = \frac{1}{r}G(r \mp vt)$$

$$= \frac{1}{r}\left[A_+ e^{+i\frac{\omega}{v}(vt\mp r)} + A_- e^{-i\frac{\omega}{v}(vt\mp r)}\right] = A_+\frac{e^{+i(\omega t\mp kr)}}{r} + A_-\frac{e^{-i(\omega t\mp kr)}}{r}$$

where

$$\frac{\omega}{v} = k = \frac{2\pi}{\lambda}$$

as in the plane wave case.

There exists a relationship between plane waves of definite frequency and spherical waves of definite frequency.

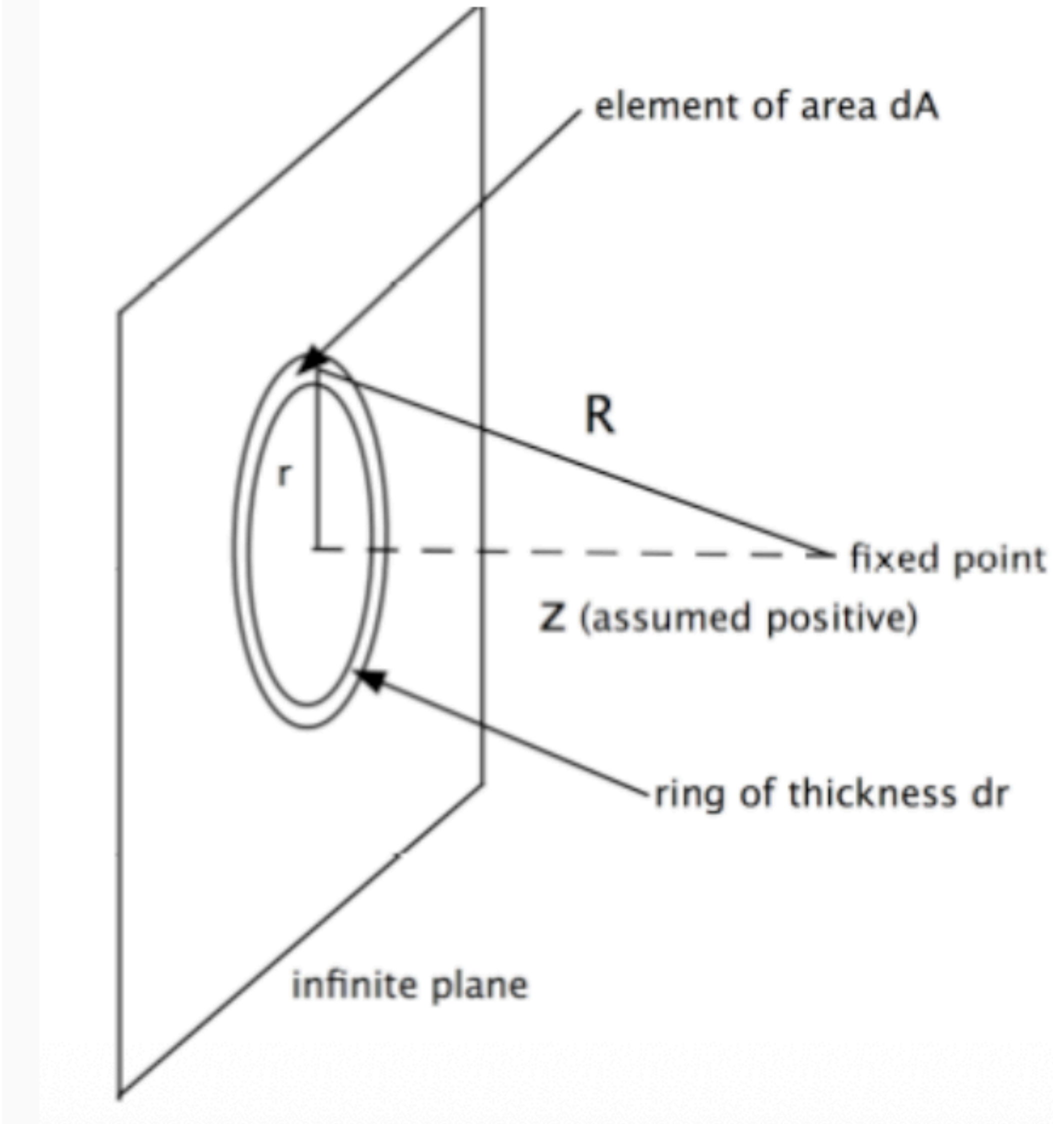
Consider the functions defined by the integrals

$$I_\pm = \int_{\substack{\text{infinite}\\ \text{plane}}} dA \frac{e^{\pm ikR}}{R} e^{-\varepsilon R} \qquad \texttt{(a definition!)}$$

where $e^{-\varepsilon R}$ is a convergence factor and we will take the limit $\varepsilon \to 0^+$ after the integration has been done.

If $\varepsilon \to 0^+$ inside the integral, $e^{-\varepsilon R} \to 1$ and the resulting integral will not exist or so it seems (as will be seen below).

Since all area elements on the circular ring shown below have the same R, we can write (using the diagram)



$$I_{\pm} = \int_{r=0}^{r=\infty} 2\pi r dr \frac{e^{\pm ikR}}{R} e^{-\varepsilon R}$$

which is just integrating over the rings.

Now $R = \sqrt{r^2 + z^2}$ with z fixed during the integration.

Therefore,

$$dR = \frac{\partial R}{\partial r} dr = \frac{1}{R} r dr$$

with R going from R = z (when r = 0) to R = ∞ (when r = ∞).

Therefore,

$$I_{\pm} = \int_{r=0}^{r=\infty} 2\pi dR e^{(\pm ik-\varepsilon)R} = \frac{2\pi}{\pm ik-\varepsilon} e^{(\pm ik-\varepsilon)R}\Bigg|_{R=z}^{R=\infty}$$

The value at the upper limit vanishes because $e^{\pm k(\infty)}e^{-\varepsilon(\infty)} = 0$ for $\varepsilon > 0$.

If we had let $\varepsilon = 0$ at the beginning of the calculation, then the value at the upper limit would have been $e^{\pm k(\infty)}$, which is undefined (it just keeps on oscillating).

Thus,

$$I_{\pm} = \lim_{\varepsilon \to 0^+} \frac{2\pi}{\pm ik-\varepsilon} e^{(\pm ik-\varepsilon)z} = \pm \frac{2\pi i}{k} e^{\pm ikz}$$

so we get the amazing result (limit $\varepsilon \to 0^+$ understood)

$$e^{\pm ikz} = \pm \frac{k}{2\pi i} \int_{\substack{\text{infinite} \\ \text{plane}}} \mathrm{dA} \frac{e^{\pm ikR}}{R} e^{-\varepsilon R}$$

This can be proven rigorously using the theory of complex variables.

where R = distance from dA to a fixed point located a distance z from the plane.

This then implies that

$$C_+ e^{+i(\omega t - kz)} + C_- e^{-i(\omega t - kz)} = \frac{k}{2\pi i} \int_{\substack{\text{entire} \\ \text{plane} \\ \perp \text{ to z-axis} \\ \text{at z=0}}} dA e^{-\varepsilon R} \left[C_+ \frac{e^{+i(\omega t - kR)}}{R} + C_- \frac{e^{-i(\omega t - kR)}}{R} \right]$$

Thus, a plane wave propogating in the +z direction is equal to a linear superposition of spherical outgoing waves emanating from each point on a plane perpendicular to the z-axis, that is,

$$e^{+i(\omega t - kz)} = \frac{k}{2\pi i} \int_{\substack{\text{entire} \\ \text{plane} \\ \perp \text{ to z-axis} \\ \text{at z=0}}} dA e^{-\varepsilon R} \left[\frac{e^{+i(\omega t - kR)}}{R} \right]$$

This is the basis of Huygens' Principle!

Note A: A classical physical quantity (for example $\vec{E}$ or $\vec{B}$) is a **real quantity**.

Therefore, $\eta_{physical}(\vec{x},t)$ = real linear superposition of $\cos(\omega t - \vec{k}\cdot\vec{x})$ and $\sin(\omega t - \vec{k}\cdot\vec{x})$ if $\eta_{physical}(\vec{x},t)$ is a plane wave of definite frequency, and $\eta_{physical}(\vec{x},t)$ = real linear superposition of $\frac{1}{r}\cos(\omega t - \vec{k}\cdot\vec{x})$ and $\frac{1}{r}\sin(\omega t - \vec{k}\cdot\vec{x})$ if $\eta_{physical}(\vec{x},t)$ is a spherical wave of definite frequency.

theFor definiteness, let us consider plane waves (similar results hold for spherical waves).

A real linear superposition of $\cos(\omega t - \vec{k}\cdot\vec{x})$ and $\sin(\omega t - \vec{k}\cdot\vec{x})$can be written as

$$\eta_{physical}(\vec{x},t) = A\cos(\vec{k}\cdot\vec{x} - \omega t + \varphi) \quad , \quad A,\varphi \;\; real$$

We let

$$\eta(\vec{x},t) = Ae^{i\varphi}e^{i(\vec{k}\cdot\vec{x}-\omega t)} = Ze^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad , \quad Z = complex\ number$$

so that

$$\eta_{physical}(\vec{x},t) = A\cos(\vec{k}\cdot\vec{x} - \omega t + \varphi) = \mathrm{Real}(\eta(\vec{x},t))$$

Therefore, it is sufficient that we consider $\eta(\vec{x},t) = Ze^{i(\vec{k}\cdot\vec{x}-\omega t)}$ in our discussions.

When $\eta_{physical}(\vec{x},t)$ is required, we need only take the real part of $\eta(\vec{x},t)$.

WE do this because $\eta(\vec{x},t)$ is simpler than $\eta_{physical}(\vec{x},t)$ to manipulate in calculations.

The analogous result for a spherical wave is obvious:

outgoing wave:

$$\eta(\vec{x},t) = Z\frac{e^{i(kr-\omega t)}}{r}$$

or incoming wave:

$$\eta(\vec{x},t) = Z\frac{e^{-i(kr-\omega t)}}{r}$$

with

$$\eta_{physical} = \mathrm{Real}(\eta)$$

The linear composition of spherical outgoing waves which gives a palne wave can therefore be used in the earlier form with $C_+ = 1$ and $C_- = 0$

$$e^{i(kz-\omega t)} = +\frac{k}{2\pi i} \int_{\substack{\text{entire} \\ z=0 \text{ plane}}} \mathrm{dA} e^{-\varepsilon R} \frac{e^{i(kR-\omega t)}}{R}$$

AsI said, we shall do our calculations with $\eta(\vec{x},t)$ and $\eta_{physical}(\vec{x},t)$ is recovered at any stage of the calculation simply by taking the real part.

Note B: If $\eta(\vec{x},t) = F(\vec{x})e^{-i\omega t}$ has a definite frequency, where $F(\vec{x}) = A(\vec{x})e^{i\psi(\vec{x})}$ and A,ψ real, then we have

$$\eta_{physical}(\vec{x},t) = \mathrm{Real}(\eta(\vec{x},t)) = A(\vec{x})\cos(\omega t - \psi(\vec{x}))$$

The intensity of a wave = magnitude of energy flow per unit time per unit area.

This means that

$$\text{intensity} \propto \left(\eta_{physical}(\vec{x},t)\right)^2 = \text{intensity at } \vec{x} \text{ and } t$$

For example, the Poynting vector $\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$ gives the energy flow per unit time per unit area or the intensity.

For a plane wave of definite frequency propagating in free space (vacuum), $|\vec{E}| = |\vec{B}|$ and $\vec{E} \perp \vec{B}$ with

$\vec{E} \times \vec{B}$ in the direction of propagation.

Therefore,

$$|\vec{S}| = \frac{c}{4\pi}|\vec{E}||\vec{B}|\sin 90^\circ = \frac{c}{4\pi}|\vec{E}|^2$$

Since the intensity $\propto \left(\eta_{physical}(\vec{x},t)\right)^2$ we have $\eta_{physical} = |\vec{E}|$.

Now $\eta_{physical}(\vec{x},t)$ oscillates with angular frequency ω so that the intensity also oscillates.

Since these oscillations are usually impossible to observe ($\omega \approx 10^{15} sec^{-1}$ for visible light), one is really interested in the time-averaged intensity (averaged over 1 cycle), designated by $\overline{\text{intensity}}$ and

$$\overline{\text{intensity}} = \frac{1}{T}\int_0^T dt\left(\text{intensity}(\vec{x},t)\right) \quad , \quad T = \frac{2\pi}{\omega}$$

Therefore,

$$\overline{\text{intensity}} \propto \frac{1}{T}\int_0^T dt\left(\eta_{physical}(\vec{x},t)\right)^2 = \frac{1}{T}\int_0^T dt\left(A(\vec{x})\cos\left(\omega t - \psi(\vec{x})\right)\right)^2$$

$$= \frac{1}{\omega T}\left(A(\vec{x})\right)^2 \int_0^{\omega T} d(\omega t)\cos^2\left(\omega t - \psi(\vec{x})\right)$$

$$= \frac{\left(A(\vec{x})\right)^2}{2\pi}\int_0^{2\pi} du\cos^2\left(u - \psi(\vec{x})\right) = \frac{\left(A(\vec{x})\right)^2}{2} = \frac{1}{2}|\eta(\vec{x},t)|^2$$

Thus,

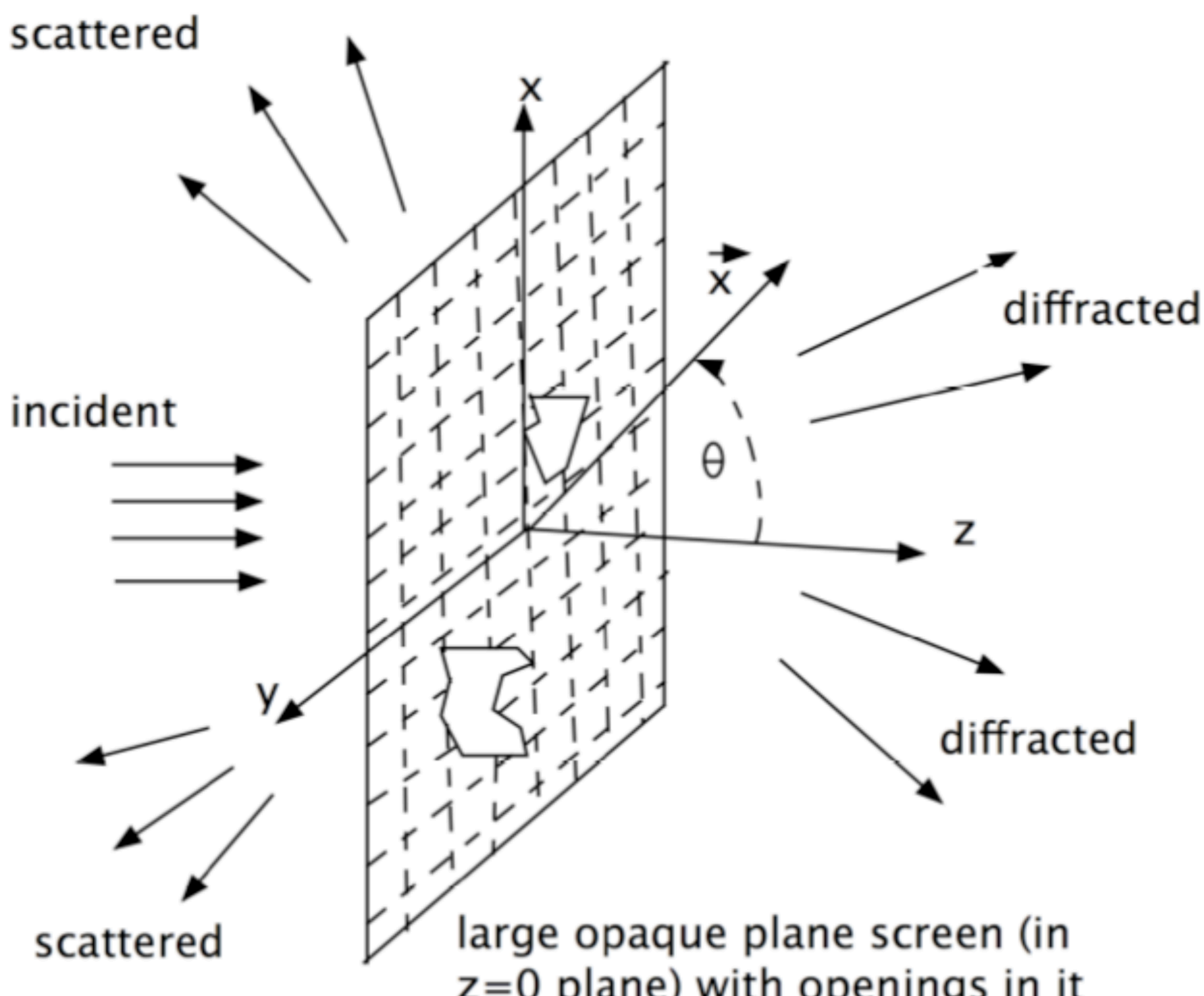
$$\overline{\left(\eta_{physical}(\vec{x},t)\right)^2} = \frac{1}{2}\left|\eta(\vec{x},t)\right|^2$$

which is independent of t since $\eta \sim e^{-i\omega t}$.

Clearly we have the result
$$\overline{\text{intensity}} \propto \left|\eta(\vec{x},t)\right|^2$$

Finally, we have all the necessary tools…………

Diffraction of Waves



Referring to the diagram above, we choose the origin of the coordinate system to lie in the region containing the aperture.

We assume a wave incident from the z < 0 region given by

$$\eta_{incident}(\vec{x},t) = e^{i(kz-\omega t)}$$

In general, the screen will affect the incident wave so that there will be scattered wave (back into the z < 0 region) and diffracted waves (waves in the z > 0 region).

Now

$$\eta_{incident}(\vec{x},t) = e^{i(kz-\omega t)} = +\frac{k}{2\pi i}\int_{\substack{\text{entire}\\ z=0\text{ plane}}} \mathrm{dA}e^{-\varepsilon R}\,\frac{e^{i(kR-\omega t)}}{R}$$

This would be the wave for z > 0 if no screen were present.

This plane wave is a linear superposition of spherical outgoing waves emanating from each point in the z = 0 plane.

The **Kirchhoff approximation** for the diffracted wave is simply a linear superposition of spherical outgoing waves emanating from each point **in the openings** in the screen with each of these spherical wave having a coefficient equal to the coefficient in the expansion of $\eta_{incident}(\vec{x},t) = e^{i(kz-\omega t)}$.

Note that the diffracted wave contains no spherical waves emanating from points on the opaque screen itself.

$$\eta_{diffracted}(\vec{x},t) = +\frac{k}{2\pi i}\int_{\substack{\text{openings}\\ \text{in the}\\ \text{opaque}\\ \text{screen}}} \mathrm{dA}e^{-\varepsilon R}\,\frac{e^{i(kR-\omega t)}}{R}$$

where R is the distance from dA to the point $\vec{x}$.

This result seems reasonable, but note that we have proved nothing!!

To prove that this gives a good approximation for $\eta_{diffracted}(\vec{x},t)$ requires an analysis of the solution to the classical wave equation and the boundary values of these solutions at the screen and in the openings.

The Kirchhoff approximation is a good one under the following conditions:

(a) $r >> \lambda \rightarrow kr >> 1$ and $r >>$ linear dimensions of the region containing the apertures. Thus, we must be far from the apertures for the above expression for $\eta_{diffracted}(\vec{x},t)$ to be valid.

(b) $\theta << 1$, that is, $\eta_{diffracted}(\vec{x},t)$ should be evaluated with the above expression only when $\vec{x}$ makes a small angle with the z-axis.

(c) $\lambda <<$ linear dimensions of the region containing the apertures (high frequency limit).

We will apply the above expression to situations in which condition (c) is sometimes violated - in such cases our results will only be qualitatively accurate.

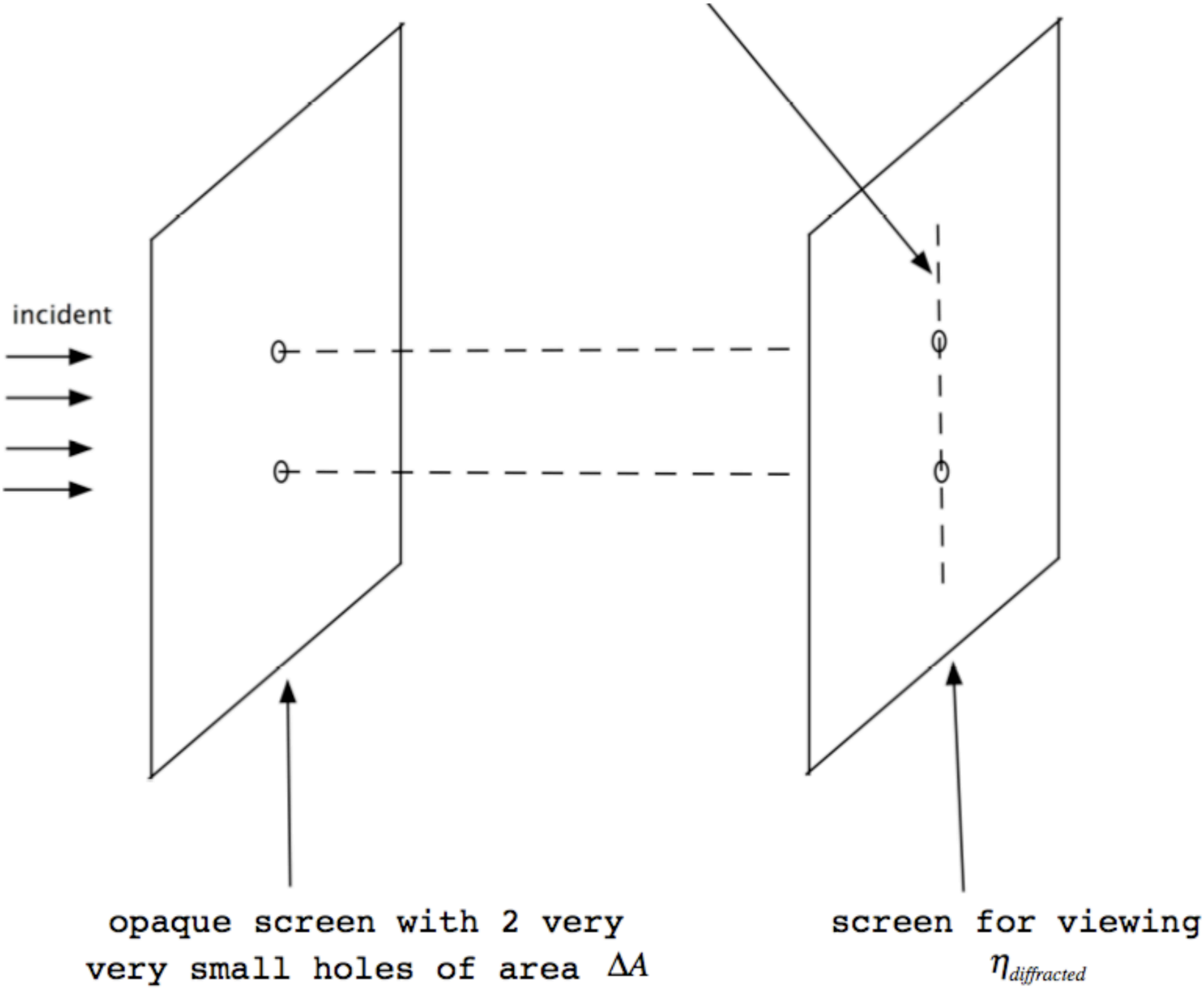
(d) If the wave is described by a vector field ($\vec{E}$ and $\vec{B}$), then there exist relationships among the various components. These relationships have not been taken into account in the Kirchhoff approximation and therefore this approximation has neglected all polarization effects.

Note: When the apertures on the screen are finite, the integral is over a finite area and the limit $\varepsilon \to 0^+$ may be taken inside the integral (with $e^{-\varepsilon R} \to 1$) because the resulting integral will exist.

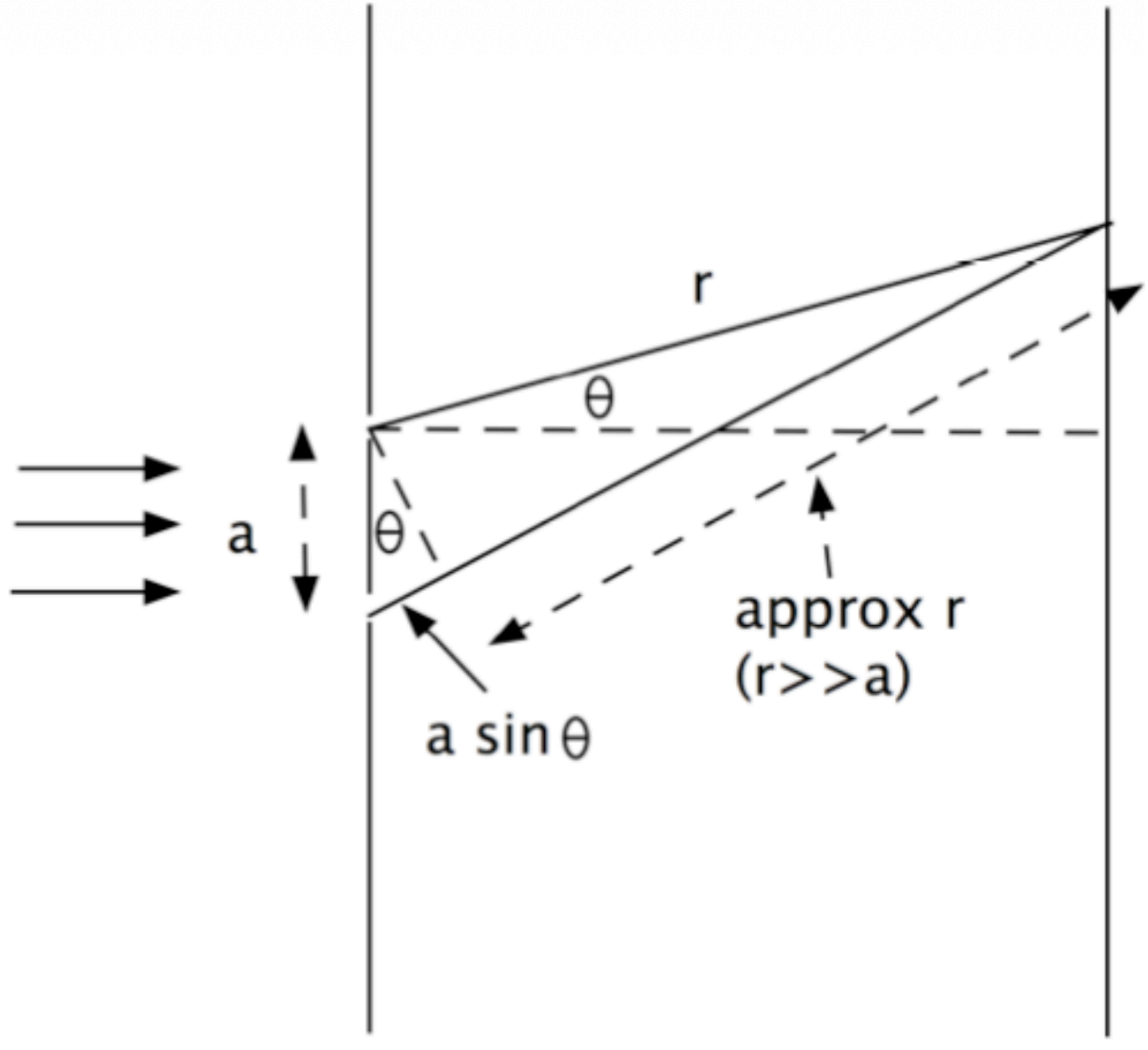
One must do the integral before letting $\varepsilon \to 0^+$ only when the integration extend over an infinite area (which is unphysical anyway!).

Application:

We shall calculate $\eta_{diffracted}$ only on this line (the line through the geometrical projection of the two given holes as shown by the arrow below) in order to simplify the calculation.


incident
opaque screen with 2 very very small holes of area ΔA
screen for viewing $\eta_{diffracted}$

Looking from the side of the diagram:



We have

$$\eta_{diffracted}(\vec{x},t) = +\frac{k}{2\pi i}\int_{\text{openings}} \mathrm{dA}e^{-\varepsilon R}\frac{e^{i(kR-\omega t)}}{R} = +\frac{k}{2\pi i}\int_{\text{openings}} \mathrm{dA}\frac{e^{i(kR-\omega t)}}{R}$$

where we have taken the limit inside the integral.

For small openings

$$\eta_{diffracted}(\vec{x},t) = +\frac{k}{2\pi i}\left[\frac{e^{i(kR-\omega t)}}{r}\Delta A + \frac{e^{i(k(r+a\sin\theta)-\omega t)}}{r+a\sin\theta}\Delta A\right]$$

$$= \frac{k}{2\pi i}\Delta A \frac{e^{i(kR-\omega t)}}{r}\left[1+\frac{e^{ika\sin\theta}}{1+\frac{a}{r}\sin\theta}\right]$$

Since $r >> a\sin\theta$ we have

$$|\eta_{diff}| = \frac{k}{2\pi}\frac{\Delta A}{r}\left|1+e^{ika\sin\theta}\right|$$

and

$$|\eta_{diff}| = \frac{k}{2\pi}\frac{\Delta A}{r}\left|1+e^{ika\sin\theta}\right|$$

$$\overline{\text{intensity}} \propto |\eta_{diff}|^2 = \frac{k^2}{4\pi^2}\frac{(\Delta A)^2}{r^2}(2+2\cos(ka\sin\theta))$$

$$\overline{\text{intensity}} \propto 1+\cos(ka\sin\theta)$$

Fourier Series Introduction

Many complicated functions can be represented by power series.

Another powerful way of representing such functions is using a sum of sine and cosine terms, which is called a Fourier series.

Unlike Taylor series, a Fourier series can describe functions that are not everywhere continuous and/or differentiable.

Derivation

The world is full of vibrations. The sound we hear is an acoustic wave, the things we see are electromagnetic waves and the surfers in Santa Cruz, CA ride on gravity waves of the ocean.

The simplest wave motion in 1-dimension is described by the 1-dimensional wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

Let us assume a solution (we will see why this is a solution later when we study partial differential equations))

$$u(x,t) = \chi(x) e^{i\omega t}$$

Substitution into the full wave equation gives an equation for the wave amplitude $\chi(x)$.

$$\left(\frac{d^2\chi(x)}{dx^2}+\frac{\omega^2}{v^2}\chi(x)\right)e^{i\omega t}=0\rightarrow\left(\frac{d^2}{dx^2}+k^2\right)\chi(x)=0\ \ ,\ \ \frac{\omega^2}{v^2}=k^2$$

This new equation(an ordinary differential equation) is easily solved(we will see why later) by the function

$$\chi(x)=a\cos kx+b\sin kx$$

which gives the shape (at fixed time t) of the vibrating system (say a string).

This is like taking a photograph of the vibrating string at one instant of time.

Let us choose a string of length π fixed at both ends.

We then have the **boundary conditions**:

$$\chi(0) = 0 = \chi(\pi)$$

which implies that

$$\chi(0) = a = 0 \quad \text{and} \quad \chi(\pi) = 0 = b \sin k\pi \to k = n = \text{ integer}$$

so that

$$\chi(x) = b \sin nx \quad , \quad n = 1, 2, 3, ...$$

which is called the n^{th} **normal mode** of the string (given by its shape).

Each such mode has a different frequency and shape.

Historically, the wave equation was first studied in the 1700s.

In 1742, Bernoulli showed that

vibrations of different modes (frequency) could coexist in the string

In 1753, D'Alembert, Euler and Bernoulli showed that

all possible shapes of a vibrating string, even when the ends were not fixed, were representable by the series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

There was great dispute about this result.... what about the cosine series, i.e., how does one represent even functions of x?

Even though $f(x)$ solves the wave equation, others disputed the claim that this was the most general solution.

In 1807, Fourier (in a paper on heat conduction) showed that

every function in the closed interval

$$[-\pi, \pi] \quad \text{or} \quad -\pi \leq x \leq \pi$$

could be represented in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

He re-derived integral formulas for the coefficients a_n, b_n had already been obtained by Euler in 1777.

Fourier, however, broke new ground by pointing out that these integral formulas were well-defined even for arbitrary functions and that the resulting coefficients were identical for different functions that agreed within the interval, but not outside it.

The paper by Fourier **was rejected** by Lagrange, Laplace and Legendre on behalf of the Academy of Sciences on the grounds that it lacked mathematical rigor.

A second version of the paper won the Academy's Grand prize in 1812.

This work has had a great impact on the development of mathematical physics in the 1800s and it is still influencing things now.

The Sine-Cosine Series

The general Fourier series expansion is sum of sine and cosine terms of the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

where the frequencies

$$\omega_n = \frac{2\pi n}{T_0}$$

are integer multiples of a fundamental frequency

$$\omega_0 = \frac{2\pi}{T_0}$$

and T_0 is determined either by the natural periodicity of $f(t)$ or possibly by an enforced periodicity of some sort.

Different treatments of this subject can have definitions of T_0 which differ by various factors.

The results of all calculations are the same.

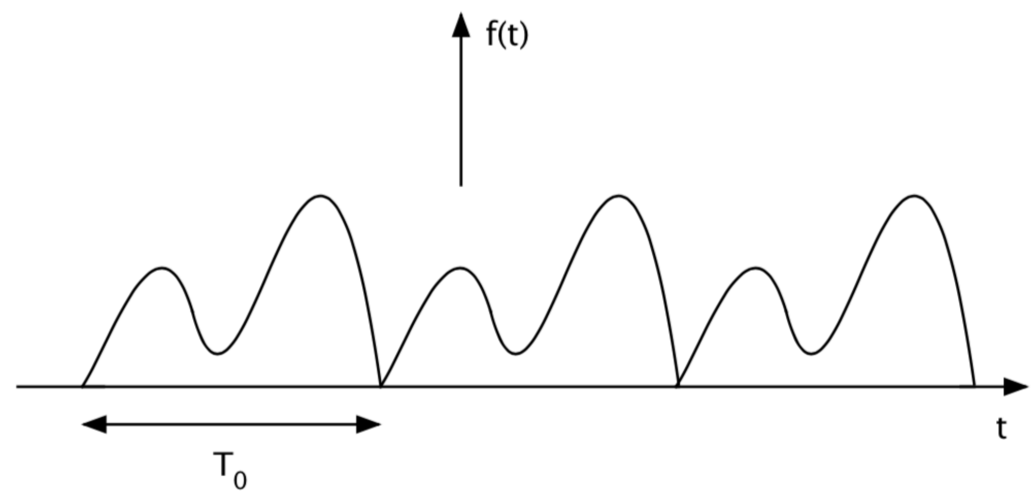
The assumed form guarantees that $f(t)$ has periodicity T_0, i.e.,

$$f(t) = f(t + T_0) \quad \text{for all} \;\; t$$

In terms of the independent variable t, $f(t)$ has this periodicity for the entire interval

$$-\infty < t < \infty$$

as shown below



The typical Fourier series problem is such that we are given a function $g(t)$ and we then determine both T_0 and the a_n and b_n coefficients so that the series expansion is equal to $g(t)$ for all t.

This requires that

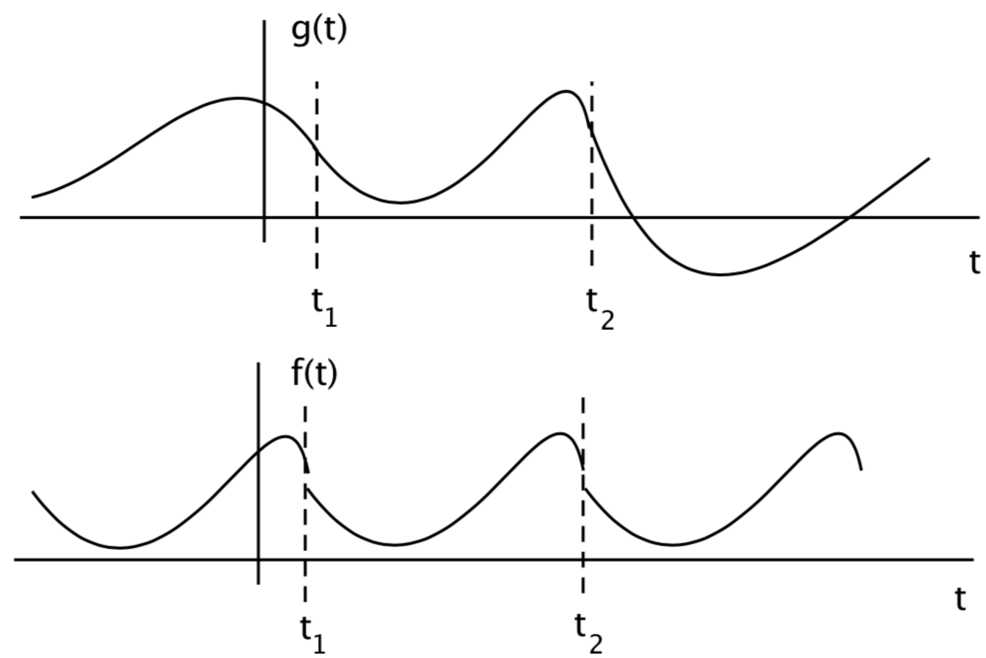
$$g(t) = g(t + T_0) \quad \text{for all} \quad t$$

The **smallest value** of T_0 that satisfies this equation is the **period** of $g(t)$.

What if $g(t)$ is not periodic?

In this case we cannot use a general Fourier series to represent $g(t)$ for **all** t.

On the other hand, we can make the series equal to $g(t)$ for some finite interval as shown below



Clearly, $g(t)$ above is not periodic.

Suppose however we define the basic period for the Fourier series to be $T_0 = t_2 - t_1$ where the interval $t_1 < t < t_2$ is as shown.

The Fourier series $f(t)$ can then be made identical to $g(t)$ in that interval.

Outside the interval, the Fourier series $f(t)$ is periodic and will not match $g(t)$ as shown.

Digression: The formal mathematical requirements that a function $f(x)$ must satisfy in order that it may be expanded in a Fourier series are known as the **Dirichlet conditions**, which are summarized as follows:

- the function must be periodic
- it must be single-valued and continuous, except possibly at a finite number of finite discontinuities
- it must have only a finite number of maxima and minima within one period
- the integral over one period of $|f(x)|$ must converge

The Orthogonality Conditions

We can determine the coefficients a_n and b_n using so-called **orthogonality conditions** (proved using calculus)

$$\int_{t_0}^{t_0+T_0} dt\, \sin\left(\frac{2\pi n}{T_0}t\right) \sin\left(\frac{2\pi m}{T_0}t\right) = \delta_{nm}\frac{T_0}{2}$$

$$\int_{t_0}^{t_0+T_0} dt\, \cos\left(\frac{2\pi n}{T_0}t\right) \cos\left(\frac{2\pi m}{T_0}t\right) = \delta_{nm}\frac{T_0}{2}$$

$$\int_{t_0}^{t_0+T_0} dt\, \sin\left(\frac{2\pi n}{T_0}t\right) \cos\left(\frac{2\pi m}{T_0}t\right) = 0$$

for **integer** values of m and n.

In addition we have

$$\int_{t_0}^{t_0+T_0} dt\, \sin\left(\frac{2\pi n}{T_0}t\right) = 0 \quad , \quad \int_{t_0}^{t_0+T_0} dt\, \cos\left(\frac{2\pi n}{T_0}t\right) = \delta_{n0}\, T_0$$

We can now evaluate the coefficients as follows.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

The integral operation

$$\begin{aligned}\int_{t_0}^{t_0+T_0} dt\, f(t) &= \int_{t_0}^{t_0+T_0} dt\, \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} dt\ \cos \omega_n t \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T_0} dt\ \sin \omega_n t \\ &= \int_{t_0}^{t_0+T_0} dt\, \frac{a_0}{2} = \frac{a_0 T_0}{2}\end{aligned}$$

determines a_0, i.e.,

$$a_0 = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} dt\, f(t)$$

The integral operation

$$
\begin{aligned}
\int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t f(t) &= \int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t \frac{a_0}{2} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t \cos\omega_n t \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t \sin\omega_n t \\
&= \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t \cos\omega_n t \\
&= \sum_{n=1}^{\infty} a_n \delta_{nm} \frac{T_0}{2} = a_m \frac{T_0}{2}
\end{aligned}
$$

determines $a_m,\ m > 0$, i.e.,

$$
a_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} dt\, \cos\omega_m t f(t)
$$

The integral operation

$$\begin{aligned}
\int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t f(t) &= \int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t \frac{a_0}{2} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t \cos\omega_n t \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t \sin\omega_n t \\
&= \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t \sin\omega_n t \\
&= \sum_{n=1}^{\infty} b_n \delta_{nm} \frac{T_0}{2} = b_m \frac{T_0}{2}
\end{aligned}$$

determines b_m, $m > 0$, i.e.,

$$b_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} dt\ \sin\omega_m t f(t)$$

Several questions arise:

(1) Do these Fourier coefficients exist?

(2) Is the Fourier series convergent?

(3) Does it converge to the original function?

The answer is **YES for all physically realizable systems!!!!**

Examples:

(1) The function $f(t)$ is

$$f(t) = \sin(t)$$

This function is periodic with period $T_0 = 2\pi \rightarrow \omega_n = n$.

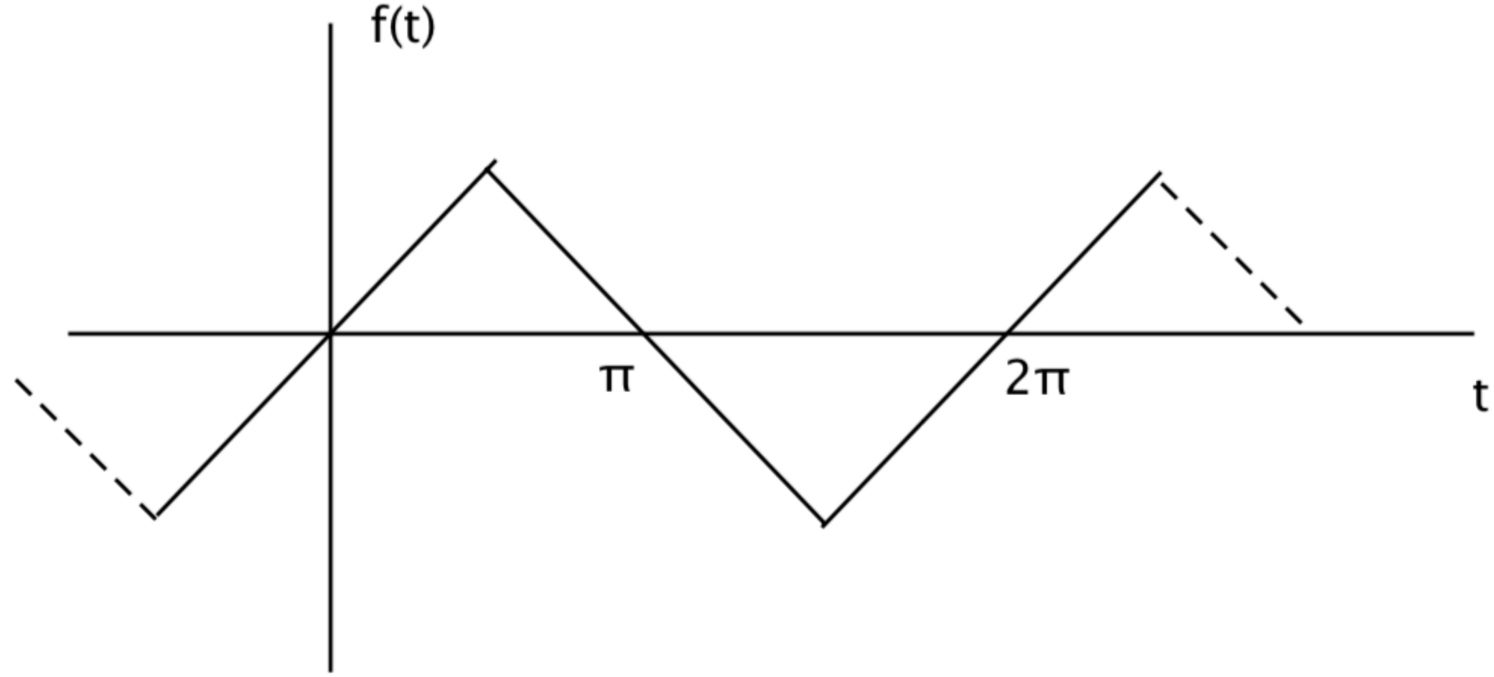
Therefore the Fourier series looks like

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin nt + \sum_{n=1}^{\infty} b_n \cos nt$$

This example requires no calculations. It is clear that

$$a_m = 0 \quad \text{for all} \quad m \quad , \quad b_m = 0 \quad \text{for all} \quad m \neq 1 \quad , \quad b_1 = 1$$

(2) The function $f(t)$ equals the periodic triangle wave shown above.



In the interval $-\pi/2 < t < 3\pi/2$ this has the functional form

$$f(t) = \begin{cases} \frac{2t}{\pi} & -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\ 2 - \frac{2t}{\pi} & \frac{\pi}{2} \le t \le \frac{3\pi}{2} \end{cases}$$

Once again the basic period is $T_0 = 2\pi$ which again gives $\omega_n = n$.

We have

$$a_m = \frac{2}{T_0} \int_{-\pi/2}^{3\pi/2} dt \, \cos \omega_n t f(t) = 0 \quad \text{for all} \quad m$$

since the integrand is the **product of an odd and an even function**.

We also have

$$b_m = \frac{2}{T_0} \int_{-\pi/2}^{3\pi/2} dt \, \sin \omega_n t f(t) = \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2}$$

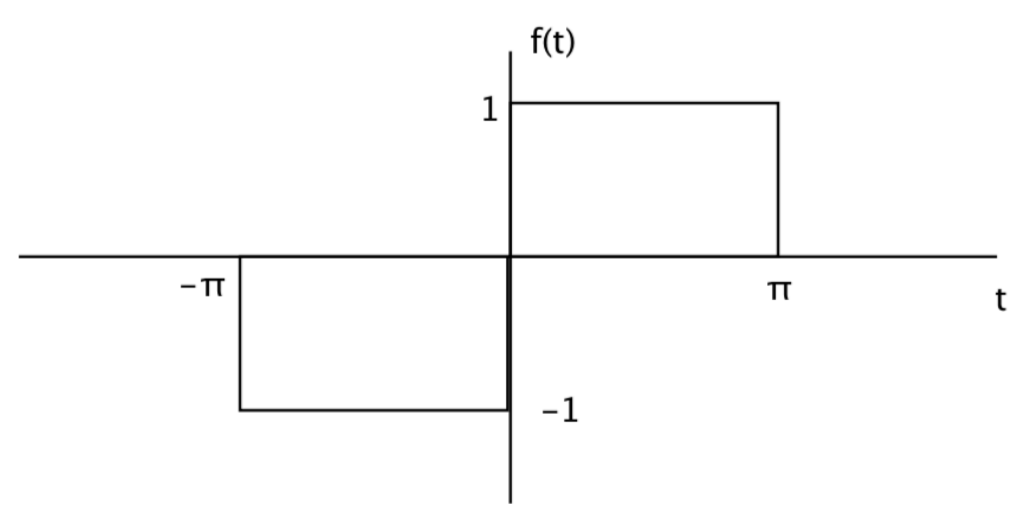
Therefore, we get

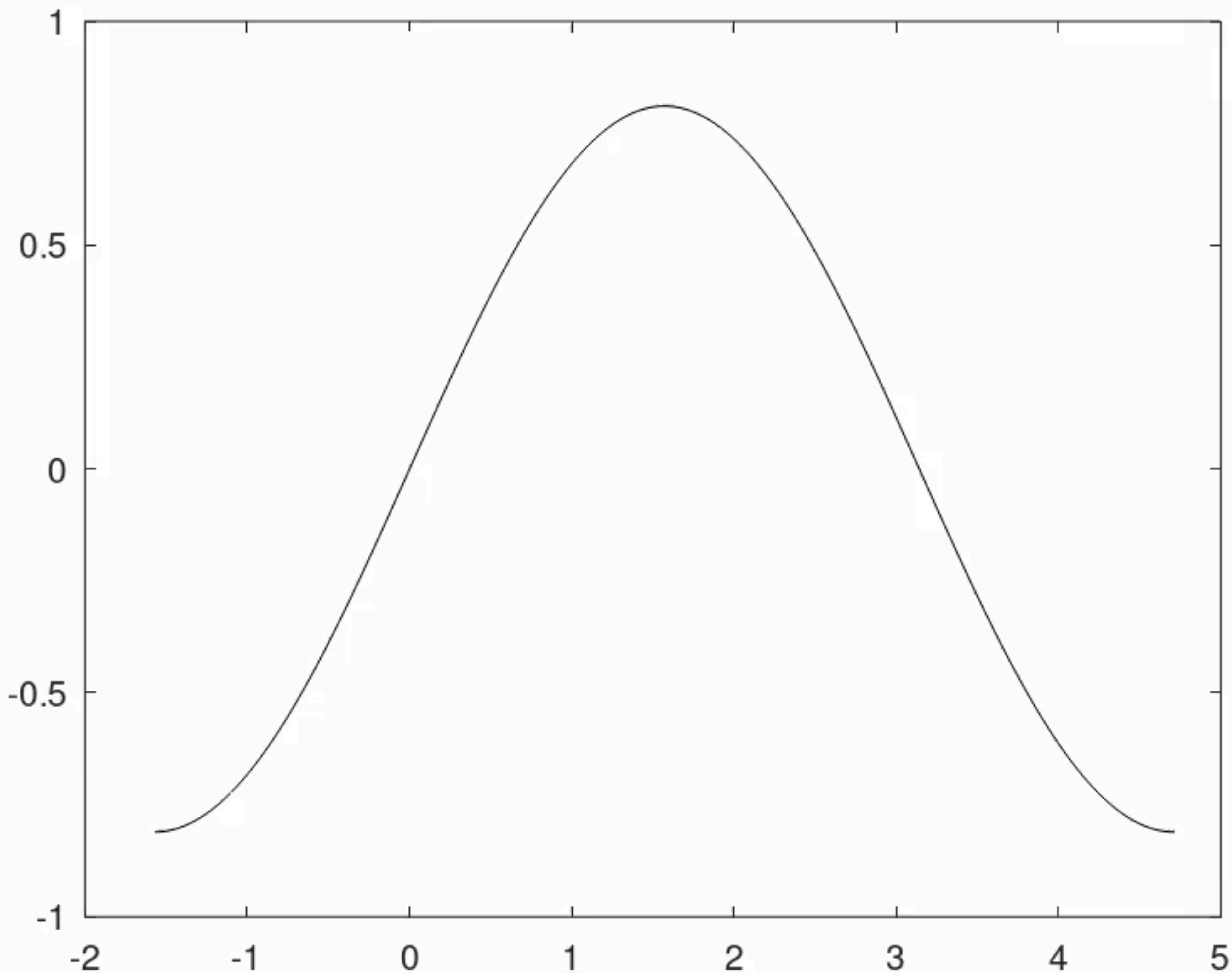
$$f(t) = \sum_{n=1,odd} \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2} \sin nx$$

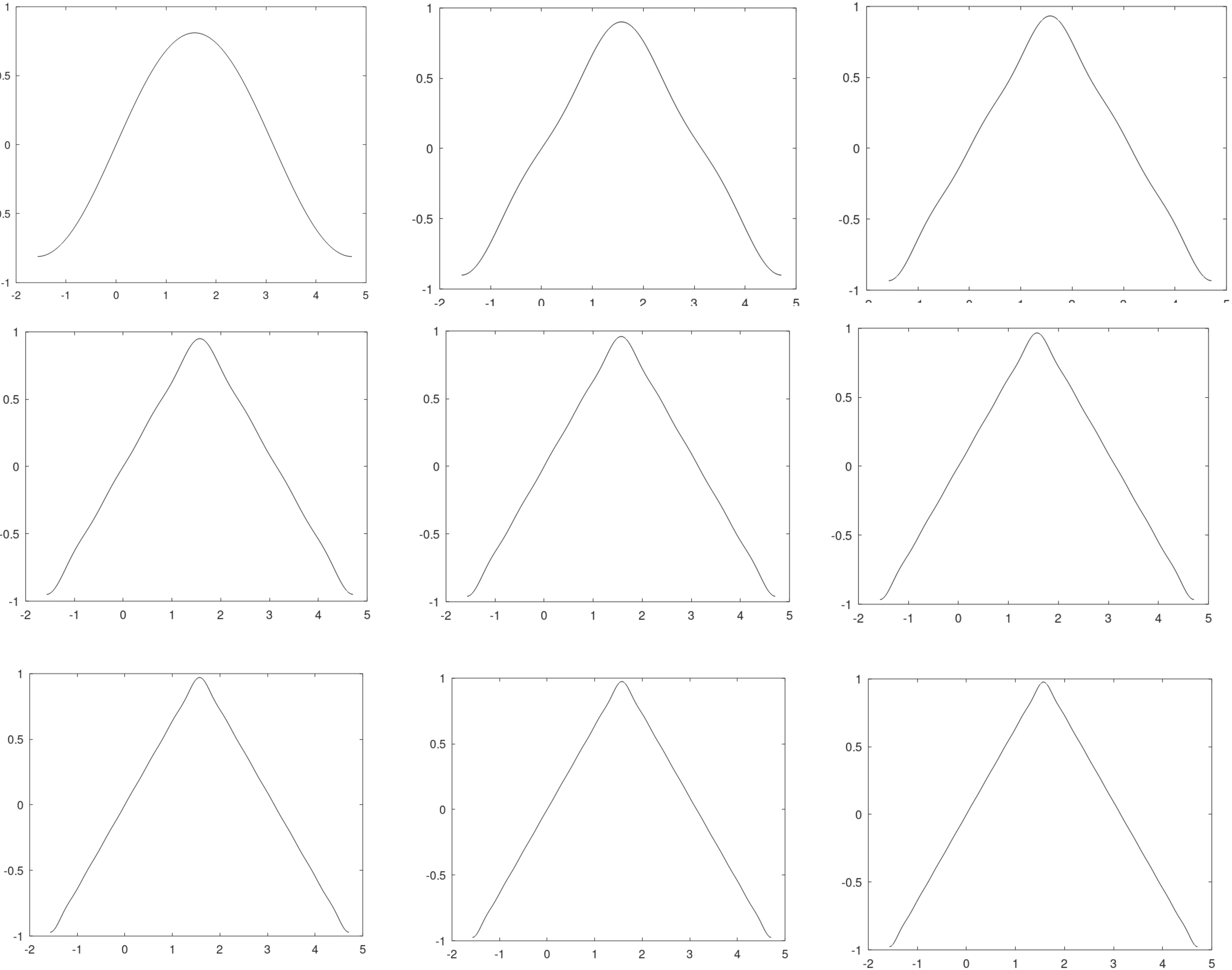
Illustrate in class with Octave program:

```
% illustrate Fourier Series
% triangle wave
% gives sequence of images building
% up to final curve
x=-pi/2:pi/100:3*pi/2;
val = zeros(1,length(x));
figure(1);
for m=1:2:100
 f=8/(m*pi)^2*sin(m*pi/2)*sin(m*x);
 val=val+f; m
 clf;
 plot(x,val, '-k');
  axis([-pi/2 3*pi/2 -1.5 1.5]);
 hold on;
 pause(1);
end
```

(3) $f(t)$ is the square wave function




1
0.5
0
-0.5
-1
-2
-1
0
1
2
3
4
5


1
0.5
0
-0.5
-1
-2
-1
0
1
2
3
4
5

In the interval $[-\pi, \pi]$ we have

$$f(t) = \begin{cases} +1 & 0 \leq t \leq \pi \\ -1 & -\pi \leq t \leq 0 \end{cases}$$

Once again the basic period is $T_0 = 2\pi$ which again gives $\omega_n = n$.

Again this is an odd function so all the $a_n = 0$ and we find

$$b_n = \begin{cases} \frac{4}{n\pi} & n = 1, 3, 5, 6, ... \\ 0 & n = 2, 4, 6, 8, ... \end{cases}$$

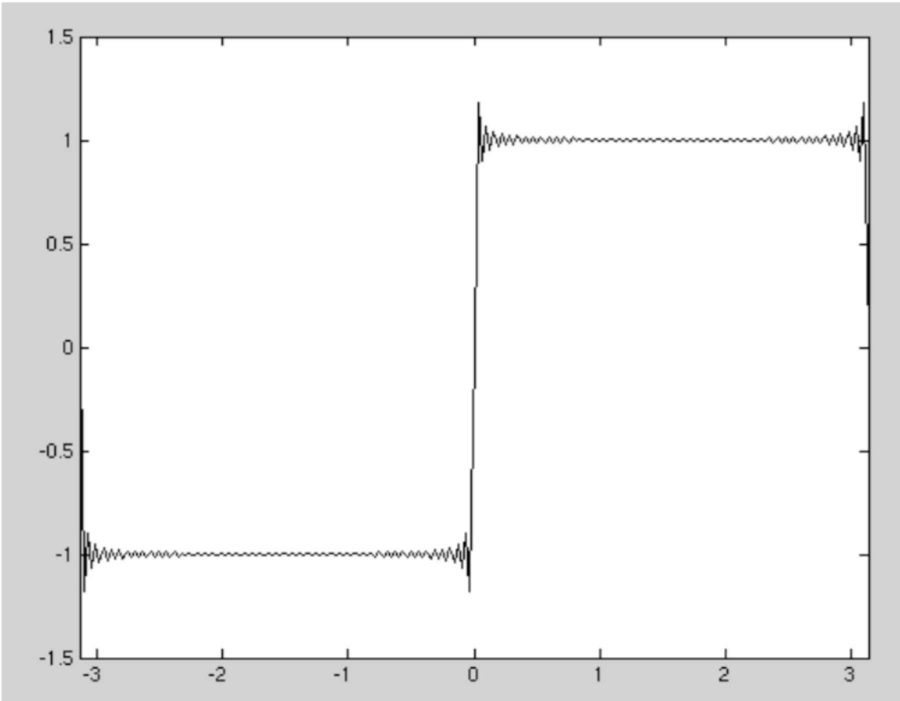
which gives

$$f(t) = \frac{4}{\pi} \sum_{n=odd} \frac{sinnt}{n}$$

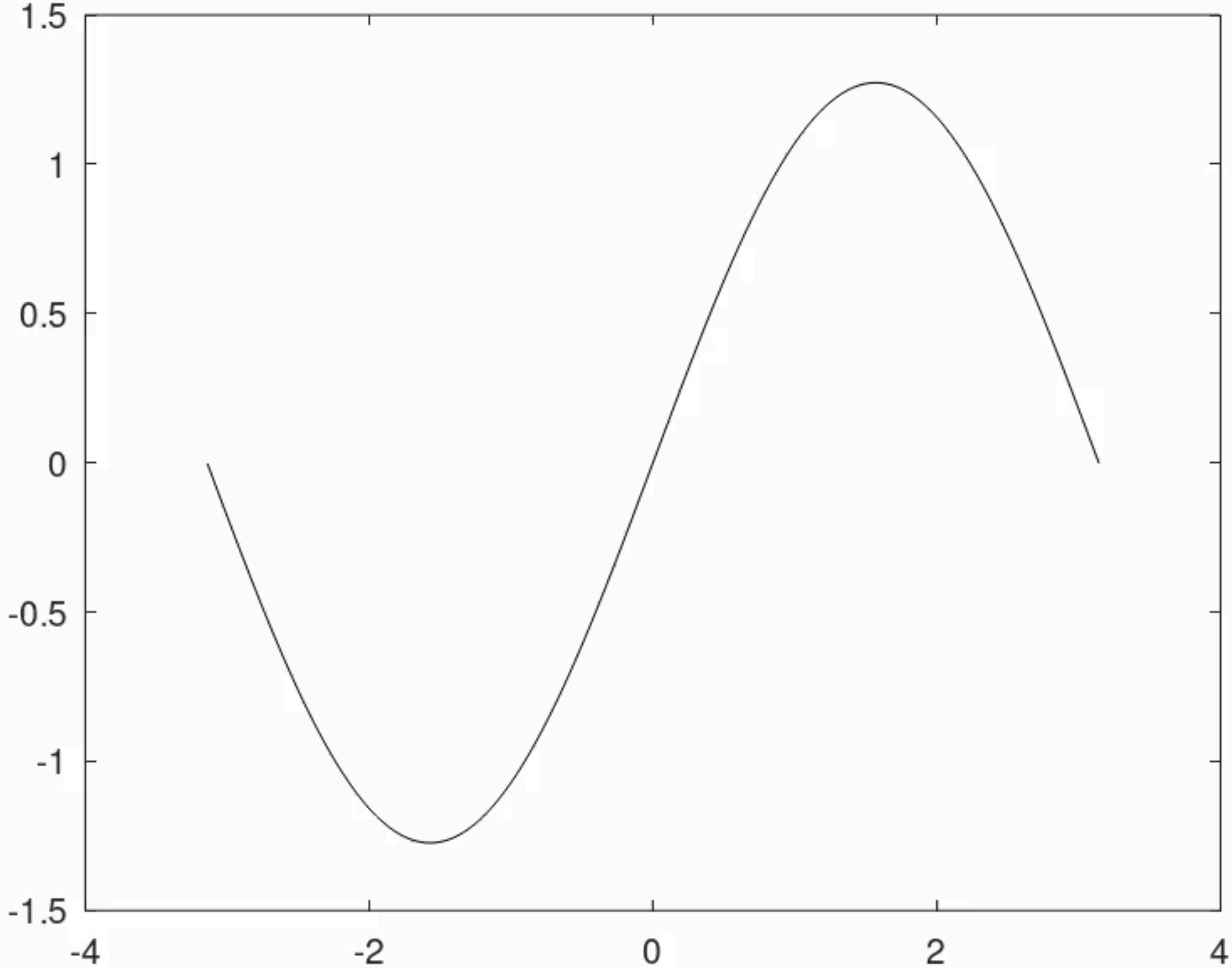
Illustrate in class with Octave program:

```
% illustrate Fourier Series for square wave pulse
% gives sequence of images building up to final curve x=-pi:pi/100:pi;
val = zeros(1,length(x));
figure(1);
for m=1:2:500
 f=4/(m*pi)*sin(m*x);
 val=val+f;
 m
 clf;
 plot(x,val,'-k');
 axis([-pi pi -1.5 1.5]);
 hold on;
 pause(1);
end
```

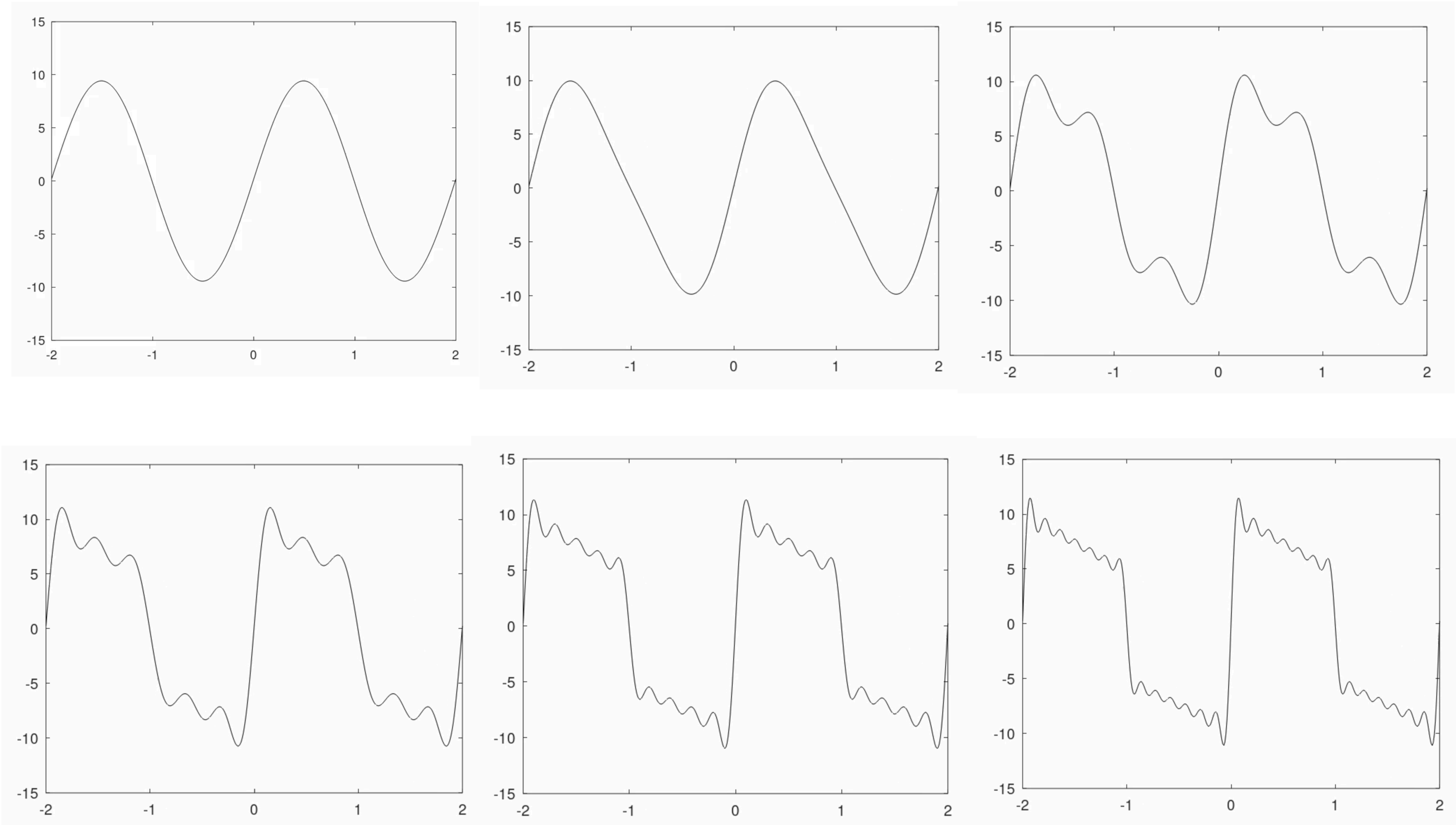
It looks like the figure below after including a 100 terms in the sum....



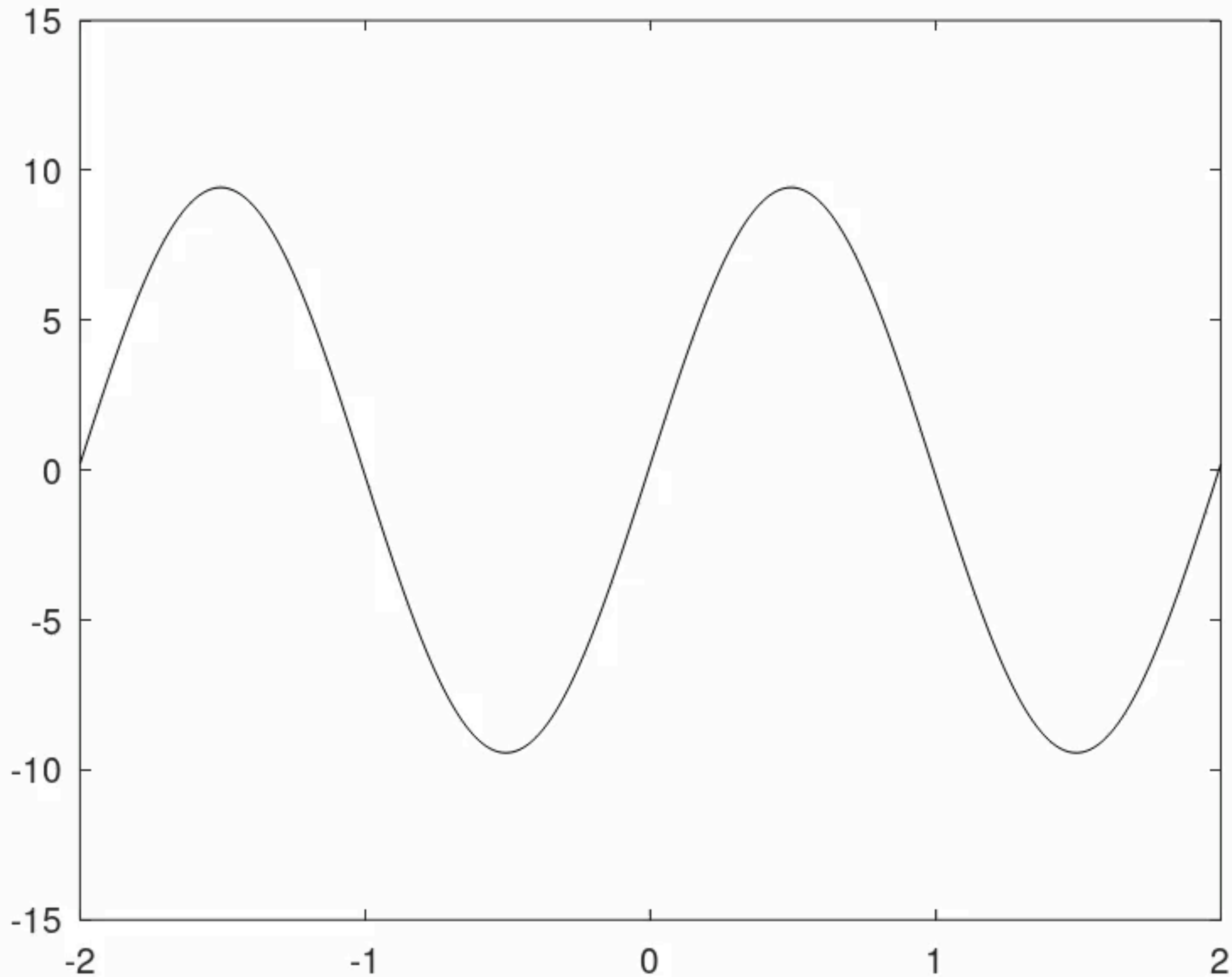
We will do better in class.



1.5
1
0.5
0
-0.5
-1
-1.5
-4
-2
0
2
4



another example - first 6 slide because it happens fast


15
10
5
0
-5
-10
-15
-2
-1
0
1
2

Even and Odd Stuff

We can always rewrite any function as:

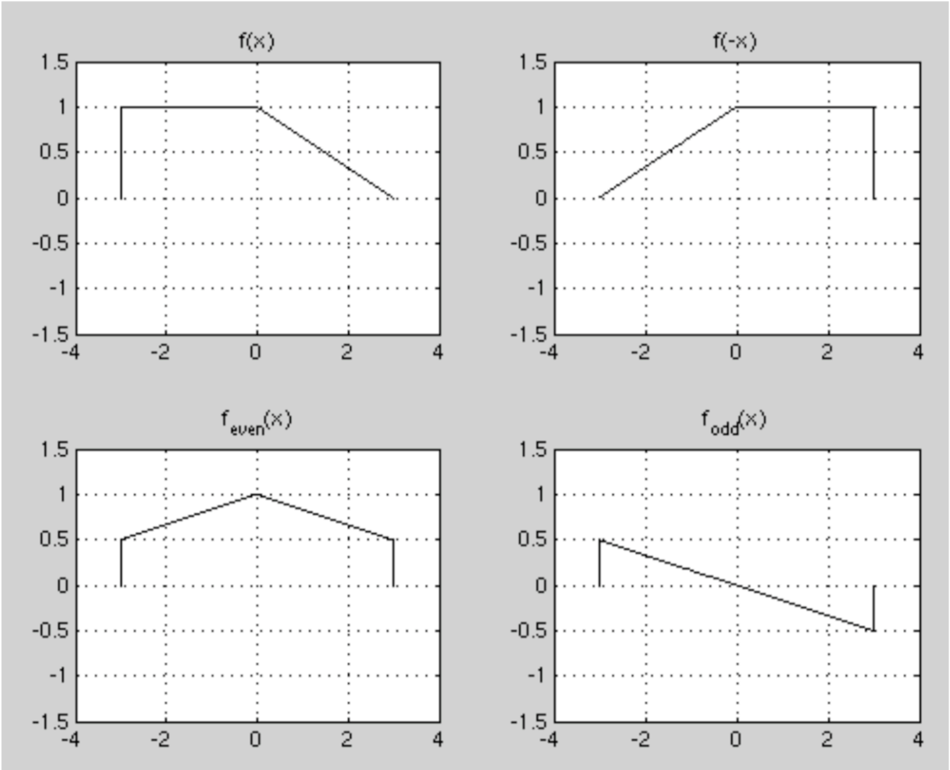
$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = f_{even}(x) + f_{odd}(x)$$

We can then use Fourier series to write:

$$f_{even}(x) = \frac{a_0}{2}\sum_n a_n \cos nx \quad , \quad f_{odd}(x) = \sum_n b_n \sin nx$$

where we have assumed that the function has a period $T_0 = 2\pi$.

Example:



Now we allow for an **Arbitrary Interval**.

If $f(x)$ is defined for an interval $[-L, L]$ of length(period) $2L$ instead of the standard interval of length (period) 2π, then a simple change of variable and integration range deals with the problem.

We have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n}{T_0}t + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n}{T_0}t$$

with

$$a_0 = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} dt\, f(t)\ ,\ a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} dt\, \cos\frac{2\pi nt}{T_0} f(t)$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} dt\, \sin\frac{2\pi nt}{T_0} f(t)$$

We let $T_0 = 2L$ to get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{\pi n}{L}x$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} dx\, f(x)\ ,\ \ a_n = \frac{1}{L}\int_{-L}^{L} dx\ \cos\frac{\pi n x}{L} f(x)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx\ \sin\frac{\pi n x}{L} f(x)$$

Another Example:

Suppose

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ h & 0 < x < \pi \end{cases}$$

The is a square pulse(wave).

We might imagine this is a signal being sent into some electronic apparatus.

We can calculate its Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ f(x)\frac{1}{\pi}\int_{0}^{\pi} h dx = h$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \cos nx f(x) = 0 \quad \text{for all} \quad n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \sin nx f(x) = \frac{h}{n\pi}(1 - \cos n\pi) = \begin{cases} 2h/n\pi & n \ \text{odd} \\ 0 & n \ \text{even} \end{cases}$$

which gives

$$f(x) = \frac{h}{2} = \frac{2h}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

We note that the terms fall off only as $1/n$, which implies physically that a square wave contains lots of **high frequency components**.

This implies that if an electronic apparatus does not pass high-frequency components well, the square wave input will emerge with corners rounded off in the best case and, possibly, an amorphous blob in the worst case.

and the corresponding Fourier series is

$$f(x) = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \ldots\ldots$$

Such expansion can also be made in terms of other special functions such as Bessel functions, etc and we will use this fact to great advantage when solving partial differential equations. **Fourier Transforms** The complex Fourier series has an important limi

Fourier Transforms

Transforms The complex Fourier series has an important limiting form when the period approaches infinity, i.e., $T_0 \to \infty$ or $L \to \infty$. Suppose that in this limit

(1) $k = \frac{n\pi}{L}$ remains large (ranging from $-\infty$ to ∞) and

(2) $c_n \to 0$ since it is proportional to L, but

$$g(k) = \lim_{\substack{L\to\infty \\ c_n\to 0}} \frac{L}{\pi} c_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \quad \text{finite}$$

then we have

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{ikx} = \lim_{\substack{L\to\infty \\ c_n\to 0}} \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\pi}{L} g(k) e^{ikx}$$

where $k - \frac{n\pi}{L}$.

The sum over n is in steps of $\Delta n = 1$.

Thus, we can write using

$$\Delta k = \frac{\pi}{L} \Delta n$$

which becomes infinitesimally small when L becomes large, as a sum over k, which becomes an integral in the limit

$$f(x) = \lim_{\substack{L\to\infty \\ c_n\to 0}} \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\pi\Delta n}{L} g(k) e^{ikx} = \lim_{\Delta k\to 0} \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \Delta k g(k) e^{ikx}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

We call $g(k)$ the Fourier Transform of $f(x)$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = F(f)$$

and the last equation is the so-called Fourier inversion formula.

We can now obtain an integral representation of the delta-function.

This corresponds to the orthogonality condition for the complex exponential Fourier series.

We substitute the definition of $g(k)$ into the inversion formula to get

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' \\
&= \int_{-\infty}^{\infty} dx' f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk \right] \\
&= \int_{-\infty}^{\infty} dx' f(x') \delta(x - x')
\end{aligned}
$$

where

$$
\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk
$$

Properties

The evaluation of the integrals involved in many Fourier transforms involves complex integration, which we shall learn later.

We will just state some properties

Examples:

(1) The Fourier transform of the box function

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| \geq a \end{cases}$$

is

$$\begin{aligned} F(f) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-kx}}{ik} \right|_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{2 \sin ka}{k} \end{aligned}$$

(2) The Fourier transform of the derivative of a function is

$$
\begin{aligned}
F\left(\frac{df}{dx}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(x)}{dx}e^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ikx}\Big|_{-\infty}^{\infty} - (-ik)\int_{-\infty}^{\infty}f(x)e^{-ikx}dx\right] \\
&= \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx = ikF(f)
\end{aligned}
$$

where we have assumed that $f(x) \to 0$ as $|x| \to \infty$.

This generalizes to

$$
F\left(\frac{d^n f}{dx^n}\right) = (ik)^n F(f)
$$

(3) Other useful properties of the Fourier transform are:

$$F(f(x)) = g(k) \ , \ \ F(f(x-a)) = e^{-ika} g(k) \ , \ \ F(f(x)e^{ax}) = g(k+ia)$$

A short table of Fourier Transforms is shown below:

$f(x)$	$g(k)$
$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
$\begin{cases} 0 & x>0 \\ e^{-\lambda x} & x<0 \end{cases}$	$\frac{1}{\sqrt{2\pi}} \frac{1}{a+ik}$
$e^{-\frac{cx^2}{2}}$	$\frac{1}{\sqrt{c}} e^{-\frac{k^2}{2c}}$
$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\lvert k \rvert}$

Fourier Transform Examples:

(1) **The Square Pulse** - Consider the function

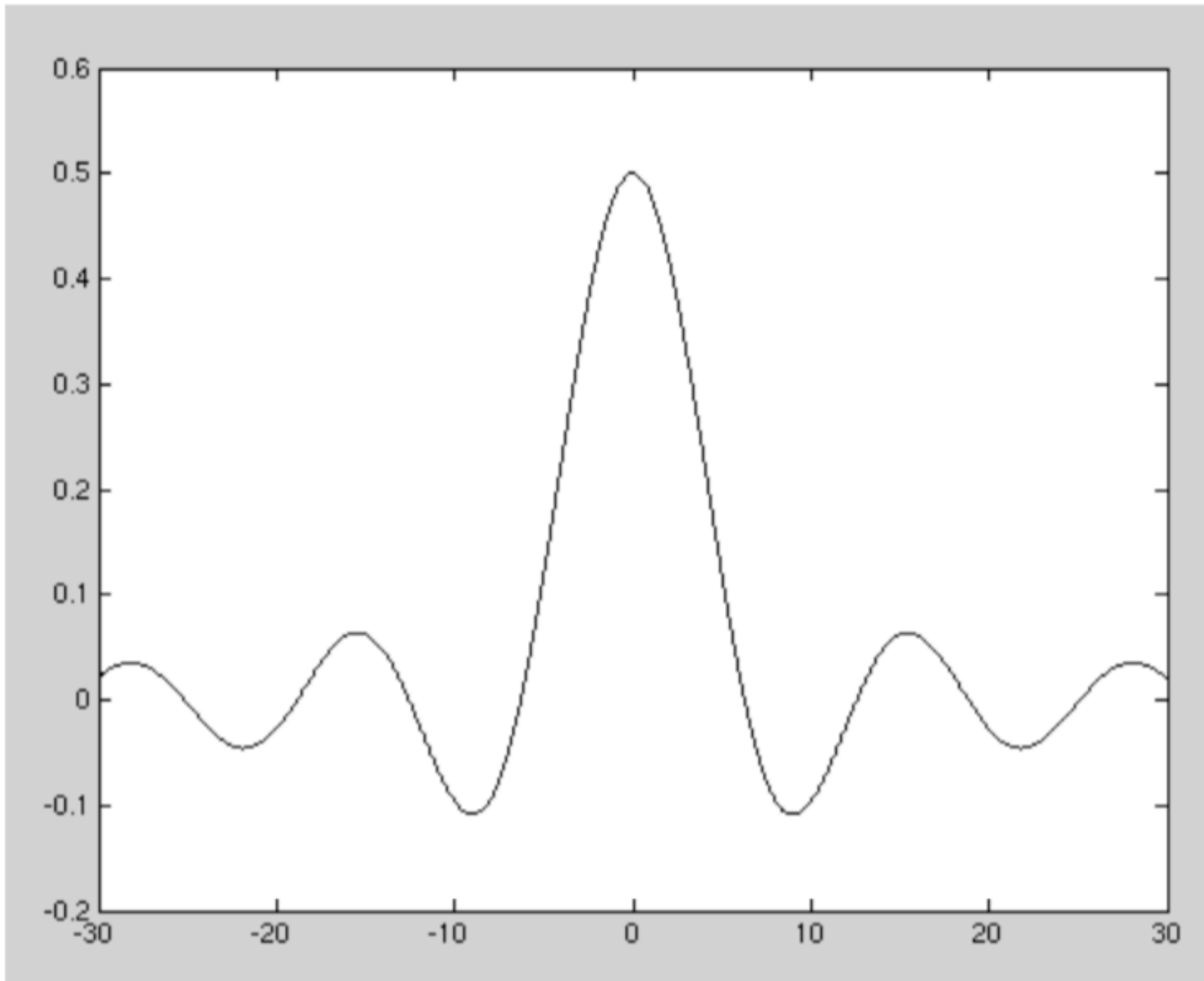
$$f(x) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

f(t) is absolutely integrable so it has a valid Fourier transform.

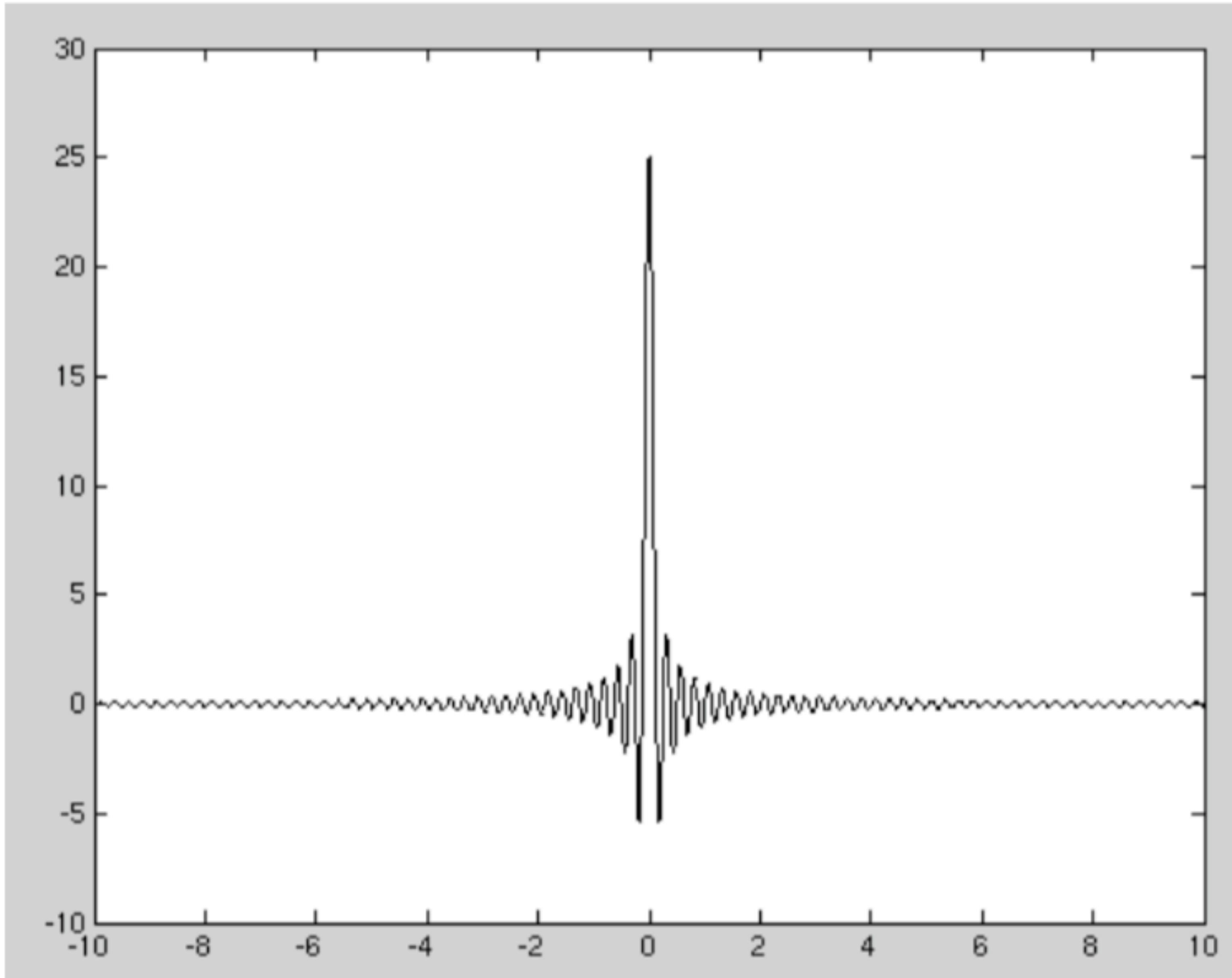
It is given by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} dt\, e^{-i\omega t} = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \frac{\omega T}{2}}{\omega} \right]$$

which looks like (for $T = 1$)



In the limit $T \to \infty$ we have ($T = 50$, in fact, here)



We get a sharp spike, but the area remains constant.

This implies that as $T \to \infty$

$$F(\omega) \to \delta(\omega)$$

Formally, we have

$$F(\omega) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} dt\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} = \sqrt{2\pi}\delta(\omega)$$

(2) **Transform of a Delta-Function** - Consider the function

$$f(t) = \delta t$$

The transform is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \delta(t) = \frac{1}{\sqrt{2\pi}}$$

The inverse transform is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\frac{1}{\sqrt{2\pi}} = \delta(t)$$

Now

$$F\left(\frac{df}{dt}\right) = i\omega F(f) = i\omega F(\omega)$$

Therefore for the square pulse we have

$$\frac{df}{dt} = \delta(t+T/2) - \delta(t-T/2)$$

$$\mathcal{I}\left(\frac{df}{dt}\right) = \mathcal{I}(\delta(t+T/2) - \delta(t-T/2))$$

But

$$F(f(t-t_0)) = e^{-i\omega t_0}F(f(t))$$

Thus,

$$F\left(\frac{df}{dt}\right) = (e^{-i\omega(-T/2)} - e^{-i\omega(T/2)} F(\delta(t)) = i\sqrt{\frac{2}{\pi}} \sin\frac{\omega T}{2} = i\omega F(\omega)$$

$$\rightarrow F(\omega) = sqrt\frac{2}{\pi}\left[\frac{\sin\frac{\omega T}{2}}{\omega}\right] \quad \text{for the square pulse (as before)}$$

Remember this only makes sense inside an integral.

(3) **Transform of a Gaussian** - Consider the function

$$f(t) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 t^2} = \quad \text{normalized Gaussian pulse}$$

We choose $\alpha = 1$.

The peak is at α/π .

The 1/2 maximum points are separated by $\Delta t = 1/\alpha$.

The area under the curve is $= 1$.

The Fourier transform is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 t^2} e^{-i\omega t} = \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{(\alpha^2 t^2 + i\omega t)}$$

We complete the square to evaluate the integral. We have

$$\alpha^2 t^2 + i\omega t = \alpha^2 t^2 + i\omega t + \gamma - \gamma = (\alpha t + \beta)^2 - \gamma$$

$$\rightarrow 2\alpha\beta = i\omega t \rightarrow \beta = \frac{i\omega}{2\alpha}$$

$$\rightarrow \gamma = \beta^2 = -\frac{\omega^2}{4\alpha^2}$$

We thus have

$$F(\omega) = \frac{\alpha}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha^2}} \int_{-\infty}^{\infty} dt\, e^{-\left(\alpha t + \frac{i\omega}{2\alpha}\right)^2}$$

Let

$$x = \alpha t + \frac{i\omega}{2\alpha} \rightarrow dx = \alpha dt$$

then

$$F(\omega) = \frac{1}{\pi\sqrt{2}} e^{-\frac{\omega^2}{4\alpha^2}} \int_{-\infty}^{\infty} dx\, e^{-x^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha^2}}$$

which is a different Gaussian.

After stating the basic properties, let us now delve deeper into Fourier theory.

Introduction

Fourier Transform theory is essential to many areas of physics including acoustics and signal processing, optics and image processing, solid state physics, scattering theory, and in the solution of differential equations in applications as diverse as weather modeling to quantum field calculations.

The Fourier Transform can either be considered as expansion in terms of an orthogonal basis set (sine and cosine), or a shift of space from real space to reciprocal(or frequency) space (x —> k).

Actually these two concepts are mathematically identical although they are often used in very different physical situations.

Notation

Unlike many mathematical fields of science, Fourier Transform theory does not have a well defined set of standard notations.

To familiarize you with other notations, the notation maintained from now on will be:

$$x,y \to \text{Real Space coordinates}$$
$$u,v \to \text{Frequency Space coordinates}$$

and lower case functions ($f(x)$), being a real space function and upper case functions ($F(u)$) , being the corresponding Fourier transform, thus:

$$F(u) = \Im\{f(x)\}$$
$$f(x) = \Im^{-1}\{F(u)\}$$

where $\Im\{\ \}$ is the Fourier Transform operator.

The character i for imaginary numbers has the property that $i^2 = -1$.

Two special functions will also be employed, these being sinc() defined as,

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x} \qquad (01)$$

giving sinc(0) = 1 and $\mathrm{sinc}(x_0) = 0$ at $x_0 = \pm\pi, \pm 2\pi$, ..., as shown in figure 1.

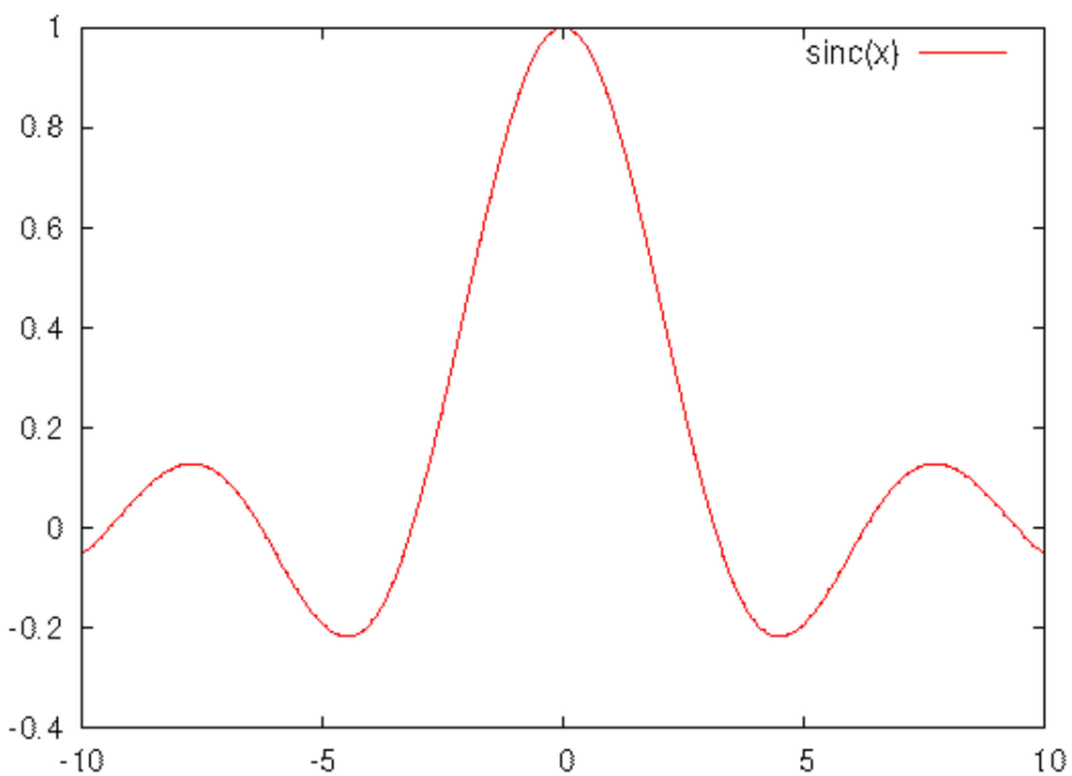


Figure 1: The sinc() function.

The top hat function $\prod(x)$, is given by,

$$\prod(x) = \begin{cases} 1 & for \ \ |x| \le 1/2 \\ 0 & otherwise \end{cases} \tag{02}$$

It is a function of unit height and width centered about x = 0 and is shown in figure 2.

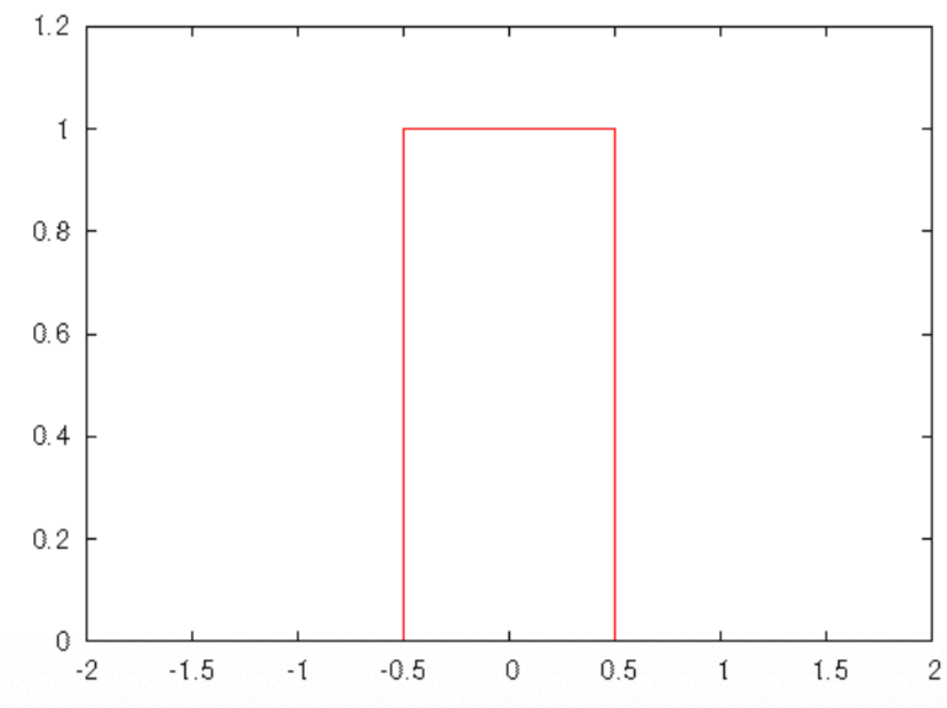


Figure 2: The $\prod(x)$ function

The Fourier Transform

Given a one dimensional continuous function $f(x)$, its Fourier transform is defined by:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx \tag{03}$$

with the inverse Fourier transform defined by;

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{2\pi iux} du \tag{04}$$

where it should be noted that the factors of 2π are incorporated into the transform kernel(exponential factor).

There are various definitions of the Fourier transform that put the 2π either inside the kernel or as external scaling factors(earlier definition).

The difference between them is whether the variable in Fourier space is a “frequency” or “angular frequency”.

The difference between the definitions are clearly just a scaling factor.

The optics and digital Fourier applications the 2π is usually defined to be inside the kernel but in solid state physics and differential equation solution the 2π is usually an external scaling factor.

Some insight to the Fourier transform can be gained by considering the case of the Fourier transform of a real signal $f(x)$.

In this case the Fourier transform can be separated to give,

$$F(u) = F_r(u) + iF_i(u) \qquad (05)$$

where we have,

$$F_r(u) = \int_{-\infty}^{\infty} f(x)\cos(2\pi ux)dx$$

the

$$F_i(u) = -\int_{-\infty}^{\infty} f(x)\sin(2\pi ux)dx$$

So the real part of the Fourier transform is the decomposition of $f(x)$ in terms of cosine functions, and the imaginary part a decomposition in terms of sine functions.

The u variable in the Fourier transform is interpreted as a frequency, for example if $f(x)$ is a sound signal with x measured in seconds then F(u) is its frequency spectrum with u measured in Hertz (s^{-1}).]

Clearly (ux) must be dimensionless, so if x has dimensions of time then u must have dimensions of time^{-1}.

This is one of the most common applications for Fourier Transforms where $f(x)$ is a detected signal (for example a sound made by a musical instrument), and the Fourier Transform is used to give the **spectral or frequency** response.

Properties of the Fourier Transform

The Fourier transform has a range of useful properties, some of which are listed below.

In most cases the proof of these properties is simple and can be formulated by use of equation (03) and equation (04).

The proofs of many of these properties are given later in examples.

Linearity: The Fourier transform is a linear operation so that the Fourier transform of the sum of two functions is given by the sum of the individual Fourier transforms.

Therefore,

$$\Im\{af(x)+bg(x)\}=aF(u)+bG(u) \tag{06}$$

where F(u) and G(u) are the Fourier transforms of $f(x)$ and g(x) and a and b are constants.

This property is central to the use of Fourier transforms when describing linear systems.

Complex Conjugate: The Fourier transform of the Complex Conjugate of a function is given by

$$\Im\left\{f^{*}(x)\right\}=F^{*}(-u) \tag{07}$$

where F(u) is the Fourier transform of $f(x)$.

Forward and Inverse: We have that

$$\Im\left\{F(u)\right\}=f(-x) \tag{08}$$

so that if we apply the Fourier transform twice to a function, we get a spatially reversed version of the function.

Similarly with the inverse Fourier transform we have that,

$$\Im^{-1}\left\{f(x)\right\}=F(-u) \tag{09}$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

Differentials: The Fourier transform of the derivative of a functions is given by

$$\Im\left\{\frac{df(x)}{dx}\right\} = i2\pi u F(u) \tag{10}$$

and of the second derivative is given by

$$\Im\left\{\frac{d^2 f(x)}{dx^2}\right\} = -(2\pi u)^2 F(u) \tag{11}$$

Power Spectrum: The Power Spectrum of a signal is defined by the modulus square of the Fourier transform, being $|F(u)|^2$.

This can be interpreted as the power of the frequency components or the spectrum of power versus frequency.

Any function and its Fourier transform obey the condition that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(u)|^2 du \tag{12}$$

which is frequently known as Parseval's Theorem.

If $f(x)$ is interpreted as a voltage, then this theorem states that the power is the same whether measured in real (time), or Fourier (frequency) space.

Strictly speaking Parseval's Theorem applies to the case of Fourier series, and the equivalent theorem for Fourier transforms is correctly, but less commonly, known as Rayleigh's theorem.

Two Dimensional Fourier Transform

When using two-dimensional scalar potentials or images, one is dealing with a two dimensional function.

We will define the two dimensional Fourier transform of a continuous function *f*(x,y) by

$$F(u,v) = \iint f(x,y) e^{-i2\pi(ux+vy)} dxdy \tag{13}$$

with the inverse Fourier transform defined by

$$f(x,y) = \iint F(u,v) e^{i2\pi(ux+vy)} dudv \tag{14}$$

where the limits of integration are taken from $-\infty \rightarrow \infty$.

Unless otherwise specified all integral limits will be assumed to be from $-\infty \rightarrow \infty$.

Again for a real two dimensional function *f*(x,y), the Fourier transform can be considered as the decomposition of a function into its sinusoidal components.

If *f*(x,y) is considered to be an image with the "brightness" of the image at point (x_0,y_0) given by f(x_0,y_0), then the variables x,y have the dimensions of length.

In Fourier space, the variables u,v therefore have the dimensions of inverse length, which is interpreted as spatial frequency.

Typically x and y are measured in mm so that u and v are in units of mm^{-1} - referred to as lines per mm.

The Fourier transform can then be taken as being the decomposition of the image into two dimensional sinusoidal spatial frequency components.

The properties of one the dimensional Fourier transforms covered earlier extend into two dimensions.

Clearly the derivatives then become

$$\Im\left\{\frac{\partial f(x,y)}{\partial x}\right\} = i2\pi u F(u,v) \qquad (15)$$

and

$$\Im\left\{\frac{\partial f(x,y)}{\partial y}\right\} = i2\pi v F(u,v) \qquad (16)$$

yielding the important result that,

$$\Im\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v)$$

where we have that $w^2 = u^2 + v^2$

So that taking the Laplacian of a function in real space is equivalent to multiplying its Fourier transform by a circularly symmetric quadratic of $-4\pi^2 w^2$.

The two dimensional Fourier Transform , of a function $f(x,y)$ is a separable operation, and can be written as,

$$F(u,v) = \int P(u,y) e^{-i2\pi vy} dy \tag{18}$$

where

$$P(u,y) = \int f(x,y) e^{-i2\pi ux} dx \tag{19}$$

where P(u,y) is the Fourier Transform of $f(x,y)$ with respect to x only.

This property of separability leads to an implementation of two dimensional discrete Fourier Transforms (DFT) in terms of one dimensional Fourier Transforms.

The Three-Dimensional Fourier Transform

In the three dimensional case we have a function $f(\vec{r})$ where $\vec{r}$ = (x, y, z), then the three-dimensional Fourier Transform is

$$F(\vec{s}) = \iiint f(\vec{r}) e^{-i2\pi \vec{r}\cdot\vec{s}} d\vec{r}$$

where $\vec{s} = (u,v,w)$ being the three reciprocal variables each with units *length*$^{-1}$.

Similarly the inverse Fourier Transform is given by

$$f(\vec{r}) = \iiint F(\vec{s}) e^{i2\pi \vec{r}\cdot\vec{s}} d\vec{s}$$

This is used extensively in solid state physics where the three-dimensional Fourier Transform of crystal structures is usually called Reciprocal Space or $\vec{k}$-space where $\vec{k} = 2\pi\vec{s}$

The three-dimensional Fourier Transform is again separable into one-dimensional Fourier Transforms.

This property is independent of the dimensionality and a multi-dimensional Fourier Transform can always be formulated as a series of one dimensional Fourier Transforms.

Examples

Rectangular Aperture

Let us figure out the two dimensional Fourier transform of a rectangle of unit height and size a by b centered about the origin.

We let a = 5mm and b = 1mm.

We can express a rectangle of size a x b by:

$$f(x,y) = \begin{cases} 1 & \mathit{for}\ |x| < a/2\ \mathit{and}\ |y| < b/2 \\ 0 & \mathit{otherwise} \end{cases}$$

The Fourier Transform is given by:

$$F(u,v) = \iint f(x,y) e^{-i2\pi(ux+vy)} dxdy$$

which can then be written as:

$$F(u,v) = \int_{-b/2}^{b/2} \left[\int_{-a/2}^{a/2} e^{-i2\pi(ux+vy)} dx \right] dy$$

Noting that the exp() term is separable, this can be written as

$$F(u,v) = \int_{-b/2}^{b/2} e^{-i2\pi vy} dy \int_{-a/2}^{a/2} e^{-i2\pi ux} dx$$

Look at one of the integrals, and we get,

$$\int_{-a/2}^{a/2} e^{-i2\pi ux} dx = \frac{1}{-i2\pi u} e^{-i2\pi ux} \Big|_{-a/2}^{a/2} = \frac{-i}{2\pi u} \left[e^{i\pi ua} - e^{-i\pi ua} \right]$$

$$= \frac{\sin(\pi ua)}{\pi u} = a\mathrm{sinc}(\pi ua)$$

The other integral is of exactly the same form, so that the Fourier transform of the rectangle is:

$$F(u,v) = ab\mathrm{sinc}(\pi ua)\mathrm{sinc}(\pi vb)$$

The zeros of this function occur at:

$$u_n = \pm\frac{n}{a} \quad for \quad n = 1,2,3,....$$

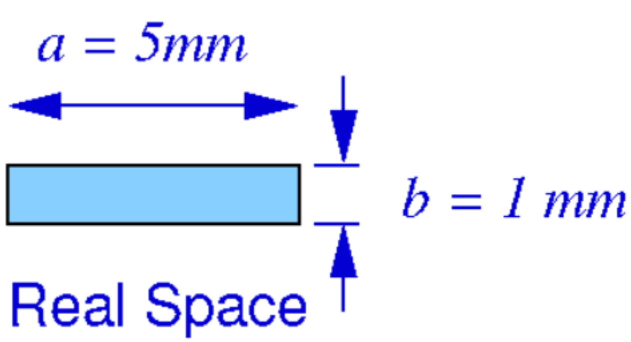
$$v_m = \pm\frac{m}{b} \quad for \quad m = 1,2,3,....$$

which if a = 5mm and b = 1mm then

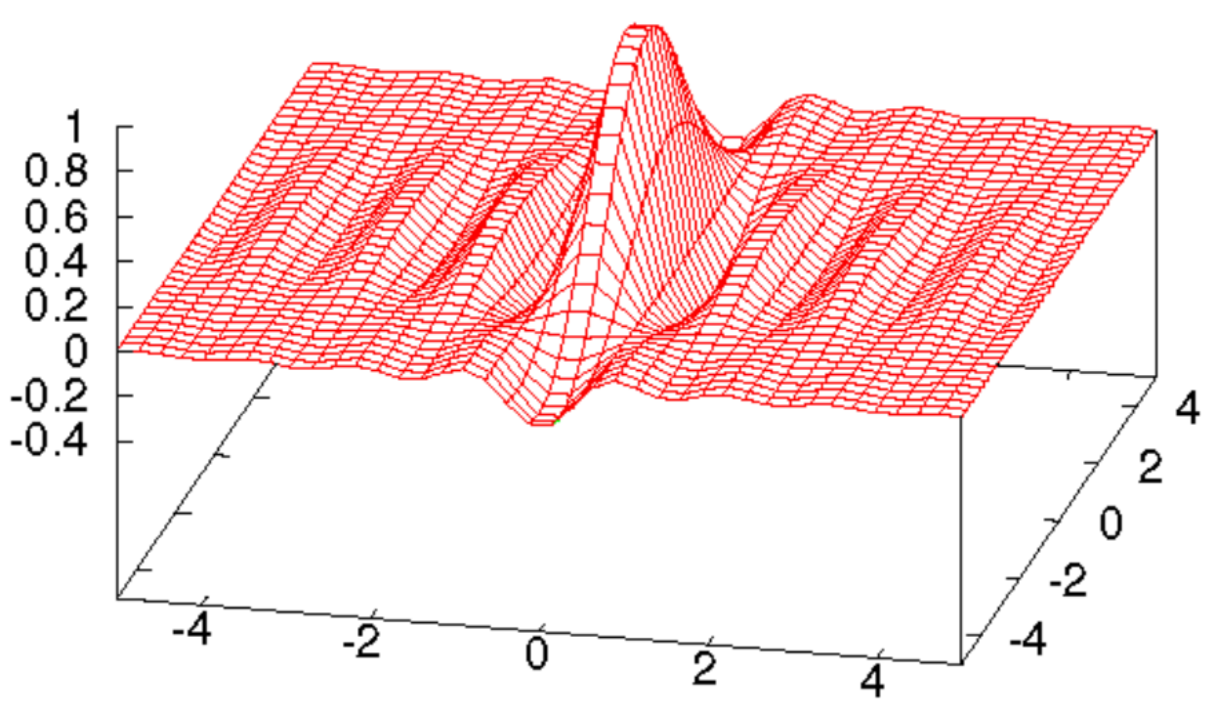
$$u_n = 0.2mm^{-1}, 0.4mm^{-1}, 0.6mm^{-1},.....$$

$$v_m = 1mm^{-1}, 2mm^{-1}, 3mm^{-1},.....$$

In diagrams we get,



so in Fourier space we get a three-Dimensional plot



Note that the long/thin shape of the rectangleFourier Transforms to tall/thin structures in the FourierTransform

Two-Dimensional Gaussian

Let us calculate the Fourier Transform of a two-dimensional Gaussian given by,

$$f(x,y) = e^{-r^2/r_0^2}$$

where $r^2 = x^2 + y^2$ and r_0 is the radius of the e^{-1} point.

We will need the standard mathematical identity that

$$\int_{-\infty}^{\infty} e^{-bx^2} e^{iax} dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

The Fourier Transform is given by:

$$F(u,v) = \iint e^{-\left(x^2+y^2\right)/r_0^2} e^{-i2\pi(ux+vy)} dxdy$$

Since the Gaussian and the Fourier kernel are separable, this can be written as

$$F(u,v) = \int e^{-x^2/r_0^2} e^{-i2\pi ux} dx \int e^{-y^2/r_0^2} e^{-i2\pi vy} dy$$

so we need only evaluate one integral.

Using the identity we have

$$\int e^{-x^2/r_0^2} e^{-i2\pi ux} dx = \frac{\sqrt{\pi}}{r_0} e^{-\pi^2 r_0^2 u^2}$$

which is also a Gaussian. The Fourier Transform of a Gaussian is a Gaussian.

It is the only function that is its own Fourier Transform.

We get exactly the same expression for the y integral, so we get that

$$F(u,v) = \frac{\pi}{r_0^2} e^{-\pi^2 r_0^2 (u^2+v^2)}$$

thewhich is more conveniently written as:

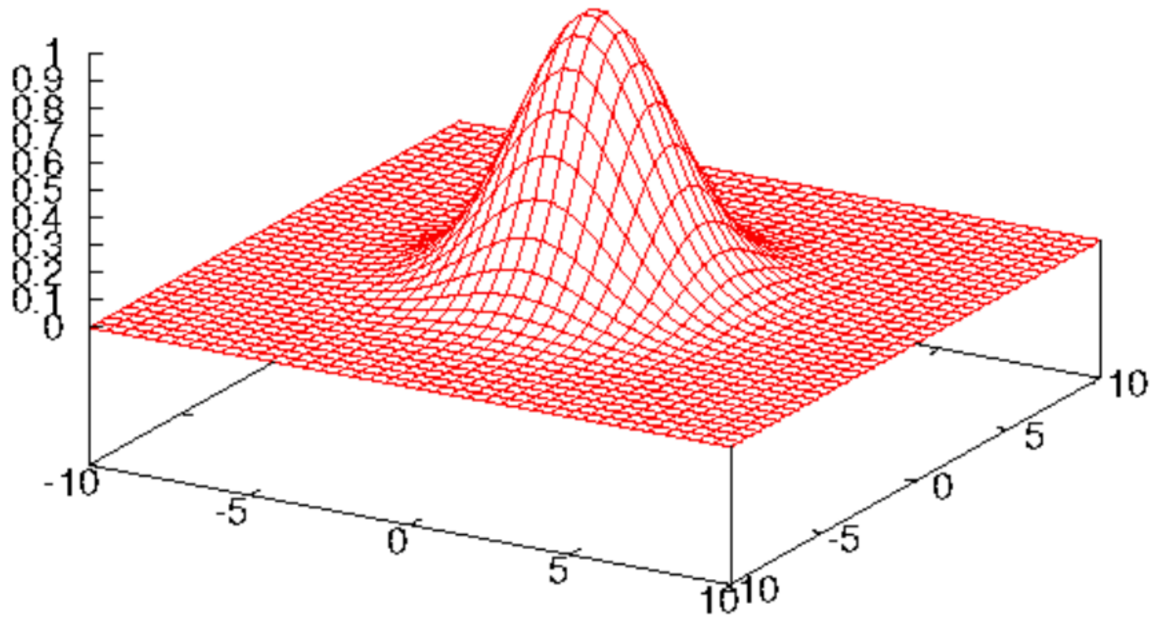
$$F(u,v) = \frac{\pi}{r_0^2} e^{-w^2/w_0^2}$$

where $w^2 = u^2 + v^2$ and $w_0 = 1/\pi r_0$, which is a circular Gaussian with e-1 point at w_0.

So the Fourier Transform of a Gaussian is a Gaussian of reciprocal width.

Or more simply, a wide Gaussian Fourier Transform gives a narrow Gaussian and vice versa.

General shape of two dimensional Gaussian with $r_0 = 3$ is given by



INTERFERENCE

For light (electromagnetic radiation) it is the electric and magnetic field which oscillate in strength and direction.

At visible frequencies these oscillations occur so fast that they are exceedingly difficult to observe directly (10^{14} Hz.)

An important characteristic of waves is that they may interfere with each other.

In simple terms interference is the result of oscillating dips and rises adding together or canceling out.

A good example of interference is seen when the circular wave patterns from two stones thrown into a pond meet.

High points from the separate waves add to make even higher points, dips from each wave combine making lower dips, and when a dip from one wave passes the same point as a high from the other they cancel.

Figure 1 below shows how two sinusoidal waves add together to produce a new wave.

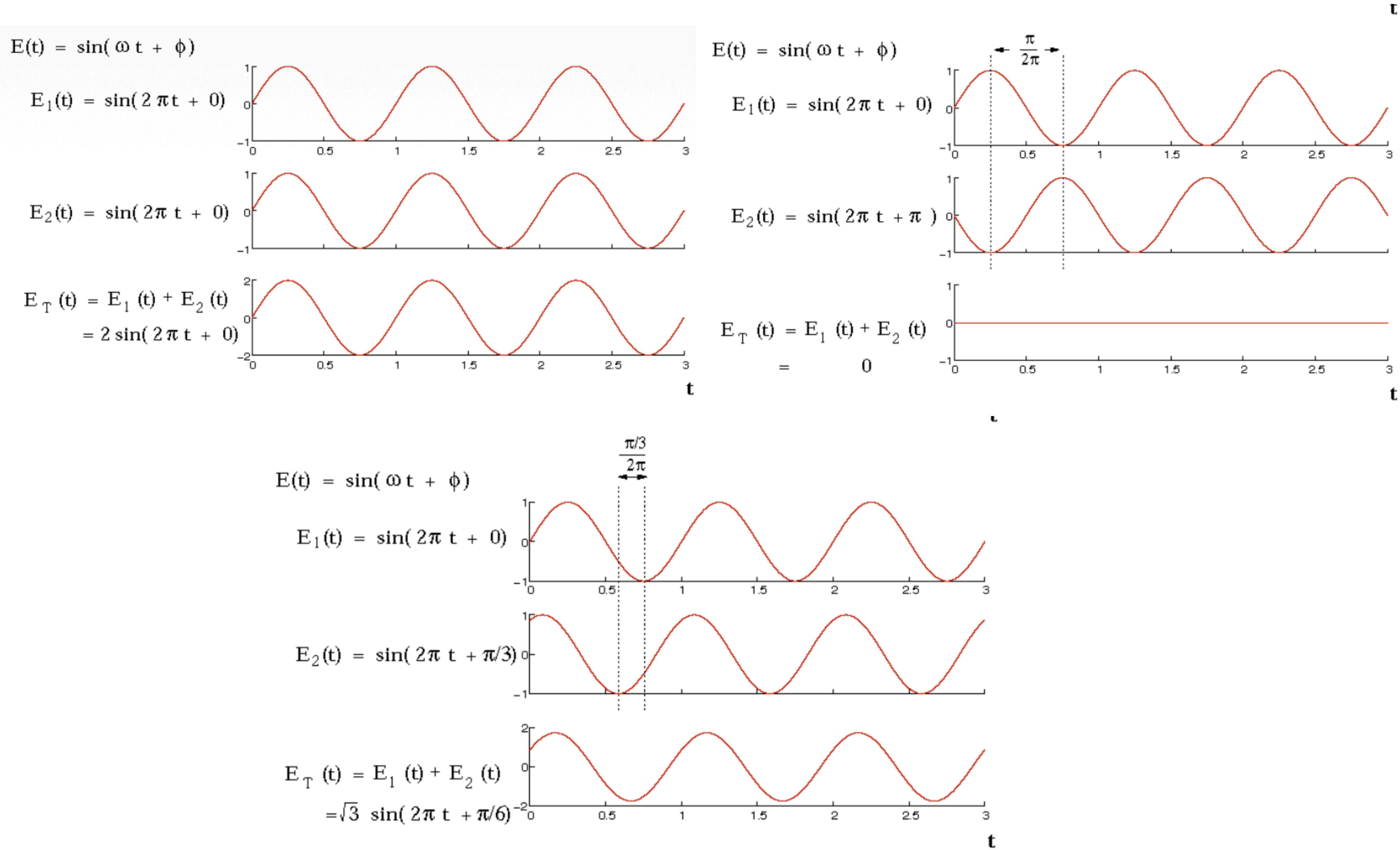


Figure 1 Addition and Interference Between Sinusoidal Waves

The most obvious optical phenomena displaying wave nature of light is interference where two, or more, beams of light add together to give fringe patterns of high and low intensity.

Before investigating how interference affects the intensity pattern of light that has passed through two narrow adjacent slits and many parallel slits (a diffraction grating), we now look at the formal description of interference to derive the key expression, which is then used to analyze a series of practical optical cases where interference is the dominant optical phenomena.

Description of Waves

Before looking at interference effects, we need to consider waves in detail.

The expression for a monochromatic linear polarized electro-magnetic wave at a point $\vec{r}$ is space to can be written as

$$\vec{\psi}(\vec{r},t) = \vec{E}\cos(\vec{k}\cdot\vec{r} - \omega t + \phi(t))$$

where

$\vec{E}$ = the electric vector
$\vec{k} = k\hat{k} = \hat{k}$ where is the unit direction
ω = the angular frequency
$\phi(t)$ = the phase, possibly time varying

However if we detect this wave by measuring its **intensity**, we do not see the wave, or vector, nature, but we actually detect

$$I = \left\langle \left|\vec{\psi}\right|^2 \right\rangle$$

where $\langle\ \rangle$ is the time average over many cycles of the wave.

For a **single** wave, as we have seen in earlier discussion, this reduces to

If we now consider the case of two electromagnetic waves given by

$$\vec{\psi}_1(\vec{r},t) = \vec{E}_1 \cos(\vec{k}_1 \cdot \vec{r} - \omega_1 t + \phi_1(t))$$
$$\vec{\psi}_2(\vec{r},t) = \vec{E}_2 \cos(\vec{k}_2 \cdot \vec{r} - \omega_2 t + \phi_2(t))$$

and we add them together and detect the intensity, we get

$$I = \left\langle \left|\vec{\psi}_1 + \vec{\psi}_2\right|^2 \right\rangle = \left\langle \left|\vec{\psi}_1\right|^2 \right\rangle + \left\langle \left|\vec{\psi}_2\right|^2 \right\rangle + 2\left\langle \vec{\psi}_1 \cdot \vec{\psi}_2 \right\rangle$$

which we can write as

$$I = I_1 + I_2 + I_{12}$$

where we have written

$$I_1 = \left\langle \left|\vec{\psi}_1\right|^2 \right\rangle = \frac{1}{2}\left|\vec{E}_1\right|^2$$ = the intensity of first wave

$$I_2 = \left\langle \left|\vec{\psi}_2\right|^2 \right\rangle = \frac{1}{2}\left|\vec{E}_2\right|^2$$ = the intensity of second wave

$$I_{12} = 2\left\langle \vec{\psi}_1 \cdot \vec{\psi}_2 \right\rangle$$ = the interference term between the two waves

The interference term, which is the one of interest, depends on both waves, and can be expanded to give the expression

$$\begin{aligned}\langle \vec{\psi}_1 \cdot \vec{\psi}_2 \rangle &= \vec{E}_1 \cdot \vec{E}_2 \left\langle \cos(\vec{k}_1 \cdot \vec{r} - \omega_1 t + \phi_1(t)) \cos(\vec{k}_2 \cdot \vec{r} - \omega_2 t + \phi_2(t)) \right\rangle \\ &= \vec{E}_1 \cdot \vec{E}_2 \frac{1}{2} \Big[\left\langle \cos(\vec{k}_1 \cdot \vec{r} + \vec{k}_2 \cdot \vec{r} - (\omega_1 + \omega_2)t + \phi_1(t) + \phi_2(t)) \right\rangle \\ &\qquad + \left\langle \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi_1(t) - \phi_2(t)) \right\rangle \Big]\end{aligned}$$

which is the summation of two cos() terms.

Now in the optical region we have that $\omega \simeq 10^{14}\ \mathrm{s}^{-1}$, so that any practical time averaging term containing

$$\left\langle \cos(\ldots\ldots\ldots\ldots - (\omega_1 + \omega_2)t + \ldots\ldots\ldots\ldots) \right\rangle = 0$$

so on time averaging the rather messy interference term reduces to

$$I_{12} = \vec{E}_1 \cdot \vec{E}_2 \left\langle \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi_1(t) - \phi_2(t)) \right\rangle$$

We will now consider special cases.

Incoherent Illumination

If ω_1 and ω_2 are significantly different, so we have two beams of significantly different wavelength, and/or $\phi_1(t)$ and $\phi_2(t)$ terms vary rapidly in time with respect to each, then the interference term

$$\left\langle \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi_1(t) - \phi_2(t)) \right\rangle = 0$$

so we get

$$I = I_1 + I_2$$

or the intensities of the two waves add and there is no interference.

This is the **fully incoherent** condition.

This is the typical condition when the two beams come from independent sources, for example two stars, two light bulbs, two apparently identical lasers, or even two different points from the same extended light source.

This is the condition we get in most everyday schemes, and it is the condition assumed in all of geometric optical imaging.

Coherent Illumination

The other extreme case is when together so that $\omega_1 = \omega_2$ and the phase of the two beams are locked together so that

$$\phi_1(t) - \phi_2(t) = \phi_0 \quad \forall\, t$$

This usually occurs when the two beams are formed by splitting a single beam into two.

In this case, the time dependence of I_{12} vanishes so there is no time averaging and we get that

$$I_{12} = \vec{E}_1 \cdot \vec{E}_2 \cos(\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} + \phi_0)$$

which depends on the spatial coordinate $\vec{r}$ where the intensity is measured.

This is the **fully coherent** condition.

The term inside the cos() is the phase difference between the two beams, which we will write as

$$\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} + \phi_0 = \delta$$

If we further assume that the two beams are in the same state of polarization, for example horizontal linear polarization, then we can write

$$\vec{E}_1 = E_1 \hat{x} \quad \textit{and} \quad \vec{E}_2 = E_2 \hat{x} \quad \textit{so that} \quad \vec{E}_1 \cdot \vec{E}_2 = E_1 E_2$$

We now also have that

$$I_1 = \frac{1}{2} E_1^2 \quad \textit{and} \quad I_2 = \frac{1}{2} E_2^2 \quad \textit{so that} \quad E_1 E_2 = 2\sqrt{I_1 I_2}$$

so that finally, under these conditions, we have

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

where I_1 and I_2 are the intensities of the two beams and δ is the phase shift between them.

This will give rise to bright and dark fringes with bright fringes where

$$\delta = 0, \pm 2\pi, \pm 4\pi, ...$$

as plotted in figure 1 where $I_1 = 1$ and $I_2 = 3$.

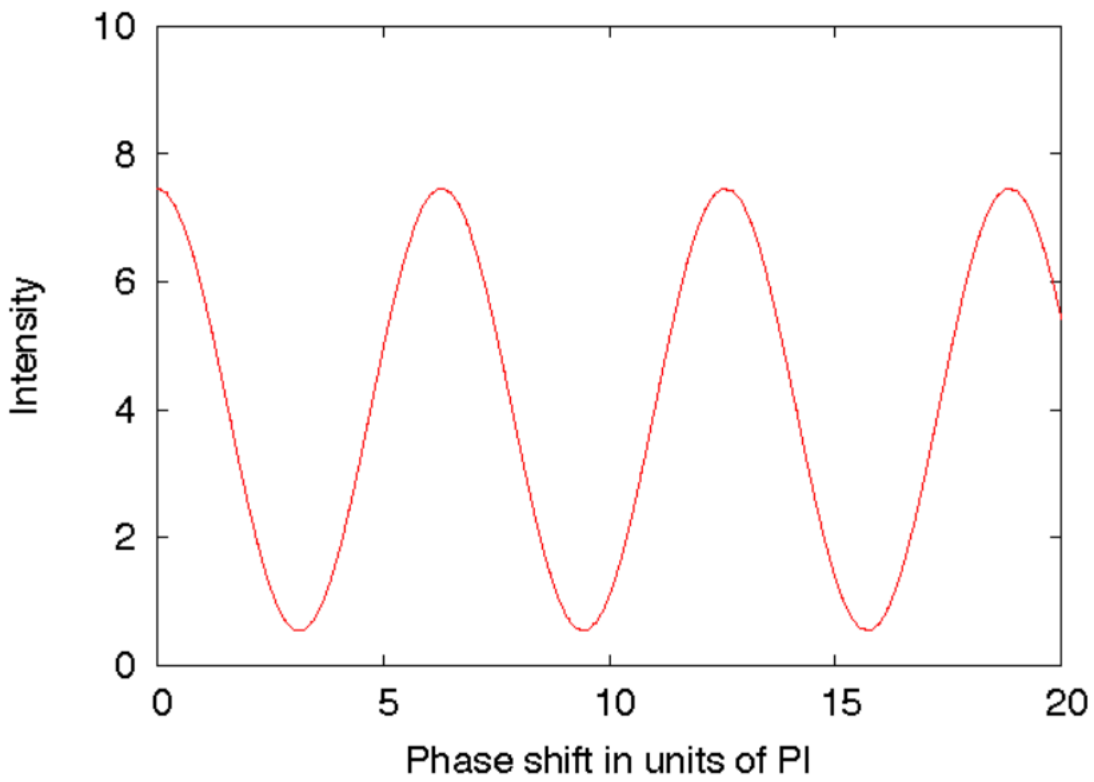


Figure 1: Plot of intensity against phase shift for $I_1 = 1$ and $I_2 = 3$

The maximum and minimum intensity of the fringe pattern will be

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2} \quad and \quad I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

with the special case $I_1 = I_2 = I_0$, where both beams have the same intensity, we get

$$I_{\max} = 4I_0 \quad and \quad I_{\min} = 0$$

In characterizing the fringe pattern it is useful to define **fringe contrast** as,

$$C = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

which is easily measured and with a little re-arrangement, is given by

$$C = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}$$

C= 1 where for the special case of $I_1 = I_2$, we get C= 1.

We get the same result with other polarization states.

Partial Coherence

We have considered the two extreme cases, either **fully incoherent** where we have no interference and **fully coherent** where we get cos() fringes.

The middle ground is where there is some interference between the beams, this is known as **partial coherence**, which is a particularly nasty theoretical region with no simple analytical results.

Fortunately most practical optical systems are either fully incoherent where we can ignore interference completely, for example most imaging systems, or fully coherent where they are specifically designed to exhibit interference, for example interferometers.

The only systems operating in the partial coherent regions are high magnification optical microscopes under specific illumination conditions and some astronomical optical interferometers specifically designed to analyze the partial coherent region.

Interference of two plane beams

Consider two horizontally polarized beams of the same wavelength in the y/z plane with an angle θ between them, as shown in figure 2.

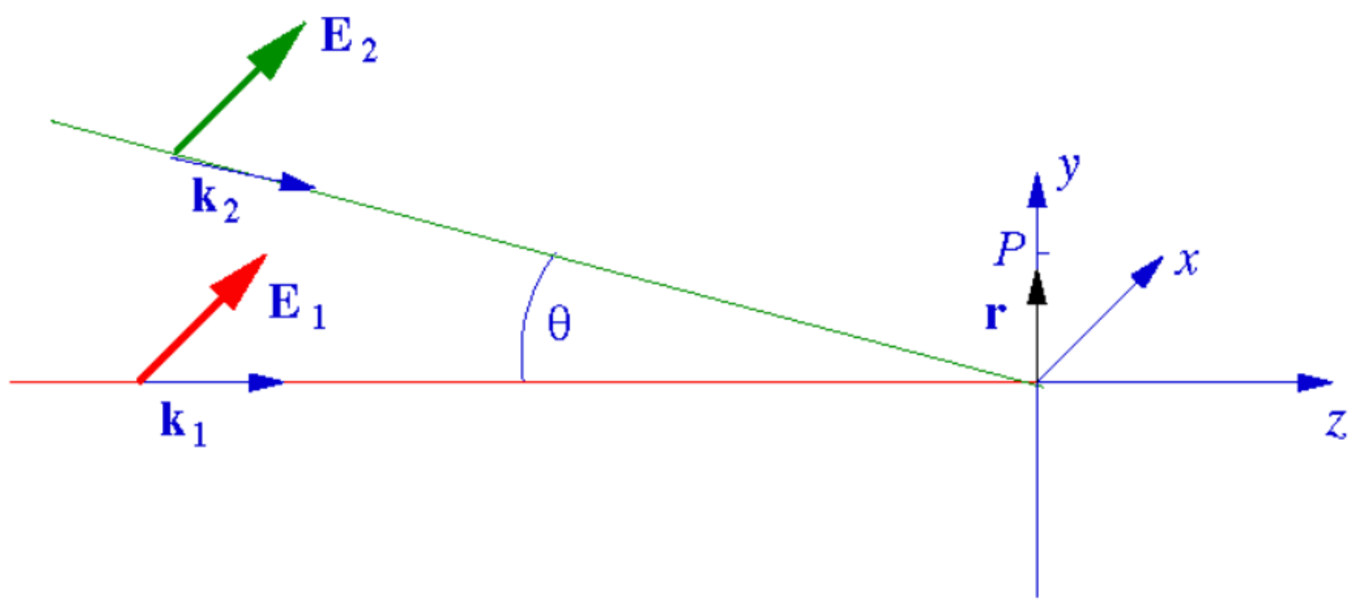


Figure 2: Interference of two horizontal polarized beams

Both beams are horizontally polarized, so we have

$$\vec{E}_1 = E_1\hat{x} \quad and \quad \vec{E}_2 = E_2\hat{x} \quad so\ that \quad \vec{E}_1 \cdot \vec{E}_2 = E_1 E_2$$

The first beam is traveling along the z-axis, and the second in the y/z plane at angle θ to the z axis, so that

$$\vec{k}_1 = k\hat{z} \quad and \quad \vec{k}_2 = k(\sin\theta\hat{y} + \cos\theta\hat{z})$$

If we observe the interference in the x/y plane at a position P, then $\vec{r} = y\hat{y}$, we get the phase difference is

$$\delta = \vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r} + \phi_0 = -ky\sin\theta + \phi_0$$

We get the intensity observed at this position in the y direction to be

$$I(y) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(-ky\sin\theta + \phi_0)$$

and in the special case of both beams with the same intensity, I_0 , we get

$$I(y) = 2I_0(1 + \cos(-ky\sin\theta + \phi_0))$$

which corresponds to horizontal fringes of unit contrast as shown in figure 3

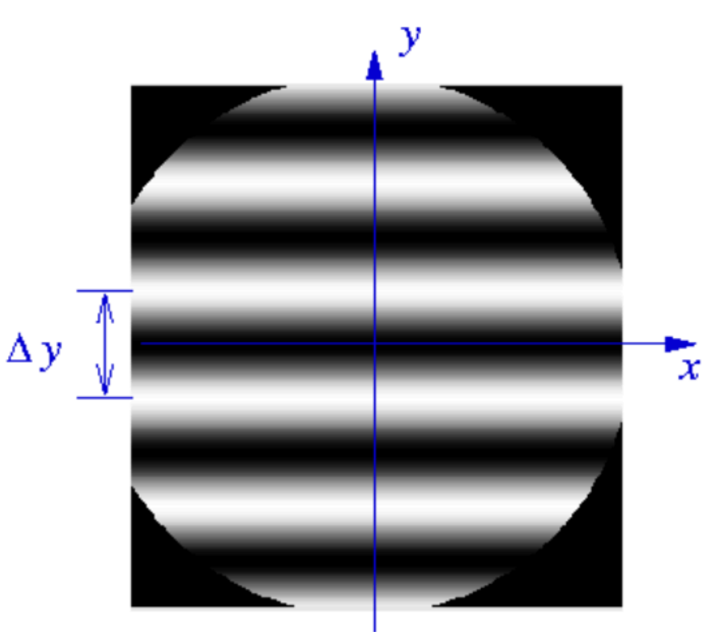


Figure 3: Intensity plot to two beam fringes

We will get a bright fringe when

$$-ky\sin\theta + \phi_0 = 0, \pm 2\pi, \pm 4\pi, ...$$

giving the separation Δy as

$$k\Delta y\sin\theta = 2\pi \quad \textit{so that} \quad \Delta y = \frac{2\pi}{k\sin\theta} = \frac{\lambda}{\sin\theta}$$

Note the term ϕ_0 gives the vertical location of the whole fringe pattern.

Let now investigate how interference affects the intensity pattern of light that has passed through two narrow adjacent slits, many parallel slits (diffraction grating), and a special bullseye pattern called a Fresnel zone plate.

Two Slit Interference

In 1801 Thomas Young devised a classic experiment for demonstrating the wave nature of light similar to the one diagrammed in Figure 4.

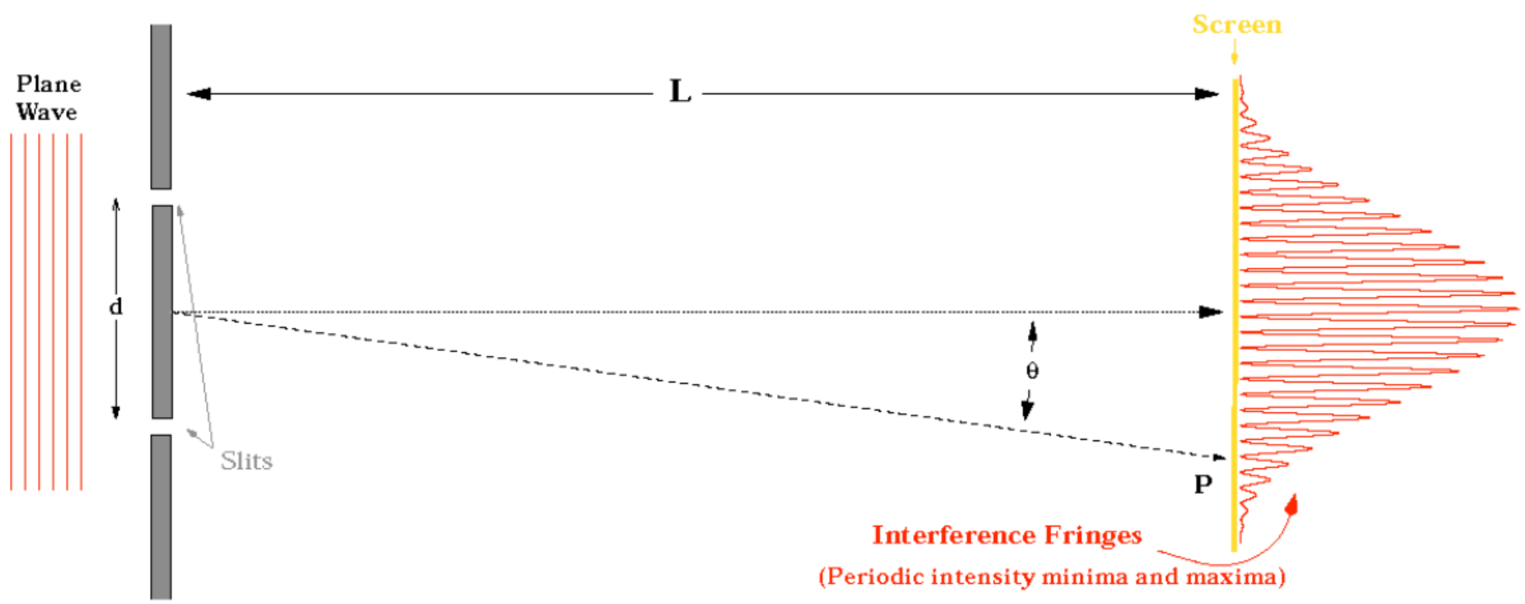


Figure 4: Young's Two SlitInterference withNarrow Slits

By passing a wide plane wave through two slits he effectively created two separate sources that could interfere with each other. (A plane wave is a wave consisting of parallel wavefronts all traveling in the same directions.)

For a given direction θ , the intensity observed at point P on the screen will be a superposition of the light originating fro each slit.

Whether the two waves add or cancel at a particular point on the screen depends upon the relative phase between the waves at that point. (The phase of a wave is the stage it is at along its periodic cycle.)

The relative phase of the two waves at the screen varies in a oscillatory manner as a function of angle causing periodic intensity minima and maxima called fringes (see Figure 4).

The spacing of the fringes depends on the distance between the two slits.

If the width of each individual slit is narrow compared to a wavelength, the intensity of the two-slit interference pattern is given by

Two Slit Interference

$$E(\theta) = E(0)\cos\beta \quad , \quad \beta = \frac{\pi d}{\lambda}\sin\theta$$
$$I(\theta) = I(0)\cos^2\beta \tag{01}$$

where *E is* the electric field. *I* is the field intensity with units of energy per area per second and is proportional to the square of the electric field E.

Equation (01) is easily derived.

The amplitude of the electromagnetic wave emanating from each slit may be written as $E = E_0 \sin(\omega t + \phi)$, where is a phase ϕ given by $2\pi / \lambda$ multiplied by the distance the wave travels from the slit.

After the wave emanating from one of the slits travels a distance ℓ the wave undergoes a phase shift $\phi = 2\pi\, \ell / \lambda$ so that the electric field is of form $E = E_0 \sin(\omega t + 2\pi\, \ell / \lambda)$.

The difference in distance light from each slit travels to a point on a distant screen is shown in Figure 5 as a function of angle.

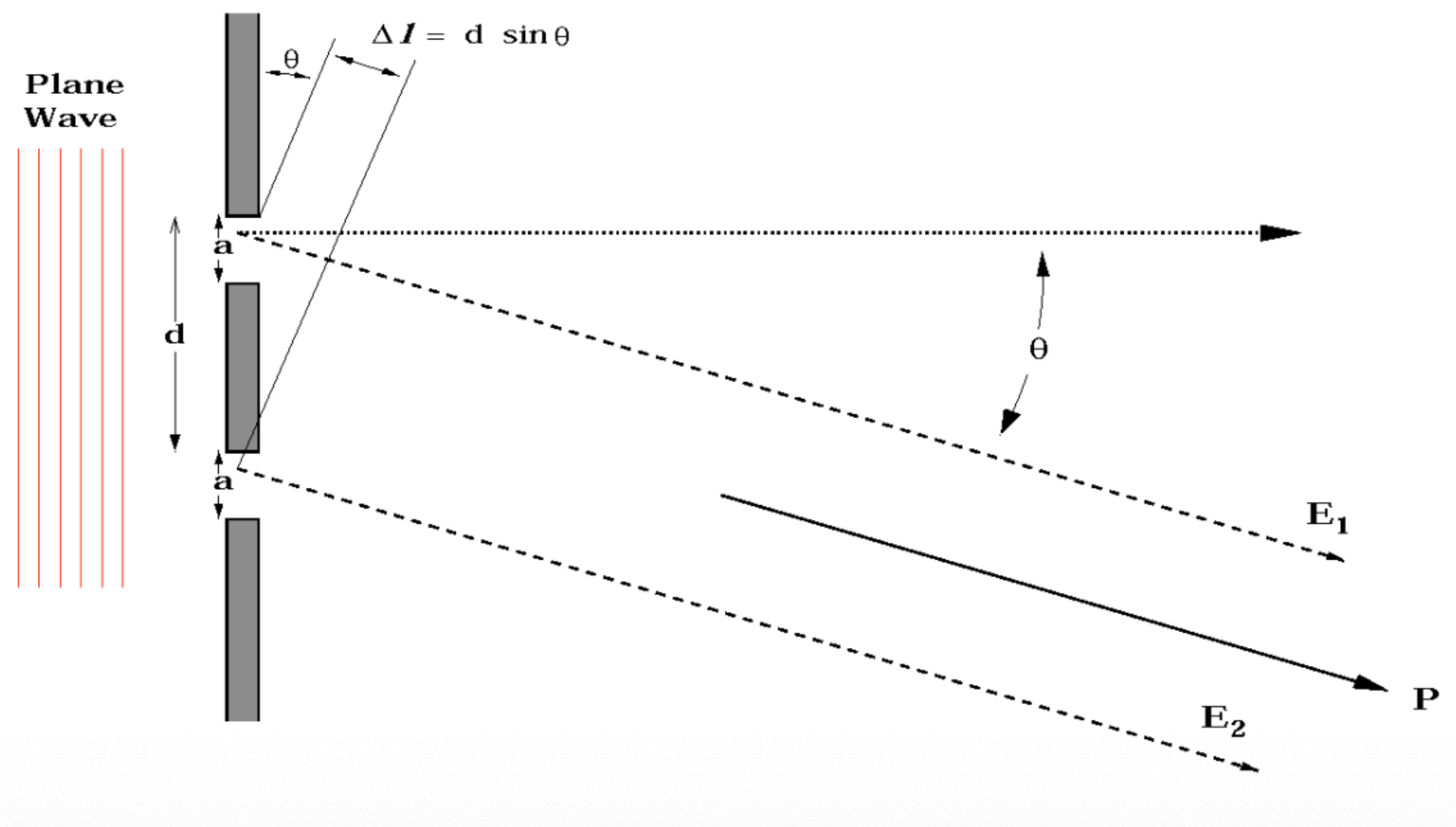


Figure 5: Phase Difference BetweenWaves Coming from each Slit

It is seen that the wave front emanating from one slit lags behind the other by $\Delta\ell = d \sin\theta$.

This gives a relative phase difference of $2\pi\,\Delta\ell/\lambda = 2\pi d\sin\theta/\lambda$.

The fields due to each slit are,

$$E_1 = E_0\sin(\omega t + 2\pi(\ell + \Delta\ell)/\lambda) \quad \textit{and} \quad E_2 = E_0\sin(\omega t + 2\pi\ell/\lambda)$$

Since we are interested only in relative phase differences, for algebraic simplicity we can write these as

$$E_1 = E_0\sin(\omega t + \pi\Delta\ell/\lambda) \quad \textit{and} \quad E_2 = E_0\sin(\omega t - \pi\Delta\ell/\lambda)$$

Using the trigonometric identity,

$$\sin A + \sin B = 2\cos\frac{A-B}{2}\sin\frac{A+B}{2}$$

gives,

$$\begin{aligned} E_T &= E_1 + E_2 = E_0\sin(\omega t + \pi\Delta\ell/\lambda) + E_0\sin(\omega t - \pi\Delta\ell/\lambda) \\ &= 2E_0\cos\frac{\pi\Delta\ell}{\lambda}\sin(\omega t) = 2E_0\cos\frac{\pi d\sin\theta}{\lambda}\sin(\omega t) \end{aligned} \tag{02}$$

which when squared and averaged over time produces Equation (1), i.e.,

$$\begin{aligned} I &= |E_T|^2 = 4|E_0|^2\cos^2\frac{\pi d\sin\theta}{\lambda}\left\langle\sin^2(\omega t)\right\rangle = 2|E_0|^2\cos^2\frac{\pi d\sin\theta}{\lambda} \\ &= I(0)\cos^2\beta \qquad , \quad \beta = \frac{\pi d}{\lambda}\sin\theta \end{aligned}$$

Maximum intensity peaks occur at angles such that the path difference taken by waves from each slit to the distant screen is a integral multiple of wavelengths $\Delta\ell = d\ \sin\theta = m\lambda$ (see Figure 4).

This translates to $\beta = m\pi$ ($m = \pm1, \pm2, \ldots$.) and thus the angles of maximum intensity satisfy,

$$\sin\theta = m\frac{\lambda}{d} \quad , \quad m = \pm1, \pm2, \ldots..$$

If the width of the slits is greater than a wavelength the interference fringes are seen to oscillate in brightness as seen in Figure 6.

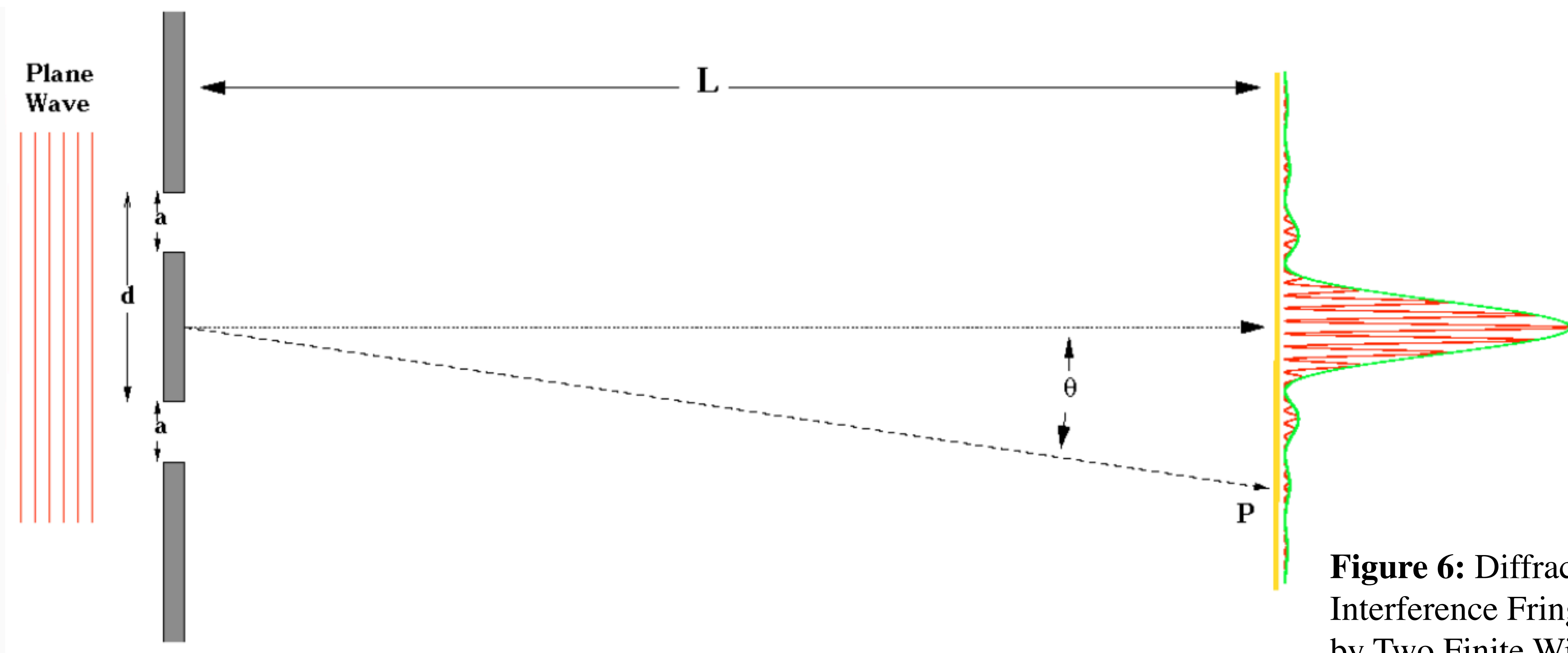


Figure 6: Diffraction Interference Fringes generated by Two Finite Width Slits

This is due to diffraction.

Diffraction is interference of one wave with itself.

According to Huygen's Principle waves propagate such that each point reached by a wavefront acts as a new point source.

The sum of the secondary waves emitted from all points on the wavefront propagate the wave forward.

Interference between secondary waves emitted from different parts of the wave front can cause waves to bend around corners and cause intensity fluctuations much like interference patterns from separate sources.

Diffraction will be discussed in more detail later.

As we will derive later, the amplitude of the field of a diffraction pattern from one slit is given by,

$$E(\theta) = E(0)\left(\frac{\sin\alpha}{\alpha}\right)\sin(\omega t) \quad , \quad \alpha = \frac{\pi a}{\lambda}\sin\theta \tag{03}$$

Just as the two waves were added above (without the diffraction term) the relative phase difference of each wave is considered, the two waves are added and squared resulting in,

$$I(\theta) = I(0)\left(\frac{\sin\alpha}{\alpha}\right)^2 \cos^2\beta \quad , \quad \alpha = \frac{\pi a}{\lambda}\sin\theta \quad , \quad \beta = \frac{\pi d}{\lambda}\sin\theta \tag{04}$$

The diffraction term

$$\left(\frac{\sin\alpha}{\alpha}\right)^2$$

is of the form of a sinc function squared, where

$$\operatorname{sinc}\alpha = \frac{\sin\alpha}{\alpha}$$

A graph of the sinc function is shown in Figure 7.

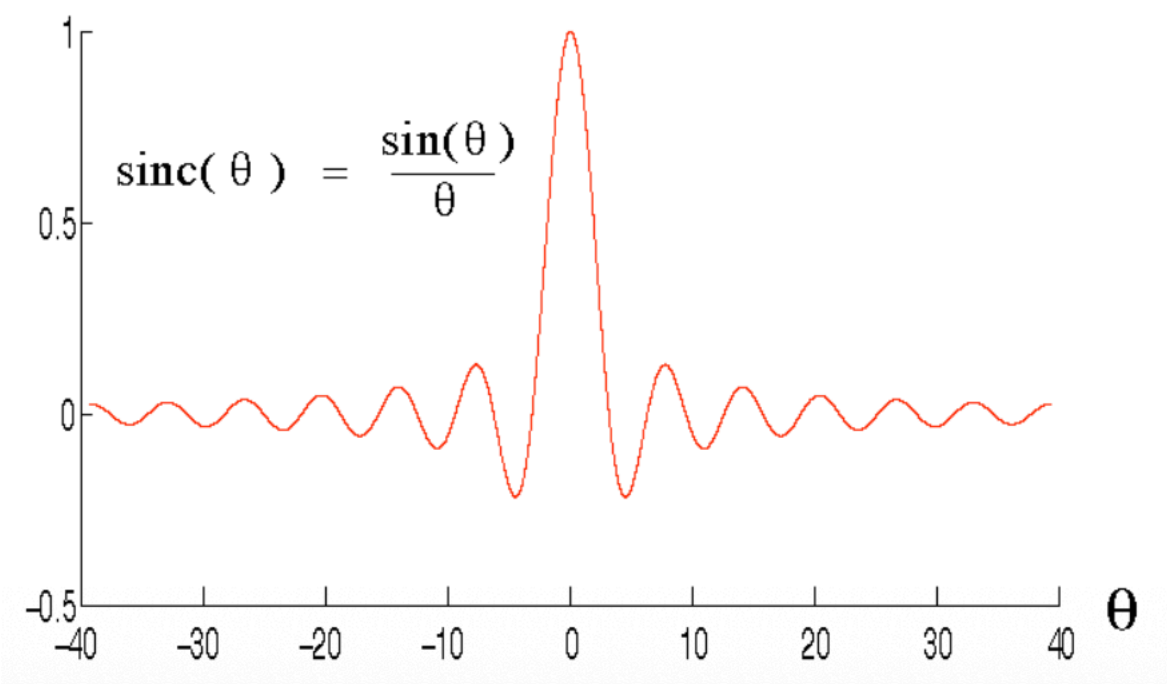


Figure 7: sinc Function

Note that if the slit width a is less than λ then α is always less than one and diffraction causes little change in intensity with angle.

In this case Equation (01) becomes a good approximation.

Figure 8 shows how the separate terms of Equation (04) influence the final intensity pattern for a particular set of slits.

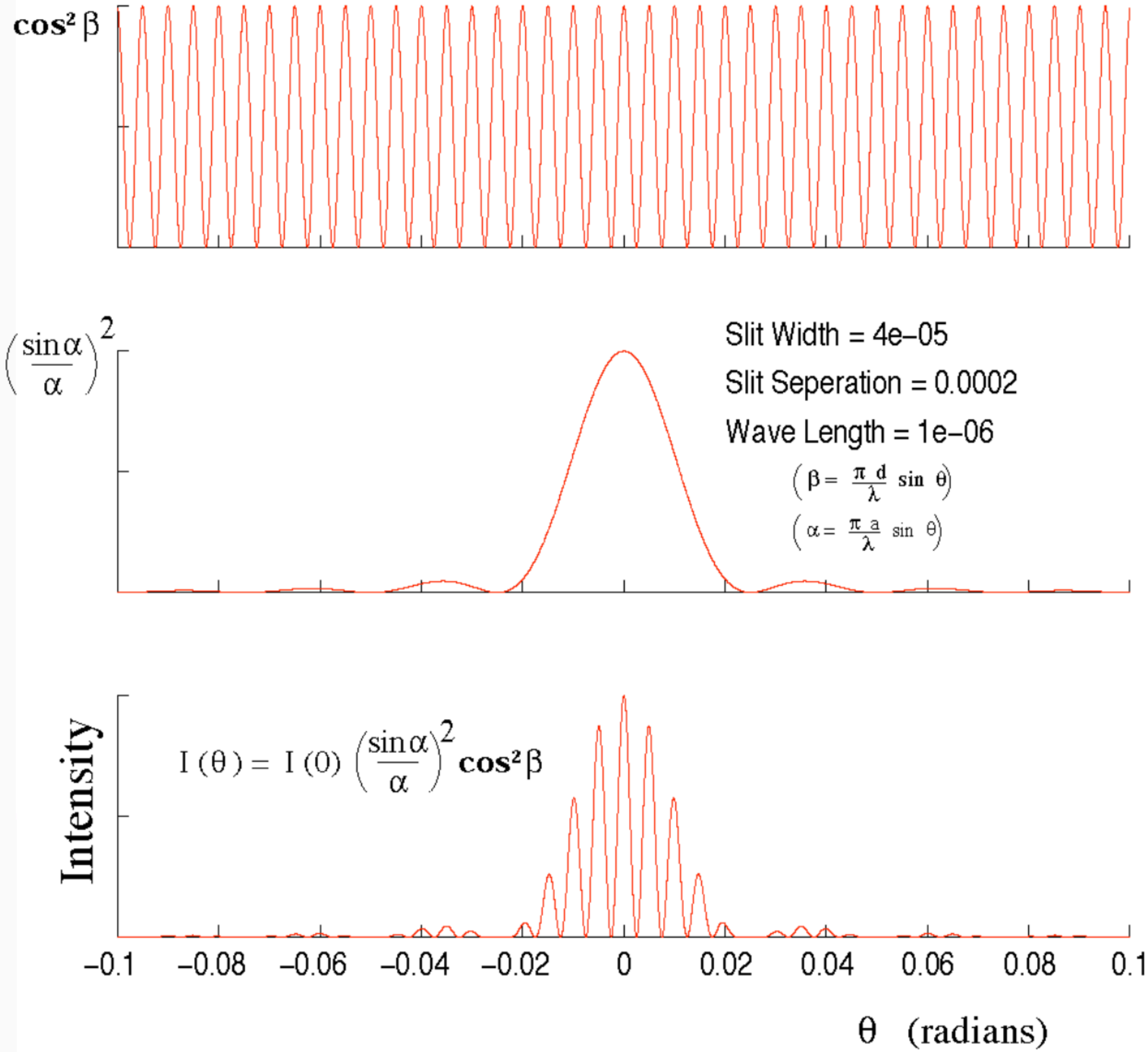


Figure 8: Two Slit Interference/Diffraction

Diffraction Gratings

A system consisting of a large number equally spaced parallel slits is called a grating.

The diffraction pattern expected from a grating can be calculated in a way similar to the two slit system.

A simple way to do this calculation is to represent the amplitude and phase of each wave originating from one of the grating slits and impinging on a point on the screen by a vector called a **phasor**.

Digression to discuss phasors

What happens when coherent light comes through more than two slits, all equally spaced a distance d apart, in a line parallel to the incoming wave fronts?

The same criterion still holds for completely constructive interference (what we will now refer to as the PRINCIPAL MAXIMA) but we no longer have a simple criterion for destructive interference: each successive slit's contribution cancels out that of the adjacent slit, but if there are an odd number of slits, there is still one left over and the combined amplitude is not zero.

Does this mean there are no angles where the intensity goes to zero?

Not at all; but it is not quite so simple to locate them.

One way of making this calculation easier to visualize (albeit in a rather abstract way) is with the geometrical aid of PHASORS:

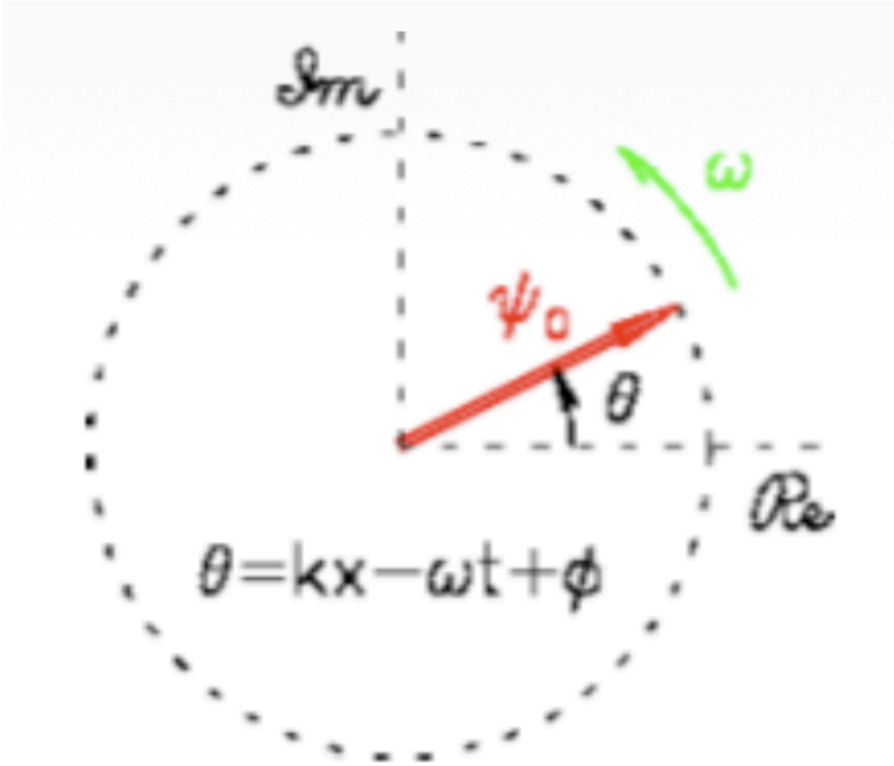


Figure 9 - A single PHASOR of length Ψ_0 (the wave amplitude) precessing at a frequency ω in the complex plane.

A single wave can be expressed as $\psi(x,t) = \psi_0 e^{i\theta}$ where $\theta = kx - \omega t + \phi$ is the **phase** of the wave at a fixed position x at a given time t.

ϕ is the **initial** phase at x=0 and t=0.

Although it is usually ignored, we retain it for completeness.

If we focus our attention on one particular location in space, this single wave's **displacement** ψ at that location can be represented geometrically as a vector of length ψ_0 (the wave amplitude) in the complex plane called a **PHASOR**.

As time passes, the **direction** of the phasor rotates at an angular frequency ω in that abstract plane.

There is not much advantage to this geometrical description for a single wave but when one goes to **add together** two or more waves with different phases, it helps a lot!

For example, two waves of equal amplitude but different phases can be added together algebraically as for BEATS:

$$\begin{aligned}\bar{\psi} &= \bar{\psi}_0 e^{i\bar{\theta}} = \psi_0\left[e^{i\theta_1} + e^{i\theta_2}\right] = \psi_0 e^{i(\theta_1+\theta_2)/2}\left[e^{i(\theta_1-\theta_2)/2} + e^{-i(\theta_1-\theta_2)/2}\right] \\ &= 2\psi_0 e^{i\bar{\theta}} \cos\left((\theta_1-\theta_2)/2\right)\end{aligned} \tag{05}$$

where

$$\bar{\psi}_0 = 2\psi_0 \cos\left(\delta/2\right) \quad , \quad \bar{\theta} = (\theta_1+\theta_2)/2 \quad , \quad \delta = \theta_1 - \theta_2 \tag{06}$$

That is, the combined amplitude ψ_0 can be obtained by adding the phasors **tip-to-tail** like ordinary vectors.

Like the original components, the whole thing continues to precess in the complex plane at the common frequency ω.

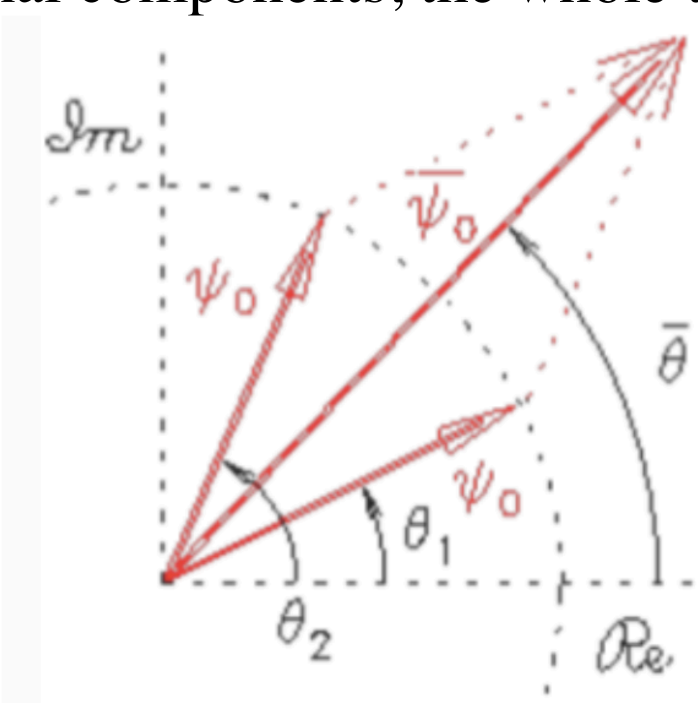


Figure 10 - Two waves of equal amplitude ψ_0 but different phases θ1 and θ2 are represented as PHASORS in the complex plane. Their vector sum has the resultant amplitude and the average phase .

We are now ready to use **PHASORS** to find the amplitude of an arbitrary number of waves of arbitrary amplitudes and phases but a common frequency and wavelength interfering at a given position.

This is illustrated in Figure 11 for 5 phasors.

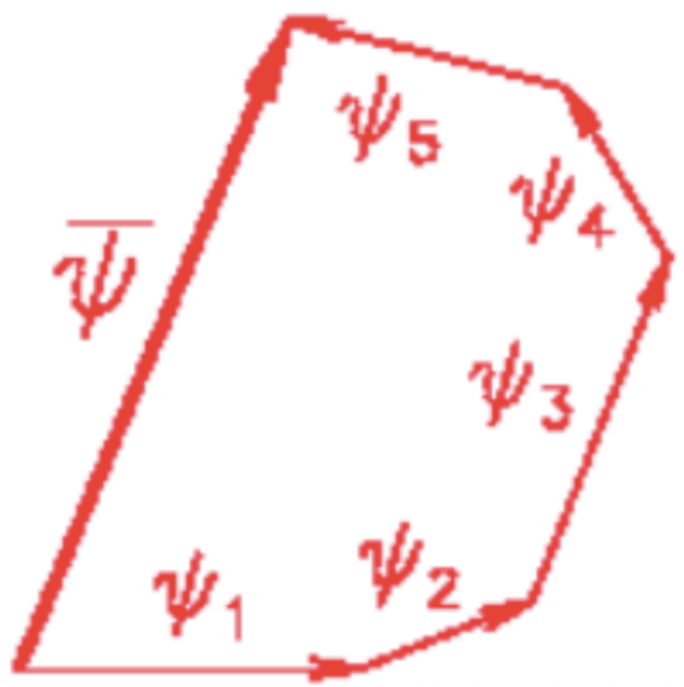


Figure 11: The net amplitude of a wave produced by the interference of an arbitrary number of other waves of the same frequency of arbitrary amplitudes and phases can in principle be calculated geometrically by tip-to-tail vector addition of the individual **PHASORS** in the complex plane.

In practice, we rarely attempt such an arbitrary calculation, since it cannot be simplified algebraically.

Instead, we concentrate on simple combinations of waves of equal amplitude with well-defined phase differences, such as those produced by a regular array of parallel slits with an equal spacing between adjacent slits.

The result is obtained by analysis of a geometrical construction like that illustrated below for 7 slits, each of which contributes a wave of amplitude a , with a phase difference of δ between adjacent slits.

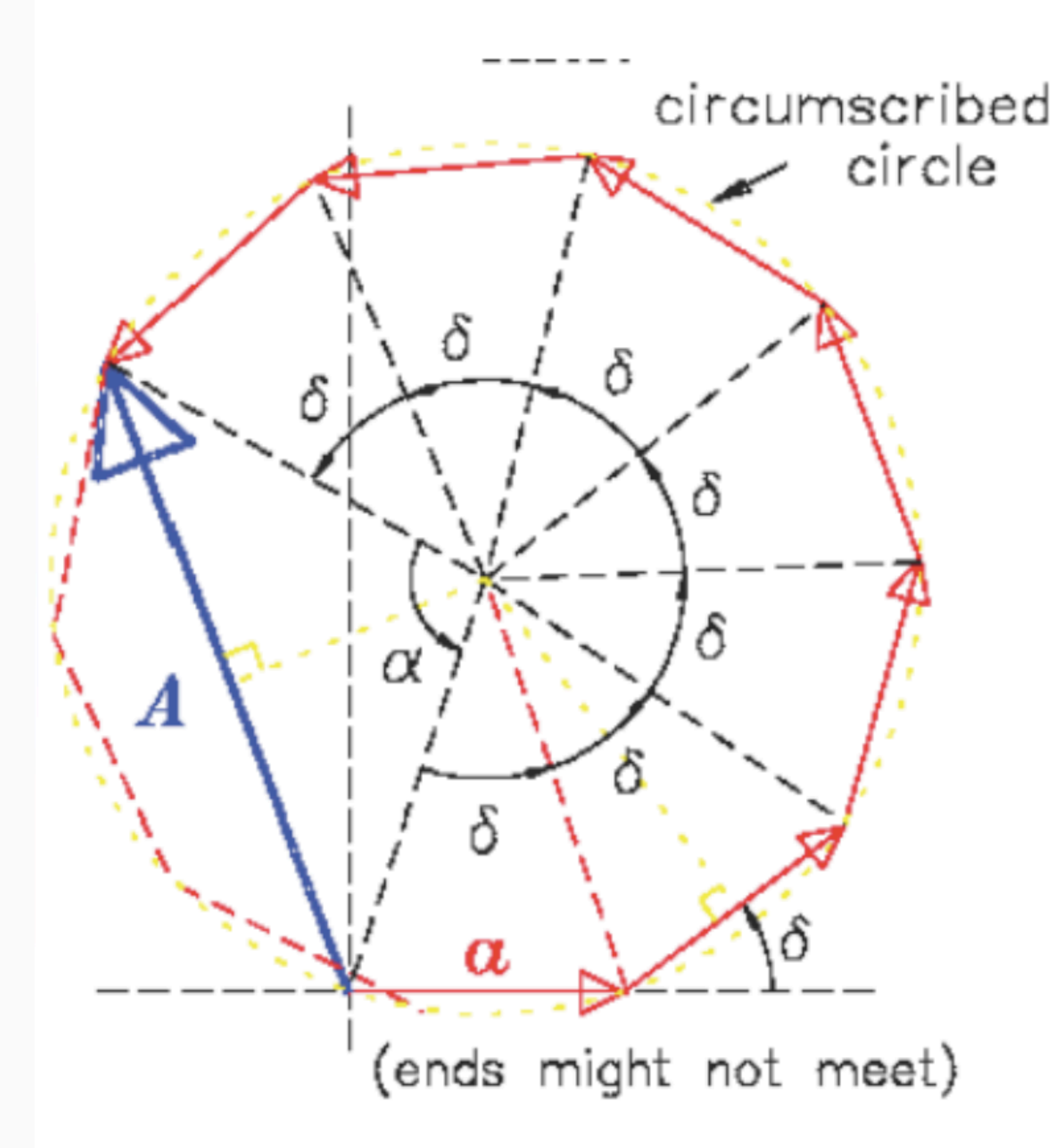


Figure 12: PHASOR DIAGRAM for calculating the intensity pattern produced by the interference of coherent light passing through 7 parallel, equally spaced slits.

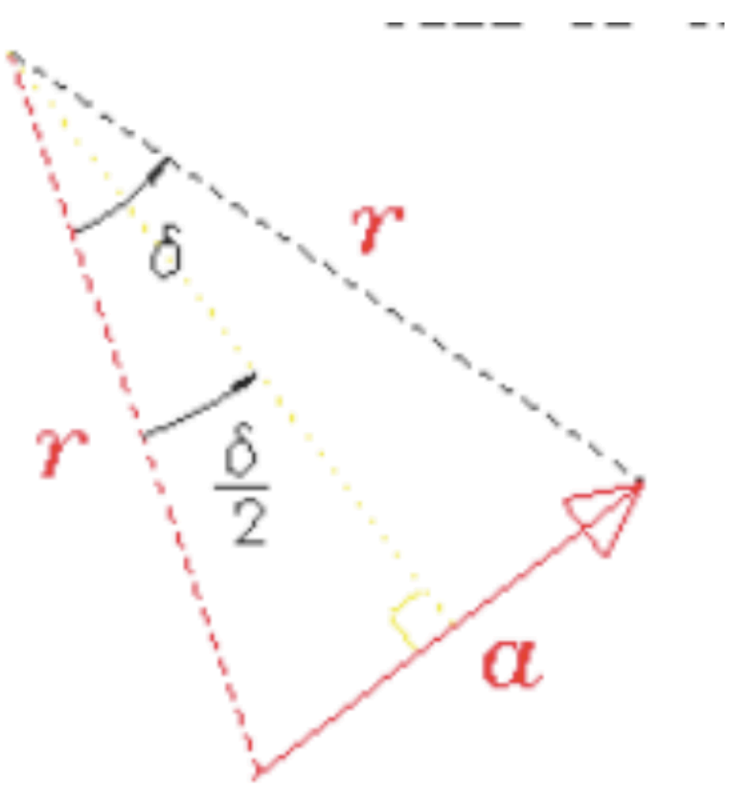


Figure 13: Blowup of one of the isosceles triangles formed by a single phasor and two radii from the center of the circumscribed circle to the tip and tail of the phasor.

After adding all 7 equal-length phasors above **tip-to-tail**, we can draw a vector from the starting point to the tip of the final phasor.

This vector has a length A (the net amplitude) and makes a chord of the circumscribed circle, intercepting an angle

$$\alpha = 2\pi - N\delta \tag{07}$$

where in this case N=7. The radius r of the circumscribed circle is given by

$$\frac{a}{2} = r\sin\frac{\delta}{2} \tag{08}$$

as can be seen from the blowup; this can be combined with the analogous

$$\frac{A}{2} = r\sin\frac{\alpha}{2} \tag{09}$$

to give the net amplitude

$$A = a\frac{\sin\frac{\alpha}{2}}{\sin\frac{\delta}{2}} \tag{10}$$

From Eq. (07) we know that $\alpha/2 = \pi - N\delta/2$, and in general $\sin(\pi-\theta) = \sin\theta$, so

$$A = a\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}} \tag{11}$$

where

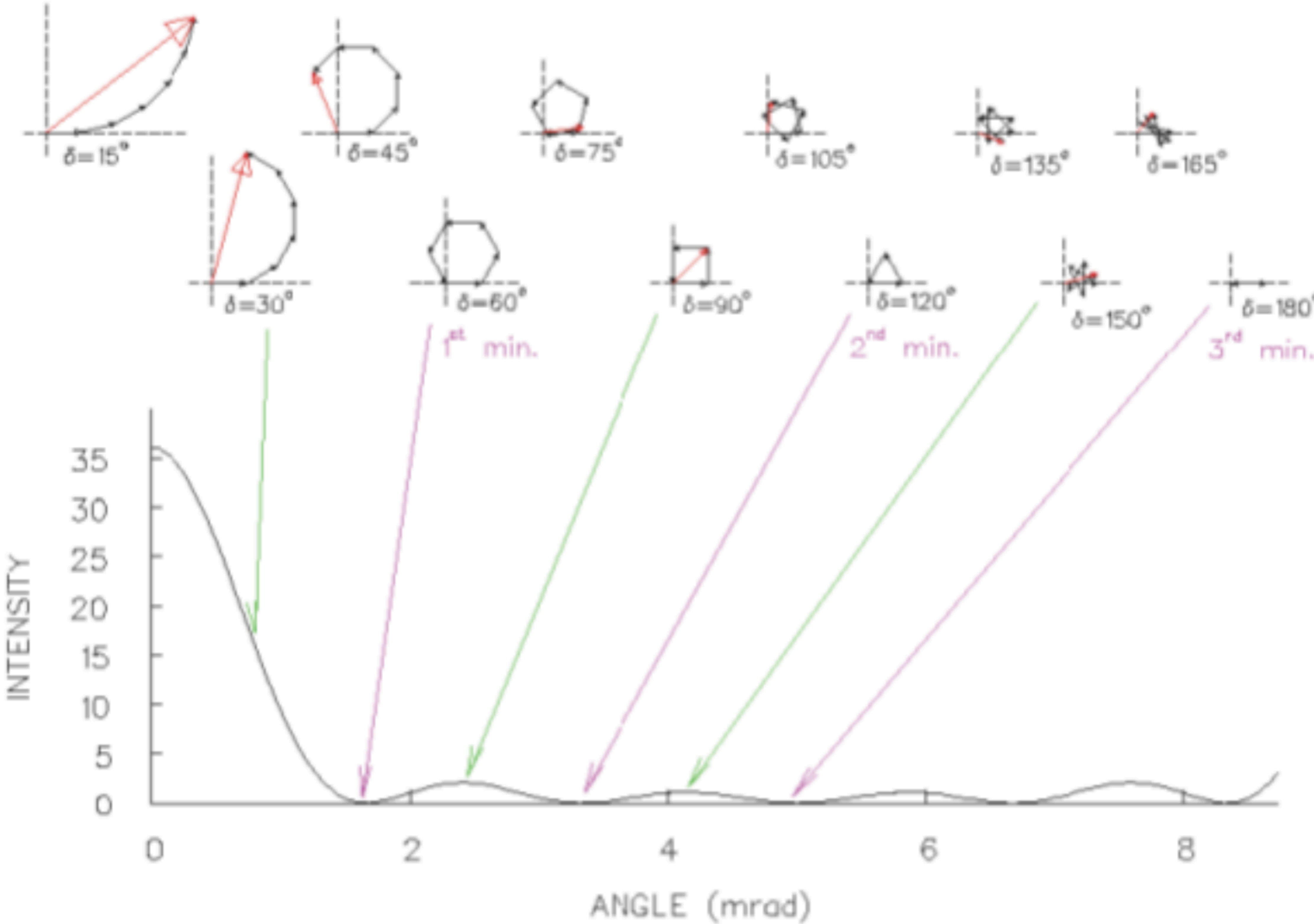
$$\delta = 2\pi\left(\frac{d}{\lambda}\right)\sin\theta \qquad \text{(from earlier discussion))} \tag{12}$$

Although the drawing shows N=7 phasors, this result is valid for an arbitrary number N of equally spaced and evenly illuminated slits.

The figure below shows an example using 6 identical slits with a spacing d=100λ.

The angular width of the interference pattern from such widely spaced slits is quite narrow, only 10 mrad (10^{-2} radians) between principal maxima where all 6 rays are in phase.

In between the principal maxima there are 5 minima and 4 secondary maxima; this can be generalized:



The interference pattern for N equally spaces slits exhibits (N-1) minima and (N-2) secondary maxima between each pair of principal maxima.

We can understand this analytically from examining Eq. (11) with calculus: the extrema (maxima and minima) occur at the physical angles θ where dA/dθ = 0.

That is, where

$$\frac{\pi a d}{\lambda \sin\frac{\delta}{2}}\left[N\tan\frac{\delta}{2}-\tan\frac{N\delta}{2}\right]\cos\theta=0$$

which is satisfied **trivially** at the **CENTRAL MAXIMUM** (θ=0) and otherwise where

$$N\tan\frac{\delta}{2}=\tan\frac{N\delta}{2} \tag{13}$$

The **almost trivial** solution is when δ/2 is a multiple of π (or δ is a multiple of 2π), making both tangents zero.

This corresponds to the PRINCIPAL MAXIMA where, as expected from Eq. (12),

$$d\sin\theta_m = m\lambda \tag{14}$$

where m is called the “diffraction order”.

What about the minima?

From the above figure we can see that the first minimum occurs when $N\delta = 2\pi$, that is, when the phasor diagram closes on itself.

For large N, δ is a rather small angle and we can use the small-angle approximation $\tan(\delta/2) \approx \delta/2$ giving $N\tan(\delta/2) = \pi = N\delta/2$ to satisfy Eq. (13).

Combined with Eq. (12) this predicts that the first minimum occurs at a laboratory angle

$$\theta\left(1^{st}\ \mathrm{min}\right) = \frac{\lambda}{N} \tag{15}$$

For large N and/or d **>>** λ, this means

> The interference pattern for N equally spaced slits has its first minimum at an angle $\theta(1_{st}\ \mathrm{min})$ inversely proportional to N.

In figure 13 above the intensity pattern is produced by the interference of coherent light passing through six parallel slits 100 wavelengths apart.

PHASOR DIAGRAMS are shown for selected angles.

Note that, while the phase angle difference δ between rays from adjacent slits is a monotonically increasing function of the angle θ (plotted horizontally) that the rays make with the **forward** direction, the latter is a real geometrical angle in space while the former is a pure abstraction in **phase space**.

The exact relationship is $\delta/2\pi = (d/\lambda)\sin\theta \approx (d/\lambda)\theta$ for very small θ.

Note the symmetry about the 3rd minimum at $\theta \approx 5$ mrad.

At $\theta \approx 10$ mrad the intensity is back up to the same value it had in the central maximum at $\theta = 0$; this is called the first **PRINCIPAL MAXIMUM**.

Then the whole pattern repeats ....

Returning to the earlier discussion ....

As we have seen, summing the phasors corresponding to each slit gives the correct amplitude and phase of the total wave.

The intensity distribution of the light scattered off a grating with N slits is,

$$I(\theta) = \frac{I(0)}{N^2}\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \tag{16}$$

This relation is very similar to the one for the diffraction of two slits (α and β defined in Equations (1) and (3)).

In fact if N = 2, then substituting the identity

$$\sin 2\beta = 2 \sin\beta \cos\beta$$

in the last term of Equation (16) gives,

$$\left(\frac{\sin 2\beta}{\sin\beta}\right)^2 = cos^2\beta$$

and it is seen to reduce to the two slit formula Equation (04).

The interference maxima are described by the function

$$\frac{\sin N\beta}{\sin\beta}$$

With increasing number of slits N this function has a rapidly varying numerator and a slowly varying denominator.

The effect of each term on the final pattern is shown in Figure 14 for a particular grating.

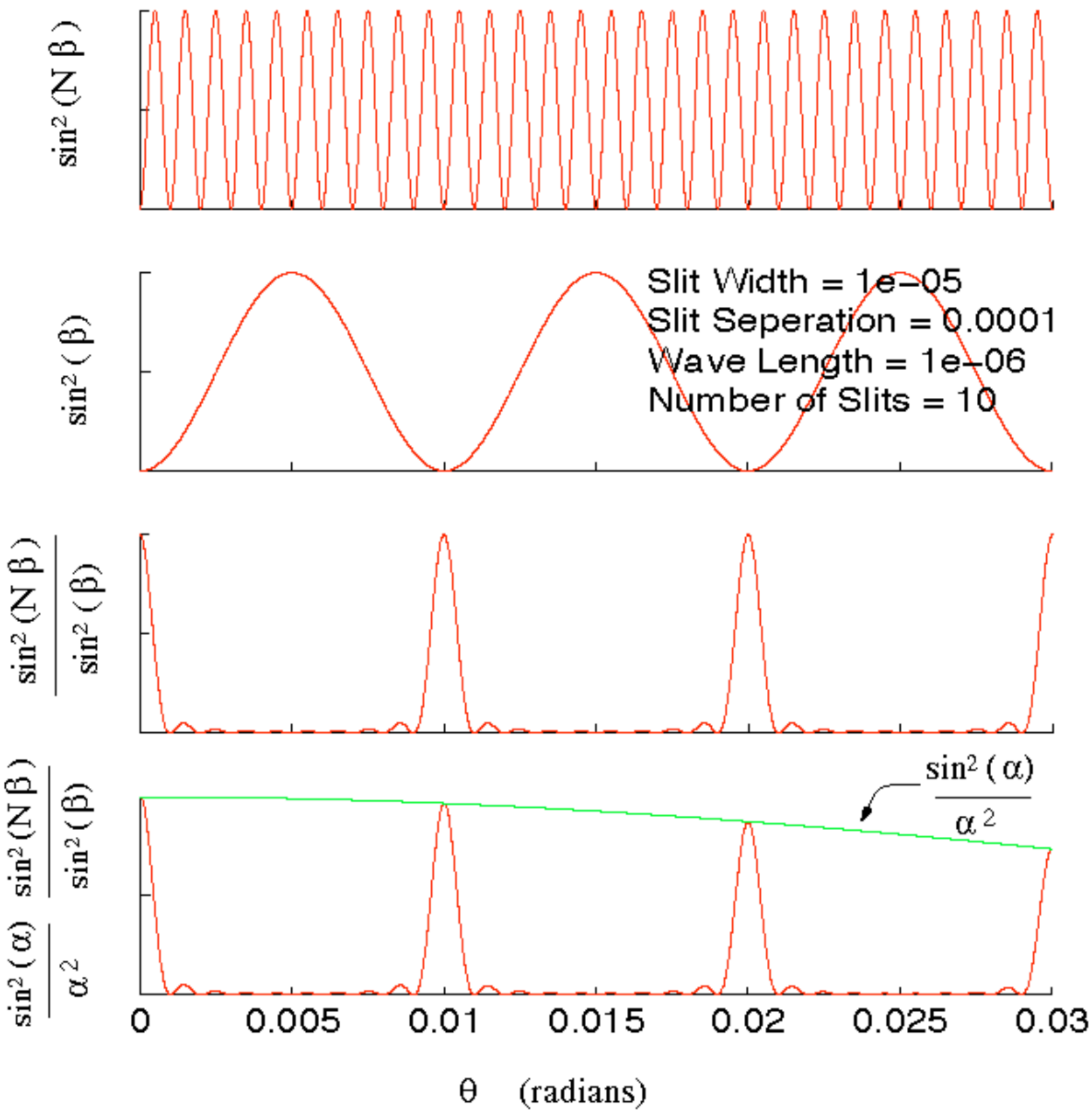


Figure 14: Interference Fringes from a Diffraction Grating

As the number of illuminated slits N increases, the intensity peaks will narrow and brighten.

The position of the fringe maxima generated by a grating is given by,

Fringe Maxima

$$\sin\theta = m\frac{\lambda}{d} \quad , \quad m = \pm 1, \pm 2, \ldots\ldots.. \tag{17}$$

where d is the grating constant (the distance between the grating slits).

Gratings are commonly used to measure the wavelength of light.

This may determined using Equation (17) and measurements of the angle between the maxima.

Diffraction and Huygens' Principle

Difference between interference and diffraction

What is the difference between an interference pattern and a diffraction pattern?

None, really. For historical reasons, the amplitude or intensity pattern produced by superposing contributions from a finite number of discrete coherent sources is usually called an interference pattern.

The amplitude or intensity pattern produced by superposing the contributions from a "continuous" distribution of coherent sources is usually called a diffraction pattern.

Thus one speaks of the interference pattern from two narrow slits, or the diffraction pattern from one wide slit, or the combined interference and diffraction pattern from two wide slits.

How an opaque screen works

All electromagnetic radiation originates from oscillating charged particles.

The total electric (and magnetic) field at any given point is a superposition of the waves produced by all the sources, i.e., all the oscillating charges.

In this case, one of the sources is the distant point source(S) that produces the plane wave incident on the screen.

Behind the opaque screen the total wave amplitude is zero (what is meant by an opaque screen).

This total wave is a superposition of the wave from S and the waves emitted by oscillating electrons in the material of the screen.

That is, the screen doesn't just gobble up the incident wave from S.

Its electrons are driven by the incident radiation (and also by the radiation emitted by the other electrons of the screen) and the superposition of all of the waves, i.e., from S and from all the electrons, gives zero behind the screen.

All electromagnetic fields come from charged particles, and such a "zero" field as that behind an opaque screen is the result of a superposition.

Shiny and black opaque screens

There are two extremes in the varieties of opaque screen.

At one extreme we can have a shiny opaque screen (such as an opaque piece of aluminum foil).

The electrons in the metal are driven by the local electric field; consequently they emit electromagnetic waves.

In the forward direction (the direction of the incident radiation), it turns out that the superposition of the incident wave and that from the electrons gives zero.

In the backward direction, it gives a reflected wave.

The motion of a given electron is entirely due to the elastic amplitude (harmonic oscillator), and thus the velocity is 90 degrees out of phase with the total electric field at its location; therefore no work is done on the electron during any complete cycle.

The electron "redirects" the radiation energy without permanently absorbing any energy. At the other extreme, we can have a black opaque screen such as black cardboard or a microscope slide painted with a layer of soot suspended in water.

Again the electrons are driven by the incident radiation.

They also suffer a resistive drag from the medium and are always at terminal velocity.

Their radiation in the forward direction is 180 degrees out of phase with the incident radiation and superposes with it to give zero (after sufficient thickness of screen).

The velocity of a given electron is always in phase with the total electric force at its location, and consequently net work is done on the electron.

The work done on the electron is transferred to the medium, which gets hotter.

There is no net rejected wave - the contributions from different layers of the screen superpose to give zero in the backward direction.

Effect of a hole in an opaque screen

Now cut a small hole (or slit) in the opaque screen.

First mark the material to be removed.

Called this slit number 1 - label the material to be removed plug 1.

The material of the screen above and below plug 1 is labeled a (above) and b (below).

The total field behind the screen, which is zero, is the superposition of the fields emitted by the source S and by the material from a, b, and plug 1.

Thus, before removing the material of slit 1, we have

$$E = 0 = E_S + E_a + E_b + E_1 \tag{18}$$

This situation is shown in figure 15.

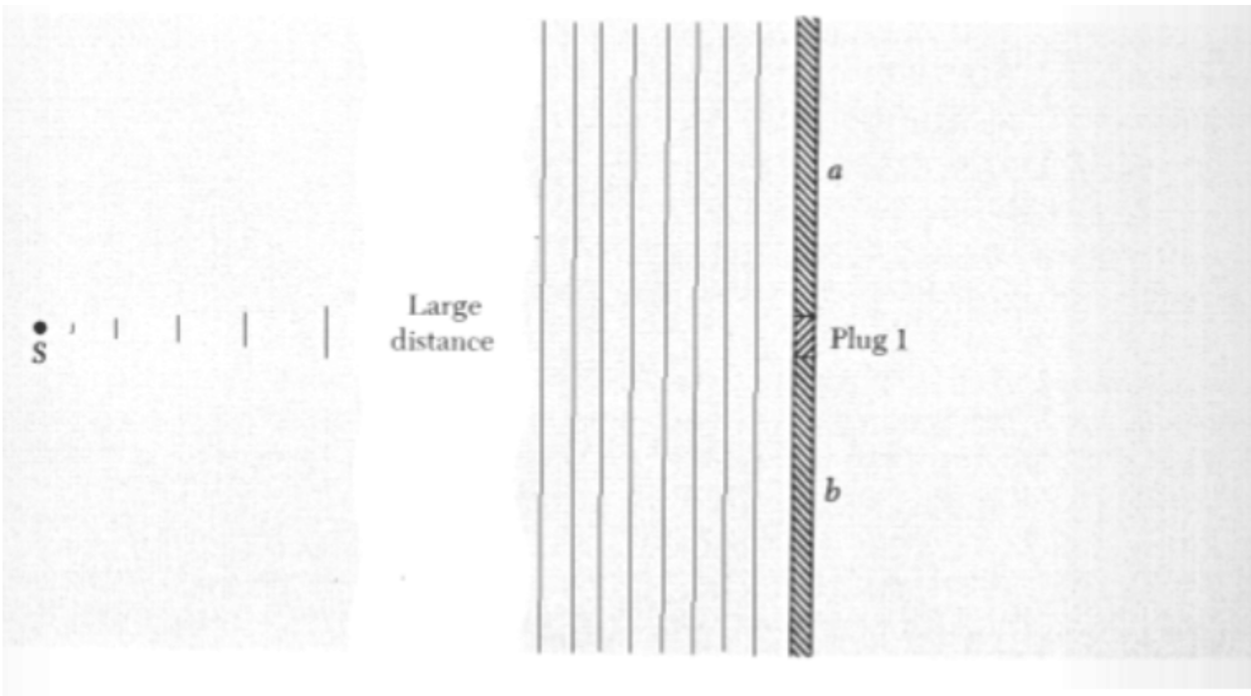


Figure 15: Planes waves from distant source S are incident on opaque screen. Superposition of fields due to charges at S, a, b and plug 1 give zero behind the screen.

Now remove the material that is plugging up slit 1.

Assume that the motion of the electrons in the regions a and b is not changed by removal of the plug.

This is an approximation.

With this assumption the total field behind the screen is no longer the superposition given by Eq. (18), which adds to zero.

Instead it is that superposition minus the contribution from plug 1:

$$E = E_S + E_a + E_b = \left(E_S + E_a + E_b + E_1\right) - E_1 \approx 0 - E_1 = -E_1 \tag{19}$$

We see that the remaining field, which is a superposition of contributions from the source S and the remaining material of the screen, a and b, is just the same (except for a minus sign) as that which was being emitted by the plug when it was in place.

Thus we can find the field behind the screen by imagining that we substitute for the source and the screen with slit the simpler system consisting of just the plug by itself, with no source S and no remaining screen, and with the electrons in the plug all oscillating with the same phase and amplitude, as they actually were doing when the plug was in place.

This gives us an easy way to calculate the interference patterns due to slits in an opaque screen.

It is easy because we make no attempt to know the variation of amplitude and phase constant of the oscillating electrons in the plug as functions of position along the beam direction.

We assume the field E_1 produced by the plug is due to an infinitesimally thin layer of oscillating chargers all oscillating in phase and with the same amplitude.

Huygens' principle

This calculation device is called Huygens' principle.

It can be used with any number of slits and also with a single broad slit.

Its basis is in Eqs. (18) and (19).

Now we assumed Ea and Eb were the same with the plug in place as without it.

This is only approximately true, as mentioned above.

If one has, for example, a single wide slit and uses Huygens' construction to calculate the fields to the right of the screen and in the slit itself, one finds the following:

If one is sufficiently far to the right of the screen and sufficiently near the forward direction, and if the screen is many wavelengths wide, then Huygens' construction gives a very good approximation to the right answer (as determined by experiment).

If one is in the vicinity of the slit itself, then Huygens' construction gives a very poor approximation to the right answer.

If you are in the slit, the most important moving charges in the remaining screen material are those nearest the edge of the slit, because they are closest.

But these are just the ones most affected by the removal of the plug.

The field pattern can be very complicated in the slit and especially near the edges of the slit, where the nearest oscillating charges dominate.

You may ask, "Why not just solve the problem exactly?"

This is very difficult.

You must use Maxwell's equations in all the vacuum regions and in the material, specifying the properties of the material precisely, and fitting everything together at the boundaries.

There are no general methods of finding solutions, and very few such problems have been solved exactly.

Calculation of single-slit diffraction pattern using Huygens' construction

We want to calculate the diffraction pattern produced when a plane wave, emitted by a distant point source, is incident on a slit.

Using Huygens' construction, we mentally replace the incident plane wave (or distant point source) and material of the screen by a slab of radiating material - the Huygens' plug.

Since we have a continuous distribution of oscillating charges across the slab, we should perform an integration (a superposition) over the contributions from infinitesimal elements of the slab.

Instead of an integration over a continuous distribution, we can (and will) consider a discrete sum over contributions from N identical equally spaced "antennas".

In the limit that N goes to infinity, we shall have a continuous distribution of radiating sources.

The advantage of using N discrete sources rather than a continuous distribution is that we thereby obtain at the same time the solution for the radiation pattern produced by N antennas or N narrow slits, for arbitrary N from N = 2 to infinity.

Let the total width of the single wide slit be D.

Then D is the width of the region that contains our linear array of N "Huygens' antennas".

Let the separation between adjacent antennas be d.

Then we have D = (N-1)d.

Suppose the incident plane wave is in the +z direction and the N slits are along x, as shown in figure 16.

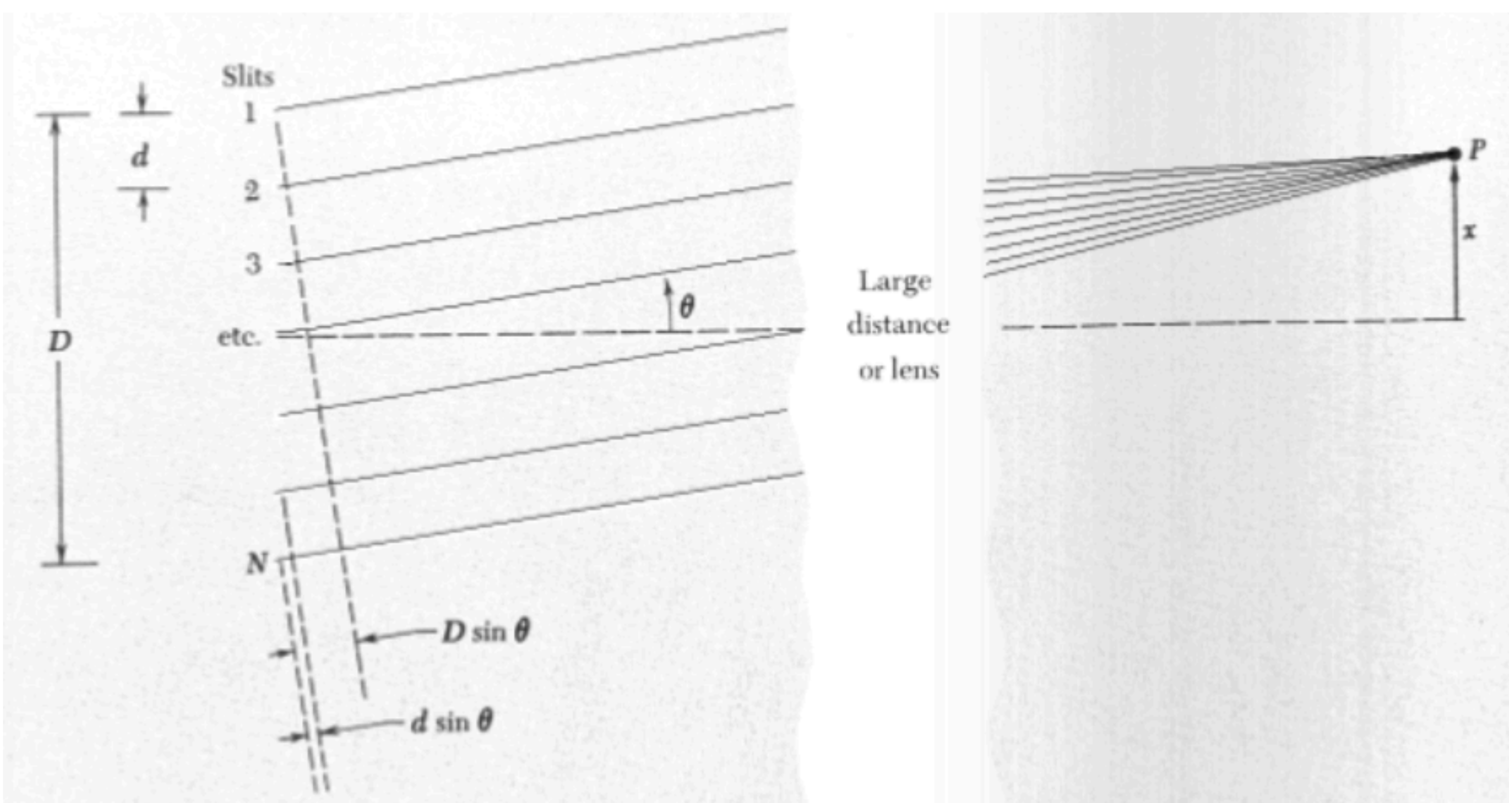


Figure 16: N antennas, N narrow slits, with charges all oscillating in phase

At a distant field point P, each antenna gives a contribution that has the same amplitude A(r) (because P is distant enough so that in the dependence of amplitude on distance we can assume the distance is approximately the same for all the antennas).

All the antennas oscillate in phase(hypothesis).

The electric field E at point P is therefore given by the superposition

$$E = A(r)\cos\left(kr_1 - \omega t\right) + A(r)\cos\left(kr_2 - \omega t\right) + \ldots\ldots\ldots + A(r)\cos\left(kr_N - \omega t\right) \tag{20}$$

We wish to re-express this superposition of N outgoing traveling waves by a single outgoing traveling wave propagating from the average position of the array and having an amplitude that is modulated as a function of emission angle.

We can simplify the algebra by using complex numbers.

The field E is the real part of the complex quantity E_c , where

$$E_c = A(r)e^{-i\omega t}\left(e^{ikr_1} + e^{ikr_2} + \ldots\ldots + e^{ikr_N}\right) \tag{21}$$

But according to figure 16,

$$\begin{aligned} r_2 &= r_1 + d\sin\theta \\ r_3 &= r_1 + 2d\sin\theta \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ r_N &= r_1 + (N-1)d\sin\theta \end{aligned} \tag{22}$$

Thus Eq. (21) becomes

$$E_c = A(r)e^{-i\omega t}e^{ikr_1}\left(1 + e^{ik(r_2 - r_1)} + \ldots\ldots + e^{ik(r_N - r_1)}\right) = A(r)e^{-i\omega t}e^{ikr_1}S \tag{23}$$

where

$$S = 1 + e^{ik(r_2 - r_1)} + \ldots\ldots + e^{ik(r_N - r_1)} = 1 + a + a^2 + \ldots\ldots + a^{N-1} \tag{24}$$

with

$$a = e^{ik(r_2 - r_1)} = e^{ikd\sin\theta} = e^{i\Delta\phi} \tag{25}$$

where

$$\Delta\phi = kd\sin\theta = \frac{2\pi}{\lambda} d\sin\theta \tag{26}$$

is the relative phase of the waves(at P) from neighboring antennas.

The geometric series S given by Eq. (24) satisfies the relation

$$S = \frac{a^N - 1}{a - 1} = \frac{e^{iN\Delta\phi} - 1}{e^{i\Delta\phi} - 1} = \frac{e^{iN\Delta\phi/2}}{e^{i\Delta\phi/2}} \frac{e^{iN\Delta\phi/2} - e^{-iN\Delta\phi/2}}{e^{i\Delta\phi/2} - e^{-i\Delta\phi/2}} = e^{i(N-1)\Delta\phi/2} \frac{\sin\frac{1}{2}N\Delta\phi}{\sin\frac{1}{2}\Delta\phi} \tag{27}$$

Then Eq. (23) becomes

$$\begin{aligned} E_c &= A(r)e^{-i\omega t}e^{ikr_1}e^{i(N-1)\Delta\phi/2}\frac{\sin\frac{1}{2}N\Delta\phi}{\sin\frac{1}{2}\Delta\phi} = A(r)e^{-i\omega t}e^{ik(r_1+(N-1)d\sin\theta/2)}\frac{\sin\frac{1}{2}N\Delta\phi}{\sin\frac{1}{2}\Delta\phi} \\ &= A(r)e^{-i\omega t}e^{ikr}\frac{\sin\frac{1}{2}N\Delta\phi}{\sin\frac{1}{2}\Delta\phi} \end{aligned} \tag{28}$$

where the quantity

$$r = r_1 + (N-1)d\sin\theta / 2 \tag{29}$$

gives the distance from P to the center of the array.

Taking the real part of Eq. (28), we obtain for the field at P

$$E(r,\theta,t) = \left[A(r) \frac{\sin \frac{1}{2} N\Delta\phi}{\sin \frac{1}{2} \Delta\phi} \right] \cos(kr - \omega t) = A(r,\theta)\cos(kr - \omega t) \tag{30}$$

Let us check that Eq. (30) gives the correct result for N = 2.

Using the identity sin2x = 2sinxcosx with x = $\Delta\phi/2$ we get

$$\begin{aligned} E(r,\theta,t) &= \left[A(r) \frac{\sin \frac{1}{2} 2\Delta\phi}{\sin \frac{1}{2} \Delta\phi} \right] \cos(kr - \omega t) = \left[A(r) \frac{2\sin \frac{1}{2} \Delta\phi \cos \frac{1}{2} \Delta\phi}{\sin \frac{1}{2} \Delta\phi} \right] \cos(kr - \omega t) \\ &= \left[2A(r) \cos \tfrac{1}{2} \Delta\phi \right] \cos(kr - \omega t) \end{aligned}$$

which agrees with our earlier result.

Single-slit diffraction pattern

In this case, we let N go to infinity while holding D constant.

The spacing d goes to zero.

The relative phase shift $\Delta\phi$ between the waves contributed by adjacent antennas goes to zero.

The total phase shift Φ between the contributions of the first and N_{th} antennas at P is exactly $(N-1)\Delta\phi$.

This is approximately $N\Delta\phi$ for N huge:

$$\Phi = (N-1)\Delta\phi = kD\sin\theta \tag{31}$$

$$\Phi \approx N\Delta\phi \quad , \quad N \gg 1 \tag{32}$$

Thus the modulated amplitude in Eq. (30) becomes

$$A(r,\theta) = A(r)\frac{\sin\frac{1}{2}N\Delta\phi}{\sin\frac{1}{2}\Delta\phi} \approx A(r)\frac{\sin\frac{1}{2}\Phi}{\sin\frac{1}{2}(\Phi/N)} \tag{33}$$

In the limit that N is sufficiently large, we can neglect all except the first term of the Taylor's series for sin(Φ/2N) in Eq. (33):

$$\sin\tfrac{1}{2}(\Phi/N) \approx \tfrac{1}{2}(\Phi/N) \tag{34}$$

$$A(r,\theta) = NA(r)\frac{\sin\frac{1}{2}\Phi}{\frac{1}{2}\Phi} \tag{35}$$

We can make one further simplification.

As N goes to infinity, we must let A(r) go to zero in such a way that NA(r) is constant, because we want the same contribution for a given infinitesimal element dx of the continuous array no matter how many antennas it contains. (Remember, we are using the antennas in a Huygens' construction).

We can eliminate specific reference to N and A(r) in Eq. (35) by noticing that as θ goes to zero,Φ goes to zero, and the ratio

$$\frac{\sin\frac{1}{2}\Phi}{\frac{1}{2}\Phi}$$

goes to unity.

Thus, A(r,0) equals NA(r) times unity, according to Eq. (35).

Finally we have

$$E(r,\theta,t) = A(r,0)\frac{\sin\frac{1}{2}\Phi}{\frac{1}{2}\Phi}\cos\left(kr - \omega t\right) \tag{36}$$

with

$$\Phi = \frac{2\pi}{\lambda} D\sin\theta \tag{37}$$

The time-averaged energy flux has angular dependence(fixed r)

$$I(r,\theta) = I_{\max}\frac{\sin^2\frac{1}{2}\Phi}{\left(\frac{1}{2}\Phi\right)^2} \tag{38}$$

as is easily seen from Eq. (36).

The amplitude and intensity patterns of Eqs. (36) and (38) are plotted in figure 17.

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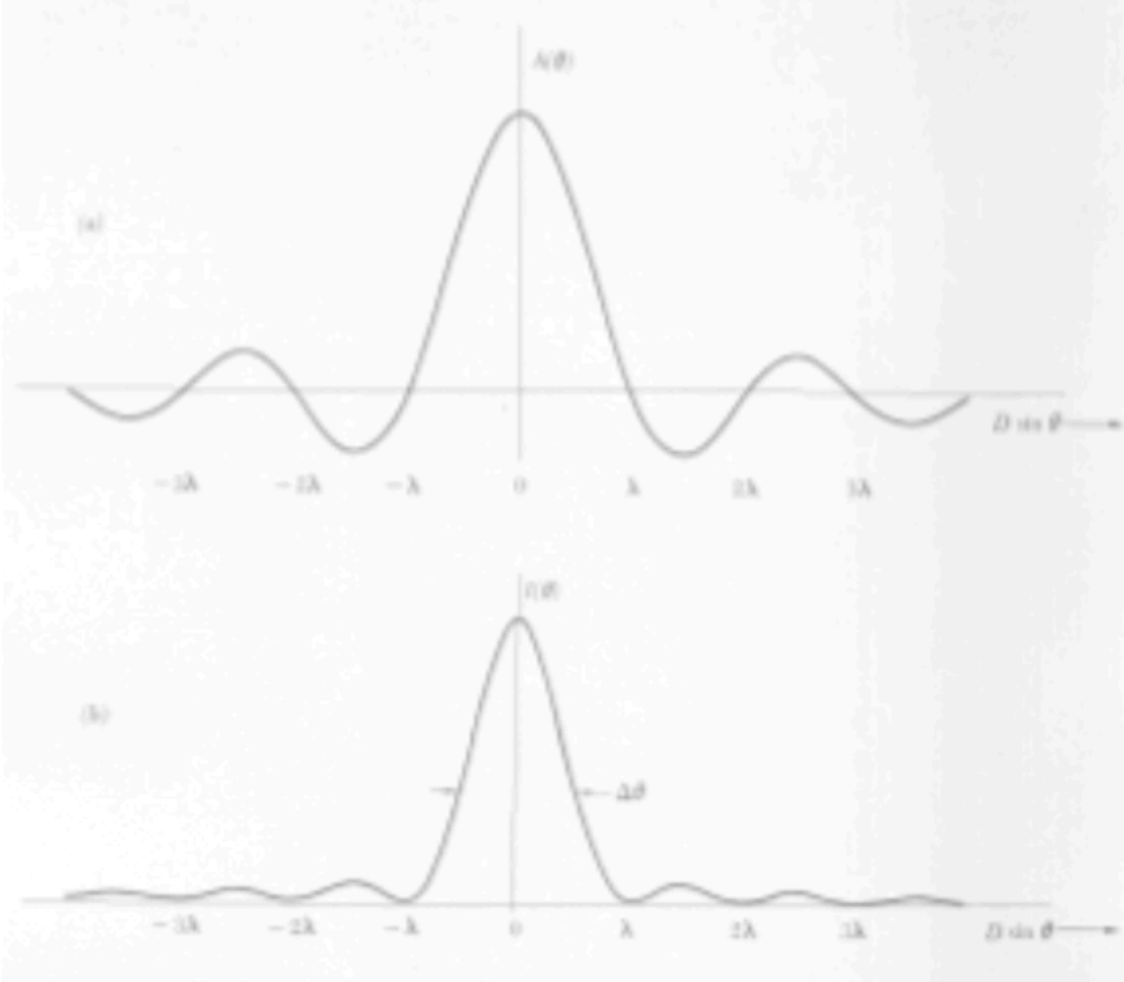


Figure 17: Single-slit diffraction pattern. (a) Amplitude. (b) Intensity. The angular band $\Delta\theta$ extending from $-\lambda/2D$ to $+\lambda/2D$ corresponds approximately (for small angles) to the "full width at half intensity". The intensity is down by a $(2/\pi)2=0.41$, rather than 0.5.

DIFFRACTION - Reviewing/expanding the ideas we have developed

Diffraction is interference of a wave with itself.

According to Huygen's Principle waves propagate such that each point reached by a wavefront acts as a new wave source.

The sum of the secondary waves emitted from all points on the wavefront propagate the wave forward.

Interference between secondary waves emitted from different parts of the wave front can cause waves to bend around corners and cause intensity fluctuations much like interference patterns from separate sources.

Some of these ideas were touched in the previous notes on interference.

The intensity distributions of monochromatic light diffracted from the described objects are based on:

a) the Superposition Principle
b) the wave nature of light
   Disturbance: $A = A_0 \sin(\omega t + \phi)$
   Intensity: $I = \left(\sum A\right)^2$
c) Huygen's Principle -- Light propagates in such a way that each point reached by the wave acts as a point source of a new light wave. The superposition of all these waves represents the propagation of the light wave.

All calculations are based on the assumption that the distance L between the slit and the viewing screen is much larger than the slit width a:, i.e. L >> a.

This particular case is called Fraunhofer scattering.

The calculations of this type of scattering are much simpler than the Fresnel scattering in which case the L >> a constraint is removed.

THEORY

A narrow slit of infinite length and width a is illuminated by a plane wave (laser beam) as as shown in Figure 18.

The intensity distribution observed (on a screen) at an angle α with respect to the incident direction is given by equation (39).

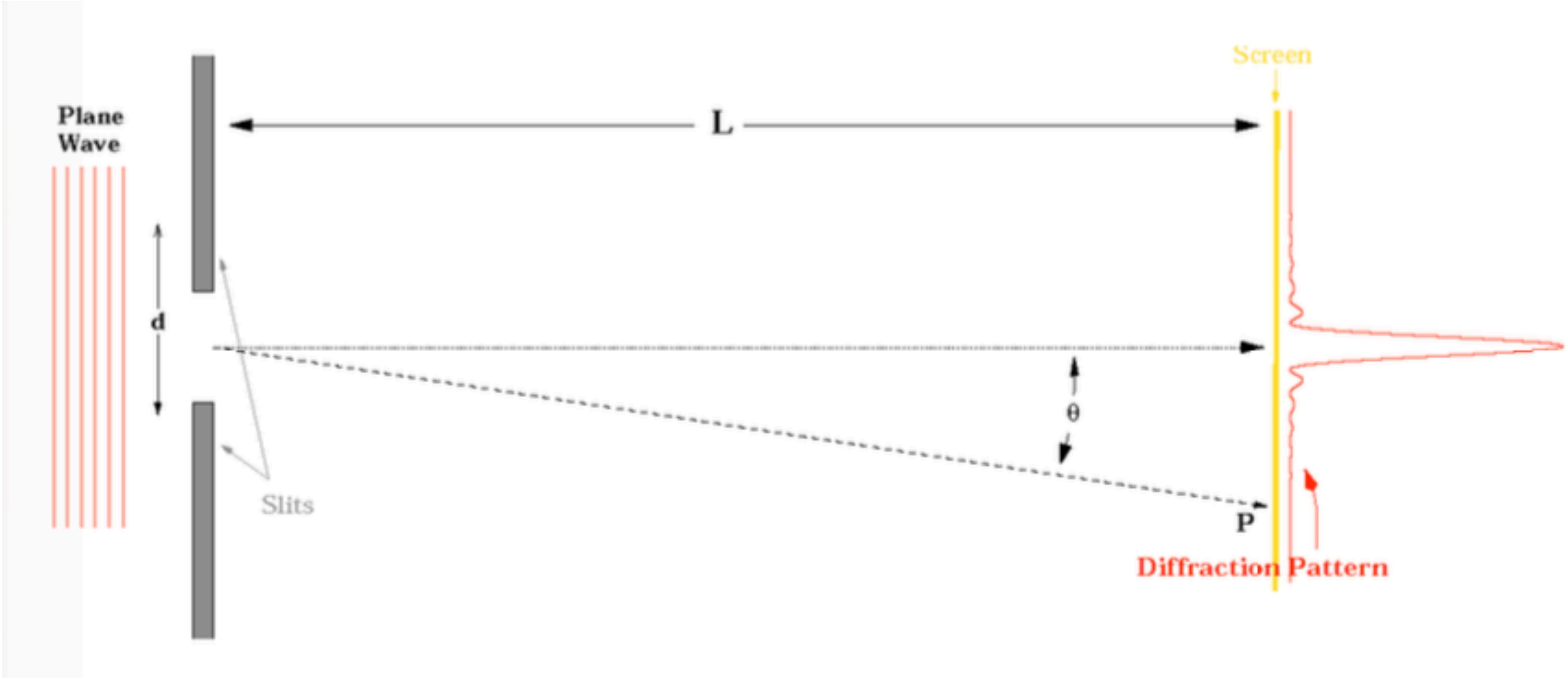


Figure 18: Intensity Profile of the Diffraction Pattern Resulting from a Plane Wave Passing Through a Single Narrow Slit

$$\text{Diffraction Single Slit:}\quad \frac{I(\theta)}{I(0)}=\frac{\sin^2\alpha}{\alpha^2}\quad,\quad \alpha=\frac{\pi a}{\lambda}\sin\theta \tag{39}$$

where λ = wavelength of incident plane wave; a = slit width; θ = angle of observation (wrt incident direction); I(θ) = intensity in direction of observation; I(0) = maximum intensity of diffraction pattern (central fringe).

Let us derive this equation in detail.

This relation is mathematics used to calculate this relation are very simple.

The contributions from the field at each small area of the slit to the field at a point on the screen are added together by integration.

Squaring this result and disregarding sinusoidal fluctuations in time gives the intensity.

The main difficulty in the calculation is determining the relative phase of each small contribution.

Figure Figure 19 shows the quantities used to calculate the diffraction from a slit of width a.

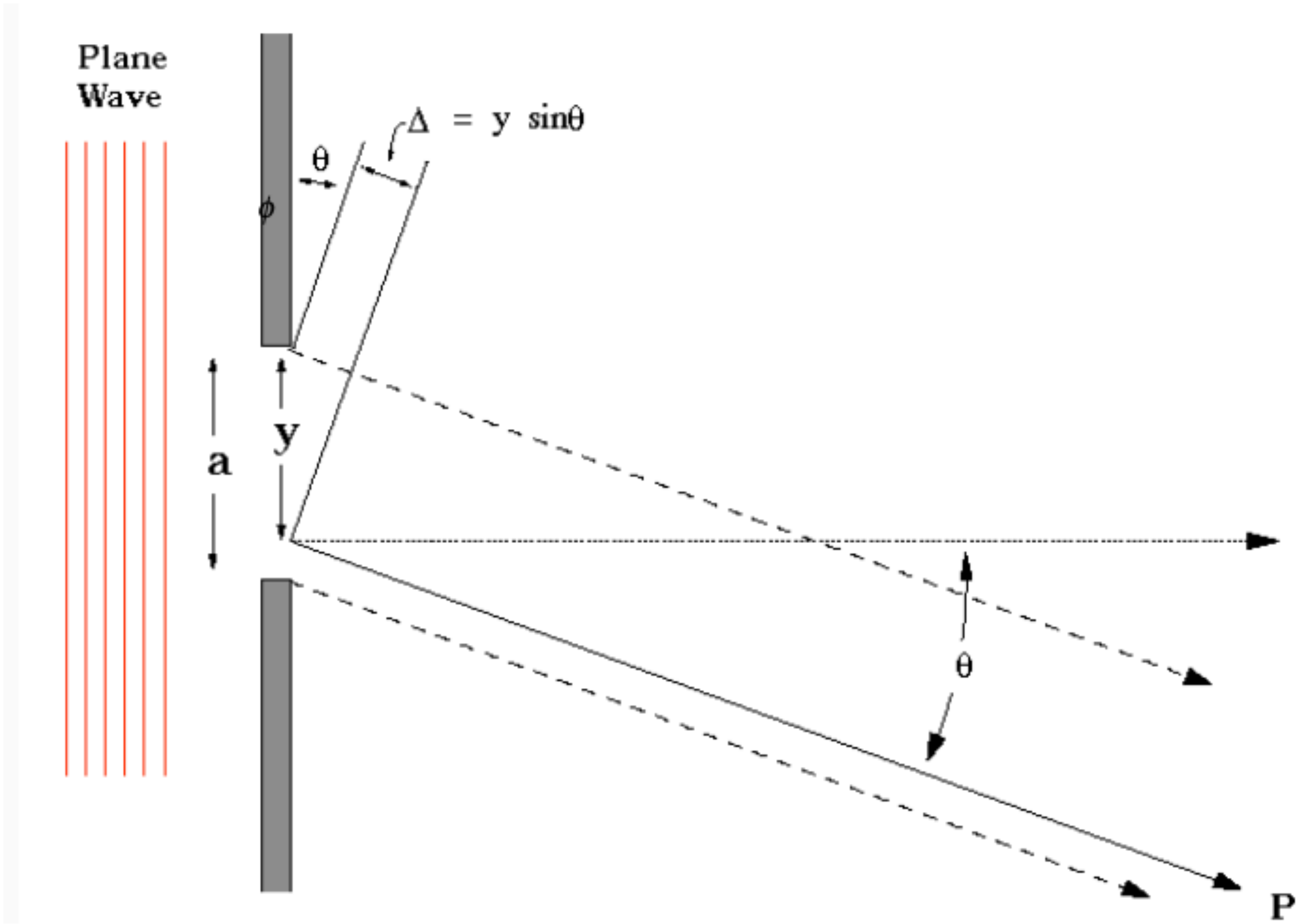


Figure 19: Diffraction by a Single Slit

Each point of the slit along the y-direction will generate its own wavelet (Huygen's principle) — and each of these wavelets will have the same wavelength as the incident plane wave.

The amplitude of a wavelet generated at point y and reaching point P will be

$$E = E_0 \sin(\omega t + \phi)$$

with

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

The phase ϕ can be arrived at by using the path difference:

$$\frac{phase\ difference}{2\pi} = \frac{\phi}{2\pi} = \frac{path\ difference}{\lambda} = \frac{\Delta}{\lambda} = \frac{y\sin\theta}{\lambda}$$

Consequently

$$\phi = \frac{2\pi}{\lambda} y \sin\theta$$

At point P, the amplitude of each wavelet generated by any point y along the slit will be

$$E(y) = E_0 \sin\left(\omega t + \frac{2\pi}{\lambda} y \sin\theta\right) \tag{40}$$

The total amplitude at P will be (according to the superposition principle) the sum of the contribution of each point along y.

This is obtained by integrating along y:

$$E_{TOT} = \int_0^a E(y)dy \tag{41}$$

This integral is straightforward, and is given by

$$E_{TOT} \propto \frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a}{\lambda}\sin\theta}$$

We are interested in the light intensity $I = (E_{TOT})2$ rather than the amplitude E_{TOT} ,

Neglecting constants, which relate the intensity at P to the intensity of the incident wave, we obtain

$$I(\theta) = I(0)\frac{\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\left(\frac{\pi a}{\lambda}\sin\theta\right)^2} \tag{42}$$

with

$$\alpha = \frac{\pi a}{\lambda}\sin\theta$$

This relation is found in the literature as

$$\text{Single Slit Diffraction:} \quad I(\theta) = I(0)\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad , \quad \alpha = \frac{\pi a}{\lambda}\sin\theta = 0 \tag{43}$$

I(0) is the intensity of the central fringe ($\theta = 0$).

It is always the maximum intensity of the diffraction pattern.

This relation will have a value of zero each time that $\sin^2\alpha = 0$.

This occurs when

$$\text{Single Slit Minima:} \qquad \sin\theta = m\frac{\lambda}{a} \quad , \quad m = \pm 1, \pm 2, \pm 3, \ldots \tag{44}$$

where m = order of diffraction.

The shape of the distribution given by (42) is shown in figure Figure 20.

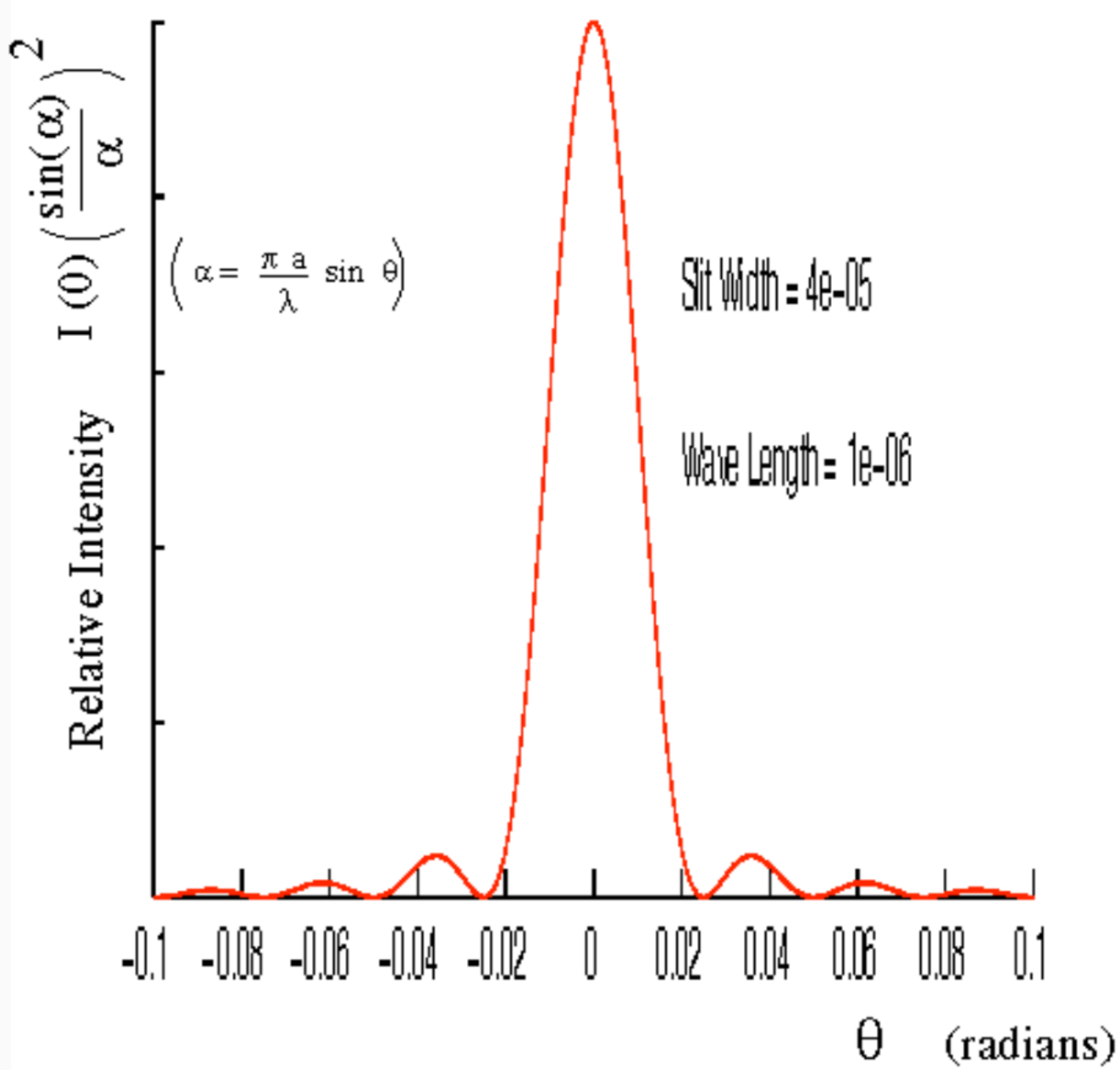


Figure 20: Intensity Profile of Single Slit Diffraction Pattern

This relation is satisfied for integer values of m.

Increasing values of m give minima at correspondingly larger angles.

The first minimum will be found for m = 1, the second for m = 2 and so forth.

If $a \sin\theta / \lambda$ is less than one for all values of θ, there will be no minima, i.e., when the size of the aperture is smaller than a wavelength($a < \lambda$).

This indicates that diffraction is strongly caused by perturbations with sizes that are about the same dimension as a wavelength.

Diffraction by a Circular Aperture

THEORY

For a circular hole of diameter d the diffraction pattern consists of concentric rings, which are analogous to the bands which we obtained for the slit.

The pattern for this intensity distribution can be calculated in the same way as for the single slit, but because the aperture here is circular, it is more convenient to use cylindrical coordinates . (z,ρ,θ)

The superposition principle requires us to integrate over a disk, and the result is a Bessel function as shown below.

The intensity of the Frauenhofer diffraction pattern of a circular aperture (Airy disk) is (derived later)

$$I(\theta) = I(0)\left(\frac{2J_1\left(\frac{2\pi d}{\lambda}\sin\theta\right)}{\frac{2\pi d}{\lambda}\sin\theta}\right)^2 = I(0))\left(\frac{2J_1(x)}{x}\right)^2 \quad , \quad x = \frac{2\pi d}{\lambda}\sin\theta$$

where θ is the angle between the line through the center of the circle and the line from the center to the observation point.

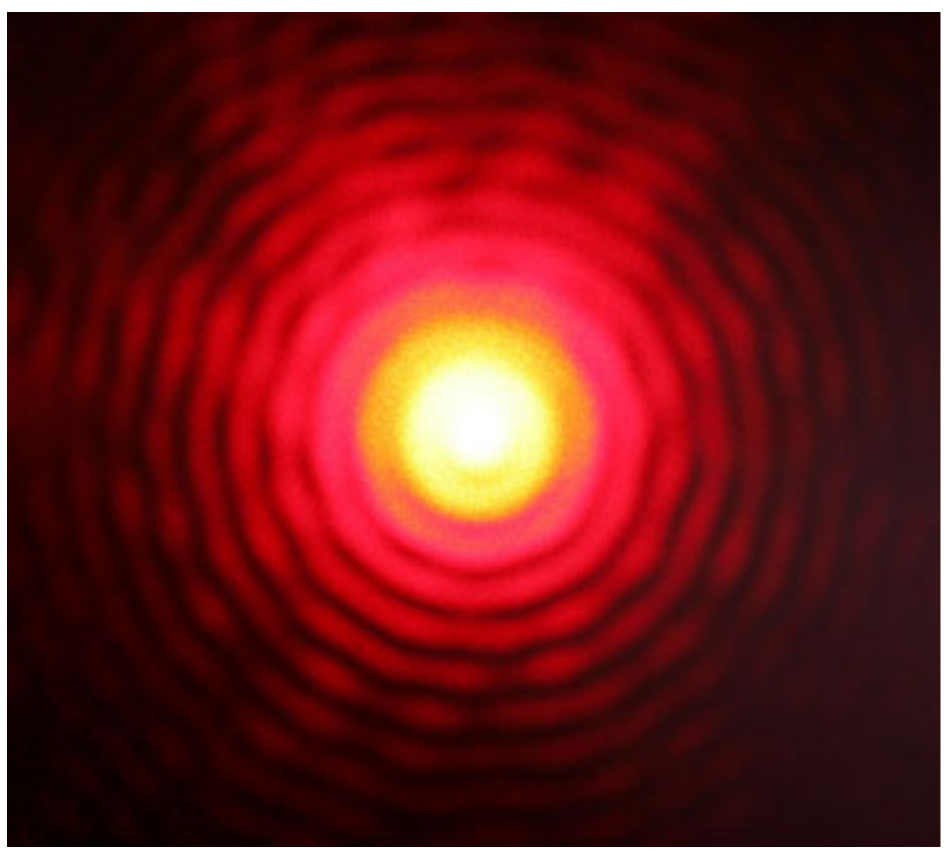
The condition for observing a minimum of intensity is found from the zeroes of the Bessel function $J_1(x)$.

The zeros are at

$$x = 0, 3.8317, 7.0156, 10.1735, 13.3268, 16.4706, \ldots$$

The first dark ring in the diffraction pattern occurs where

$$\sin\theta = \frac{3.83}{\frac{2\pi d}{\lambda}} = \frac{3.83}{2\pi}\frac{\lambda}{d} = 1.22\frac{\lambda}{d}$$



Diffraction by a circular aperature

Light as Matter or Waves: The Poisson Spot

THEORY

In 1818 Fresnel entered a competition sponsored by the French Academy.

His paper was on the theory of diffraction.

He showed that if light is to be described as a wave phenomenon, then a bright spot would be visible at the center of the shadow of a circular opaque obstacle, a result which he felt proved the absurdity of the wave theory of light.

This surprising prediction, fashioned by Poisson as the death blow to wave theory, was almost immediately verified experimentally by Dominique Argo.

The spot actually existed.

The experimental setup in Figure 21 can easily settle a controversy that preoccupied the minds of the brightest philosophers and scientists for centuries.

Is light described as a stream of particles or as a wave phenomena?

Or in other words, does the Poisson spot exist?

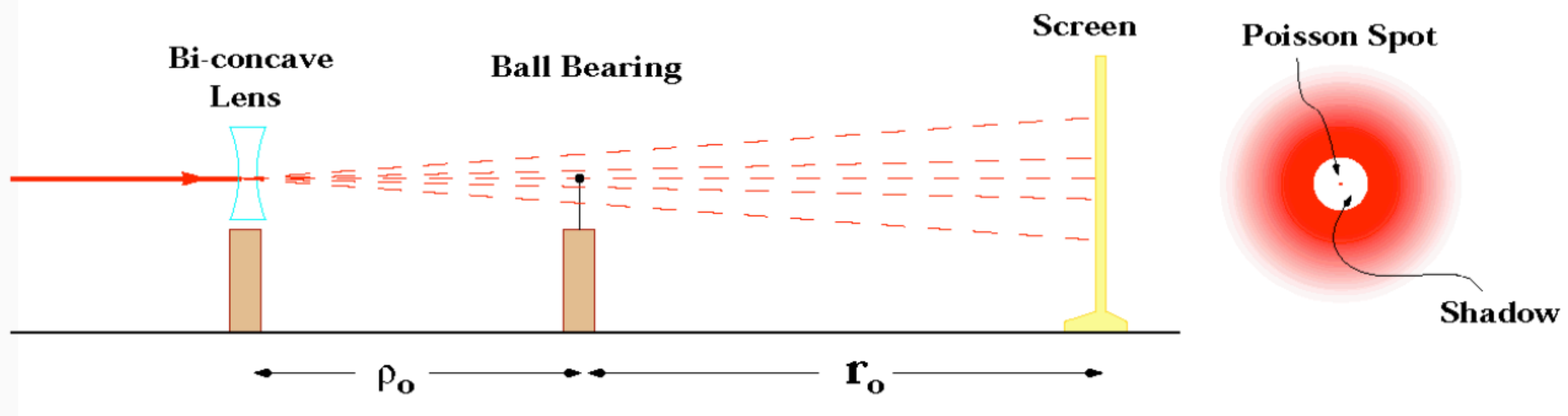
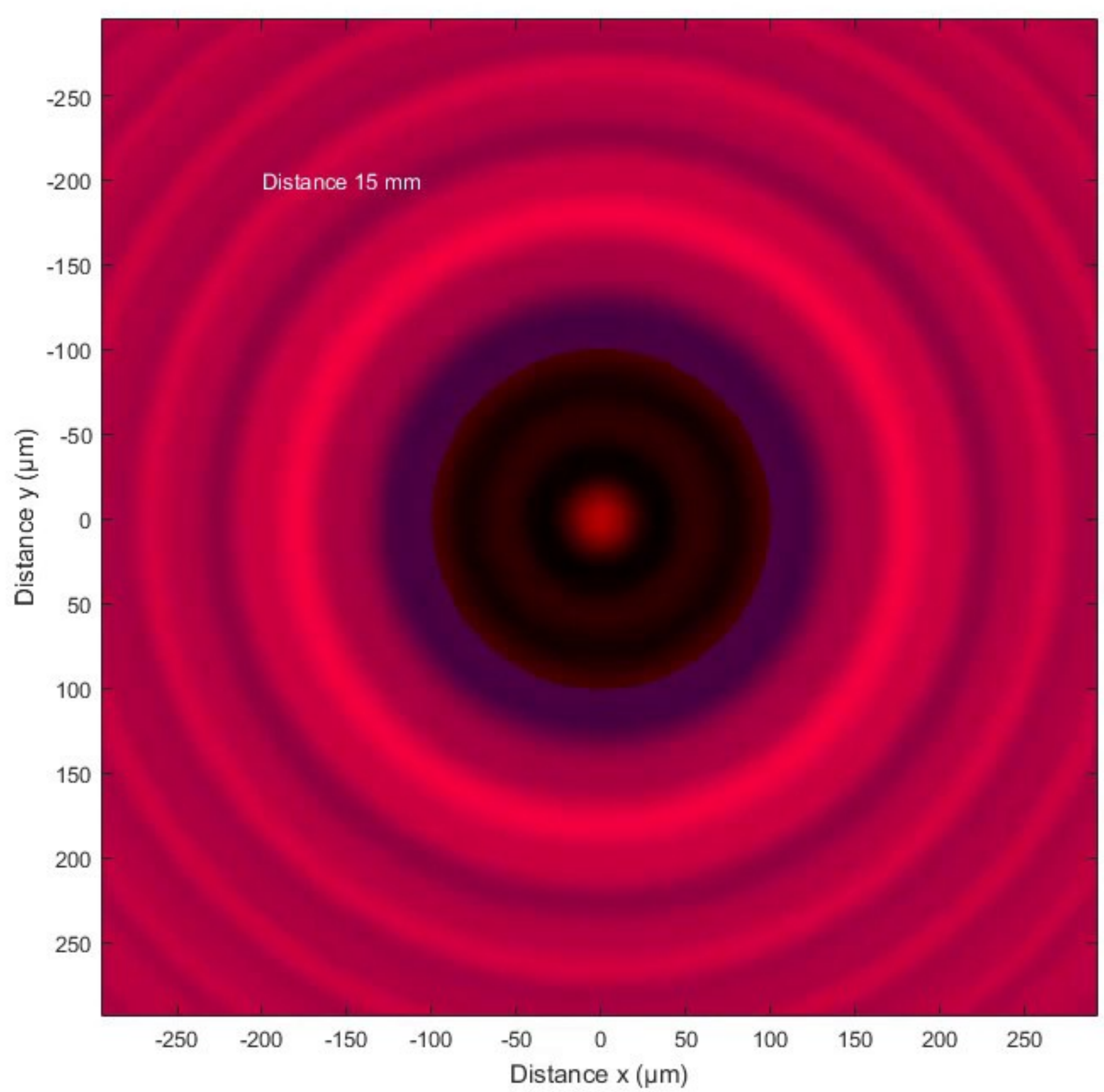


Figure 21: Poisson Spot Experiment Setup



Diffraction - More thoughts.......

Introduction

So far we have considered a range of models for light, each one appropriate to the phenomena we are considering.

The models so far have always taken the direction of the propagation to be given by Snell's Law, so that it only changes at refractive index boundaries.

To go further we need to consider the spatial wave nature of light, which results in diffraction.

We will start with a simplified scalar model where we will ignore the polarization effects of light.

Consider a ideal point source where wave of wavelength λ spread out in three-dimensions from the point (0,0,0) drawn from the y/z plane as shown in figure 22.

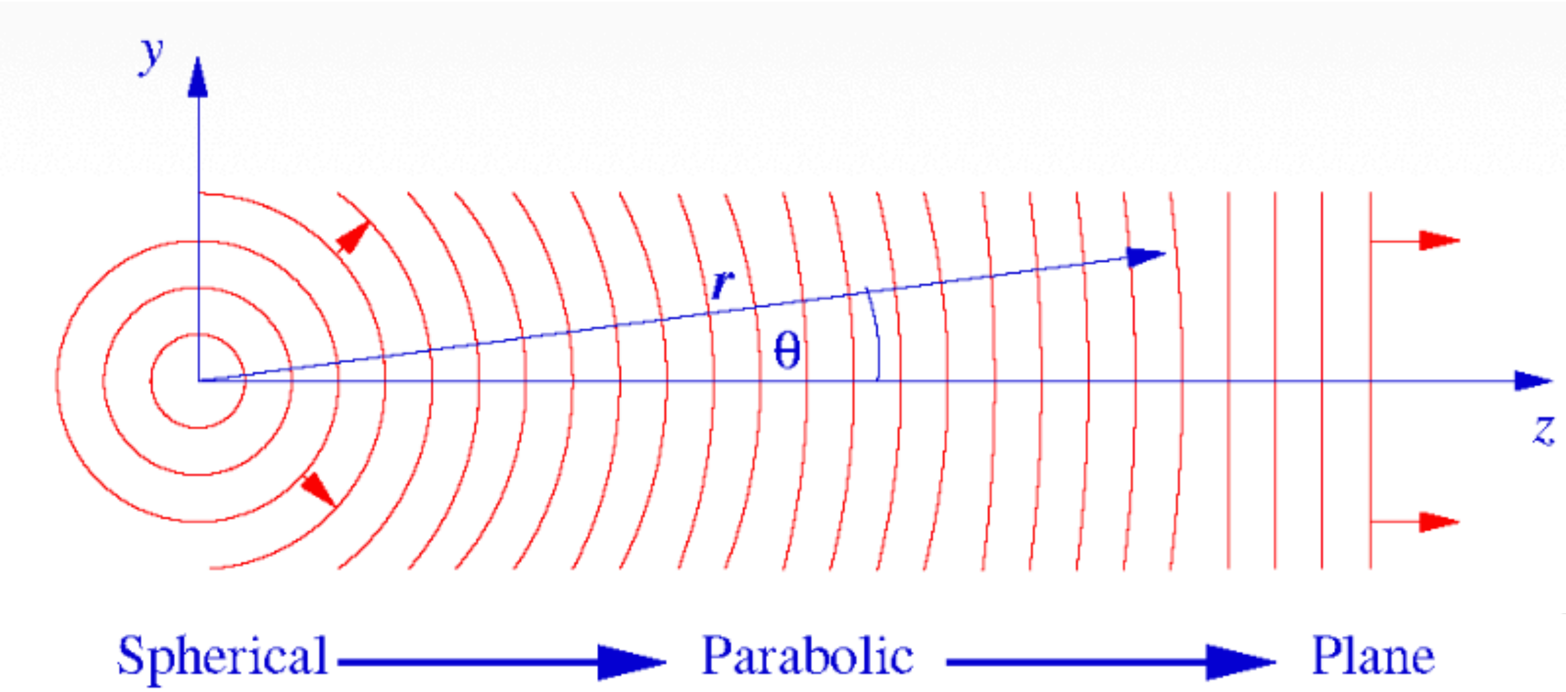


Figure 22: Spherical wave spreading from a point source.

Ignoring the time varying part, the scalar amplitude at a point a distance r from the origin is,

$$E(x,y,z) = \frac{E_0}{r}\cos(kr) \quad where \quad k = \frac{2\pi}{\lambda}$$

and

$$r^2 = x^2 + y^2 + z^2 \quad so\ that \quad r = z\left(1 - \frac{x^2 + y^2}{z^2}\right)^{1/2}$$

with the intensity given by

$$I(r) = \frac{1}{2}|E(r)|^2 = \frac{1}{2r^2}|E_0|^2 = \frac{1}{r^2}I_0$$

which is the inverse square relation, so the apparent intensity of the source drops off as the square of the distance.

When very close to the point source, we have to use the full expression for r which corresponds to expanding spherical waves, however as z becomes large we can start taking approximations that simplify the calculations

Parabolic Approximation:

If z << x/y , we can expand the expression for r to give

$$r = z + \frac{x^2 + y^2}{z}$$

which corresponds to parabolic wave approximation, which will result in **Fresnel** or **near field** diffraction.

Plane Wave Approximation:

The further approximation is that the wavefronts are locally plane as shown in figure 23.

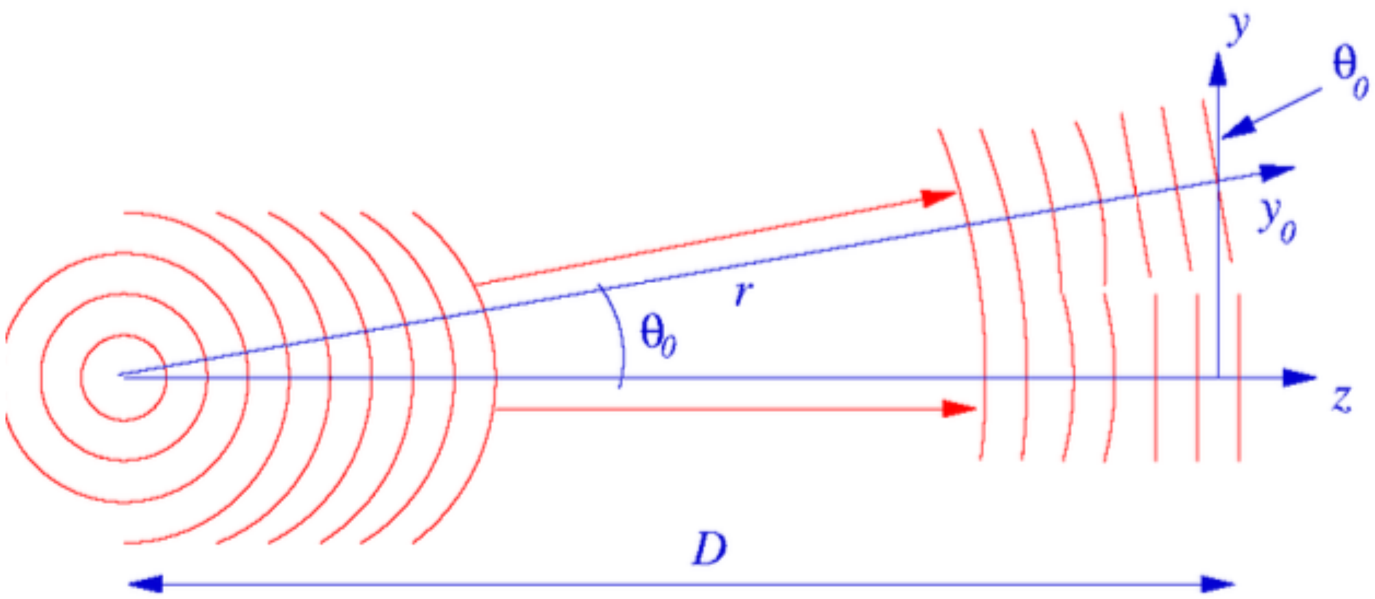


Figure 23: Far field approximation from a point source

At a distance D from the point source at angle θ_0 , about the point y_0 *we have*

$$r = \frac{D}{\cos\theta_0} + (y - y_0)\sin\theta_0 \quad \textit{where} \quad y_0 = D\tan\theta_0$$

This approximation is equivalent to expressing the wavefront as a series of plane waves, which will result in **Far Field diffraction**.

As we will see, this apparently rather drastic approximation results in many more usable results than you would expect.

Here, we will only deal with **far field** diffraction, which can be formulated in a Fourier scheme.

This will allow us to understand the effects of diffraction on imaging systems.

Consider a single slit of width d in plane P_0 being illuminated by a scalar wave of wavelength λ.

In plane P_1 a large distance D along the z axis, we consider a point s as shown in figure 24, where the line from the center of the slit to s makes an angle θ with the z axis.

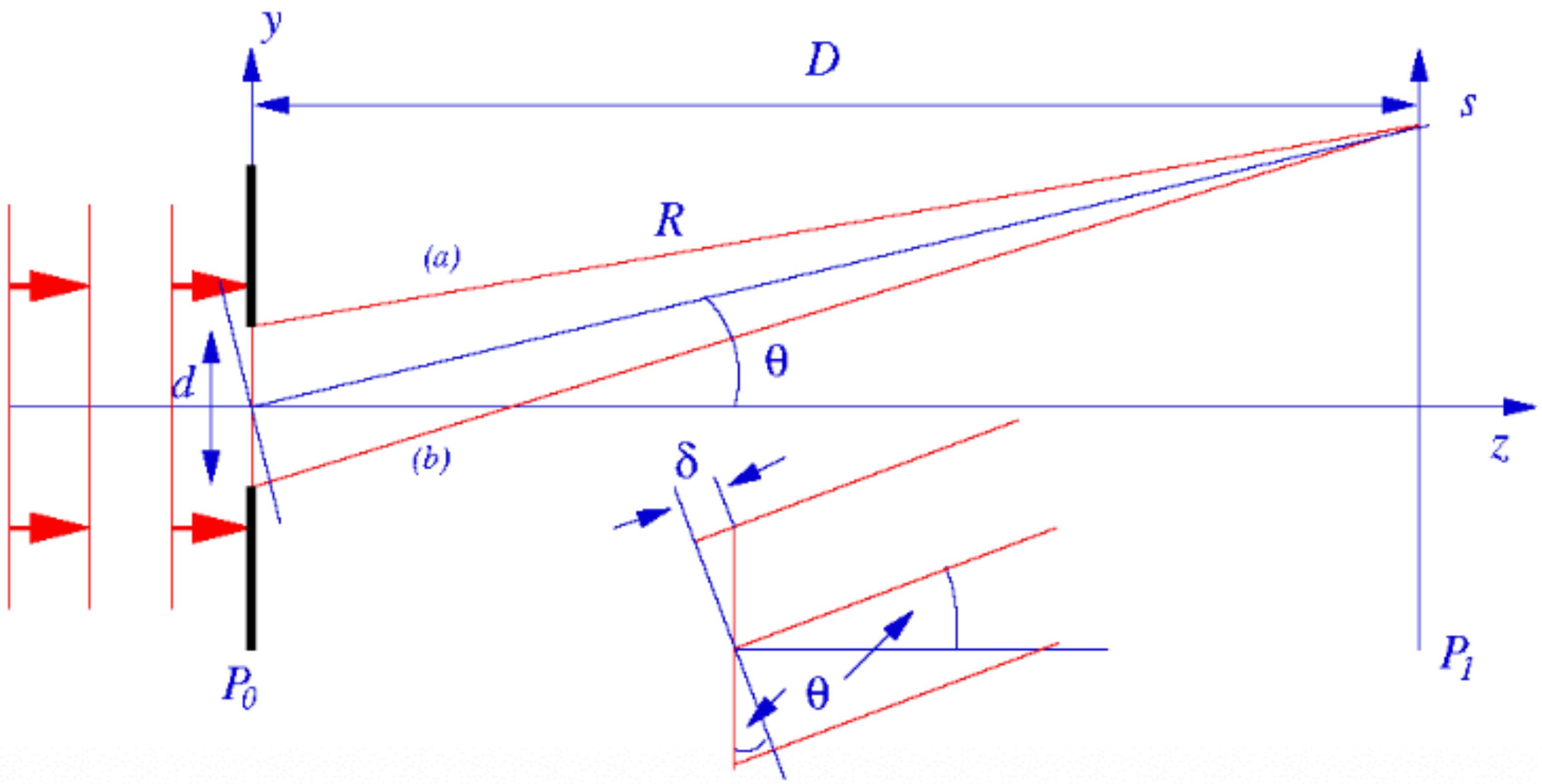


Figure 24: Layout and construction for far-field diffraction from a single slit

If we assume that D >> d, the width of the slit, then the angle of the rays (a) from the **top** of the slit, and (b) from the **bottom**, to s is also θ.

Then, as seen from s there appears to be a phase shift $\delta(y)$ across the slit given by

$$\delta(y) = ky\sin\theta \quad \textit{where} \quad k = \frac{2\pi}{\lambda}$$

The distance R from the slit to the point s can also be taken as constant and given by, $$R = \frac{D}{\cos\theta}$$

The scalar amplitude at s will then be a summation of the components coming from the slit and will be given by

$$E(\theta) = \frac{E_0}{R} \int_{-d/2}^{d/2} \cos(ky\sin\theta)dy$$

where E_0 is the scalar amplitude at the slit.

This can be integrated to give

$$E(\theta) = \frac{E_0}{Rk\sin\theta} \sin(ky\sin\theta)\Big|_{-d/2}^{d/2} = \frac{2E_0 \sin(\frac{kd}{2}\sin\theta)}{Rk\sin\theta}$$

To simplify this expression let

$$\beta = \frac{kd}{2}\sin\theta = \frac{\pi d}{\lambda}\sin\theta$$

we can then write E(θ) as

$$E(\theta) = \frac{E_0 d}{2R}\frac{\sin\beta}{\beta}$$

The intensity, which is what we would actually measure, is now given by

$$I(\theta) = \frac{1}{2}|E(\theta)|^2 = I_p\left(\frac{\sin\beta}{\beta}\right)^2 = I_p \mathrm{sinc}^2\beta \quad where \quad I_p = I_0 \frac{d^2}{4R^2}$$

where $I_0 = |E_0|^2 / 2$ is the intensity at the slit.

For small θ, the shape of this function is dominated by the $\mathrm{sinc}^2\beta$ term, which is plotted in figure 25.

The $\mathrm{sinc}^2\beta$ term is zero for $\beta = \pm m\pi$, giving zeros of the pattern when

$$\sin\theta_m = \frac{m\lambda}{d} \quad for \quad m = \pm 1, \pm 2, \pm 3, ...$$

This gives the angular shape of the diffracted pattern when D >> d.

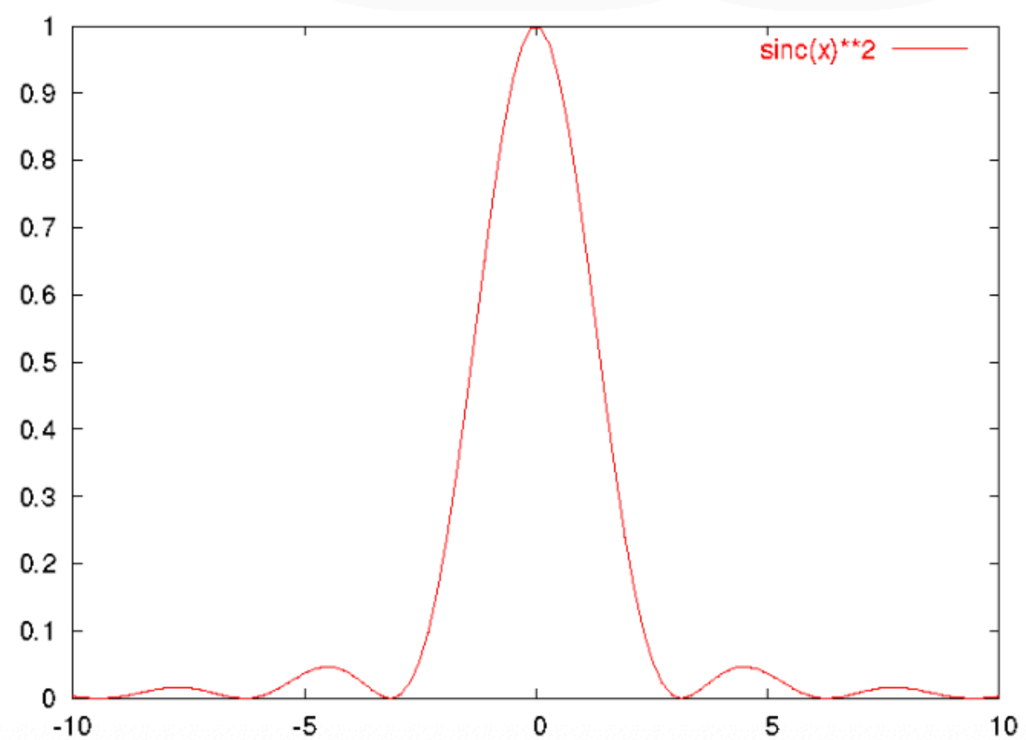


Figure 25: Shape of $\mathrm{sinc}^2(x)$

Diffraction Pattern from Slit and Lens

The above formulation assumes that the diffracted wavefronts after the slit are plane which result in a diffraction pattern formed at infinity.

A more useful geometry is to have a slit of width d immediately followed by a lens of focal length f as shown in figure 26.

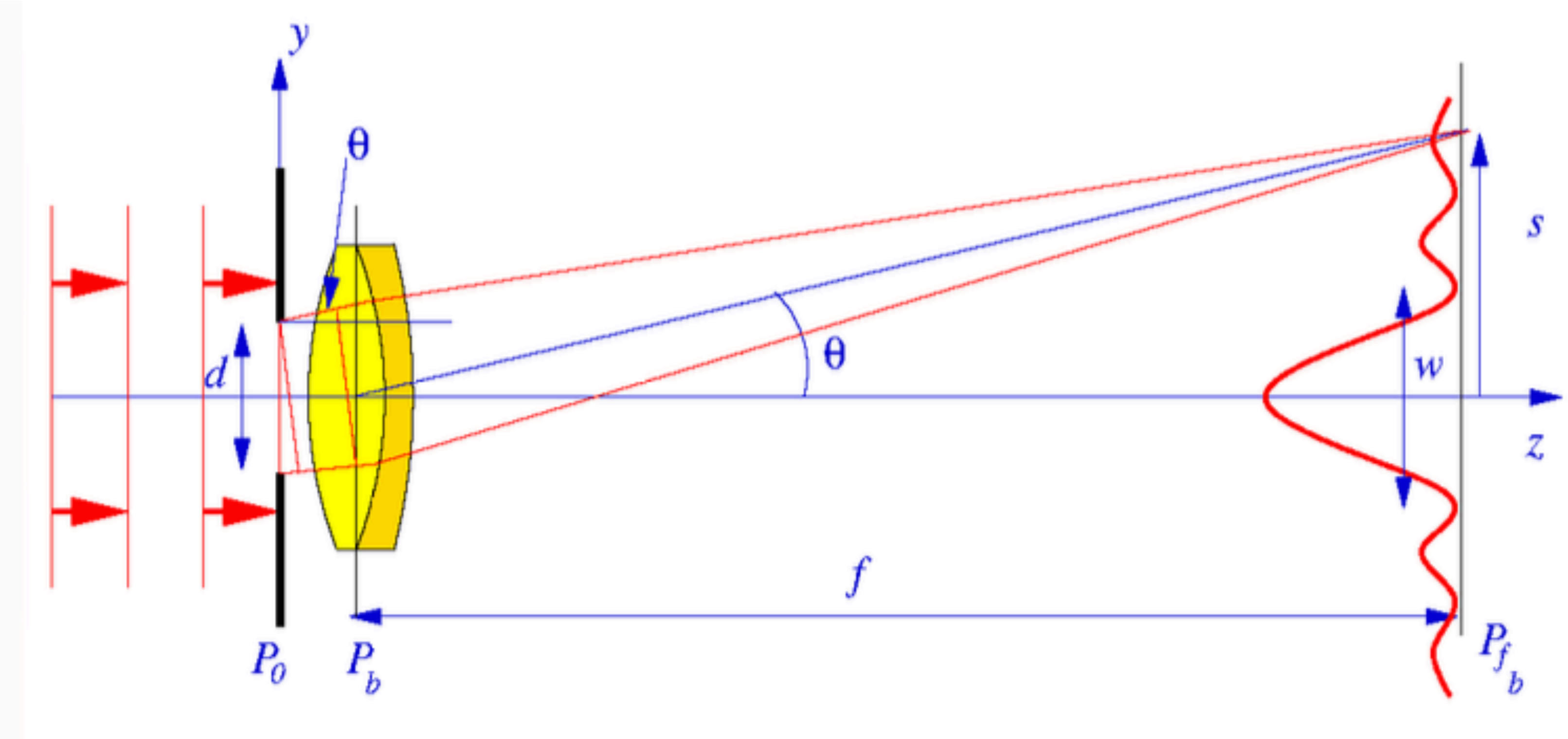


Figure 26: Diffraction from a slit imaged by a lens

We know that a collimated beam at angle θ striking the front of the lens is imaged to a point in the back focal plane P_{fb}, which is a distance f from the back principal plane of the lens.

Thus a plane wave, diffracted in direction θ will be focused as s, where

$$s = f\tan\theta \approx f\theta \quad for\ small \quad \theta$$

so the intensity in the back focal plane will be

$$I(s) = I_p \mathrm{sinc}^2\left(\frac{\pi d}{\lambda f} s\right)$$

which has zeros at

$$s_m = m\frac{\lambda f}{d} \quad for \quad m = \pm 1, \pm 2, \pm 3 \ldots\ldots$$

The distance or width between the ±1 zeros is

$$w = \frac{2\lambda f}{d}$$

Note the reciprocal relations between d and w, as as the slit get narrower then it diffraction pattern gets wider.

The Fourier Approach - 1st Try

The Fourier transform of a function f(x) is given by,

$$F(u) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-2i\pi ux} dx$$

which for a real function f(x) we can write as

$$F(u) = \int_{-\infty}^{\infty} f(x)\cos(2\pi ux)dx - i\int_{-\infty}^{\infty} f(x)\sin(2\pi ux)dx$$

If we now define a function p(x) which represents the slit as

$$p(x) = \begin{cases} 1 & for\ |x| < d/2 \\ 0 & for\ |x| \geq d/2 \end{cases}$$

then from our prior work

$$E(\theta)=\frac{E_0}{R}\int_{-d/2}^{d/2}\cos(ky\sin\theta)dy$$

can be written as

$$e^{iky\sin\theta}$$

$$E(\theta)=\frac{E_0}{R}\int_{-\infty}^{\infty}p(y)\cos(ky\sin\theta)dy=\frac{E_0}{R}\mathrm{Re}\{P(k\sin\theta)\}\quad where\quad k=\frac{2\pi}{\lambda}$$

where $P(u)=\Im\, p(x)$, or more importantly, if we have a lens of focal length f after the slit, then in the back focal plane of the lens we have

$$E(s)=\frac{E_0}{f}\int_{-\infty}^{\infty}p(y)\cos\left(\frac{2\pi}{\lambda f}ys\right)dy=\frac{E_0}{f}\mathrm{Re}\left\{P\left(\frac{1}{\lambda f}s\right)\right\}\qquad \text{using}\quad \sin\theta\approx\theta=\frac{s}{f}$$

where

$$P(u)=\Im\{p(x)\}=d\,\mathrm{sinc}(\pi du)$$

and d is the the width of the slit and the intensity is

$$I(s)=\frac{1}{2}|E(s)|^2$$

or

$$I(s)=\frac{1}{2}\left(\frac{E_0}{f}\right)^2\left|P\left(\frac{s}{\lambda f}\right)\right|^2$$

So we have the important result that the intensity in the back focal plane is related to the modulus squared of the scaled Fourier transform of p(x), which represents the slit.

The intensity is then

$$I(s) = I_0 \mathrm{sinc}^2\left(\frac{\pi ds}{\lambda f}\right)$$

the scheme is easily extended to a two-dimensional slit, being a rectangle of sides a x b, by defining a rectangle function,

$$p(x,y) = \begin{cases} 1 & for\ |x| < a/2 \quad and \quad |y| < b/2 \\ 0 & elsewhere \end{cases}$$

The two-dimensional Fourier transform is given by

$$P(u,v) = \Im\{p(x,y)\} = ab\,\mathrm{sinc}(\pi au)\mathrm{sinc}(\pi bv)$$

since both the Fourier transform and p(x,y) are separable.

Then following the procedure above the intensity diffracted from a rectangle, in the back focal plane of a lens of focal length f is given by

$$I(s,t) = I_0 \mathrm{sinc}^2\left(\frac{\pi as}{\lambda f}\right)\mathrm{sinc}^2\left(\frac{\pi bt}{\lambda f}\right)$$

where I_0 is the intensity at the center of the pattern.

This shape is plotted as a surface plot in figure 27, for an aperture where a = 3b.

It shows two $\mathrm{sinc}^2()$ functions at right angles, one being the three times wider than the other.

The log() of the function is also plotted which makes the structure much more obvious.

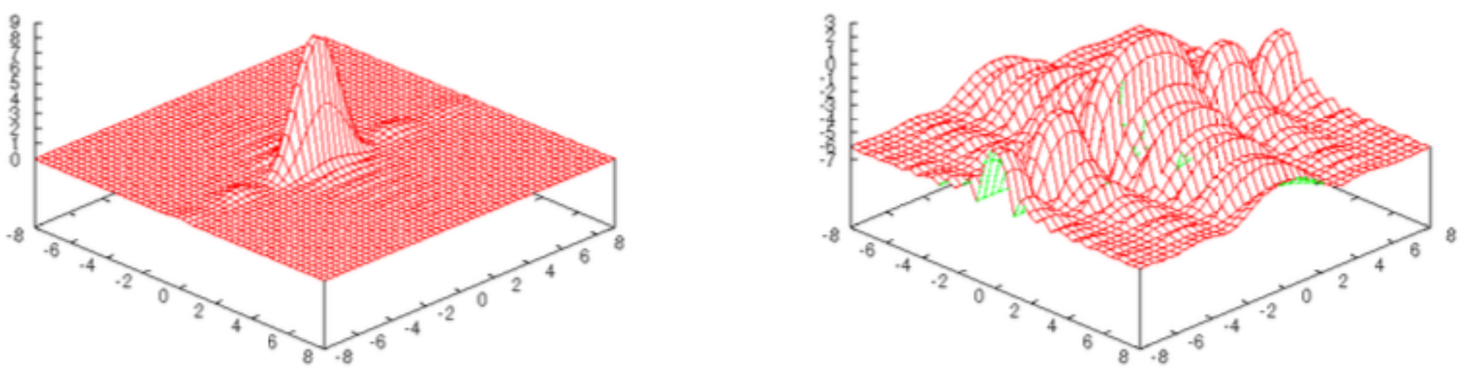


Figure 27: Plot of diffraction pattern from a rectangular aperture with of size a x b where a = 3b, and its log().

Diffraction from Two Slits

The power of the Fourier approach really becomes obvious when the diffracting object become more complex.

Consider diffraction from two slits of width a separated by distance b.

If, as above, we define p(x) to be the transmission function for one slit, being given by

$$p(x) = \begin{cases} 1 & for\ |x| < a/2 \\ 0 & for\ |x| \geq a/2 \end{cases}$$

then two slits, separated by b can be written as

$$f(x) = p(x) \odot \left[\delta\left(x - \frac{b}{2} \right) + \delta\left(x + \frac{b}{2} \right) \right]$$

where $\delta(x)$ is the δ-function, and $\odot$ is the convolution operator given by

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)h(x-s)ds$$

This is shown schematically in figure 28.

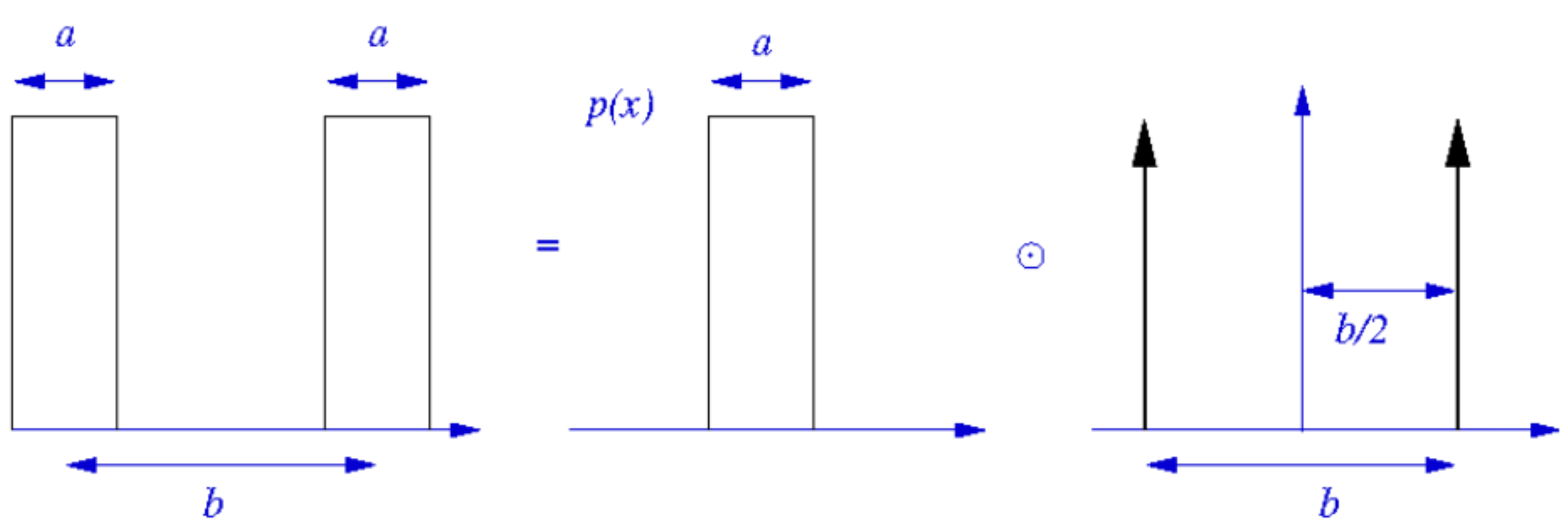


Figure 28: Representation of two slits by convolution.

We then know from the convolution theorem, that the Fourier transform is just the product of the Fourier transform of the two function, so in this case we get that

$$F(u) = \mathrm{sinc}(\pi a u)\cos(\pi b u)$$

The diffracted intensity behind a lens of focal length f is now just

$$I(s) = I_0 \mathrm{sinc}^2\left(\frac{\pi a s}{\lambda f}\right)\cos^2\left(\frac{\pi b s}{\lambda f}\right)$$

where I_0 is the intensity at the center of the pattern.

A plot of this intensity for b = 6a is shown in figure 29, along with a log() plot to make the outer peaks more visible.

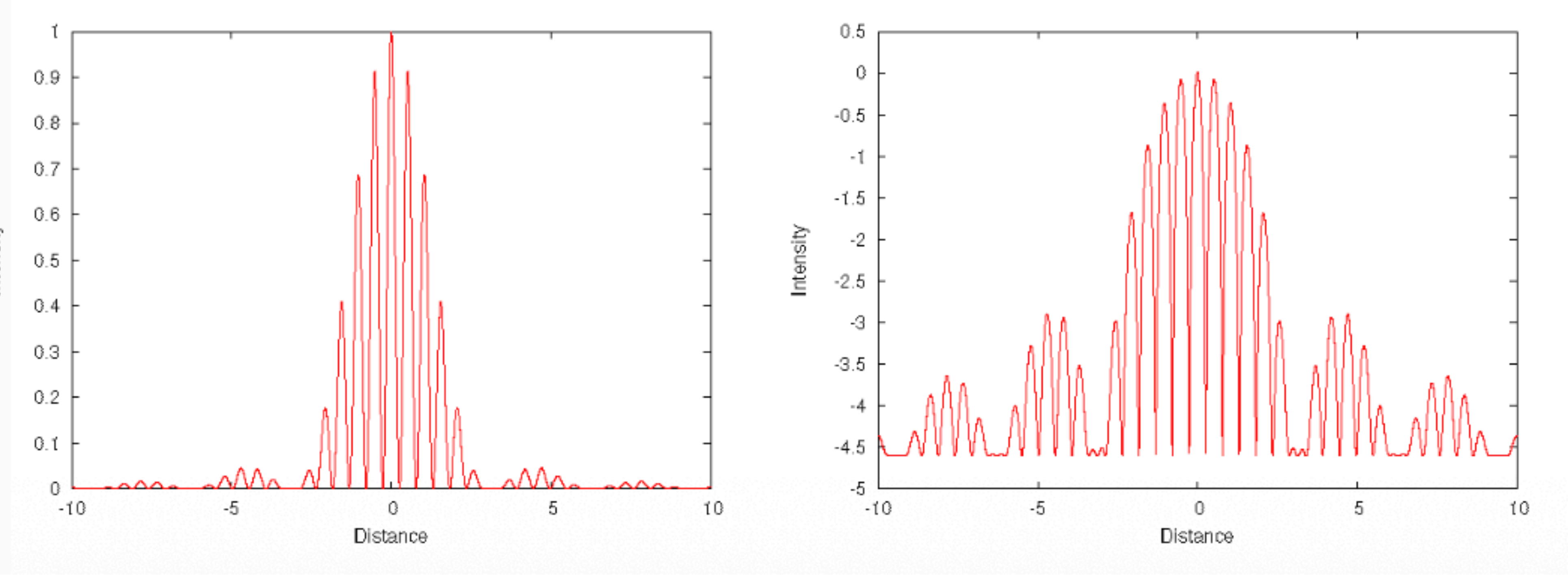


Figure 29: Plot of intensity diffracted from two slits separated by six times their width, and its log().

The shape is cos() fringes from the two slits modulated by the sinc²() resulting from the finite thickness of the slits.

If you compare this analysis with the traditional scheme in the textbooks, the power of the Fourier techniques is obvious!

The Diffraction Grating

The diffraction grating is a series of thin slits all separated by a distance d, which images by a lens of focal length f as shown in figure 30.

The traditional and simple analysis is that at angle θ, the phase delay between light diffracted from adjacent slits is,

$$\delta = \frac{2\pi}{\lambda} d \sin\theta$$

They will all sum together coherently, when $\delta = 2m\pi$ giving a series of peaks when

$$d \sin\theta_m = m\lambda \quad for \quad m = 0, \pm 1, \pm 2, ...$$

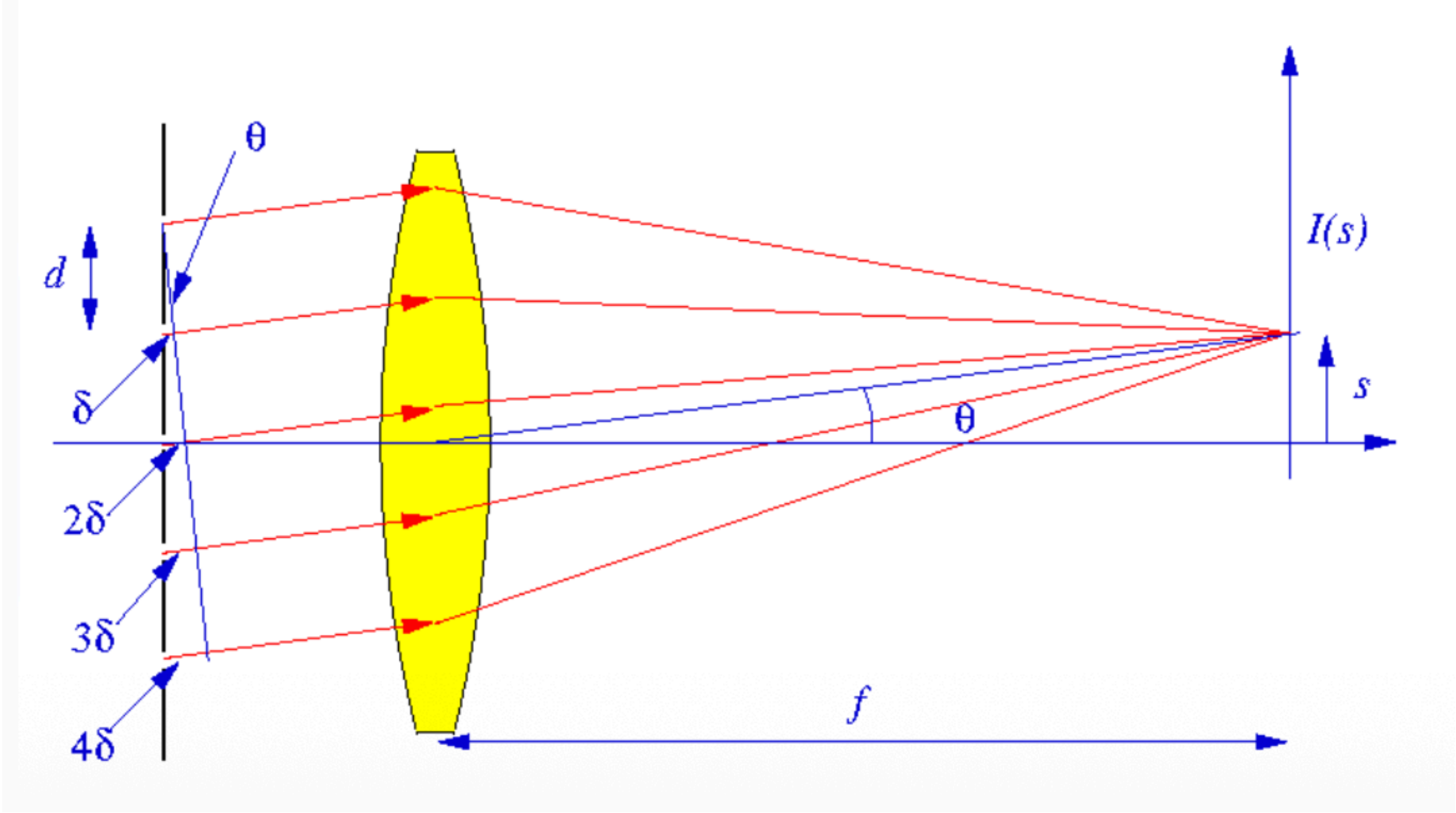


Figure 30: Diffraction grating images by a lens.

In the back focal plane of a lens of focal length f which, for small θ, will give a series peaks at,

$$s_m = m\frac{\lambda f}{d} \quad for \quad m = 0, \pm 1, \pm 2, \ldots\ldots\ldots$$

However this does not tell us anything about the shape of the peaks, or what happens when the slits are of finite width, for this we need Fourier methods.

We can represent a series of very thin slits as a series of δ -functions, each separated by distance d, by

$$c(x) = \sum_{m=-\infty}^{\infty} \delta(x - md)$$

which is the Comb() function as shown in figure 31.

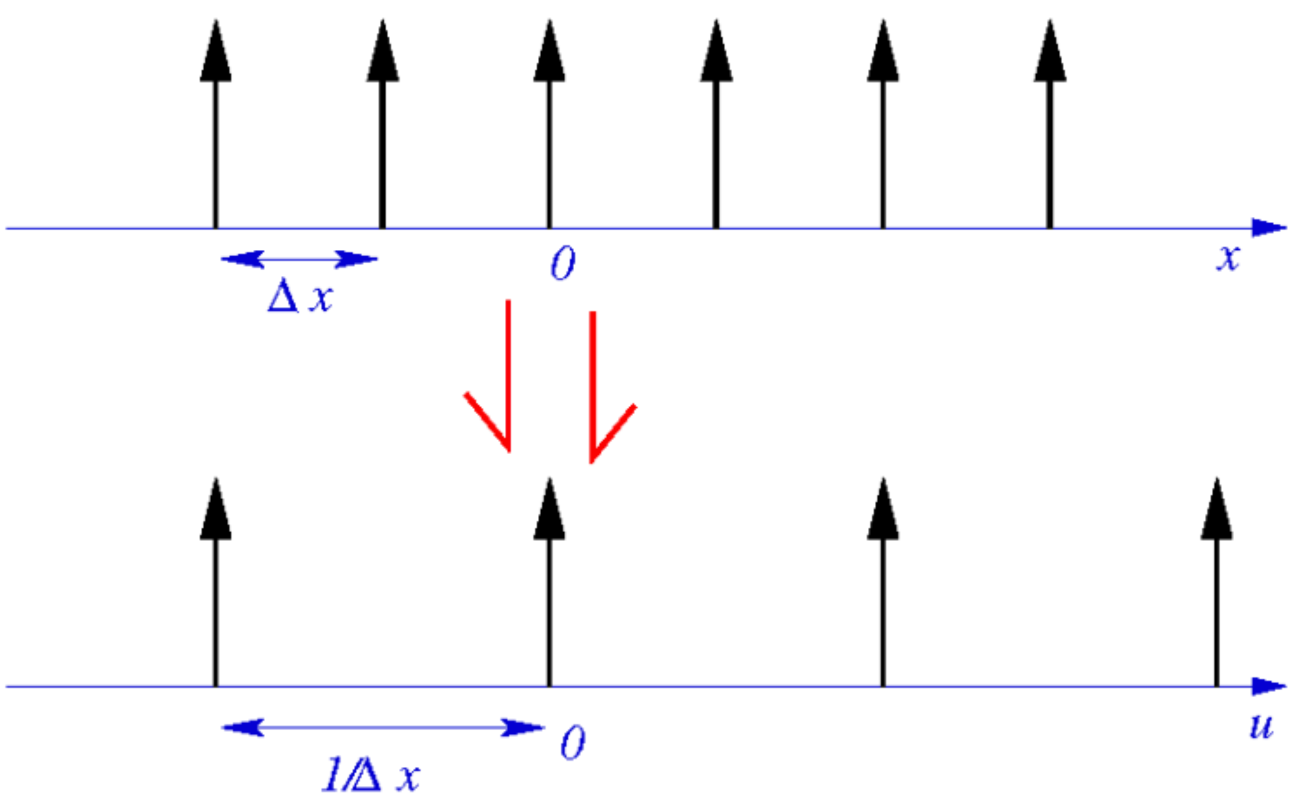


Figure 31: A Comb() function and its Fourier Transform.

Its Fourier transform is also a Comb(), but of reciprocal spacing, so

$$C(u) = \sum_{m=-\infty}^{\infty} \delta\left(u - \frac{m}{d}\right)$$

The intensity output behind a lens of focal length f is then given by

$$I(s) = I_0 \left| \mathrm{C}\left(\frac{\mathrm{s}}{\lambda \mathrm{f}}\right)\right|^2$$

which is just a set of δ-functions located at

$$s_m = m\frac{\lambda f}{d} \quad for \quad m = 0, \pm 1, \pm 2, \ldots\ldots\ldots$$

Now consider a full description of a grating with period d, slit width a, and finite length D = Nd, as shown in figure 32.

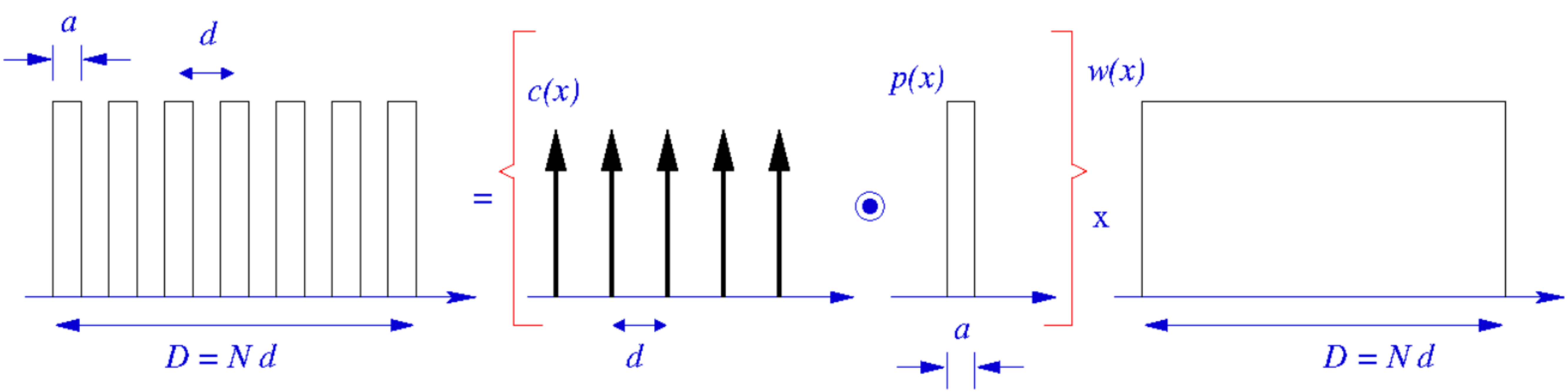


Figure 32: Full description of a grating of period d, slit width a and finite extent D = Nd

Using the notation in the figure, the grating can be described by a function

$$f(x) = \{c(x) \odot p(x)\} w(x)$$

which again from the convolution theorem, has Fourier transform

$$F(u) = \{W(u) \odot C(u)\} P(u)$$

where

$$C(u) = \sum_{m=-\infty}^{\infty} \delta\left(u - \frac{m}{d}\right) \quad , \quad P(u) = \mathrm{sinc}(\pi a u) \quad , \quad W(u) = \mathrm{sinc}(\pi N d u)$$

The intensity is then just,

$$I(s) = I_0 \left| F\left(\frac{s}{\lambda f}\right)\right|^2$$

which is plotted for a = 1, d = 5 and N = 10 in figure 33 which shows a series of sharp sinc²() functions separated by a constant distance and weighted by an overall sinc²().

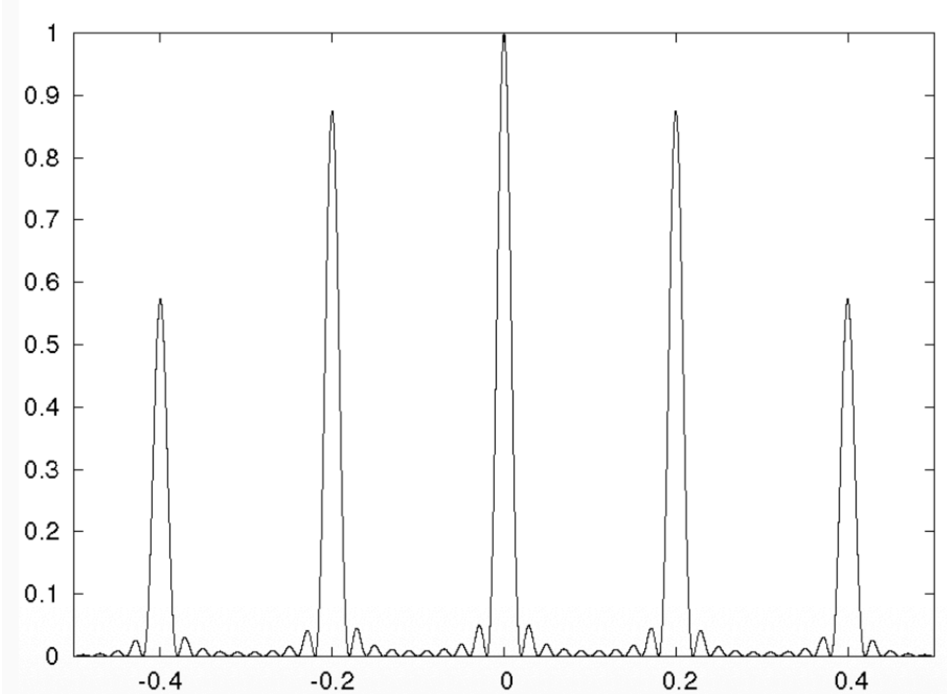


Figure 33. Output of a grating of period d, slit width a and finite extent D=Nd

DIFFRACTION AND FOURIER OPTICS

I. Introduction

Let us examine some of the main features of the Huygens-Fresnel scalar theory of optical diffraction

This theory approximates the vector electric and magnetic fields with a single scalar function, and adopts a simplified representation of the interaction of an electromagnetic wave with matter.

As you will see, the model accounts for a number of optical phenomena rather well.

If one accepts the Huygens-Fresnel theory, it becomes possible to manipulate images by altering their spatial frequency spectrum in much the same way that electronic circuits manipulate sound by altering the temporal frequency spectrum.

II. Theoretical considerations

A. Huygens-Fresnel scalar theory

Imagine an opaque screen with a hole in it, illuminated by monochromatic light.

Upon close examination, one finds an intricate pattern of light and dark behind the screen, not simply a geometric shadow as would be predicted by a particle theory of light.

The first attempt to explain the observed pattern was made by Christiaan Huygens, in 1678, on the basis of his wave theory of light.

In 1818, Augustin Fresnel combined Huygens' intuitive ideas with Young's principle of interference to produce a reasonably quantitative wave theory of optics.

Briefly put, the model assumes that each point within the illuminated aperture of a screen is the source of a spherical wave.

The amplitude of the optical field at any point beyond the screen is found by adding the spherical waves arriving from each fictitious point source.

Additional assumptions are needed to insure that the point sources only radiate in the forward direction, and that the edge of the aperture makes no special contribution.

Huygens and Fresnel did their work before Maxwell, so they did not really have a proper description of the electromagnetic field and were forced to resort to rather arbitrary assumptions.

To the modern student of optics it seems more sensible to start with Maxwell's equations, appropriate boundary conditions at the screen, and a description of the source of electromagnetic waves.

The $\vec{E}$ and $\vec{B}$ fields can then be calculated at all points in space, and the optical intensity found from $|\vec{E}|^2$.

It turns out that this approach is nearly impossible to carry through to a successful conclusion for any reasonably interesting geometry.

The most important difficulty is that $\vec{E}$ and $\vec{B}$ are coupled vector fields, so the basic equations are difficult to solve except for highly symmetric cases.

Then, even when the geometry is simple, a proper description of the response of the screen material to the electromagnetic wave is difficult.

As a result only a few simplified geometries have been calculated rigorously.

Fortunately a number of scientists, including Kirchoff, Rayleigh and Sommerfeld, were able to develop a simplified theory of optical diffraction between about 1880 and 1900.

The results of this theory agree well with experiment, and with complete calculations in those cases where the latter can be carried out.

Accordingly, we will work with this simplified model in what follows

The first step in the simplification is to replace the vector wave equation for $\vec{E}$ scalar equation for one component, E(x,y,z,t) :

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{01}$$

By treating only one component of the field, we are assuming that interactions with the aperture cannot affect the polarization of the incident wave.

Since we already know the explicit time dependence of the wave, we write

$$E(x,y,z,t) = \mathrm{Re}\left[\mathrm{E}\exp(-i\omega t)\right]$$

and substitute into (01) to obtain the desired scalar wave equation

$$\left(\nabla^2 + k^2\right)\mathrm{E} = 0 \tag{02}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{03}$$

The solution of (02), with appropriate boundary conditions, will be our description of the diffracted wave.

Equation (02) is to be solved for the geometry shown in Fig. 1, which depicts a planar aperture (the object) in an opaque screen illuminated from the left, and an image plane.

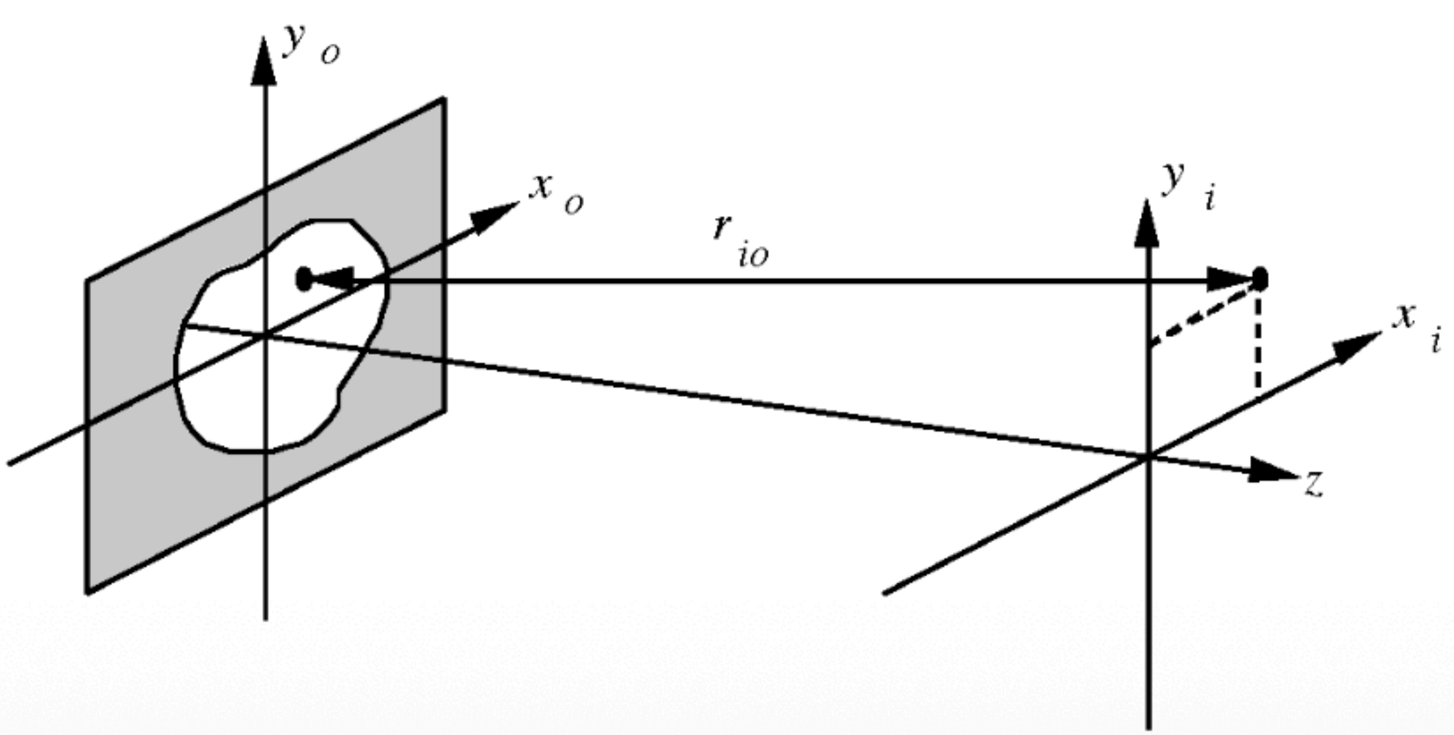


Fig. 1 - Sketch of object and image plane coordinates showing the distance rio between image and object points. The line r_{io} makes an angle θ with the z-axis.

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We assume that the aperture acts as the source of a field $E_0(x_0, y_0)$ which may be attenuated and phase shifted relative to the incident field. E_0 is assumed to be identically zero beyond the opaque portions of the screen.

We will derive rather generally that $E(x_i, y_i)$ is given by

$$E_i = \frac{1}{i\lambda} \iint E_o \frac{\exp(ikr_{io})}{r_{io}} \cos(\theta) ds \tag{04}$$

From our assumptions, E_0 is zero except directly behind the aperture, so the surface integral effectively runs over the aperture in the screen.

Although Eq. (04) appears complicated, it is really nothing more than a formal statement of the Huygens-Fresnel argument.

To see this, recall that a spherical wave of amplitude A diverging from a point source is described by

$$E = A\frac{\exp(ikr)}{r} \tag{05}$$

The integral (04) is a sum of such waves, of amplitude E_0, originating from the object points (x_o, y_o) within the aperture.

The $\cos(\theta)$ term corresponds to Fresnel's assumption that only the forward-propagating part of the spherical waves is to be retained.

The screen is characterized by a complex transmission function $T_0(x_0, y_0)$ which describes the phase shift and attenuation produced by the aperture.

If, as is frequently the case, the barrier either totally transmits or totally blocks the incident beam, $T_0(x_0, y_0)$ becomes identically 1 for (x_o, y_o) within the aperture and zero otherwise.

The object amplitude E_0 is found by multiplying the incident wave $T_0(x_0, y_0)$ by product is particularly simple for a point-source on the coordinate axis a distance z_s from the aperture, provided z_s is large compared to the dimensions of the aperture.

In that limit we can replace r by z_s in the denominator of Eq. (05).

We must be more careful with the r in the exponential, because kr is likely to be a big number which we must determine to a fraction of 2π .

A simple expansion of r gives

$$r \approx z_s\left[1+\frac{1}{2}\left(\frac{x_i - x_o}{z_s}\right)^2+\frac{1}{2}\left(\frac{y_i - y_o}{z_s}\right)^2\right] \tag{06}$$

which can be inserted into (05) to yield

$$\mathrm{E} = \frac{A}{z_s}\exp(ikz_s)\exp\left[\frac{ik}{2z_s}\left(x_o^2 + y_o^2\right)\right]T_o(x_o, y_o) \tag{07}$$

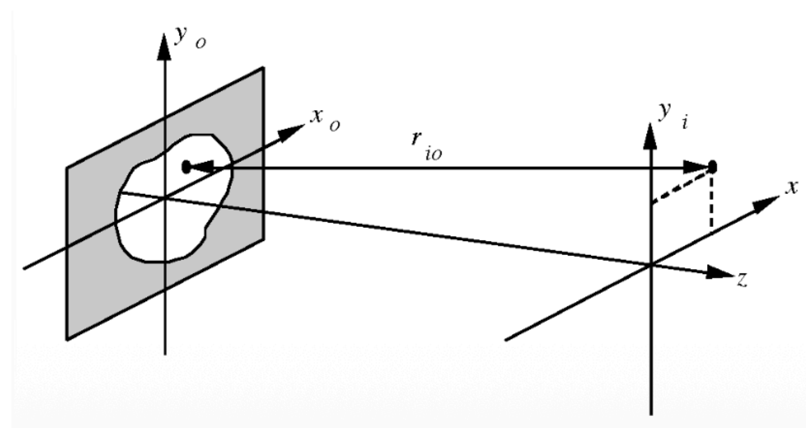
Further simplification is obtained for plane wave illumination normal to the aperture, which is obtained by letting $z_s \rightarrow \infty$ holding $A/z_s \rightarrow A'$ finite.

The varying phase factors then vanish, leaving $E_0 = A'T_0(x_0, y_0)$.

At this point, most of the physics is done and the content of the model has been presented.

The remainder of our job is to make some geometrical approximations that will let us evaluate (04) in the cases of interest.

B. Fresnel and Fraunhofer diffraction



Referring again to Fig. 1, we see that θ in Eq. 04 is determined by the distance z between aperture and image planes, and by the distance from the image point to the origin.

If we restrict our attention to a finite region of the image plane, corresponding to θ less than a few degrees, it is adequate to take $\cos\theta \approx 1$.

Exactly the same considerations allow us to replace r_{i0} in the denominator by z and by the expansion (06) in the exponential.

Substituting into (04) we get

$$E_i = \frac{\exp(ikz)}{i\lambda} \iint E_o \exp\left\{ \frac{ik}{2z_s} \left[\left((x_i - x_o)^2 + (y_i - y_o)^2 \right) \right] \right\} dx_o dy_o \qquad (08)$$

This is known as the **Fresnel** approximation to the scalar diffraction theory.

It is useful when z is very large compared to a wavelength, but not necessarily much bigger than the linear dimensions of the aperture. (The expansion (06) seems to require , $z >> (x_i - x_0)$ but most of the contribution to the integral (08) comes from regions where $(x_i - x_0) \approx 0$.

Outside of those regions the exponential oscillates rapidly, leading to a cancellation of positive and negative contributions.)

A further rearrangement of the Fresnel diffraction expression will be computationally convenient.

Expanding the quadratic terms in the exponential, we get

$$E_i = \frac{\exp(ikz)}{i\lambda z}\exp\left\{\frac{ik}{2z_s}\left(x_i^2 + y_i^2\right)\right\}$$
$$\iint E_o \exp\left\{\frac{ik}{2z_s}\left(x_o^2 + y_o^2\right)\right\}\exp\left\{-\frac{ik}{z_s}\left(x_i x_o + y_i y_o\right)\right\} dx_o dy_o \qquad (09)$$

This tells us that the Fresnel diffraction pattern can be found, within phase factors, by computing the Fourier transform of

$$E_o \exp\left\{\frac{ik}{2z_s}\left(x_o^2 + y_o^2\right)\right\}$$

As we know, a very efficient algorithm, the Fast Fourier Transform or FFT, exists to do this computation.

The physical significance of the transform is discussed later.

If we move farther away from the aperture, so that $z >> k\ (x_0^2+y_0^2)$, the quadratic phase factor in (09) is approximately unity over the entire aperture.

This infinite-distance limit is called the **Fraunhofer** regime, and is the case usually considered in elementary treatments.

The diffraction pattern is explicitly given by

$$\mathrm{E}_i = \frac{\exp(ikz)}{i\lambda z}\exp\left\{\frac{ik}{2z_s}\left(x_i^2 + y_i^2\right)\right\}\iint \mathrm{E}_o \exp\left\{-\frac{ik}{z_s}\left(x_i x_o + y_i y_o\right)\right\} dx_o dy_o \tag{10}$$

which is simply the Fourier transform of the aperture illumination.

We will usually want to know the optical intensity, which is proportional to $|E_0|^2$, so the phase factor in front is irrelevant.

C. Thin lenses and spatial filtering

The geometrical-optics analysis of lenses is already familiar to you.

Here we are concerned with the phase of the optical waves, in order to treat interference effects properly, so we must reexamine the effect of a lens on an incoming wave.

We will eventually find that lenses do indeed form images, but with a phase shift which depends on position.

Fortunately, we are usually concerned with intensity, not amplitude, so the phase shift is unimportant, and we can expect to use lenses as usual to form magnified images of objects.

More surprisingly, we will also find that a lens can be used as an optical Fourier transformer.

This will allow us to observe the Fraunhofer diffraction pattern at a finite distance from the diffracting object, and to do the Fourier transform required for spatial filtering experiments.

1. Lens geometry

The type of lenses we need to understand consist of optically dense material bounded by spherical surfaces, as shown in Fig. 2a.

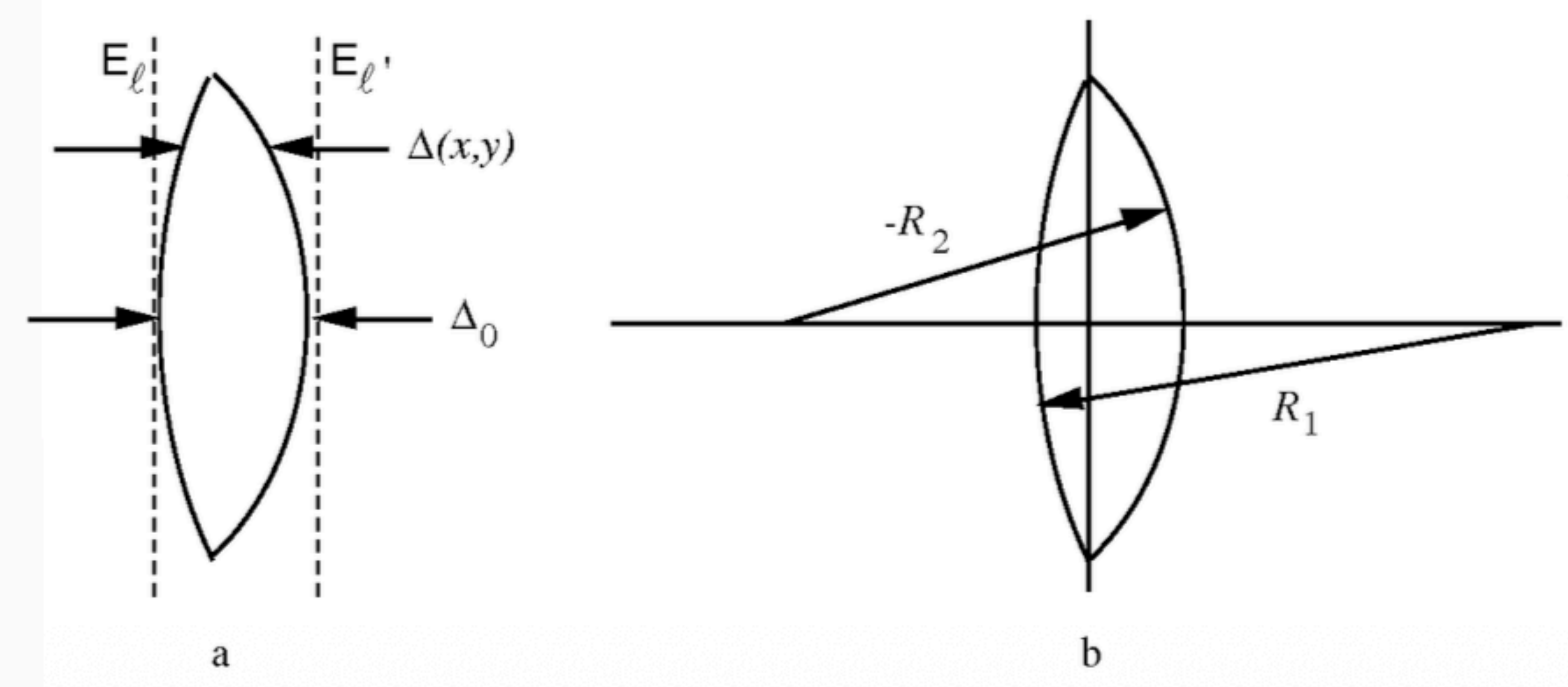


Fig. 2 Definition of the geometry for a thin lens

We will treat only thin lenses, in the sense that a ray entering at (x,y) on one face emerges at nearly the same coordinate on the other face.

A thin lens therefore serves only to delay the incident wavefront by an amount proportional to the thickness of the lens at each point.

Denote the thickness at any point by $\Delta(x, y)$, and the maximum thickness by Δ_0.

The total phase delay between the input wave E_ℓ and the output wave $E_{\ell'}$ is

$$\phi(x,y) = kn\Delta(x,y) + k\left[\Delta_0 - \Delta(x,y)\right] \tag{11}$$

where n is the index of refraction.

Rearranging, we get

$$E_{\ell'} = E_\ell \exp\left[ik\Delta_0\right]\exp\left[ik(n-1)\Delta(x,y)\right] \tag{12}$$

To calculate $\Delta(x,y)$, we imagine splitting the lens, as shown in Fig. 2b, and work out the geometry to obtain

$$\Delta(x,y) = \Delta_0 - R_1\left[1-\left(1-\frac{x^2+y^2}{R_1^2}\right)^{1/2}\right] + R_2\left[1-\left(1-\frac{x^2+y^2}{R_2^2}\right)^{1/2}\right] \tag{13}$$

This simplifies enormously if we stay near the lens axis, since then we can expand the square roots to obtain:

$$\Delta(x,y) = \Delta_0 - \frac{x^2 + y^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{14}$$

Eq. (14) limits us to paraxial rays, but that is the normal approximation in geometric optics anyway.

Using the lens-maker's formula to relate the focal length f to the lens parameters, we substitute (14) into (12) to arrive at the desired equation for the output

$$\mathrm{E}_{\ell'} = \mathrm{E}_{\ell} \exp\left[-\frac{ik}{2f}\left(x_\ell^2 + y_\ell^2\right)\right] \tag{15}$$

where we have suppressed a constant phase factor.

As a check on the derivation, you might wish to convince yourself that an incident plane wave E_ℓ

will be transformed into a spherical wave converging on a point a distance f behind the lens, as you would expect.

2. Arbitrary d_0 and d_i

We now have all the machinery we need to handle the situation shown in Fig. 3.

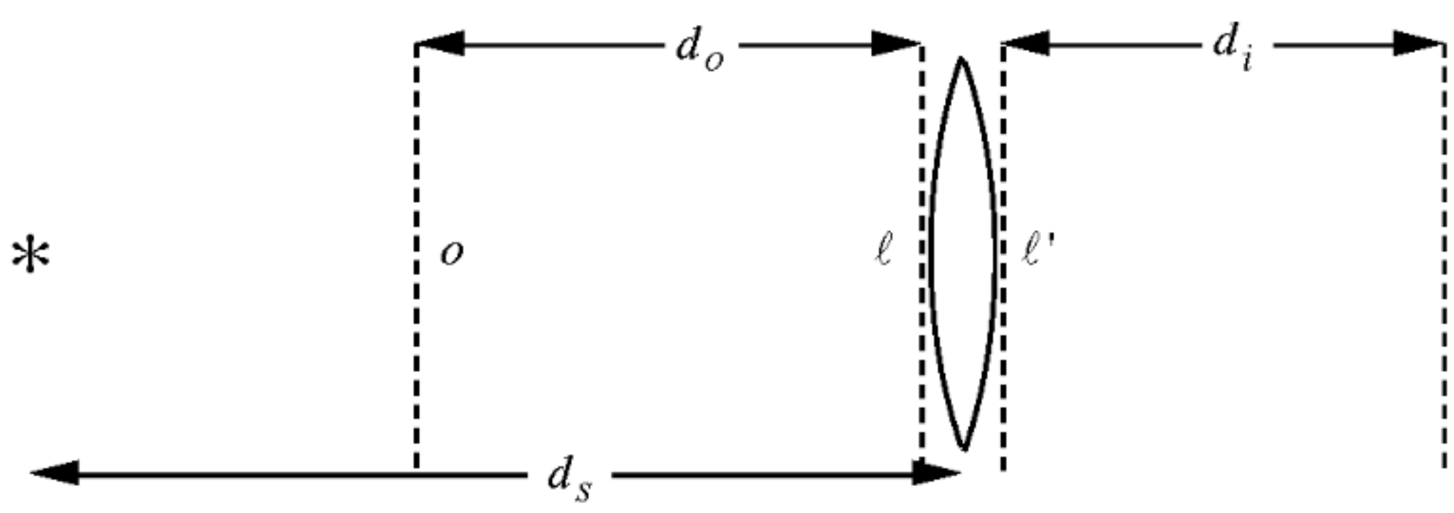


Fig. 3 Object, lens and image planes for a thin lens. A point-source is located on the axis a distance d_s from the lens.

An object a distance d_o from the lens is illuminated by a point source.

The object transmits a wave $E_0(x_0,y_0)$ which is modified by the lens and then travels a distance d_i to the "image" plane, where it is described by $E_i(x_i,y_i)$.

To calculate E_i we start from the Huygens-Fresnel description of the object as a sum of point sources, each emitting a spherical wave of strength $E_0(x_0,y_0)$.

The superposition of all the spherical waves will give an expression for E_ℓ of the form of Eq. (08).

This wave is transformed to $E_{\ell'}$ by the lens according to Eq. (15).

If we consider $E_{\ell'}$ to be a new superposition we can describe E_i in the form of Eq. (08) again.

Writing out these steps is messy but necessary.

From (08), dropping the constant phase $\exp(ikz)$, we have

$$\mathrm{E}_\ell = \frac{1}{i\lambda d_0}\iint \mathrm{E}_o \exp\left[\frac{ik}{2d_0}\left((x_i - x_o)^2 + (y_i - y_o)^2\right)\right] dx_o dy_o \tag{16}$$

We then get $\mathrm{E}_{\ell'}$ from (15), and put it into (08) again to get

$$\mathrm{E}_i = \frac{1}{i\lambda d_i}\iint \mathrm{E}_{\ell'} \exp\left[\frac{ik}{2d_i}\left((x_{\ell'} - x_i)^2 + (y_{\ell'} - y_i)^2\right)\right] dx_{\ell'} dy_{\ell'} \tag{17}$$

It is convenient to introduce a new function $h(x_i,y_i;x_0y_0)$ defined by

$$\mathrm{E}_\ell = \iint h(x_i, y_i; x_o, y_o)\mathrm{E}_o dx_o dy_o \tag{18}$$

The explicit form of h is found from (15), (16), and (17):

$$\begin{aligned} h(x_i, y_i; x_o, y_o) = {} & \frac{1}{\lambda^2 d_0 d_i} \exp\left[\frac{ik}{2d_i}\left(x_i^2 + y_i^2\right)\right] \exp\left[\frac{ik}{2d_o}\left(x_o^2 + y_o^2\right)\right] \\ & \int \exp\left[\frac{ik}{2}\left(\frac{1}{d_i} + \frac{1}{d_o} - \frac{1}{f}\right) q_1^2\right] \exp\left[-ik\left(\frac{x_o}{d_o} + \frac{x_i}{d_i}\right) q_1\right] dq_1 \\ & \int \exp\left[\frac{ik}{2}\left(\frac{1}{d_i} + \frac{1}{d_o} - \frac{1}{f}\right) q_1^2\right] \exp\left[-ik\left(\frac{y_o}{d_o} + \frac{y_i}{d_{oi}}\right) q_2\right] dq_2 \end{aligned} \tag{19}$$

Clearly, this equation is too messy to be useful, and we need to set about evaluating some special cases.

Before proceeding, note that we could have gotten h directly by finding the image of a point source located at (x_0,y_0) since the definition (18) is just the Huygens-Fresnel superposition integral again.

3. Image condition

As a first case, let us look at the geometry for forming an ordinary image.

Recall from geometrical optics that a real image is formed by a positive lens when $(1/d_0) + (1/d_i) = 1/f$.

For that geometry the quadratic phase factors in (19) vanish leaving two integrals over $\exp(iqx)$, which are delta functions.

Substituting into Eq. (18) we get, after some algebra,

$$\mathrm{E}_i = \frac{1}{M}\exp\left[\frac{ik}{2d_i}\left(1+\frac{1}{M}\right)\left(x_\ell^2 + y_\ell^2\right)\right]\mathrm{E}_o\left(-\frac{x_i}{M},-\frac{y_i}{M}\right) \tag{20}$$

where $M = d_i/d_0$ is the geometric magnification.

Except for a phase shift, E_i is a scaled copy of E_0.

The phase factor will vanish when we take $|E_i|^2$ to get the intensity at d_i, so **we have shown that we do indeed get an image of the input object when we satisfy the geometrical-optics condition for imaging.**

4. Fourier transform condition

Although it is not obvious, a simple Fourier transform can describe the "image" in the plane at d_t shown in Fig. 4.

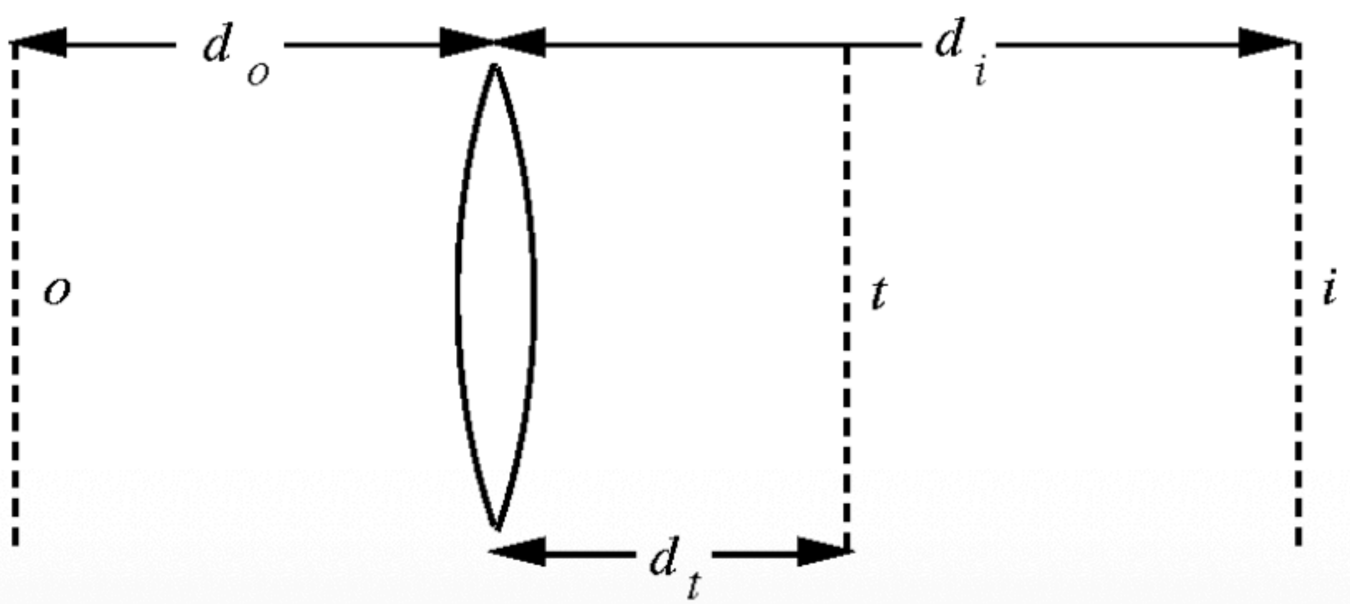


Fig. 4 The lens forms the Fourier transform at plane t and a geometric image at plane i. The source at distance d_s is not shown.

This is the plane conjugate to the source, that is $(1/d_s)+(1/d_t) = 1/f$.

When E_0 is given by Eq. (07) and we set $d_i = d_t$ in Eq. (19) the integrand in Eq. (18) becomes

$$h\mathrm{E}_o = \frac{iA(d_s - f)}{\lambda f(d_s - d_o)} \exp\left[-\frac{ik}{2f^2}\frac{(d_o - f)(d_s - f)}{(d_s - d_o)}\left(x_t^2 + y_t^2\right)\right] \\ T_0(x_o,y_o)\exp\left[-\frac{ik}{f}\frac{(d_s - f)}{(d_s - d_o)}\left(x_o x_t + y_o y_t\right)\right] \tag{21}$$

where (x_t,y_t) are coordinates in the plane at d_t.

This is almost the transform of the aperture transmission function.

The quadratic phase factors can be eliminated by setting $d_o = f$ and using plane-wave illumination for which $d_s \rightarrow \infty$ and $d_t = f$.

These conditions lead to

$$\mathrm{E}_t(x_t, y_t) = \frac{iA'}{\lambda f} \iint T_0(x_o, y_o) \exp\left[-\frac{ik}{f}\left(x_o x_t + y_o y_t\right) \right] dx_o dy_o \qquad (22)$$

which indicates that under these conditions the image formed at d_t is exactly the Fourier transform of the object transmission function.

One sees from (21) and (22) that the image formed in the plane conjugate to the point source can have some interesting properties.

Comparison with Eq. (10) shows that the intensity distribution in the transform plane is the same as in the Fraunhofer diffraction pattern of the object.

This is true even though the transform plane is rather close to the object, a potential advantage in setting up experiments.

The image is also closely related to the spatial frequency spectrum of the object, a point we now consider in detail.

5. Spatial filtering

Equation (22) shows that the complex amplitude E_t in the plane at d_t is the spatial-frequency spectrum of the object transmission function T_0.

Specifically, the observed amplitude at a point (x_t,y_t) in the "transform plane" is proportional to the spectral amplitude at frequency $(v_x,v_y)=$ $(kx_t/2\pi f, ky_t/2\pi f)$.

By inserting an appropriate barrier (filter) in the optical path at the transform plane, we can remove or phase shift any desired part of the spatial frequency spectrum.

If we pick the filter correctly, the processed image may be modified in any of several well-defined ways.

For example, a filter which removes the higher spatial frequencies (the parts of the transform farthest from the optic axis) will blur all sharp edges in the final image.

Unfortunately, the condition $d_0 = f$, required to completely eliminate the phase factors in the transform plane, leads to an image at infinity.

This is not very satisfactory for experiments.

Suppose instead that we consider the more general case of $d_0 > f$ to obtain a filtered image at a finite distance.

Taking the limit $d_s \rightarrow \infty$ in (21) then yields an expression like (22) but with a multiplicative phase factor

$$\exp\left[-\frac{ik}{2f^2}(d_o - f)\left(x_t^2 + y_t^2\right)\right]$$

A filter in the transform plane t will modify this complex amplitude according to $E_{t'} = T_f(x_t, y_t)E_t$.

The modified wave then propagates to the final image plane at di according to Eq. (09), with the result

$$E_i = \frac{A'}{\lambda^2 fz}\exp\left[ikz + \frac{ik}{2z_i}\left(x_\ell^2 + y_\ell^2\right)\right] \tag{22}$$

$$\iint T_f F\{T_o\}\exp\left[-\frac{ik}{z}\left(x_i x_t + y_i y_t\right)\right]dx_t dy_t$$

where

$$F\{T_o\} = \iint T_o \exp\left[-\frac{ik}{f}\left(x_o x_f + y_o y_f\right)\right]dx_o dy_o \tag{23}$$

is the Fourier transform of T_0 and $z = d_i - f$.

Note that the quadratic phase factors in the transform plane have vanished, so we can retain all of our previous discussion based on Eq. (22).

Also note that, when $T_f = 1$, Ei is the Fourier transform of the Fourier transform of T_0.

Double-transforming a function results in an inverted version of the original function, so we recover the geometric image, consistent with Eq. (11), as we must.

Diffraction from a Circular Aperture

As most optical systems are circular, the next most useful diffraction to consider is the circular aperture, which is going to give up the shape of the spot formed by a lens.

Consider the circular aperture of radius a, being defined by,

$$p(x,y) = \begin{cases} 1 & when \quad x^2 + y^2 \leq a^2 \\ 0 & \text{otherwise} \end{cases}$$

First we need to calculate the Fourier transform P(u,v), which using

$$x = \rho\cos\theta \quad , \quad y = \rho\sin\theta$$

thewe have that

$$P(u,v) = \int_0^a \int_0^{2\pi} \exp(-i2\pi(u\rho\cos\theta + v\rho\sin\theta)\rho d\rho d\theta$$

where the limits of integration are across a circle of radius a.

The circular aperture p(x,y) is clearly circularly symmetric, so it Fourier transform must also be circularly symmetric.

Therefore we only need to calculate P(u,v) along one radial line; select the line with v = 0, to give

$$P(u,0)=\int_0^a\int_0^{2\pi}\exp(-i2\pi u\rho\cos\theta)\rho d\rho d\theta$$

Now we have the standard special function identity that

$$\int_0^{2\pi}\exp(ir\cos\theta)d\theta=2\pi J_0(r)$$

where $J_0(r)$ is the zero order Bessel function, we get that

$$P(u,0)=2\pi\int_0^a J_0(2\pi u\rho)\rho d\rho$$

We now need the second identity that

$$rJ_0(r)=\frac{d}{dr}\left(rJ_1(r)\right)\quad \textit{so that}\quad \int_0^r J_0(t)t=rJ_1(r)$$

so if we let t = 2πuρ, we get

$$P(u,0)=\frac{1}{2\pi}\int_0^{2\pi ua}J_0(t)\frac{1}{u^2}tdt$$

which we can then integrate to get

$$P(u,0)=\frac{2\pi ua}{u^2}J_1(2\pi ua)=4\pi a^2\,\frac{J_1(2\pi ua)}{2\pi ua}$$

Using the fact that it is circularly symmetric, we can then write this in two dimensions as

$$P(u,v) = 4\pi a^2 \frac{J_1(2\pi a w)}{2\pi a w} \quad where \quad w^2 = u^2 + v^2$$

We then know from earlier work that the intensity in the back focal plane of a lens of focal length f is the square modulus of the Fourier transform scaled by λf, giving

$$I(s,t) = 4I_0 \left| \frac{J_1\left(\frac{2\pi}{\lambda f} ar\right)}{\frac{2\pi}{\lambda f} ar} \right|^2 \quad where \quad r^2 = s^2 + t^2$$

where we have incorporated the various constants into I_0 and this is the intensity at the center of the pattern.

To analyze this we need to consider the shape of $J_1(x)/x$ and $| J_1(x)/x |^2$ the which are plotted in figure 05.

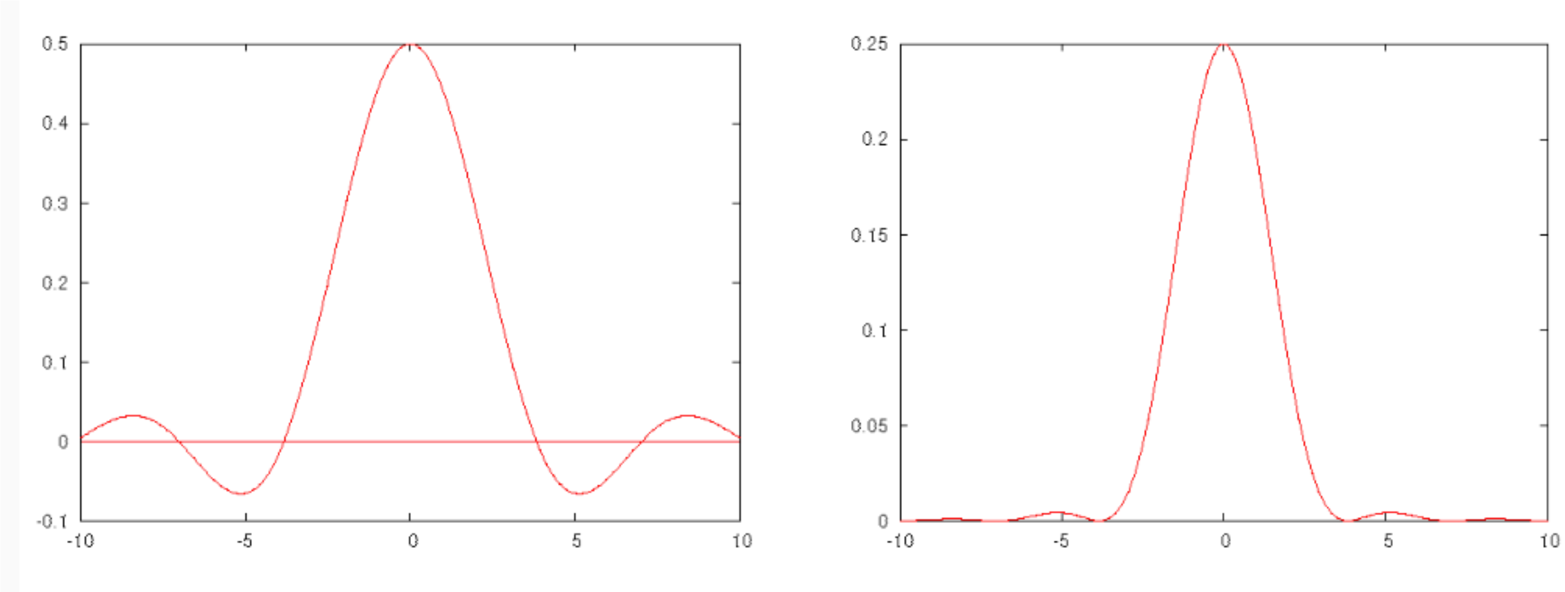


Figure 05: Plot of $J_1(x)/x$ and $|J_1(x)/x|^2$

These functions have a similar shape to sin() and sinc²(), but have

1. Peak value of $J_1(x)/x$ at x = 0 is 1/2.
2. Zeros located at 3.832, 7.016, 10.174, 13.324
3. Secondary maximas are lower that a sinc, or sinc² respectively.

The two dimensional function $|J1(r)/r|^2$ is circularly symmetric and is plotted in figure 06 along with its log(), which makes the ring patterns more obvious.

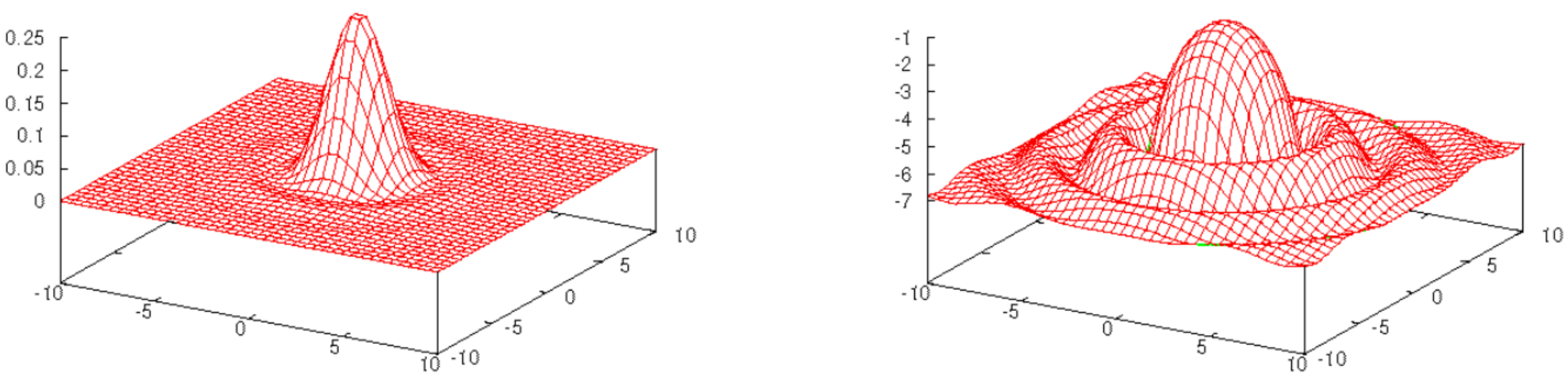


Figure 06: Two-dimensional surface plot of $|J1(r)/r|^2$ and its log()

This distribution is known as the Airy Pattern.

The key feature of this pattern is the location of first zero, being at 3.832.

If we have lens of focal length f and diameter d, then if we view a distant point object, the front of the lens will be illuminated by approximately plane waves.

The aperture of the lens is circular, so in the back focal plane we will get a diffraction pattern from the aperture.

The distant object will therefore be imaged as the distribution I(s,t) defined above as shown in figure 07.

Note that a compound lens can be represented be a simple ideal lens located in the back principal plane.

The key feature of this distribution is the location of the first zero, being at radius

$$r_0 = \frac{1.22\lambda f}{2a} = \frac{1.22\lambda f}{d}$$

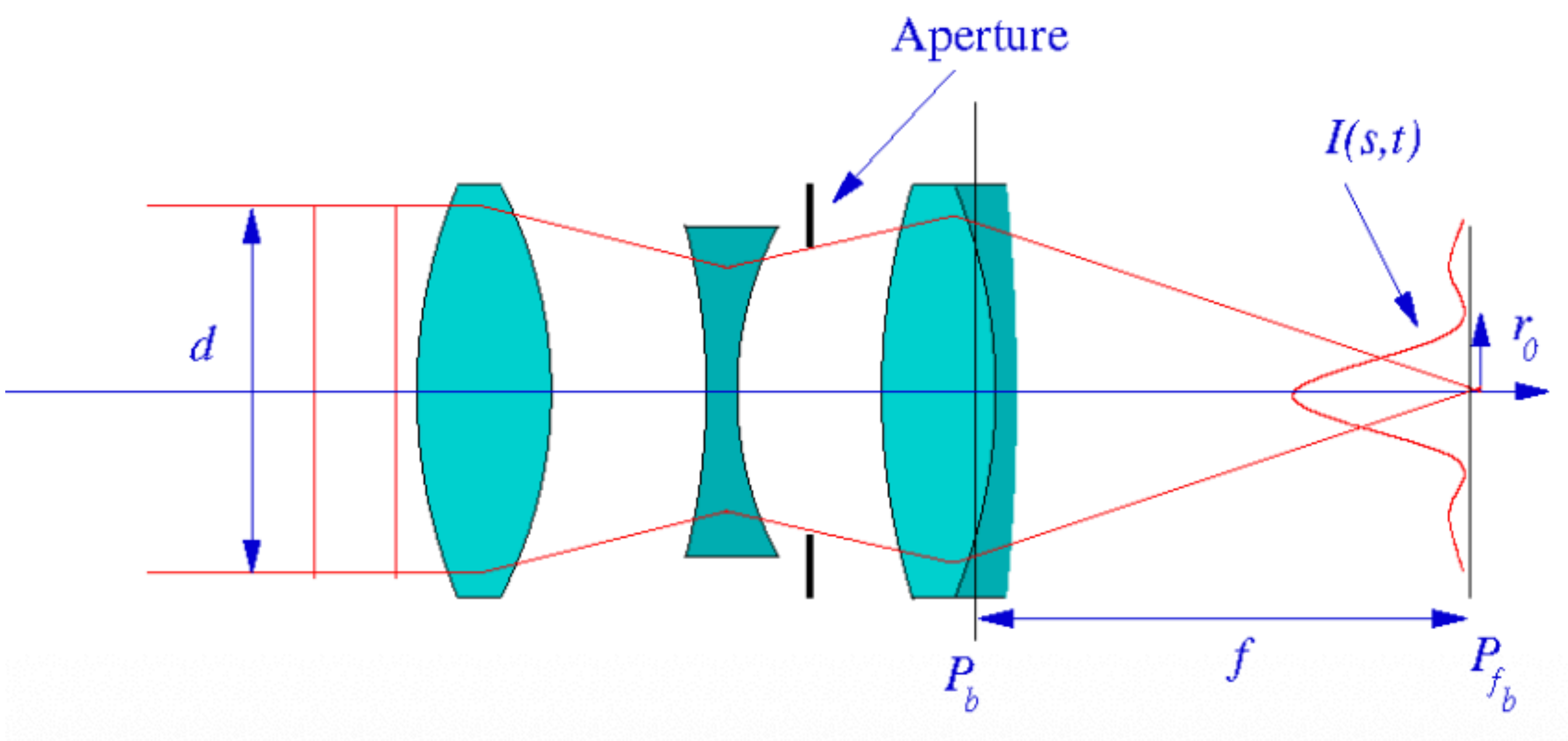


Figure 07: Point spread function of a a lens

It is the same of all systems with the same ratio f/d.

This shows that the image of a distant point object will be imaged as a bright central spot surrounded by a series of rings, known as the Airy Rings, the whole pattern being known as the **Point Spread Function** of the system.

Spatial Resolution of an Optical System

Just as the size of the grating limited the spectral resolution of a spectrometer, then the Point Spread Function limits the spatial resolution of an imaging system.

If we consider two distant point objects with angular separation $\Delta\theta$, then as shown in figure 08, in the back focal plane of an imaging system with focal length f, we get two point spread functions with, for small $\Delta\theta$, their peaks separated by

$$s = f\Delta\theta$$

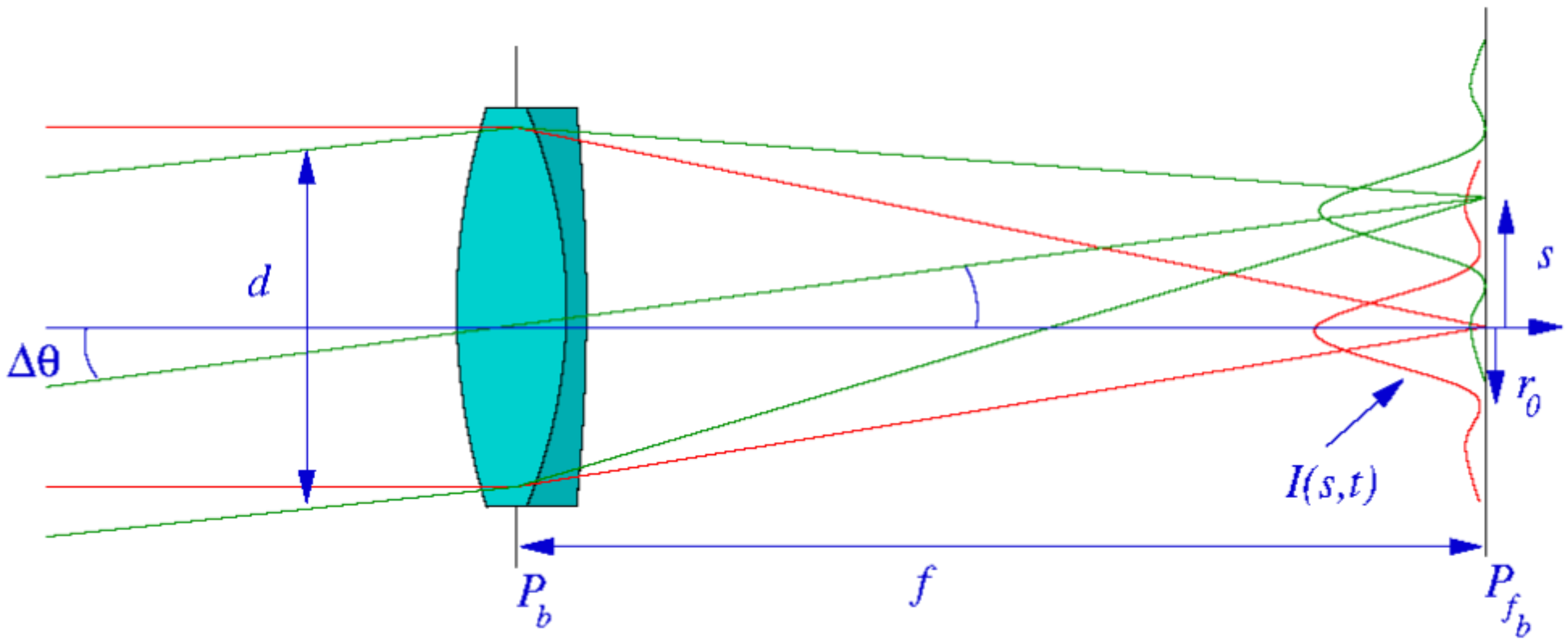


Figure 08: Image of two distant point objects.

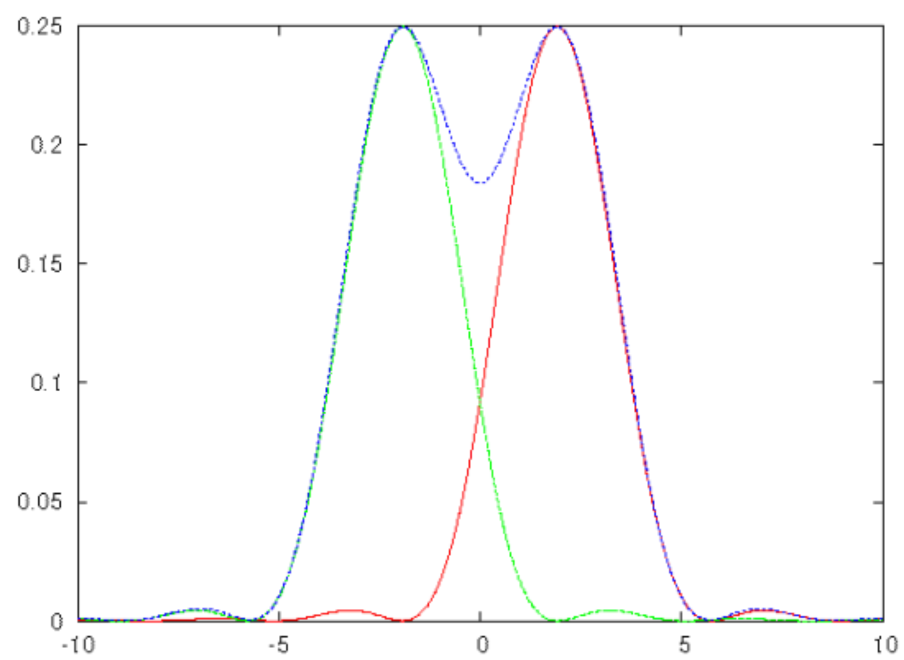
There are three possible conditions, these are:

1. $s \gg r_0$, two well separated point spread functions, distant points are resolved.
2. $s \ll r_0$, the two point spread functions merge into one, and the distant points are not resolved,
3. $s \approx r_0$, there will be a limit where the distant points are just resolved.

There are a range of resolution limits.

The most useful and practical is the Rayleigh Limit, when $s = r_0$, so the peak of one point spread function is at the zero of the other.

This results in a twin-peak with a dip of about 20% between them as shown in figure 09.



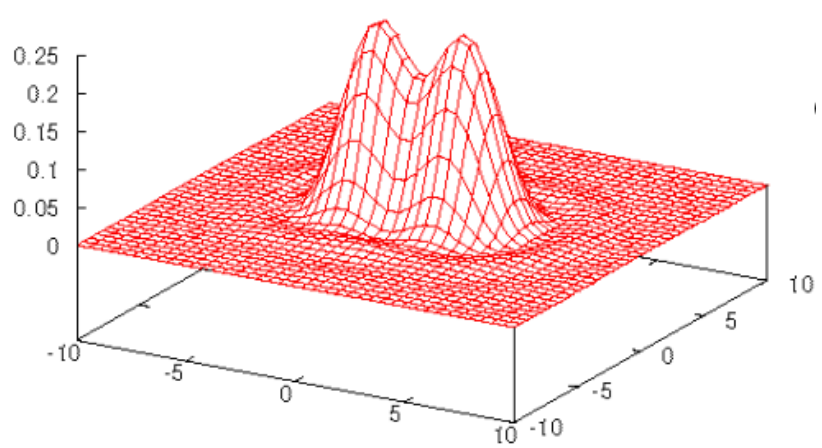


Figure 09: Plot of point spread functions at the Rayleigh resolution limit on one and two dimensions

In terms of angular resolution, we therefore get the Rayleigh Limit to be

$$\Delta\theta_0 = 1.22\frac{\lambda}{d}$$

It depends only of the diameter of the imaging system and the wavelength of light being imaged.

This limit applies to a whole range of optical system, including the eye, telescopes, microscopes and cameras.

This basic theory also applies to all other waves phenomena, including radar, microwaves, and even in acoustics!

When using this analysis for imaging system, for example a microscope, or camera, the image is not formed in the back focal plane, but rather in the image plane, a distance v from the back Principal plane, where v is given by the Gaussian Lens formula.

Under these conditions, all the above analysis is still valid, but the scaling from Fourier Transform to diffracted intensity become λv rather than λf.

The Annular Aperture

Most large astronomical telescopes have a fairly large central obstruction where the secondary mirror is located giving an annular aperture rather than a circular one.

As in the previous discussions, the diffraction pattern will be the scaled square modulus of the Fourier transform of the pupil, so we need to consider the Fourier transform of an annular aperture as shown in figure 10.

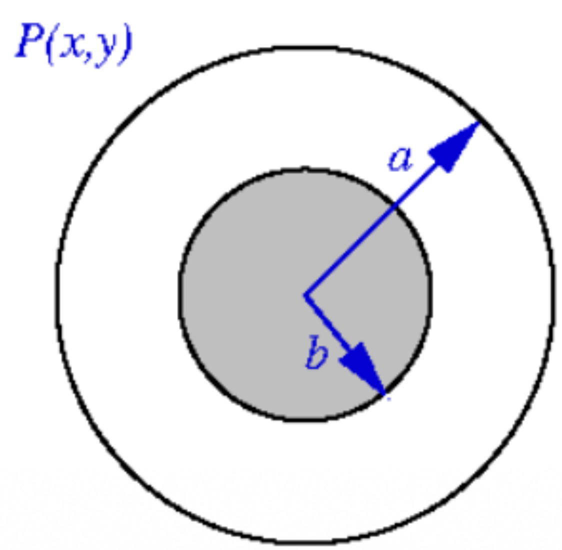


Figure 10: layout of an annular pupil

This can be mathematically written as,

$$p(x,y) = \begin{cases} 1 & when \quad a^2 < x^2 + y^2 < b^2 \\ 0 & \text{otherwise} \end{cases}$$

Using polar coordinates as earlier, the Fourier transform of this is given by

$$P(u,v) = \int_a^b \int_0^{2\pi} \exp(-i2\pi(u\rho\cos\theta + v\rho\sin\theta)\rho d\rho d\theta$$

which only differs from the circular case in the limits of the radial integration.

Again p(x,y) is radially symmetric, so the Fourier transform will the radially symmetric.

The angular integration is identical to the circular aperture, so that along one radial direction,

$$P(u,0) = 2\pi\int_a^b J_0(2\pi u\rho)\rho d\rho = 2\pi\int_0^b J_0(2\pi u\rho)\rho d\rho - 2\pi\int_0^a J_0(2\pi u\rho)\rho d\rho$$

since $J_0()$ is a symmetric function.

This is just the difference between the Fourier transform of two circular apertures, one of radius b and one of radius a, so we get

$$P(u,v) = 4\pi\left[b^2\frac{J_1(2\pi bw)}{2\pi bw} - a^2\frac{J_1(2\pi aw)}{2\pi aw}\right] \quad where \quad w^2 = u^2 + v^2$$

The intensity of the point spread function is then just the scaled modulus squared of this, giving the rather complicated expression of

$$I(s,t) = B\left| b^2 \frac{J_1\left(\frac{2\pi}{\lambda f} br\right)}{\frac{2\pi}{\lambda f} br} - a^2 \frac{J_1\left(\frac{2\pi}{\lambda f} ar\right)}{\frac{2\pi}{\lambda f} ar} \right| \quad where \quad r^2 = s^2 + t^2$$

where the constant B is given by

$$B = \frac{4I_0}{b^2 - a^2}$$

and I_0 is the intensity at the center of the pattern.

The significance of this complicated expression only becomes apparent when it is plotted and compared to the intensity from an unobstructed aperture of the same outer diameter as in figure 11, which is plotted for a = 0.7b.

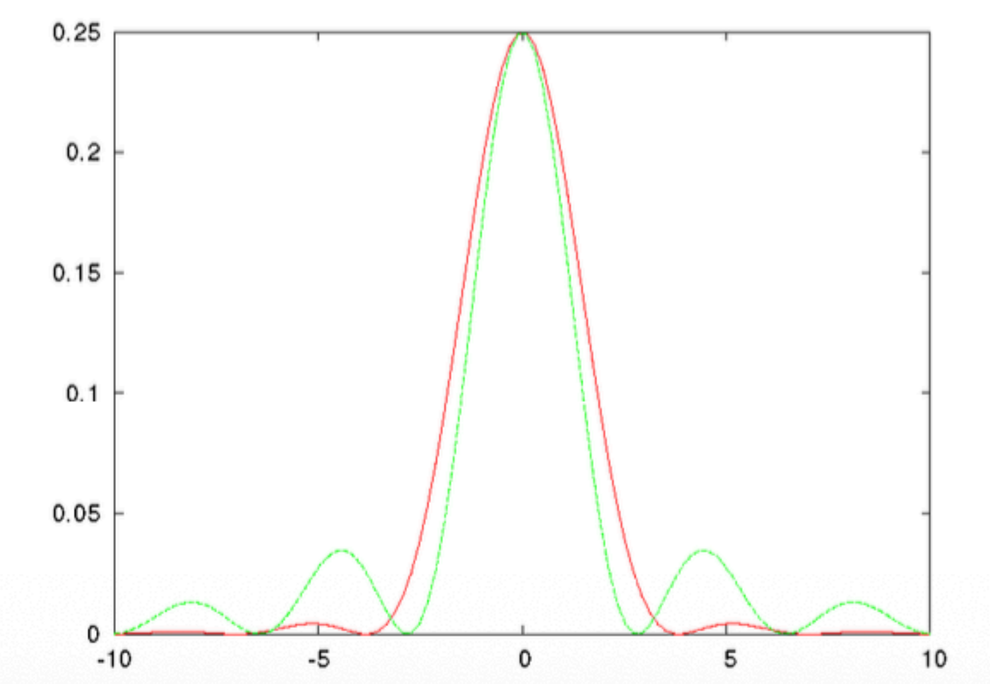


Figure 11: Comparison of annular point spread function with point spread function for unobstructed aperture.

The important feature are:

1. Both have the basic shape with a large central peak and a series of secondary maximas.

2. The secondary maximas for the obstructed aperture are higher than for the unobstructed case.

3. The central peak is narrower for the obstructed aperture, which is not what is expected.

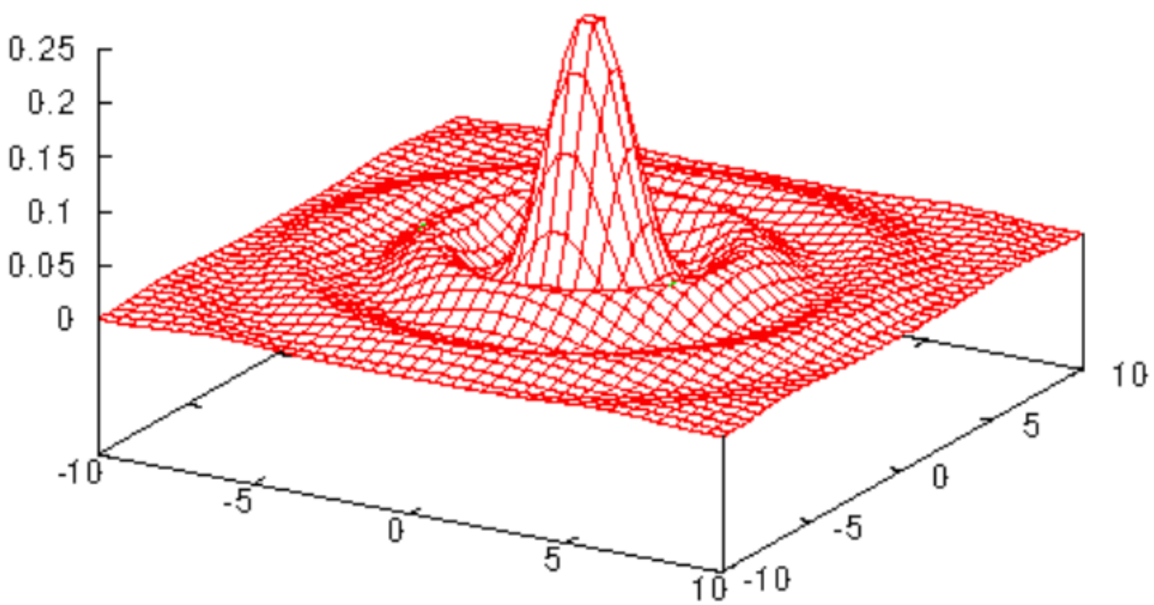


Figure 12: Surface plot of point spread function of an annular aperture with a = 0.7b.

As we have seen earlier, the angular resolution for a telescope of focal length f is given by

$$\Delta\theta_0 = \frac{r_0}{d}$$

where is the radius of the first zero of the point spread function, so since the annular aperture has a narrow point spread function, its angular resolution will be smaller and thus improved.

This effect is shown graphically in figure 13 for an annular aperture with a = 0.7b.

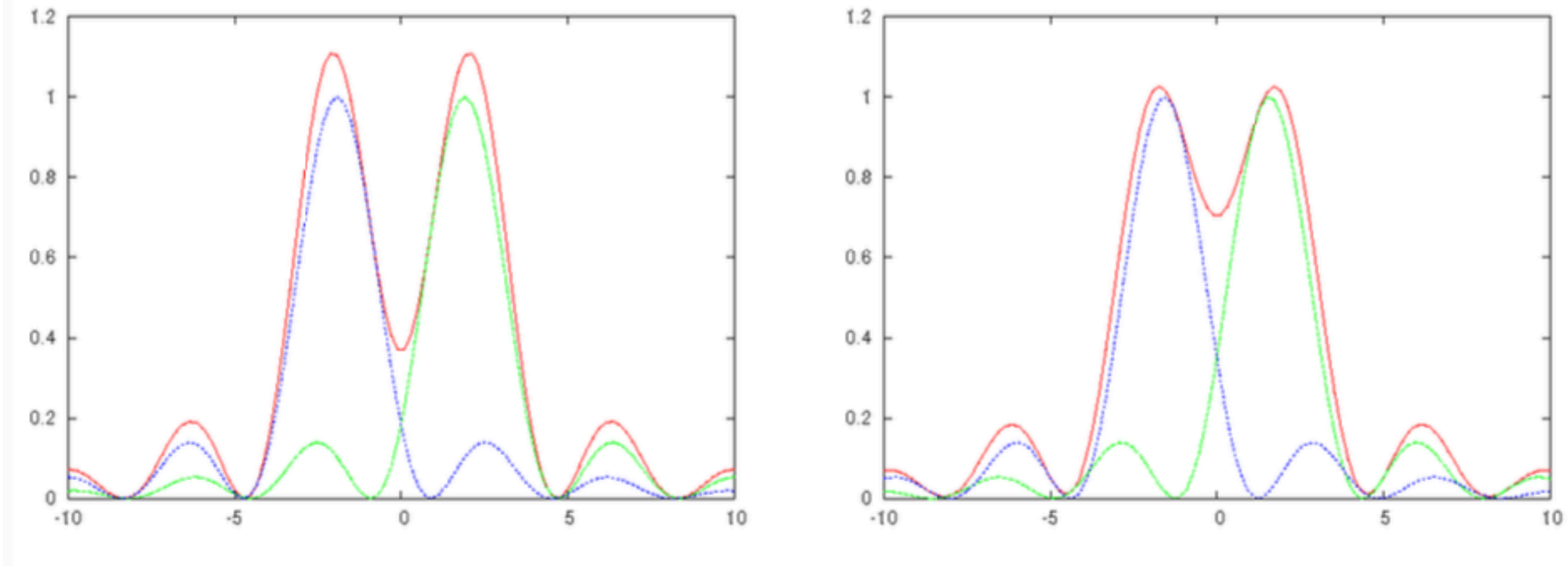


Figure 13: Plot of intensity distribution or annular aperture with two stars (a) at the Rayleigh resolution criteria of Δθ = 1.22 λ/d and (b) below the Rayleigh criteria at Δθ = λ/d

The intensity distribution at the Rayleigh limit is plotted on the left, showing a very clear dip, so the points are well resolved.

The plot on the right shows the stars separated by

$$\Delta\theta = \frac{\lambda}{d}$$

his improved resolution does have a cost, that being the absolute peak intensity is proportional to the open area of the pupil which is clearly reduced by the presence of a central stop.

Resolution in Real Astronomical Telescopes

The above analysis assumes that a distant point source results in plane wavefronts across the input aperture of the system, with the resolution of the system limited only by diffraction.

This is valid for a small high quality system, for example microscopes, the human eye, small telescopes, and high quality camera systems.

When we consider large astronomical telescopes, there is another very significant effect, that of the atmosphere.

The refractive index of a gas depends on it local density, thus for air, it depends on pressure and temperature.

Thus pressure and temperature gradients in the atmosphere result in local refractive index, and thus optical path length variations, as illustrated in figure 14.

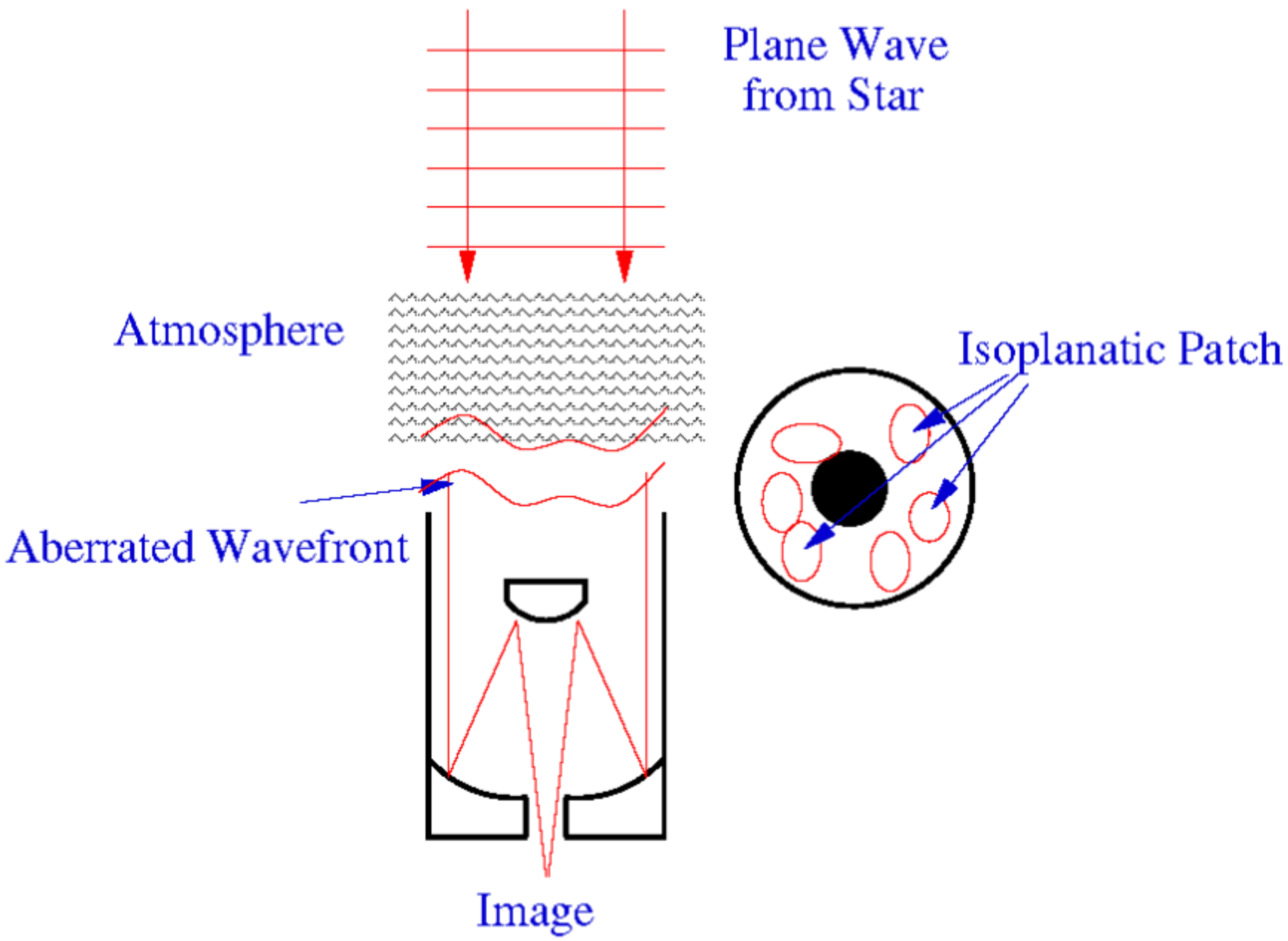


Figure 14: Schematic of a real terrestrial telescope.

This is also why start twinkle.

As the atmosphere moves, this introduces a time varying phase aberration across the aperture of the telescope with the speed and extent of the variation depending on the local atmospheric conditions.

This is known as the seeing conditions.

In conditions of good seeing, the phase aberration can be considered constant for approximately 1/10th of second and plane over a region, known as the isoplanatic patch.

Under these conditions, each isoplanatic patch acts as an individual telescope, with the long, compared to the time constant of the atmosphere, exposure.

The image of the star is an intensity summation of the image from each patch.

The resolution is therefore limited by the size of the isoplanatic and not the overall aperture of the telescope.

In areas of good seeing, the isoplanatic patch is 100 - 150 mm in diameter, giving an effective angular resolution of

$$\Delta\theta_e \approx 6\times10^{-6}\,Rad \approx 1\sec\,of\;arc$$

for λ = 500nm and d $\approx$100mm.

Since the isoplanatic patches are time-varying, and not circular, the shape of the point spread function will be time averaged and will approximate a Gaussian with it radius given by the e^{-2} intensity position, given, for a telescope of focal length f approximately by $f\,\Delta\theta e$.

There are a range of schemes to minimize this effect:

Telescope Location: Locate the telescope in areas of good seeing, typically high on a mountain plateau in an area of stable atmospheric conditions, Hawaii, Tenerif, Arizona are typical examples.

The ultimate is the remove the atmosphere, hence the Hubble and James Webb Space Telescopes!

Short Exposures: Use exposure times short compared to the movement of the atmosphere and combine after digital processing.

Adaptive Optics: Analyze the input distorted wavefront and correct its shape as shown in figure 15.

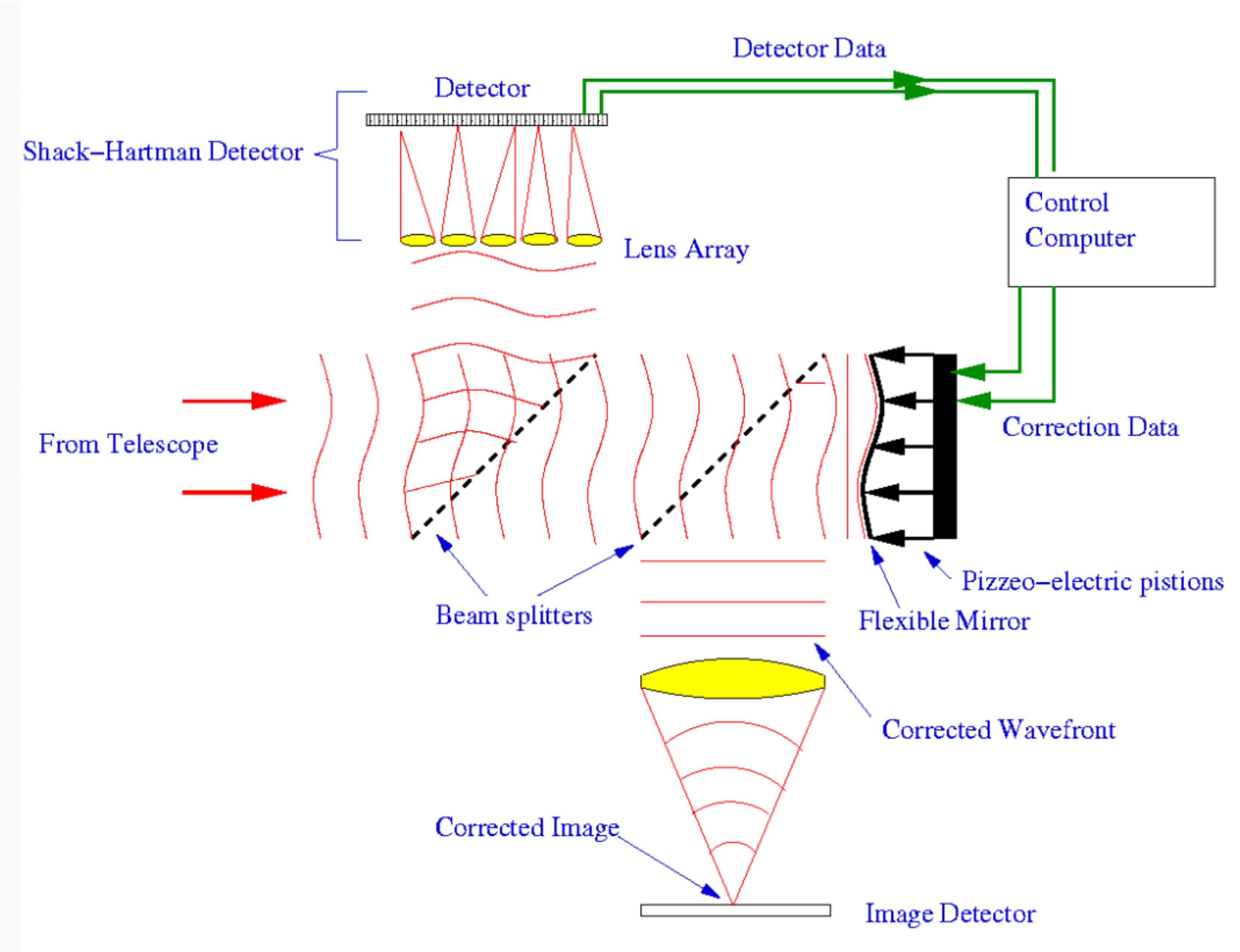


Figure 15: Layout of adaptive optical system for wavefront correction

1. Part of the beam is split off and send to a detector.

2. The shape of this distorted wavefront is analyzed by a detector system, typically consisting or an array of micro-lenses.

3. The position on the image from each lens give the local gradient of the wavefront, allowing the shape to be digitally reconstructed.

4. The wavefront shape is used to distort a flexible mirror, which typically is glass mirror that can be distorted by piezoelectric stacks.

5. Reflected, corrected wavefront is split-off and used to form the image, free of the input aberration.

6. The wavefront sensing and correction run continuously at a rate comparable to the movement time of the atmosphere, typically 1/10th second.

This system assumes that there is a single bright guide star in the field of view which acts as a point source, and hence a bright source of plane waves.

The correction is then valid over the whole field of view so that dimmer objects close to the guide star are also corrected.

This system gives significant improvement in resolution, and under optimal conditions, with a bright guide star we can get within 20% of the Rayleigh diffraction limit of the full telescope aperture.

The Fresnel Zone Plate

In the Fraunhofer formalism light is represented by plane waves, and the distance between light-source and scattering object or scattering object and observer is assumed to be very large compared to the dimensions of the obstacle in the light path.

If you drop these conditions, the light phenomena observed are described by the Fresnel formalism and the light waves are represented by spherical waves rather than plane waves. Figure 16 shows the spherical surface corresponding to the primary wave front at some arbitrary time t after it has been emitted from S at t = 0.

As illustrated the wave front is divided into a number of annular regions.

The boundaries of these regions correspond to the intersections of the wave front with a series of spheres centered at P of radius $r_0 + \lambda/2$, $r_0 + \lambda$, $r_0 + 3\lambda/2$, etc.

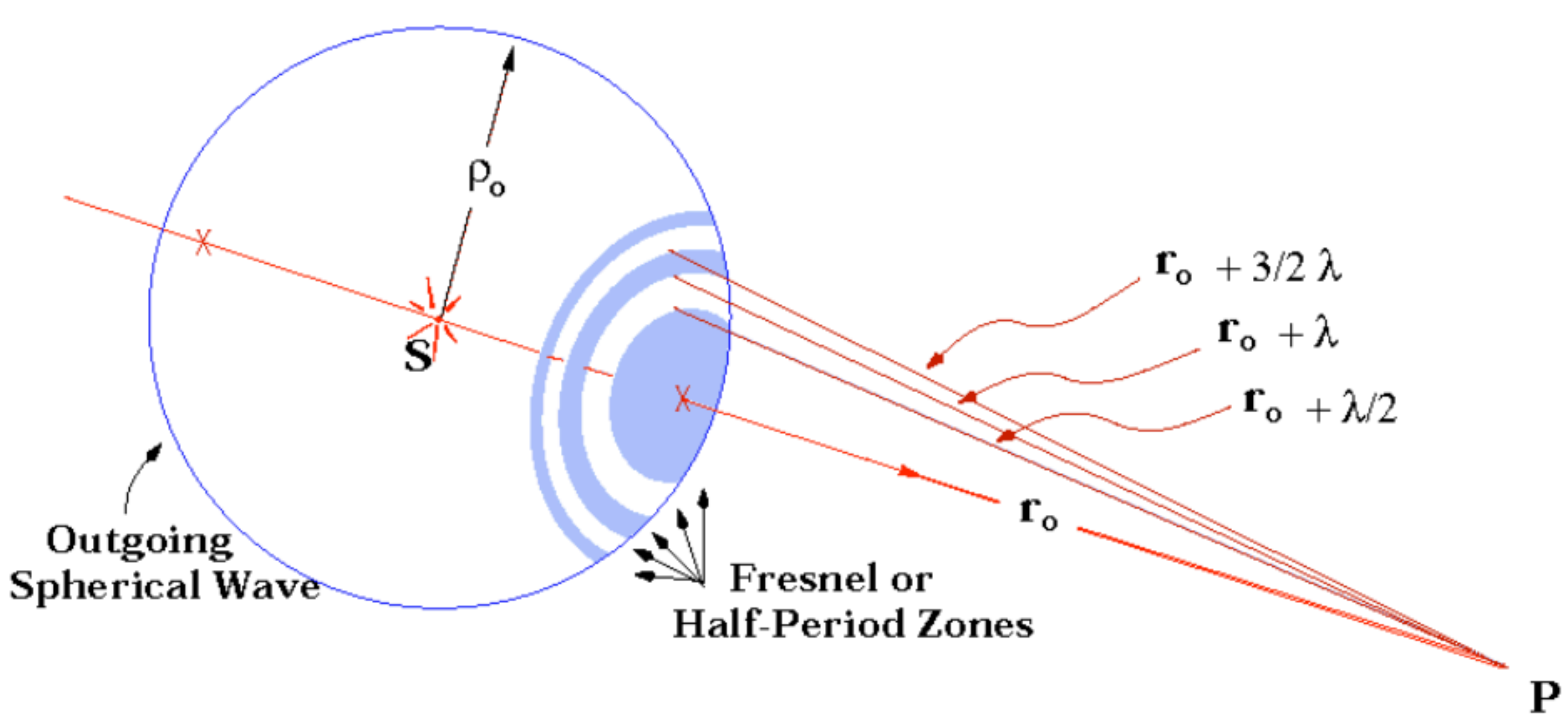


Figure 16: Fresnel Zone - The surface of the spherical wavefront generated at point S in the figure above has been divided into several **Fresnel Zones**. Each area is comprised of points that are close to the same distance from point P and thus all the secondary wavelets emanating from within the same Fresnel Zone will add constructively at P.

These are Fresnel or half-period zones.

The sum of the optical disturbances from all m zones of P is

$$E = E_1 + E_2 + E_3 + E_4 + \ldots\ldots\ldots\ldots + E_m \tag{24}$$

Because of the phase of the light passing through each consecutive zone increases by $\lambda/2$ (= change in phase by π) the amplitude E_P of the zone alternates between positive and negative values depending on whether m is odd or even.

As a result contributions from adjacent zones are out of phase and tend to cancel.

This suggests that we would observe a tremendous increase in irradiance (intensity) of P if we remove all of either the even or odd zones.

A screen which alters the light, either in amplitude or phase, coming from every other half-period is called a zone plate.

Examples are shown in Figure 17.

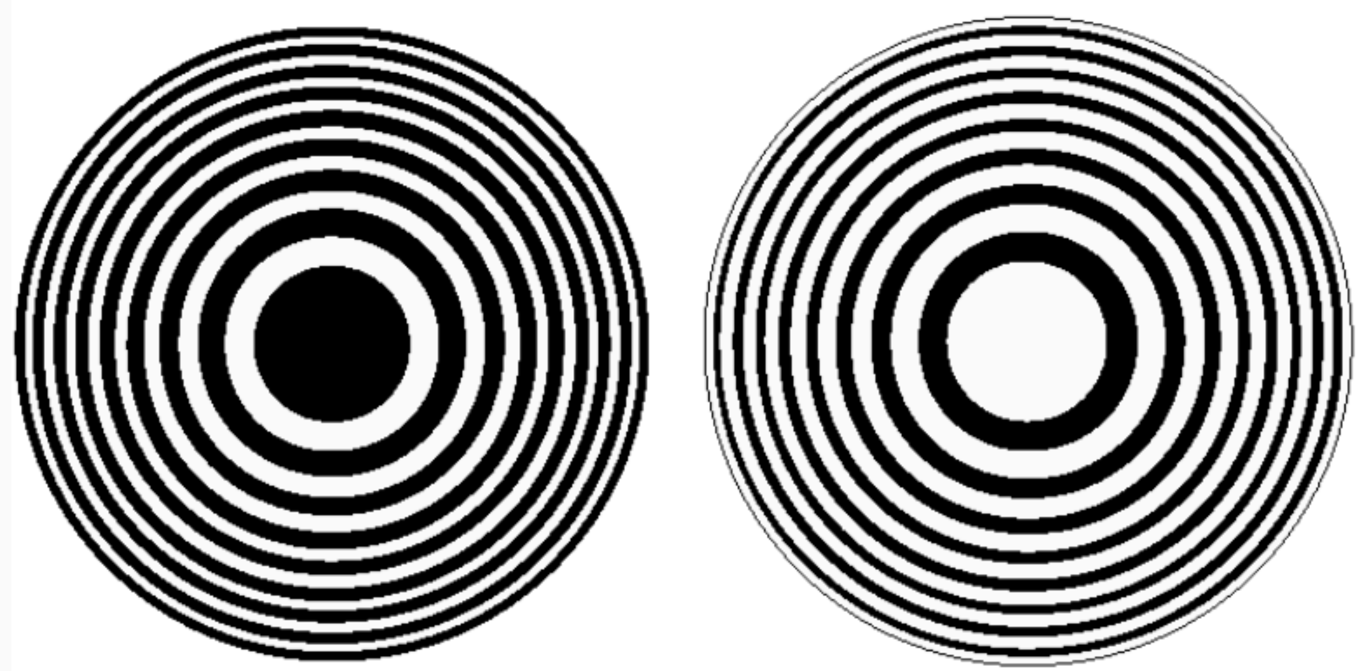


Figure 17: Fresnel Zone Plates

Suppose that we construct a zone plate which passes only the first 20 odd zones and obstructs the even ones,

$$E = E_1 + E_2 + E_3 + E_P + \dots\dots\dots + E_{39} \qquad (25)$$

and that each of these terms is approximately equal.

For a wavefront passing through a circular aperture the size of the fortieth zone, the disturbance at P (demonstrated by Fresnel but not obvious) would be $E_1/2$ with corresponding intensity $I = (E_1/2)^2$.

However, with the zone plate in place $E = 20E_1$ at P and $I = (20E_1)^2$.

The intensity I of the light at P has been increased by a factor of

$$\frac{(20E_1)^2}{(E_1/2)^2} = 1600$$

The zone plate acts as a lens with the focusing being done by interference rather than by refraction!

To calculate the radii of the zones shown in Figure 17, refer to Figure 18.

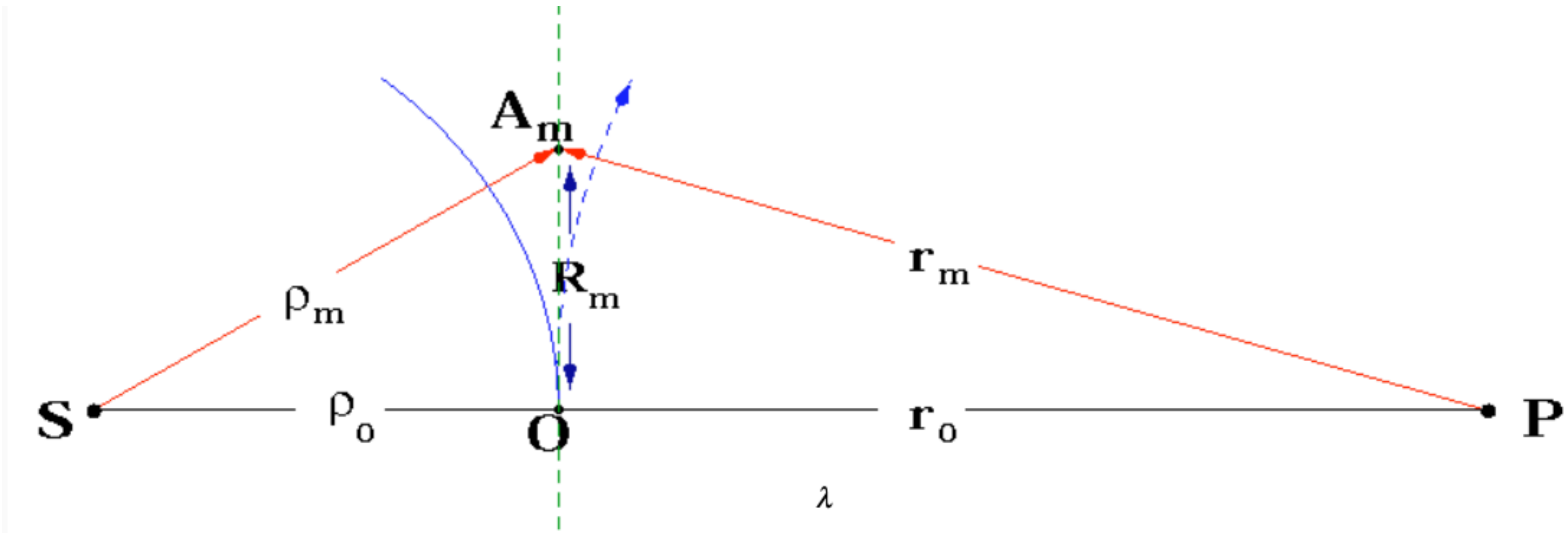


Figure 18. Fresnel Zone Radii for Fresnel Zone Plates

The outer edge of the mth zone is marked with the point A_m.

By definition, a wave which travels the path S-A_m-P must arrive out of phase by m/2 with a wave which travels the path S-O-P, that is,

$$\text{phase difference} = \left(\rho_m + r_m\right) - \left(\rho_0 + r_0\right) = m\lambda / 2 \qquad (26)$$

According to the Pythagorean theorem $\rho_m = (R_m^2 + \rho_0^2)^{1/2}$ and $r_m = (R_m^2 + r_0^2)^{1/2}$.

Expand both these expressions using the binomial series and retain only the first two terms ($\rho_m \approx \rho_0 + R_m^2 / 2\rho_0$ and $r_m \approx r_0 + R_m^2 / 2r_0$).

Substituting into Equation (25) gives the criterion the zone radii must satisfy to maintain the alternating λ/2 phase shift between zones.

$$R_m^2 = \frac{m\lambda}{\frac{1}{\rho_0} + \frac{1}{r_0}} \tag{27}$$

The width of zone m is proportional to $\sqrt{m}$.

Rewriting Equation (27) as,

$$\frac{1}{\rho_0} + \frac{1}{r_0} = \frac{m\lambda}{R_m^2} = \frac{1}{f_1} \tag{28}$$

puts it in a form identical to the thin lens equation ($1/o + 1/i = 1/f$ with primary focal length f_1,

$$f_1 = \frac{R_m^2}{m\lambda} \quad , \quad m = 1,2,3,.... \tag{29}$$