

Problems and Solutions

Classical Mechanics

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Assignment 1 - Math Review

This week we will review some important mathematical concepts.

Questions while reading (make sure you can answer these!):

- How is a vector defined in terms of its transformation properties?
- What is a direction cosine? What is an orthogonal matrix?
- How can rotations be used to demonstrate the non-commutativity of matrix multiplication?
- What's the difference between a finite and an infinitesimal rotation?
- How do you write velocity and acceleration vectors in spherical and cylindrical coordinates?
- How do you take the derivative of a vector? How do you integrate a vector?

Problem 1-1 - Simple rotation matrix

Find the transformation matrix that rotates the axis x_3 of a rectangular coordinate system 45° toward x_1 around the x_2 -axis.

Problem 1-3 - Another rotation matrix - think symmetry!

Find the transformation matrix that rotates a rectangular coordinate system through an angle of 120° about an axis making equal angles with the original three coordinate axes.

Problem 1-5 - Determinant of a transformation, i.e., orthogonal, matrix

Show by direct expansion that $|\boldsymbol{\lambda}|^2 = I$. For simplicity, take $\boldsymbol{\lambda}$ to be a two-dimensional orthogonal transformation matrix. **Problem 1-9** - Vector manipulations

For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad , \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

- (a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$
- (b) component of $\mathbf{B}$ along $\mathbf{A}$
- (c) angle between $\mathbf{A}$ and $\mathbf{B}$
- (d) $\mathbf{A} \times \mathbf{B}$
- (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

Problem 1-10 - Velocity and acceleration for an elliptical orbit

A particle moves in a plane elliptical orbit described by the position vector

$$\mathbf{r} = 2b \sin \omega t \mathbf{i} + b \cos \omega t \mathbf{j}$$

- (a) Find $\mathbf{v}$, $\mathbf{a}$, and the particle speed
- (b) What is the angle between $\mathbf{v}$ and $\mathbf{a}$ at time $t = \pi/2\omega$?

Problem 1-11 - Triple scalar product

Show that the *triple scalar product* $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ can be written as

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Show also that the product is unaffected by the interchange of the scalar and vector product operations or by a change in the order of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, as long as they are in cyclic order; that is,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})) = (\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})) = (\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})), \text{ etc.}$$

We may therefore use the notation $\mathbf{ABC}$ to denote the triple scalar product. Finally, give a geometric interpretation of $\mathbf{ABC}$ by computing the volume of a parallelepiped defined by the three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

Problem 1-12 - Analytic geometry

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three constant vectors drawn from the origin to the point A, B, C . What is the distance from the origin to the plane defined by the

points A, B, C ? What is the area of the triangle ABC ?

Problem 1-17 - Law of cosines

Obtain the cosine law of plane trigonometry by interpreting the product $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$ and the expansion of the product.

Problem 1-20 - Fun with ε_{ijk}

Show that

(a) $\sum_{i,j} \varepsilon_{ijk} \delta_{ij} = 0$

(b) $\sum_{j,k} \varepsilon_{ijk} \varepsilon_{ljk} = 2\delta_{il}$

(c) $\sum_{i,j,k} \varepsilon_{ijk} \varepsilon_{ik} = 6$

Problem 1-21 - Triple scalar product via ε_{ijk}

Show (see also Problem 1-11) that

$$\mathbf{ABC} = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j C_k$$

Problem 1-22 - An important ε_{ijk} relationship and a vector identity

Evaluate the sum

$$\sum_k \varepsilon_{ijk} \varepsilon_{lmk}$$

(which contains 3 terms) by considering the result for all possible combinations of i, j, l, m ; that is,

(a) $i = j$

(b) $i = l$

(c) $i = m$

(d) $j = l$

(e) $j = m$

(f) $l = m$

(g) $i \neq l$ or m

(h) $j \neq l$ or m

Show that

$$\sum_k \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

and then use this result to prove that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Problem 1-25 - Acceleration in spherical coordinates

Find the components of the acceleration vector $\mathbf{a}$ in spherical coordinates.

Problem 1-26 - Velocity and acceleration for constrained motion

A particle moves with $v = \text{constant}$ along the curve $r = k(1 + \cos \theta)$ (a *cardioid*). Find $\ddot{\mathbf{r}} \cdot \mathbf{e}_r = \mathbf{a} \cdot \mathbf{e}_r$, $|\mathbf{a}|$, and $\dot{\theta}$.

Problem 1-37 - Surface integral via Gauss's theorem

Find the value of the integral

$$\int_S \mathbf{A} \cdot d\mathbf{a}$$

where $\mathbf{A} = (x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and the surface S is defined by the sphere $R^2 = x^2 + y^2 + z^2$. Do the integral directly and also by using Gauss's theorem.

Problem 1-38 - Surface integral via Stoke's theorem

Find the value of the integral

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$$

if the vector $\mathbf{A} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the surface defined by the paraboloid $z = 1 - x^2 - y^2$, where $z \geq 0$.

Assignment 2 - Review of Newtonian Mechanics

This week we'll review the Newtonian approach to classical mechanics. The program is summarized in Newton's second law $\mathbf{F} = m\mathbf{a}$ (or, more correctly, as given above). A determination of the forces gives the accelerations; velocities and positions are then computed via integration. Newton's laws completely describe all of the phenomena of classical mechanics and, at this point in your careers, I'm sure you all are intimately familiar with them. What you may not be aware of is that the Newtonian approach is neither the most convenient nor the most general formalism. We will be learning other quite elegant formulations of classical mechanics in this course. However, there are still many cases where good old $\mathbf{F} = m\mathbf{a}$ remains the method of choice. Note that all formulations of mechanics, from the pedestrian to the polished, lead to the same equations of motion. Unfortunately, these differential equations are often intractable and thus one must resort to perturbation techniques or numerical methods. You learned about numerical integration of ODE's in Mathematical Methods and we'll review some of that here. You should get used to turning to the computer as a tool in solving problems for this class. One can only get so far with analytic techniques and the "real" world is a complicated place! But never fear, I won't deny you the delight of elegant analytical solutions. Note however that even for these, it is often quite instructive to make plots of your final results to get a better feeling for the physical behavior described by your equations.

Problem 2-12 - Particle in drag

A particle is projected vertically upward in a constant gravitational field with an initial speed v_0 . Show that if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it returns to the initial position is

$$\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

Problem 2-13 - Weird resistive force but elementary integrals

A particle moves in a medium under the influence of a retarding force equal to $mk(v^3 + a^2v)$, where k and a are constants. Show that for any value of

the initial speed the particle will never move a distance greater than $\pi/2ka$ and that the particle comes to rest only for $t \rightarrow \infty$.

Problem 2-14 - Shooting uphill

A projectile is fired with initial speed v_0 at an elevation angle of α up a hill of slope β ($\alpha > \beta$).

- (a) How far up the hill will the projectile land?
- (b) At what angle α will the range be maximum?
- (c) What is the maximum range?

Problem 2-17 - Home run hitter

A strong softball player smacks the ball at a height of 0,7 m above home plate. The ball leaves the player's bat at an elevation angle of 35° and travels toward a fence 2 m high and 60 m away in center field. What must the initial speed of the softball be to clear the center field fence? Ignore air resistance.

Problem 2-18 - Home runs harder with air friction, numerics required

Include air resistance proportional to the square of the ball's speed in the previous problem. Let the drag coefficient be $c_W = 0.5$, the softball radius be 5 cm and the mass be 200g.

- (a) Find the initial speed of the softball needed to clear the fence.
- (b) For this speed, find the initial elevation angle that allows the ball to most easily clear the fence. By how much does the ball now vertically clear the fence.

Problem 2-22 - Charged particle in an EM field, plot part (d) results

The motion of a charged particle in an electromagnetic field can be obtained from the *Lorentz equation* for the force on a particle in such a field. If the electric field vector is $\mathbf{E}$ and the magnetic field vector is $\mathbf{B}$, the force on a particle of mass m that carries a charge q and has a velocity $\mathbf{v}$ is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

where we assume the $v \ll c$ (speed of light).

- (a) If there is no electric field and the particle enters the magnetic field in a direction perpendicular to the lines of magnetic flux, show that the trajectory is a circle with radius

$$r = \frac{mv}{qB} = \frac{v}{\omega_c}$$

where $\omega_c \equiv qB/m$ is the *cyclotron frequency*.

- (b) Choose the z -axis to lie in the direction of $\mathbf{B}$ and let the plane containing $\mathbf{E}$ and $\mathbf{B}$ be the yz -plane. Thus

$$\mathbf{B} = B\mathbf{k} \quad , \quad \mathbf{E} = E_y\mathbf{j} + E_z\mathbf{k}$$

Show that the z component of the motion is given by

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m} t^2$$

where

$$z(0) = z_0 \quad \text{and} \quad \dot{z}(0) = \dot{z}_0$$

- (c) Continue the calculation and obtain expressions for $\dot{x}(t)$ and $\dot{y}(t)$. Show that the time averages of these velocity components are

$$\langle \dot{x} \rangle = \frac{E_y}{B} \quad , \quad \langle \dot{y} \rangle = 0$$

(Show that the motion is periodic and then average over one complete period.)

- (d) Integrate the velocity equations found in (c) and show (with the initial conditions $x(0) = -A/\omega_c$, $\dot{x}(0) = E_y/B$, $y(0) = 0$, $\dot{y}(0) = A$) that

$$x(t) = \frac{-A}{\omega_c} \cos \omega_c t + \frac{E_y}{B} t \quad , \quad y(t) = \frac{A}{\omega_c} \sin \omega_c t$$

These are the parametric equations of a trochoid. Sketch the projection of the trajectory on the xy -plane for the cases

- (i) $A > \left| \frac{E_y}{B} \right|$
- (ii) $A < \left| \frac{E_y}{B} \right|$

(iii) $A = \left| \frac{E_y}{B} \right|$ (The last case yields a cycloid.)

Problem 2-23 - Complicated force but elementary integrals, plot results

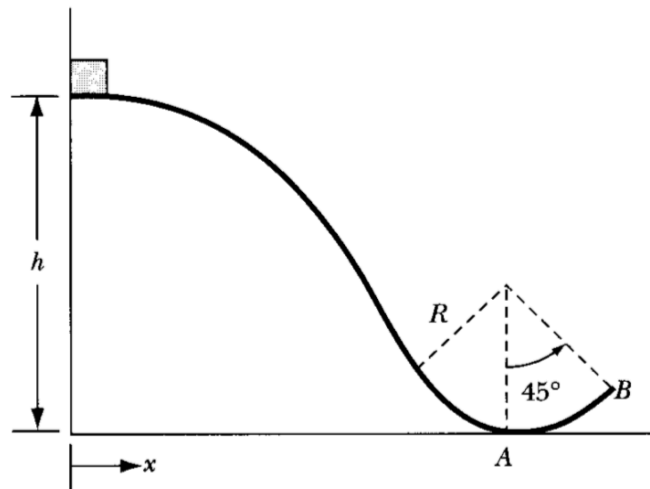
A particle of mass $m = 1$ kg is subjected to a one-dimensional force

$$F(t) = kte^{-\alpha t}$$

where $k = 1 \text{ N/s}$ and $\alpha = 0.5 \text{ s}^{-1}$. If the particle is initially at rest, calculate and plot with the aid of a computer the position, speed, and acceleration of the particle as a function of time.

Problem 2-25 - Block on a track

A block of mass $m = 1.62$ kg slides down a frictionless incline as shown in the figure below.



The block is released a height $h = 3.91$ m above the bottom of the loop.

- What is the force of the inclined track on the block at the bottom (point A)?
- What is the force of the track on the block at point B?
- At what speed does the block leave the track?

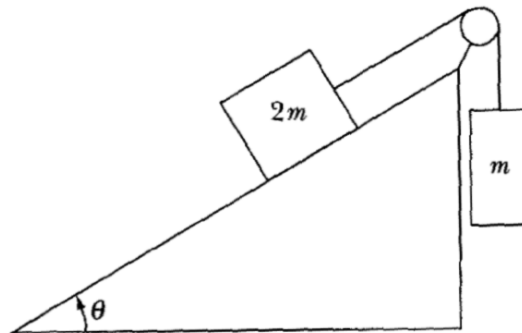
- (d) How far away from point A does the block land on level ground?
- (e) Sketch the potential energy $U(x)$ of the block. Indicate the total energy on the sketch.

Problem 2-28 - The superball and the marble

A superball of mass M and a marble of mass m are dropped from a height h with the marble just on top of the superball. A superball has a coefficient of restitution of nearly 1 (i.e., its collision is essentially elastic). Ignore the sizes of the superball and the marble. The superball collides with the floor, rebounds, and smacks the marble, which moves back up. How high does the marble go if all the motion is vertical? How high does the superball go?

Problem 2-32 - Blocks and an incline plane

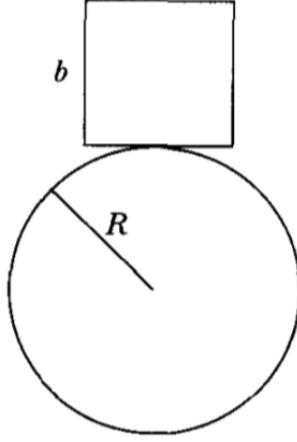
Two blocks of unequal mass are connected by a string over a smooth pulley. See figure.



If the coefficient of kinetic friction is μ_k , what angle θ of the incline allows the masses to move at a constant speed?

Problem 2-42 - Stability of cube on sphere

A solid cube of uniform density and sides b is in equilibrium on top of a cylinder of radius R . See figure.



The planes of the four sides of the cube are parallel to the axis of the cylinder. The contact between the cube and the sphere is perfectly rough. Under what conditions is the equilibrium stable or not stable?

Problem 2-43 - Determine the potential, plot $U(x)$

A particle is under the influence of a force

$$F = -kx + \frac{k}{\alpha^2}x^3$$

where k and α are constants and k is positive. Determine $U(x)$ and discuss the motion. What happens when $E = (1/4)k\alpha^2$?

Problem 2-50 - Falling water droplet

A water droplet is falling under gravity and accumulating mass as it falls. At time t it has a radius $r(t)$ and a mass $m(t)$. The rate of increase of mass is proportional to the surface area of the drop (assume it is a sphere).

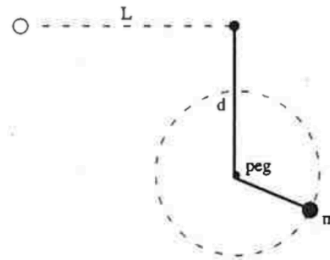
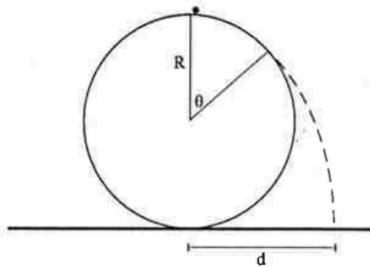
- (a) Determine the function $r(t)$ if the droplet had a size $r(0) = r_0$ at time $t = 0$.
- (b) Determine the velocity of the droplet $v(t)$. Do not forget that it is accumulating mass and its size is increasing. Assume that the drop has initial velocity $v(0) = v_0$ at $t = 0$.

Problem 2-52 - Rocket launch

Consider the launch of a single-stage rocket. Let the initial mass be m_0 and the mass at burnout be m_b . Let the exhaust speed of the ejected material be u ($u > 0$) relative to the rocket. Finally, let the fuel burn rate be $\alpha = -dm/dt$ ($\alpha > 0$). The only external force on the rocket is gravity.

- (a) Determine the velocity function $v(t)$ in the time interval $0 \leq t \leq t_b$, where t_b is the burnout time. Assume $v(0) = 0$.
- (b) Determine the position function $y(t)$ in the time interval $0 \leq t \leq t_b$. Assume $y(0) = 0$.
- (c) How much additional height occurs after burnout?

Problem A1 - Sliding off a sphere: A smooth sphere rests on a horizontal plane. A point particle slides without friction down the sphere (starting from the top). Determine the angle θ at which the particle loses contact with the sphere, the horizontal travel distance d , and the particle's velocity vector as it strikes the ground. (See Figure below left)



Problem A2 - Peg and the pendulum: A pendulum with mass m and length L is released from rest from a horizontal position. A peg located a distance d below the pivot point causes the pendulum to move along the circular path indicated. Find the minimum distance d such that the pendulum will swing completely around the circle. (See Figure above right)

Computational problems: Please work the following two problems using your favorite math package (IDL, Matlab, etc) or a programming language (Python or C). Plot your results.

Problem C(omputational)1 - Test the algorithms: Implement both the Euler and Runge-Kutta algorithms described in the numerical methods handout(following Problem C2) or your Mathematical Methods notes. Test your numerical scheme by integrating the equation of motion from Example 2-5 in the text. This is for 1-d motion with $F = -mg - kmv$ (falling in uniform gravity with a linear drag force). Compare your Euler and Runge-Kutta results with results from the analytic solution given in the text. Plot $v(t)$ and $x(t)$ for $0 < t < 10$ s. Let $m = 1$ kg and $k = 1 \text{ s}^{-1}$. What happens if the initial velocity is -9.8 m/s ?

Problem C2 - The plummeting parachutist: A parachutists of mass 70 kg jumps from a plane at an altitude of 32 km above the surface of the earth. Unfortunately, the parachute fails to open. (In the following parts, neglect horizontal motion and assume the initial velocity is zero).

- (a) Calculate the time of fall (accurate to 1 s) until ground impact, given no air resistance and a constant value of g . (Check your numerical answer with the known analytic result).
- (b) Calculate the time of fall (accurate to 1 s) until ground impact, given a constant g and a force of air resistance given by

$$F(v) = -av|v|$$

where, for a falling man (or woman), $a = 0.5 \text{ N s}^2/\text{m}^2$.

- (c) Calculate the time of fall (accurate to 1 s) until ground impact, given a scales with atmospheric density as

$$a = 0.5e^{-y/H}$$

where $H = 8 \text{ km}$ is the scale height of the atmosphere and y is the height above the ground. Furthermore, assume that g is no longer constant but is given by

$$g = \frac{9.8 \text{ m/s}^2}{\left(1 + \frac{y}{R_e}\right)^2}$$

where $R_e = 6370 \text{ km}$ is the radius of the earth.

In each case plot the acceleration, velocity, and altitude of the parachutist as a function of time. For part (c) explain why the acceleration becomes positive as the parachutist falls.

Extra Notes - Numerical Methods: Integrating the Equations of Motion

The first step in solving a mechanics problem using the method of Newtonian mechanics is to identify all the forces contributing to the $\mathbf{F}$ in Newton's second law $\mathbf{a} = \mathbf{F}/m$. This is a vector equation and each component will be of the following form:

$$a_i = \frac{dv_i}{dt} = \frac{d^2x_i}{dt^2} = F_i(\mathbf{x}, \mathbf{v}, t)$$

where, as indicated, the force $\mathbf{F}$ may depend on position $\mathbf{x}$, velocity $\mathbf{v}$, and time t . (The force will also probably depend on one or more constants such as the acceleration due to gravity g , a spring constant, or a drag coefficient). These are the equations of motion for the system and in one sense we have solved the problem once we've written down these equations. In another sense our problems have only begun as we usually want to know the velocity and position as functions of time and this requires solution of the above set of differential equations. For simple cases we can explicitly solve this set of equations. However, the "real" world is a complicated place and the equations of motion for most systems are analytically intractable. Thus numerical solutions are required. In either case, a set of initial conditions $(\mathbf{x}_0, \mathbf{v}_0, t_0)$ needs to be specified.

Numerical Techniques

Let's consider the simple case of 1D motion in the x -direction. Given a force $F(x, v, t)$, and thus an acceleration $a(x, v, t)$, our goal is to determine the functions $v(t)$ and $x(t)$ by numerically integrating the equations of motion:

$$\frac{dv}{dt} = a(x, v, t) \quad \text{and} \quad \frac{dx}{dt} = v(x, t)$$

We do this by starting at x_0, v_0, t_0 and solving the equations step by step as we increment time by an amount Δt at each step. (Note that in many discussions the time step is called h). **Euler's Method**

The simplest method is the Euler method which proceeds as follows:

$$v_{n+1} = v_n + a(x_n, v_n, t_n)\Delta t$$

$$x_{n+1} = x_n + v_n \Delta t$$

This method is easy to understand and easy to implement but it is never actually used in practice as it is neither very accurate or very stable.

Runge-Kutta Method

An only slightly more complicated, but much more accurate and stable algorithm is the fourth-order Runge-Kutta formula:

$$v_{n+1} = v_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_{n+1} = x_n + \frac{1}{6}(f_1 + 2f_2 + 2f_3 + f_4)$$

where

$$k_1 = a(x_n, v_n, t_n) \Delta t$$

$$f_1 = v_n \Delta t$$

$$k_2 = a(x_n + f_1/2, v_n + k_1/2, t_n + \Delta t/2) \Delta t$$

$$f_2 = (v_n + k_1/2) \Delta t$$

$$k_3 = a(x_n + f_2/2, v_n + k_2/2, t_n + \Delta t/2) \Delta t$$

$$f_3 = (v_n + k_2/2) \Delta t$$

$$k_4 = a(x_n + f_3, v_n + k_3, t_n + \Delta t) \Delta t$$

$$f_4 = (v_n + k_3) \Delta t$$

Note that this Runge-Kutta algorithm requires four evaluations of the function $a(x, v, t)$ to advance the solution from time t_n to time $t_{n+1} = t_n + \Delta t$. This algorithm is the accepted “work horse” method for solving ODE’s.

To implement the above algorithms you will need to specify the time step Δt . In principle a short time step means a more accurate solution, but you will need many more iterations to solve the equations out to some final time t_f . Also, at each iteration there will be some error introduced due to the approximate nature of the algorithm (and also possibly due to the finite precision of computer calculations). This error tends to accumulate so the more iterations, the greater the final error. The Euler method is accurate only to first order in Δt (which means you will potentially be accumulating an error

of order $(\Delta t)^2$ at each iteration). The above Runge- Kutta method is accurate to fourth order in Δt (so an error of order $(\Delta t)^5$ may be accumulated at each iteration).

For more information consult your notes from Mathematical Methods. Additional details and computer code can be found in "Numerical Recipes" by Press, Flannery, Teukolsky, and Vetterling. This book is an invaluable reference for all sorts of numerical techniques. It's written by physicists and is very much directed to a physics audience. You'll see it widely referred to in the physics research literature.

Assignment 3 - Linear Oscillation

$$[\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A \cos(\omega_d t)]$$

It's still Newtonian mechanics (and, for the most part, it's still review) but here we will focus on the force law $F = -kx$ which results in simple harmonic motion. If energy is somehow dissipated in the system (e.g., due to friction $\propto \dot{x}$) the oscillatory motion is damped and eventually dies out. To keep the oscillations going we can pump energy into the system by driving it. The general equation for a linearly damped, sinusoidally driven system is given above. This equation is linear since neither the coefficients on the LHS or the driving term on the RHS depend on x . The solutions to linear equations obey the principle of superposition. This property is very important as it allows us to solve the above equation for an arbitrary driving force by Fourier decomposing that force and summing the solutions for the separate (sinusoidal) Fourier components. This week we restrict our discussion to such linear systems. Most real world systems are in fact nonlinear, but the linear approximation is often a good one. Furthermore, we often analyze nonlinear systems via a perturbation expansion about the related linear system. Later in the term we will take up the very interesting topic of nonlinear systems. Phase space diagrams will be a key tool in that analysis.

Questions while reading (be sure you can answer these):

- Why does the linear force law $F = -kx$ seem so prevalent in nature?
- What is the physical behavior of systems which are under-, over-, and critically-damped?

- What is the transient solution for a driven oscillator? In what sense is it transient?
- What is the meaning of Q ? Can you explain the phase difference of $\delta = 180^\circ$ for $\omega_d \gg \omega_0$?
- Why does a discussion of Fourier series come up in our study of linear oscillations?
- What is Green's method of solution of inhomogeneous differential equations? Why is it restricted to linear equations?

Problem 3-1 - An oscillator - Part 1

A simple harmonic oscillator consists of a 100-g mass attached to a spring whose force constant is 10^4 dyne/cm. The mass is displaced 3 cm and released from rest. Calculate

- the natural frequency ν_0 and the period τ_0
- the total energy
- the maximum speed

Problem 3-2 - An oscillator - Part 2

Allow the motion in the preceding problem to take place in a resisting medium. After oscillating for 10 s, the maximum amplitude decreases to half the initial value. Calculate

- the damping parameter β
- the frequency ν_1 (compare with the undamped frequency ν_0)
- the decrement of the motion

Problem 3-3 - An oscillator - Part 3

The oscillator of Problem 3-1 is set into motion by giving it an initial velocity of 1 cm/s at its equilibrium position. Calculate

- the maximum displacement

(b) the maximum potential energy

Problem 3-6 - Coupled masses)

Two masses $m_1 = 100$ g and $m_2 = 200$ g slide freely in a horizontal frictionless track and are connected by a spring whose force constant is $k = 0.5$ N/m. Find the frequency of oscillatory motion for this system.

Problem 3-7 - Bobbing period of a floating body

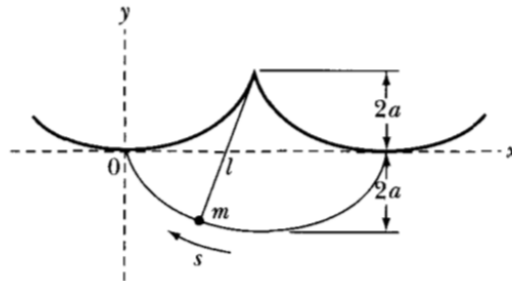
A body of uniform cross-sectional area $A = 1$ cm² and of mass density $\rho = 0.8$ g/cm³ floats in a liquid of density $\rho_0 = 1$ g/cm³ and at equilibrium displaces a volume $V = 0.8$ cm³. Show that the period of small oscillations about the equilibrium position is given by

$$\tau = 2\pi\sqrt{\frac{V}{gA}}$$

where g is the gravitational field strength. Determine the value of τ .

Problem 3-8 - An isochronous pendulum... or how to build a better clock

A pendulum is suspended from the cusp of a cycloid (look it up) cut in a rigid support (see figure below).



The path described by the pendulum bob is cycloidal and is given by

$$x = a(\phi - \sin \phi) \quad , \quad y = a(\cos \phi - 1)$$

where the length of the pendulum is $l = 4a$, and where ϕ is the angle of rotation of the circle generating the cycloid. Show that the oscillations are

exactly isochronous with a frequency $\omega_0 = \sqrt{g/l}$, independent of the amplitude.

Problem 3-10 - Amplitude fall-off for a damped oscillator

If the amplitude of a damped oscillator decreases to $1/e$ of its initial value after n periods, show that the frequency of the oscillator must be approximately $[1 - (8\pi^2 n^2)^{-1}]$ times the frequency of the corresponding undamped oscillator.

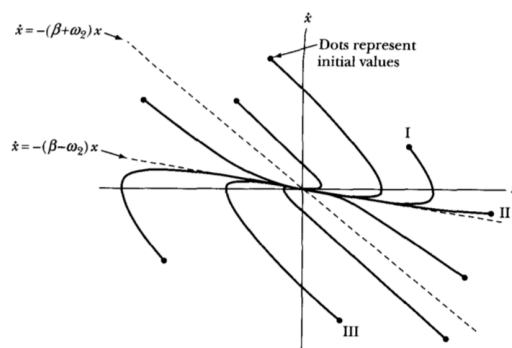
Problem 3-18 - The meaning of Q

Show that, if a driven oscillator is only lightly damped and driven near resonance, the Q of the system is approximately

$$Q \approx 2\pi \left(\frac{\text{total energy}}{\text{energy loss during one period}} \right)$$

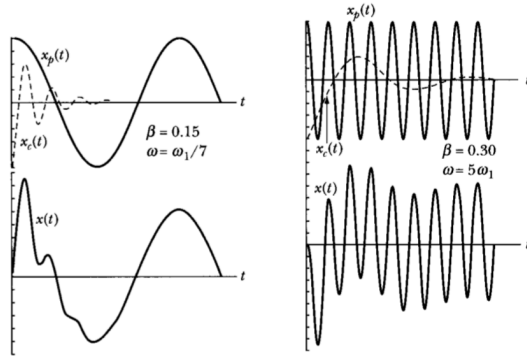
Problem 3-21 - Phase space diagram for critical damping

Use a computer to produce a phase space diagram similar to



for the case of critical damping. Show analytically that the equation of the line that the phase paths approach asymptotically is $\dot{x} = -\beta x$. Show the phase paths for at least three initial positions above and below the line. **Problem 3-24** - Transient and steady-state solutions for a driven oscillator)

For $\beta = 0.2 \text{ s}^{-1}$, produce computer plots like those shown below



for a sinusoidal driven, damped oscillator where $x_p(t)$, $x_c(t)$, and the sum $x(t)$ are shown. The left side figures have $\omega > \omega_1$ and the right side figures have $\omega < \omega_1$. Do this for values of ω/ω_1 of $1/9$, $1/3$, 1.1 , 3 , and 6 . For the solution

$$x_c(t) = Ae^{-\beta t} \cos(\omega_1 t - \delta)$$

let the phase angle $\delta = 0$ and the amplitude $A = -1m$. For the solution

$$x_p(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta)$$

let $A = 1m/s^2$ but calculate δ . What do you observe about the relative amplitudes of the two solutions as ω increases? Why does this occur? For $\omega/\omega_1 = 6$, let $A = 20m/s^2$ for $x_p(t)$ and produce the plot again.

Problem 3-28 -Fourier expansion

Obtain the Fourier expansion of the function

$$F(t) = \begin{cases} -1 & -\pi/\omega < t < 0 \\ +1 & 0 < t < \pi/\omega \end{cases}$$

in the interval $-\pi/\omega < t < \pi/\omega$. Take $\omega = 1rad/s$. In the periodic interval, calculate and plot the sums of the first two terms, the first three terms, and the first four terms to demonstrate the convergence of the series. **Problem 3-29** - Another Fourier expansion

Obtain the Fourier expansion of the function

$$F(t) = \begin{cases} 0 & -2\pi/\omega < t < 0 \\ \sin \omega t & 0 < t < 2\pi/\omega \end{cases}$$

Problem 3-31 - Response function

A damped linear oscillator, originally at rest in its equilibrium position, is subjected to a forcing function given by

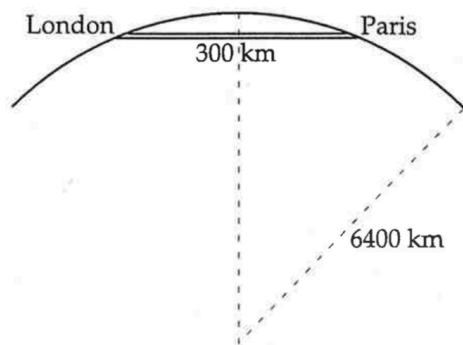
$$\frac{F(t)}{m} = \begin{cases} 0 & t < 0 \\ a \frac{t}{\tau} & 0 < t < \tau \\ a & t > \tau \end{cases}$$

Find the response function. Allow $\tau \rightarrow 0$ and show that the solution becomes that for a step function.

Problem 3-38 - Green's function method for a forced oscillator

Use Green's method to obtain the response of a damped oscillator to a forcing function of the form

$$F(t) = \begin{cases} 0 & t < 0 \\ F_0 e^{\gamma t} \sin \omega t & t > 0 \end{cases}$$



Problem A1 - Chunnel transit, London - Paris

As shown in the above figure, London and Paris are now connected by a straight subway tunnel. A train travels between the two cities powered only by the gravitational force of the earth. Calculate the maximum speed of the train and the time taken to travel between the two cities. The distance between London and Paris (as the mole burrows) is 300 km and the radius of the earth is 6400 km. Neglect friction and the rotation of the earth.

Problem C1 - Exact pendulum equation - numerical solution

The exact equation of motion for a simple pendulum of length L is given by

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

where $\omega^2 = g/L$. We generally linearize this equation by making the small angle approximation $\sin \theta \approx \theta$. This leads to isochronous behavior. Let's see how the linear approximation stands up.

- (a) Find $\theta(t)$ by numerically integrating this equation of motion (using fourth order Runge-Kutta). Let $L = 1.0m$ and take for initial conditions $\theta(0) = \pi/2$ rad and $\dot{\theta}(0) = 0$ rad/s. Plot $\theta(t)$ from $t = 0$ to 4.00 s. For comparison, plot the analytic solution to the linearized equation on the same graph.
- (b) Repeat (a) with $\theta(0) = 3.10$ rad.
- (c) Plot the period of the pendulum as a function of the amplitude $\theta(0)$ from $0 \leq \theta(0) \leq 3.10$ rad. Determine the amplitude at which the period first deviates by more than 2% from $2\pi\sqrt{L/g}$.
- (d) Show that the dependence of period T on amplitude Θ for the simple pendulum can be expressed exactly, in terms of an elliptic integral, as follows

$$T = \sqrt{L/g} \int_0^{2\pi} \frac{du}{(1 - K^2 \sin^2 u)^{1/2}}$$

where $K = \sin \Theta/2$.

Assignment 4: - Gravitation $\left[\Phi = -G \int_V \frac{\rho(r')}{r} dV' \right]$ and Calculus of Variations $\left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \right]$

Reading: Marion & Thornton: Chaps 5 + 6. (pay special attention to Secs. 5.2, 5.4, 6.3, and 6.6)

Overview This week we will cover two shorter chapters in Marion. The formal description of classical gravity, developed in Chapter 5, should look familiar to you from E&M. We define a scalar gravitational potential Φ from which we can compute the gravitational field $\mathbf{g}$ via $\mathbf{g} = -\nabla\Phi$. The force acting on a mass m in this field is $\mathbf{F} = m\mathbf{g}$. In analogy with electrostatics, the gravitational potential obey a Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, where G is the universal gravitational constant and ρ is the mass density. A key problem for us will be, given an arbitrary mass distribution $\rho(r)$, determine Φ via the integral relation given above. We will also take a careful look at the theory for ocean tides. Then we will turn our attention to the calculus of variations which provides the key tools necessary to develop the Lagrangian approach to mechanics. Lagrangian mechanics is based on Hamilton's principle, which asserts that, either by design or fancy, nature always chooses a path in phase space which minimizes the time integral of a certain functional, $L(x, \dot{x}; t)$, of the system. Thus, mechanics is reduced to a problem of minimization of a functional integral in phase space. To understand the mathematical techniques involved we consider a path in space described by the functional $f[y, dy/dx; x]$. Given a specific form of $f[]$ we seek the function $y(x)$ which minimizes the integral $\int f[y, y'; x] dx$ between points x_1 and x_2 . The famous brachistochrone problem is of this type. Euler's equation, given above, provides a required condition on $f[]$ that the integral be an extremum. We will examine a number of functional minimization problems including ones in which additional constraints are placed on the allowed family of functions. We can deal with such constraints using the method of Lagrange multipliers.

Problem 5-2 - Find $\rho(r)$ giving constant $g(r)$ inside a sphere.

If the field vector is independent of the radial distance within a sphere, find the function describing the density $\rho = \rho(r)$ of the sphere.

Problem 5-5 - Time to fall.

A particle falls to Earth starting from rest at a great height (many times Earth's radius). Neglect air resistance and show that the particle requires approximately $\frac{9}{11}$ of the total time of fall to traverse the first half of the distance.

Problem 5-7 - Φ for a thin rod (along the $\perp$ -bisector).

Calculate the gravitational potential due to a thin rod of length l and mass M at a distance R from the center of the rod and in a direction perpendicular to the rod.

Problem 5-8 - Φ along the symmetry axis of a finite cylinder.

Calculate the gravitational field vector due to a homogeneous cylinder at the exterior points on the axis of the cylinder. Perform the calculation (a) by computing the force directly and (b) by computing the potential first.

Problem 5-14 - Gravitational self-energy of a sphere.

Show that the gravitational self-energy (energy of assembly piecewise from infinity) of a uniform sphere of mass M and radius R is

$$U = -\frac{3}{5} \frac{GM^2}{R}$$

Problem 5-16 - Force between a spherical body and an infinite sheet.

A uniformly solid sphere of mass M and radius R is fixed a distance h above a thin infinite sheet of mass density ρ_s (mass/area). With what force does the sphere attract the sheet?

Problem 6-1 - Variations on a straight line.

Consider the line connecting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 1)$. Show explicitly that the function $y(x) = x$ produces a minimum path length by using the varied function $y(\alpha, x) = x + \alpha \sin \pi(1 - x)$. Use the first few terms in the expansion of the resulting elliptic integral to show the equivalent of Equation 6.4.

Problem 6-2,3 - Shortest distance in flat space ... counts as one problem.

Show that the shortest distance between two points on a plane is a straight line.

Show that the shortest distance between two points in (three-dimensional) space is a straight line.

Problem 6-6 - The brachistochrone revisited.

Reexamine the problem of the brachistochrone (Example 6.2) and show that the time required for a particle to move (frictionlessly) to a *minimum* point of the cylinder is $\pi\sqrt{a/g}$, *independent* of the starting point.

Problem 6-8 - Maximum volume of a constrained parallelepiped.

Find the dimensions of the parallelepiped of maximum volume circumscribed by (a) a sphere of radius R ; (b) an ellipsoid with semiaxes a, b, c .

Problem C1 - Φ for a circular ring ... analytic and numerical results.

Here we consider a thin uniform circular ring of radius a and mass M as discussed in example 5.2 in the text. As shown in the text, the gravitational potential within the plane of the ring is given by

$$\Phi(r) = -\frac{GM}{\pi} \int_0^\pi \frac{d\phi}{(a^2 + r^2 - 2ar \cos \phi)^{1/2}}$$

where r is the distance from the center of the ring. This is an elliptic integral and can not be evaluated in terms of elementary functions. The usual treatment is to expand the integrand such that you can integrate term by term and thereby express $\Phi(r)$ as a power series in r . This is done in Example 5.2 for small r (i.e., $r < a$) where it is shown that

$$\Phi(r) = -\frac{GM}{a} \left(1 + \frac{r^2}{4a^2} + \dots \right) \quad , \quad (r < a)$$

- (a) Carry out the analogous expansion appropriate for large r (i.e., $r > a$) and show that the potential is given by

$$\Phi(r) = -\frac{GM}{r} \left(1 + \frac{a^2}{4r^2} + \dots \right) \quad , \quad (r > a)$$

- (b) What is $\Phi(r)$ at $r = a$? (No calculation required, just write down the answer)
- (c) Evaluate the integral expression for $\Phi(r)$ given above numerically (using the built in functions in MATLAB or R, or by some other method of your choice). Plot your numerical results and the above two analytic approximations on the same plot to see how well the approximations do. In particular, make the following plot:

$$\frac{a\Phi}{GM} \text{ versus } \frac{r}{a} \text{ for } -2.0 \leq \frac{a\Phi}{GM} \leq 0.0, 0.0 \leq \frac{r}{a} \leq 5.0$$

Assignment 5 - Lagrangian Mechanics $\left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \right]$

Reading Assignment: Marion & Thornton: Chapt. 7. Sections 1-7.

Overview: Lagrangian mechanics is one of several possible reformulations of Newtonian mechanics. The Lagrangian approach will always yield the same equations of motion as we would have gotten from Newton but in many cases the Lagrangian method is more convenient. Although one can arrive at Lagrange's equation (given above) starting from Newton's second law, our starting point will be the very general and extremely important Hamilton's principle. This principle asserts that of all possible paths accessible to a dynamical system, the one chosen by nature is that which minimizes the time integral of the scalar quantity $L = T - U$, where T and U are the kinetic and potential energies of the system and L is defined as the "Lagrangian". From our study of the calculus of variations we immediately recognize the equation given above as Euler's equation for the functional $L(q, \dot{q}; t)$ and indeed, this equation is referred to as the Euler-Lagrange equation. Notice that while a Newtonian analysis considers the net force F acting on a system (the system is "told" what to do), a Lagrangian analysis considers a property L of the system (the system "knows" what to do). This raises philosophical issues, but we leave those to the philosophers since we prefer numbers to words. Since L is a scalar quantity, it is invariant with respect to coordinate transformations and thus we have complete freedom in our choice of coordinates. This is a very powerful feature of the Lagrangian method. We can work with any set of variables which completely specify the state of the system; these are our generalized coordinates. We require f coordinates to describe a system with

f degrees of freedom and we construct an Euler-Lagrange equation for each coordinate. Constraints on the system simply reduce the number of degrees of freedom and thus the number of required generalized coordinates. Alternatively, constraints can be handled explicitly using Lagrange multipliers.

Problem Assignment

Problem 7-3 Sphere rolling inside a cylinder

A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R . Determine the Lagrangian function, the equations of constraint, and Lagrange's equations of motion. Find the frequency of small oscillations.

Problem 7-4 Particle in a 2D central potential

A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α ($\alpha > 0$) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

Problem 7-6 Hoop rolling down a moving incline

A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M , which makes an angle α with the horizontal. Find the Lagrangian equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

Problem 7-9 Pendulum on the axle of a wheel rolling downhill (yikes!)

A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α . The disk has a short weightless axle of negligible radius. From this axle is suspended a simple pendulum of length $l < R$ and whose bob has a mass m . Consider that the motion of the pendulum takes place in the plane of the disk, and find Lagrange's equations for the system.

Problem 7-10 Blocks, pulley, and a string ... the usual stuff

Two blocks, each of mass M , are connected by an extensionless, uniform string of length l . One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system (a) when the mass of the string is negligible and (b) when the string has a mass m .

Problem 7-12 Motion on a tilting inclined plane

A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

Problem 7-14 Vertically accelerated simple pendulum

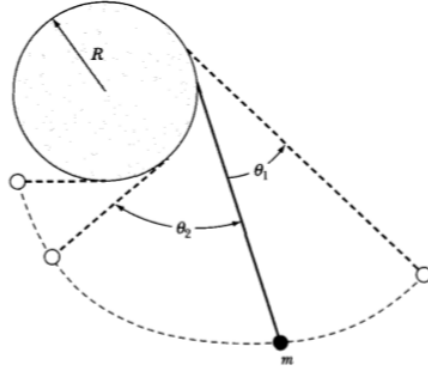
A simple pendulum of length b and bob with mass m is attached to a massless support moving horizontally with constant acceleration a . Determine (a) the equations of motion and (b) the period for small oscillations.

Problem 7-16 Simple pendulum with oscillating suspension point

The point of support of a simple pendulum of mass m and length b is driven horizontally by $x = a \sin \omega t$. Find the pendulum's equation of motion.

Problem 7-18 Pendulum on a disk

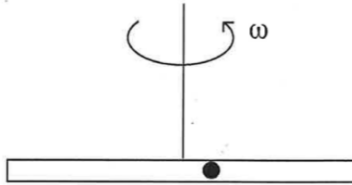
A pendulum is constructed by attaching a mass m to an extensionless string of length l . The upper end of the string is connected to the uppermost point on a vertical disk of radius ($R < l/\pi$) as in figure below.



Obtain the pendulum's equation of motion, and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e., $\theta_1 = \theta_2$).

Problem A1 Bead in a rotating tube

A hollow cylindrical tube of length $2a$ rotates with constant angular velocity ω about a vertical axis through the center of the tube as shown below.

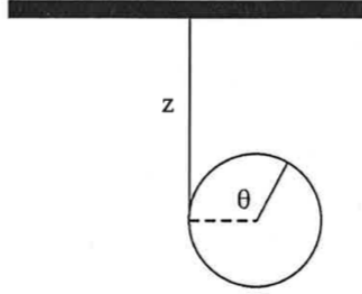


A particle of mass m is initially ($t = 0$) at rest with respect to the tube a distance b from the center. The tube rotates in a fixed plane and there is no friction in the system.

- Determine the Lagrangian for this system and construct the appropriate Euler-Lagrange equation(s) to describe the motion of m .
- Solve the differential equation(s) and find the position and velocity of the mass m (in the reference frame of the rotating tube) for $t > 0$.
- Show that the mass exits the tube at time $T = (1/\omega) \cosh^{-1}(a/b)$

Problem A2 Constraint forces for the falling yo-yo

Consider a disk that has a string wrapped around it, with one end attached to a fixed support, as shown below.



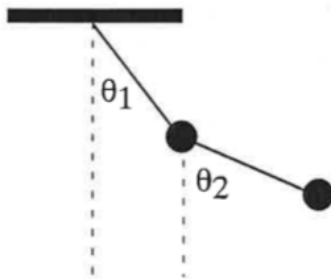
The disk is allowed to fall with the string unwinding as it falls (i.e., a yo-yo). Take the mass and radius of the disk to be M and R , respectively, and ignore the mass of the string.

- (a) Determine the Lagrangian for this system using the generalized coordinates z and θ and construct the Euler-Lagrange equations using a Lagrange multiplier to account for the constraint condition $\Delta z - R\theta$.
- (b) Determine the equations of motion for the falling disk and the two forces of constraint in the problem. Identify the physical meaning of these two forces of constraint.

Problem A3 The famous double pendulum

The double pendulum consists of a mass m suspended by a massless rod of length ℓ , from which is suspended another such rod and mass. Assume the motion is confined within a plane - figure below

- (a) Write down the Lagrangian for this system in terms of the coordinates θ_1 and θ_2 as shown in figure.
- (b) Derive the equations of motion for this system.
- (c) Simplify the part (b) equations of motion by making the small angle approximation $\theta_1, \theta_2 \ll 1$ (noting that $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \theta^2/2$ in the small angle limit).



- (d) By substituting the trial solutions $\theta_1 = A \exp(i\omega t)$ and $\theta_2 = B \exp(i\omega t)$ into your part (c) equations demonstrate that the system has two eigenfrequencies of oscillation given by $\omega^2 = (2 \pm \sqrt{2})g/\ell$.

Assignment 6 - Hamiltonian Mechanics $\left[\dot{q} = \frac{\partial H}{\partial p} ; \dot{p} = -\frac{\partial H}{\partial q} \right]$

Reading Assignment: Marion & Thornton: Chapt. 7. Sections 8-12 (and please reread Sections 1-7). **Overview:** This week we will proceed on two fronts, the purely practical and the deeply theoretical. On the practical side we will solve more problems using the Lagrangian approach. As you know by now, this is really a rather mechanical procedure (no pun intended), but therein lies its power ... we have a recipe for attacking even the most complicated mechanics problems (granted that things get ugly if we have non-conservative forces or non-holonomic constraints). On the theoretical side we will study symmetry and transformation properties of the Lagrangian function. The familiar, and powerful, conservation laws of physics can all be associated with symmetries of the Lagrangian function. In fact, the mathematician Emmy Noether proved in 1918 that every symmetry of the Lagrangian function implies a conservation law. Noether's theorem is of fundamental importance in all physics. In the Lagrangian approach we describe a system in terms of generalized position and velocity variables $\{q, \dot{q}\}$. In some cases it may be preferable to describe a system using other sets of variables and this is readily accomplished via a so called Legendre transformation of the Lagrangian function. In particular, the variable change $\{q, \dot{q}\} \rightarrow \{q, p\}$, where $p \equiv \partial L / \partial \dot{q}$ is a generalized momentum, leads to the Hamiltonian function $H(q, p, t) = p\dot{q} - L(q, \dot{q}, t)$ and Hamilton's equations of motion, given above. These equations are known as the canonical equations

of motion (*canonical*: simplest or clearest; Webster). These two first order equations replace the second order Euler-Lagrange equation. The variables $\{q, p\}$ are treated symmetrically in Hamilton's equations which simplifies our analysis of systems with so called cyclic variables. For actually solving most classical mechanics problems the Lagrangian formalism is preferred, however, the Hamiltonian formalism is of central importance to quantum and statistical mechanics. In particular, the commutators of quantum theory derive from the Poisson brackets of classical Hamiltonian mechanics. Finally, notice that variable change $\{q, \dot{q}\} \rightarrow \{p, \dot{p}\}$, and associated Legendre transform function $K(p, \dot{p}, t) = p\dot{q} + q\dot{p} - L(q, \dot{q}, t)$, provides a purely momentum space representation of classical mechanics. Such a representation is often used in quantum mechanics.

Problem Assignment:

Problem 7-21 Whirling hoop with sliding bead

A particle is constrained to move on a circular wire rotating with constant angular speed ω about a vertical diameter. Find the equilibrium position of the particle, and calculate the frequency of small oscillations around this position. Find and interpret physically a critical angular velocity $\omega = \omega_c$ that divides the particle's motion into two distinct types. Construct phase diagrams for the two cases $\omega < \omega_c$ and $\omega > \omega_c$.

Problem 7-22 L and H for a time dependent force

A particle of mass m moves in one dimension under the influence of a force

$$F(x, t) = \frac{k}{x^2} e^{-t/\tau}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

Problem 7-24 L and H for a simple pendulum with changing string length

Consider a simple plane pendulum consisting of a mass m attached to a string of length l . After the pendulum is set in motion, the length of

the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

Problem 7-26 Hamilton's equations for a pendulum and an Atwood machine

Determine the Hamiltonian and Hamilton's equations of motion for (a) a simple pendulum and (b) a simple Atwood machine (single pulley).

Problem 7-30 Properties of Poisson brackets

Consider any two continuous functions of the generalized coordinates and momenta $g(q_k, p_k)$ and $h(q_k, p_k)$. The **Poisson brackets** are defined by

$$[g, h] = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$$

Verify the following properties of the Poisson brackets:

(a)

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$$

(b)

$$\dot{q}_j = [q_j, H] \quad , \quad \dot{p}_j = [p_j, H]$$

(c)

$$[p_i, p_j] = 0 \quad , \quad [q_i, q_j] = 0$$

(d)

$$[q_i, p_j] = \delta_{ij}$$

where H is the Hamiltonian.

If the Poisson bracket of two quantities vanishes, the quantities are said to *commute*. If the Poisson bracket of two quantities equals unity, the quantities are said to be *canonically conjugate*.

Finally, show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion of the system.

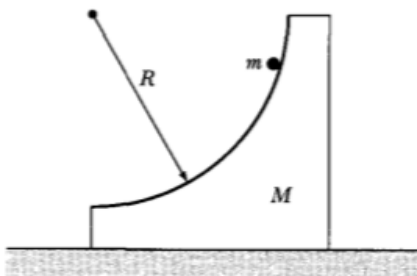
The Poisson-bracket formalism is of considerable importance in some derivations of quantum mechanics.

Problem 7-32 Hamiltonian for central force motion

A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface $H = \text{constant}$ in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional $(r, \theta, p_r, p_\theta)$, but only the first three are nontrivial. Calculate the projection of the phase path on the $r - p$ plane, then take into account the variation with θ .

Problem 7-34 Normal force via Lagrange multipliers

A particle of mass m slides down a smooth circular wedge of mass M as shown in figure:



The wedge rests on a smooth horizontal table. Find (a) the equations of motion of m and M and (b) the reaction force of the wedge on m .

Problem A1 - Effect of spring mass on oscillation period

A heavy elastic spring of uniform stiffness and density supports a mass

M . If the spring has a total mass of m and spring constant k , determine the Lagrangian of the system and show that period of vertical oscillation is

$$T = 2\pi \left(\frac{M + (m/3)}{k} \right)^{1/2}$$

(HINT: To set up the Lagrangian function for this system, assume that the velocity of any part of the spring is proportional to its distance from the point of suspension.)

Problem A2 - Momentum space Lagrangian: $K(p, \dot{p}, t) = p\dot{q} - L(q, \dot{q}, t)$

- (a) The Hamiltonian for a 1D simple harmonic oscillator (SHO) is given by:

$$H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Use this Hamiltonian to construct the SHO Lagrangian in both real space, $L(q, \dot{q})$, and in momentum space, $K(p, \dot{p})$. (Note that $\dot{q} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial q$)

- (b) Derive the equations of motion for this system using both $L(q, \dot{q})$ and $K(p, \dot{p})$. Note that the momentum space Euler-Lagrange equation is:

$$\frac{\partial K}{\partial p} - \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{p}} \right) = 0$$

Problem A3 - Poisson brackets for angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (Do Problem 7-30 first!)

- Demonstrate that $[L_x, L_y] = L_z$ for all cyclic permutations of L_x, L_y, L_z .
- Calculate all Poisson brackets for the components of $\mathbf{r}$ and $\mathbf{p}$ with the angular momentum $\mathbf{L}$ (i.e., $[x, L_z], [p_x, L_z]$, etc....there are 18 in all).
- Construct the Hamiltonian $H(x, y, p_x, p_y)$ for a 2D simple harmonic oscillator with mass m and spring constants k_x and k_y .
- Use the Poisson bracket formalism to determine the time evolution of L_z (i.e., find dL_z/dt) and demonstrate that L_z is a constant of

the motion (i.e., is conserved) only in the case of an isotropic oscillator: $k_x = k_y$.

Assignment 7: Central Force Motion and Orbital Dynamics $\left[\frac{\alpha}{r} = 1 + \varepsilon \cos \theta\right]$

Reading Assignment: Marion & Thornton: Chapter 8 and R codes “orbit.R” and “orbit1.R”.

Overview: This week we consider motion in a central potential $U(r)$. In particular we consider two bodies interacting through a central force (i.e., a force directed along the line connecting the bodies). This problem can be mapped to a one-body problem of an object of reduced mass μ moving in an effective potential $V_{\text{eff}}(r) = \ell^2/2\mu r^2 + U(r)$, where ℓ is the angular momentum. Angular momentum is always conserved for central force motion. This fact restricts motion to a plane (why?) and leads directly to Kepler’s 2nd law (equal areas). The two body problem can be formally solved (in terms of a 1D integral) for any central force law. We will focus on the inverse-square law for which we can obtain an analytical solution for the orbital motion. The exact $r(\theta)$ solution is concisely summarized in the form of the conic sections equation given above. The orbit is completely specified by the two parameters α and ε , where α is set by the angular momentum ℓ , and the eccentricity ε is set by the total energy where $E = \mu \dot{r}^2/2 + V_{\text{eff}}(r)$. When $E < 0$, the orbits are closed and elliptical (Kepler’s 1st law). Deviations from an inverse square law results in non-closed orbits and a precession of the apsides, or turning points, of the orbit. The presence of more than two bodies in our solar system cause such deviations and the effects on the planetary orbits were calculated long ago (without computers!!!). The orbital precession of Mercury was known to be anomalously large but there was no accepted explanation for this until Einstein’s theory of gravity (although I always preferred the extra planet explanation ... Vulcan). [You’ll get a chance to tinker with non-inverse-square force laws and investigate orbital precession using the R codes “orbit.R” and “orbit1.R”]. With the solution to the Kepler problem in hand we can plan some space missions using Hohmann orbital transfers and gravity assisted boosts. We’ll also discuss the stability of circular orbits. The latter are always possible in a central potential (why?)

but they are not necessarily stable.

Problem Assignment:

Problem 8.5 - Time to collide

Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period τ . If they are suddenly stopped in their orbits and allowed to gravitate towards each other, show that they will collide after a time $\tau/4\sqrt{2}$.

Problem 8.8 - Orbit for a repulsive force

Investigate the motion of a particle *repelled* by a force center according to the law $F(r) = kr$. Show that the orbit can only be hyperbolic.

Problem 8.9 - Radial kick to a circular orbit

A communications satellite is in circular orbit around Earth at radius R and velocity v . A rocket accidentally fires quite suddenly, giving the rocket an outward radial velocity v in addition to its original velocity.

- (a) Calculate the ratio of the new energy and angular momentum to the old.
- (b) Describe the subsequent motion of the satellite and plot $T(r)$, $V(r)$, $U(r)$, and $E(r)$ after the rocket fires.

Problem 8.10 - Sun loses half its mass ... does the earth escape?

Assume Earth orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does earth then have? Will Earth escape the solar system?

Problem 8.33 - Circular orbit inside a paraboloid

Consider a particle of mass m constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0}}$$

Problem 8.35 - Condition for a closed orbit

An almost circular orbit (i.e., $\varepsilon \ll 1$) can be considered to be a circular orbit to which a small perturbation has been applied. Then the frequency of the radial motion is given by

$$\omega_0^2 = \frac{3g(\rho)}{\rho} + g'(\rho) \quad , \quad g(\rho) = \frac{l^2}{\mu^2 \rho^3}$$

Consider a case in which the force law is $F(r) = -k/r^n$ (where n is an integer), and show that the apsidal angle is $\pi/\sqrt{3-n}$. Thus, show that a closed orbit generally results only for the harmonic oscillator force and the inverse-square-law force (if values of n equal to or smaller than -6 are excluded).

Problem 8.36 - Precession in a screened Coulomb type force field

A particle moves in an almost circular orbit in a force field described by $F(r) = -(k/r^2) \exp(-r/a)$. Show that the apsides advance by an amount approximately equal to $\pi\rho/a$ in each revolution, where ρ is the radius of the circular orbit and where $\rho \ll a$.

Problem A1 - Return of the comets

Remarkable as it seems (to me at least), by making only a few observations of a comet one can determine the perihelion and eccentricity, and thus the complete orbit. Let's do an easy calculation just for fun:

- (a) The comet Hyakutake, which appeared in March-May 1996, has the following data associated with it: Its eccentricity is 0.999846 and its perihelion is 0.230123 AU (from Sky and Telescope, May 1996). Using this information, and assuming the comet is small enough that its mass is negligible compared to the sun, calculate when we can predict to see the comet again. Also calculate the aphelion and compare it with Pluto's aphelion.
- (b) Repeat the calculation for the comet Hale-Bopp, which was very bright in March-May 1997. Hale-Bopp's eccentricity is 0.995075 and its perihelion is 0.913959 AU.

Problem A2 - Circularization of a satellite orbit due to drag

Consider a satellite initially placed in a highly elliptical orbit about the earth such that at perigee it just grazes Earth's upper atmosphere. Assume that the drag experienced by the satellite can be modeled by a small impulsive force that reduces its velocity by a small fraction δ as it skims the atmosphere at perigee on each of its orbits. Thus the position of the perigee will remain fixed but the distance to apogee decreases on each successive orbit.

- (a) Find an approximate analytical solution for how many orbits, n , it will take to circularize the satellites path. (We will define circular to be an eccentricity below $\varepsilon_f = 1.7$. At this point the ratio of semi-minor to semi-major axes exceeds 0.99. Clearly the model breaks down here as the drag becomes fairly continuous throughout the orbit ... but that's another problem!). Express your answer in terms of ε_0 , ε_f and δ .
- (b) Give a numerical result for n , assuming $\delta = 0.01$, the eccentricity of the initial orbit is $\varepsilon_0 = 0.9656$, and that the distance to perigee is $r_0 = 6.60 \times 10^3$ km (about 200 km altitude). (This orbit takes the satellite all the way out to the moon!!).
- (c) Using the above numerical values, calculate the speed of the satellite at apogee on its last orbit. Compare this with the speed at apogee on its initial orbit. Explain why the speed of the satellite increases at each successive apogee when it is being acted upon by a resistive force.

Problem C1 - Numerical solution for 2D orbits

Use the R codes “orbit.R” and “orbit1.R” (or write your own code) to study orbital motion about the earth.

- (a) Play with the codes and understand the calculation ... it's very simple!
- (b) Do Marion 8-41 and generate (and print) the transfer orbit using the R codes “orbit.R” and “orbit1.R”. (You'll need to increase the xrange and yrange of the plots by a factor of 5 or so).
- (c) Do Marion 8-42 and generate (and print) both a Hohmann transfer orbit and the final synchronous orbit using the R codes “orbit.R” and “orbit1.R”.

- (d) Modify the R codes “orbit.R” and “orbit1.R” to study central force motion with a non-inverse-square force law. Find conditions for which you get a well-defined precession of the apsides. Print out such an orbit. (Try an exponent of 2.05 or so). (One proposal to account for the anomalous precession of Mercury’s orbit was to change the exponent in Newton’s force law from 2 to 2.0000001612. Einstein thought this modification had “little probability” of being correct).

orbits.R code

```
x1 <- c()
y1 <- c()
m <- 1000
Me <- 5.98*10^24
rearth <- 6.38*10^6
G <- 6.67*10^-11
gm <- G*Me
k <- gm*m
mu <- m*Me/(m+Me)
x0 <- 0
y0 <- rearth
th <- pi/6
vesc <- 11.186*10^3
v <- 0.9*vesc
vx <- v*sin(th)
vy <- v*cos(th)
x <- x0
y <- y0
r=sqrt(x*x+y*y)
x1 <- append(x1,x)
y1 <- append(y1,y)
h <- 2
nstep <- 400000
count <- 0
chcnt <- 0
accx <- function(gm,r,x,y){-gm*(x/r^3)}
accy <- function(gm,r,x,y){-gm*(y/r^3)}
for (j in 0:nstep){
```

```

count <- count+1
r <- sqrt(x*x+y*y)
rold <- r
fx1 <- vx
gx1 <- accx(gm,r,x,y)
fy1 <- vy
gy1 <- accy(gm,r,x,y)
fx2 <- vx+h*gx1/2
gx2 <- accx(gm,r,x+h*fx1/2,y+h*fy1/2)
fy2 <- vy+h*gy1/2
gy2 <- accy(gm,r,x+h*fx1/2,y+h*fy1/2)
fx3 <- vx+h*gx2/2
gx3 <- accx(gm,r,x+h*fx2/2,y+h*fy2/2)
fy3 <- vy+h*gy2/2
gy3 <- accy(gm,r,x+h*fx2/2,y+h*fy2/2)
fx4 <- vx+h*gx3
gx4 <- accx(gm,r,x+h*fx3,y+h*fy3)
fy4 <- vy+h*gy3
gy4 <- accy(gm,r,x+h*fx3,y+h*fy3)
x <- x+h*(fx1+2*fx2+2*fx3+fx4)/6
vx <- vx+h*(gx1+2*gx2+2*gx3+gx4)/6
y <- y+h*(fy1+2*fy2+2*fy3+fy4)/6
vy <- vy +h*(gy1+2*gy2+2*gy3+gy4)/6
x1 <- append(x1,x)
y1 <- append(y1,y)
r <- sqrt(x*x+y*y)
if (r < rearth) break
if (r > 60*rearth) break
check <- r-rold
if (chcnt < 1){
  if (check < 0){
vold <- sqrt(vx*vx+vy*vy)
v <- sqrt(gm/r)
    vx <- vx*v/vold
    vy <- vy*v/vold
chcnt <- chcnt+1
cnt1 <- count
print(c(count,r/rearth))

```

```

    }
  }
  if (chcnt == 1){
    if (count == 15*cnt1){
      ra <- r
      rf <- 10*r
      vold <- v
      v <- sqrt((2*gm/ra)*(rf/(ra+rf)))
      vx <- vx*v/vold
      vy <- vy*v/vold
      chcnt <- chcnt+1
      print(c(count,r/rearth))
    }
  }
  if (chcnt == 2){
    if (check < 0){
      vold <- sqrt(vx*vx+vy*vy)
      v <- sqrt(gm/r)
      vx <- vx*v/vold
      vy <- vy*v/vold
      chcnt <- chcnt+1
      print(c(count,r/rearth))
    }
  }
  if (count == 100000) print(c(count,r/rearth))
  if (count == 200000) print(c(count,r/rearth))
  if (count == 300000) print(c(count,r/rearth))
}
plot(x1,y1,pch=".",col="red",xlim=c(-rearth*60,rearth*60),ylim=c(-rearth*60,rearth*60))
th <- 2*pi*(1:100)/100
x2 <- rearth*cos(th)
y2 <- rearth*sin(th)
points(x2,y2,col="green",cex=.1)

orbits1.R

x1 <- c()
y1 <- c()
m <- 1000

```



```

Me <- 5.98*10^24
rearth <- 6.38*10^6
G <- 6.67*10^-11
gm <- G*Me
k <- gm*m
mu <- m*Me/(m+Me)
x0 <- 0
y0 <- 2*rearth
r0 <- sqrt(x0*x0+y0*y0)
v <- sqrt(gm/r0)
vx <- v
vy <- 0
x <- x0
y <- y0
r=sqrt(x*x+y*y)
x1 <- append(x1,x)
y1 <- append(y1,y)
h <- 0.5
nstep <- 5000
count <- 0
accx <- function(gm,r,x,y){-gm*(x/r^3)}
accy <- function(gm,r,x,y){-gm*(y/r^3)}
for (j in 0:nstep){
  count <- count+1
  r <- sqrt(x*x+y*y)
  rold <- r
  fx1 <- vx
  gx1 <- accx(gm,r,x,y)
  fy1 <- vy
  gy1 <- accy(gm,r,x,y)
  fx2 <- vx+h*gx1/2
  gx2 <- accx(gm,r,x+h*fx1/2,y+h*fy1/2)
  fy2 <- vy+h*gy1/2
  gy2 <- accy(gm,r,x+h*fx1/2,y+h*fy1/2)
  fx3 <- vx+h*gx2/2
  gx3 <- accx(gm,r,x+h*fx2/2,y+h*fy2/2)
  fy3 <- vy+h*gy2/2
  gy3 <- accy(gm,r,x+h*fx2/2,y+h*fy2/2)

```

```

fx4 <- vx+h*gx3
gx4 <- accx(gm,r,x+h*fx3,y+h*fy3)
fy4 <- vy+h*gy3
gy4 <- accy(gm,r,x+h*fx3,y+h*fy3)
x <- x+h*(fx1+2*fx2+2*fx3+fx4)/6
vx <- vx+h*(gx1+2*gx2+2*gx3+gx4)/6
y <- y+h*(fy1+2*fy2+2*fy3+fy4)/6
vy <- vy +h*(gy1+2*gy2+2*gy3+gy4)/6
x1 <- append(x1,x)
y1 <- append(y1,y)
r <- sqrt(x*x+y*y)
if (r < rearth) break
if (r > 5*rearth) break
}
plot(x1,y1,pch=".",col="red",xlim=c(-rearth*5,rearth*5),ylim=c(-rearth*5,rearth*5))
th <- 2*pi*(1:100)/100
x2 <- rearth*cos(th);
y2 <- rearth*sin(th)
points(x2,y2,col="green",cex=.1)

```

Assignment 8 - Dynamics of Many-Body Systems $[\mathbf{P} = M\dot{\mathbf{R}}; \mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_{\alpha} \mathbf{r}'_{\alpha} \times \mathbf{p}'_{\alpha}]$

Reading Assignment: Marion & Thornton: Chapter 9.

Overview: So far in this course we've been dealing with the motion of a single particle. (Note that our solution of the central force two-body problem was via reduction to a one-body problem). Now we'll turn to many-body systems. These will include systems of particles “glued” together (rigid bodies), loosely linked (rope and chain), or independent but mutually interacting (particle scattering). When dealing with many-body systems it is generally convenient to describe the dynamics in terms of center-of-mass (CM) motion and internal motion relative to the CM. The total linear momentum $\mathbf{P}$ of a many-body system is simply equal to the CM linear momentum while the total system angular momentum $\mathbf{L}$ about some origin is equal to the angular momentum of the CM about that origin plus the internal angular momentum

of the system about the CM. The kinetic energy can be separated into similar CM and internal contributions. Conservation laws are key to simplifying the analysis of multi-body systems. For two particle elastic scattering, energy and momentum conservation provide strong constraints on the possible final states for a scattering event. For inelastic scattering, energy is not conserved but momentum conservation still applies. In either case, a more detailed and predictive analysis of the scattering process requires information about the interaction between the particles. Conversely, examination of the initial and final states of scattering events provides information about these interactions ... this is central to experimental particle physics.

Problem Assignment

Problem 9.3 - CM of an ice cream cone

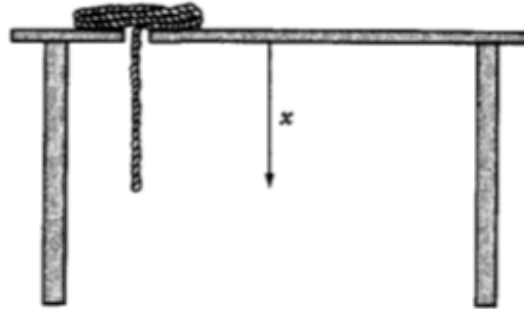
Find the center of mass of a uniformly solid cone of base diameter $2a$ and height h and a solid hemisphere of radius a where the two bases are touching (like an ice cream cone).

Problem 9.9 - Projectile breakup

A projectile is fired at an angle of 45° with initial kinetic energy E_0 . At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down. What is the velocity (magnitude and direction) of the second fragment of mass m_2 and the velocity of the first? What is the ratio of m_1/m_2 when m_1 is maximum?

Problem 9.15 - Rope slipping off a table (answer: energy lost = $mgL/6$)

A smooth rope is placed above a hole in a table, see figure.



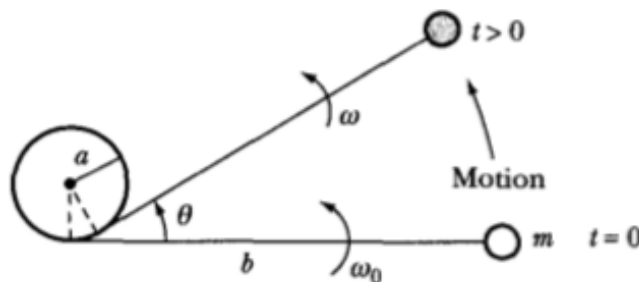
One end of the rope falls through the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity and acceleration of the rope as a function of the distance to the end of the rope x . Ignore all friction. The total length of the rope is L .

Problem 9.20 - Rope sliding off a peg

A uniform rope of total length $2a$ hangs in equilibrium over a smooth nail. A very small impulse causes the rope to slowly roll off the nail. Find the velocity of the rope as it just clears the nail. Assume that the rope is prevented from lifting off the nail and is in free fall.

Problem 9.24 - Particle on a string

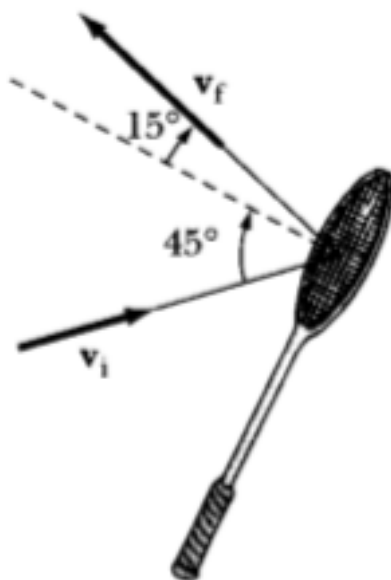
A particle of mass m at the end of a light string wraps itself around a fixed vertical cylinder of radius a , see figure.



All the motion is in the horizontal plane (disregard gravity). The angular velocity of the cord is ω_0 when the distance from the particle to the point of contact of the string and cylinder is b . Find the angular velocity and tension in the string after the cord has turned through an additional angle θ .

Problem 9.30 - Tennis racket impulse

A tennis player strikes an incoming tennis ball of mass 60 g as shown in figure.



The incoming tennis ball velocity is $v_i = 8$ m/s, and the outgoing velocity is $v_f = 16$ m/s.

- (a) What impulse was given to the tennis ball?
- (b) If the collision time was 0.01 s, what was the average force exerted by the tennis racket?

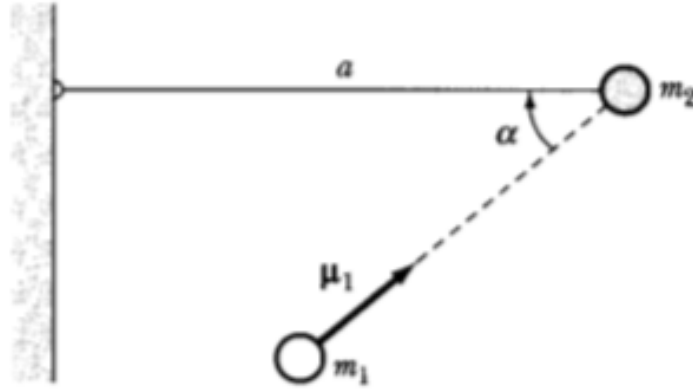
Problem 9.34 - Billiard ball collision

A billiard ball of initial velocity u_1 collides with another billiard ball

(same mass) initially at rest. The first ball moves off at $\psi = 45^\circ$. For an elastic collision, what are the velocities of both balls after the collision? At what LAB angle does the second ball emerge?

Problem 9.38 - Collision with a tethered particle

A particle of mass m_1 and velocity u_1 strikes head-on a particle of mass m_2 at rest. The coefficient of restitution is ε . Particle m_2 is tied to a point a distance a away as in figure



Find the velocity (magnitude and direction) of m_1 and m_2 after the collision.

Problem 9.41 - Proton-Helium inelastic collision (answer: 55.8°)

A proton (mass m) of kinetic energy T_0 collides with a helium nucleus (mass $4m$) at rest. Find the recoil angle of the helium if $\psi = 45^\circ$ and the inelastic collision has $Q = -T_0/6$.

Problem 9.44 - Hard-sphere scattering

Calculate the differential cross section $\sigma(\theta)$ and the total cross section σ_t for the elastic scattering of a particle from an impenetrable sphere; the potential is given by

$$U(r) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$

Assignment 9 - Non-Inertial Frames

$[\mathbf{F}_{\text{fictitious}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r]$ and Rigid-Body Dynamics I $[I_{ij} = \int \rho(\mathbf{r}) (\delta_{ij} \sum_k x_k^2 - x_i x_j) dv]$

Reading Assignment: Marion & Thornton: Chap. 10 (it's short) and Chap. 11. Sec. 1-2.

Overview: Newton's laws, which are the basis for our construction of classical physics, are only valid in inertial reference frames. If we want to do physics within a non-inertial (i.e., accelerating or rotating) frame, new "fictitious" forces must be introduced to account for the observed dynamics. The most common of these is the so called centrifugal force which always crops up when describing circular motion. A real "center-pointing" or centripetal force is required to cause an object to execute circular motion and this real force will be observed by someone in a frame rotating with the object. This observer wonders why the force doesn't cause the object to fall towards the center (as Newton would predict) and invokes a compensating centrifugal force (the first term in the equation given above) to rationalize this observation. Of course the centrifugal force is not a "true" force at all, but it and other so called fictitious forces are a convenient artifice to extend the Newtonian approach to non-inertial frames. In rotating frames we also have the more mysterious Coriolis force which is a velocity dependent fictitious force (yikes indeed ... the second term in the equation given above). The surface of the earth is a non-inertial frame but the required corrections due to the earth's rotation are small enough to be ignored when doing tabletop physics. For large scale motion described relative to the surface of the earth we introduce the above fictitious forces to account for the non-inertial effects. These fictitious forces enable us to describe large scale weather patterns, the anomalous trajectories of long range projectiles, and the precession of the Foucault pendulum. Finally, we will begin our study of rigid body dynamics by computing the inertia tensor (defined above) for a few objects.

Problem Assignment:

Problem 10.6 - Surface of a rotating liquid

A bucket of water is set spinning about its symmetry axis. Determine the shape of the water in the bucket.

Problem 10.8 - Apparent deflection of a particle projected straight up

If a particle is projected vertically upward to a height h above a point on Earth's surface at a northern latitude λ , show that it strikes the ground at a point $\frac{4}{3}\omega \cos \lambda \sqrt{8h^3/g}$ to the west. (Neglect air resistance, and consider only small vertical heights).

Problem 10.9 - Apparent deflection of a particle projected due east

If a projectile is fired due east from a point on the surface of the Earth at a northern latitude λ with a velocity of magnitude V_0 and at an angle of inclination to the horizontal of α , show that the lateral deflection when the projectile strikes Earth is

$$d = \frac{4V_0^2}{g^2} \omega \sin \lambda \sin^2 \alpha \cos \alpha$$

where ω is the rotation frequency of Earth.

Problem 10.12 - Deviation of a plumb-line from vertical

Show that the small angular deviation ε of a plumb line from the true vertical (i.e., toward the center of the Earth) at a point on Earth's surface at a latitude λ is

$$\varepsilon = \frac{R\omega^2 \sin \lambda \cos \lambda}{g_0 - R\omega^2 \cos^2 \lambda}$$

where R is the radius of Earth. What is the value (in seconds of arc) of the maximum deviation? Note that the entire denominator in the answer is actually the effective g , and g_0 denotes the pure gravitational component.

Problem 11.2 - Inertia tensor for a cone

Calculate the moments of inertia I_1 , I_2 , and I_3 for a homogeneous cone of mass M whose height is h and whose base has a radius R . Choose the x_3 -axis along the axis of symmetry of the cone. Choose the origin at the apex of the cone, and calculate the elements of the inertia tensor. Then make a transformation such that the center of mass of the cone becomes the origin, and find the principal moments of inertia.

Problem A1 - Inertia tensor for a rectangular plate

Consider a thin homogenous rectangular plate of mass M with sides, of length a and b , aligned with the x and y coordinate axes, respectively.

- (a) Determine the inertia tensor for the plate when the origin of the coordinate system is located at a corner of the plate.
- (b) Determine the inertia tensor for the plate when the origin of the coordinate system is located at the center of the plate.

Problem A2 - Hamiltonian in a rotating system

The Hamiltonian formalism can be very useful when working in a rotating system. The form of the Hamiltonian written in reference to the non-inertial frame is simply related to the form in a corresponding inertial frame. To demonstrate this we will consider doing physics on a rotating turntable. We will work in Cartesian coordinates with (x, y) referring to life on the rotating turntable and (x_{lab}, y_{lab}) referring to the external fixed lab frame. The coordinate systems are related by:

$$x_{lab} = x \cos \omega t - y \sin \omega t \quad \text{and} \quad y_{lab} = y \cos \omega t + x \sin \omega t \quad (\text{with } z_{lab} = z)$$

where we have the turntable rotating counterclockwise about the z -axis at a constant angular speed ω . We start from the Lagrangian which must be defined in the inertial (fixed) frame:

$$L = \frac{1}{2} m v_{lab}^2 - U(x_{lab}, y_{lab})$$

- (a) Rewrite the Lagrangian in terms of the (x, y) coordinates and $(\dot{x}, \dot{y})$ velocities. (Assume you can write $U(x, y)$ for the potential).
- (b) Determine the canonical momenta *in the rotating system*: p_x and p_y .
- (c) Construct the Hamiltonian *in the rotating system*: $H(x, y, p_x, p_y)$.
- (d) Demonstrate that $H_{\omega \neq 0} = H_{\omega=0} - \omega l_z$, where l_z is the z component of the angular momentum as seen in the rotating frame. [Thus we have a simple prescription for writing the Hamiltonian in a rotating frame. All the non-inertial dynamics are contained in this Hamiltonian.]

- (e) Write Hamilton's equations *in the rotating system*.

Problem C1 - Force free motion in a rotating system

Consider motion on a frictionless rotating turntable as described in example 10.2. The motion is “force free” but to an observer in the rotating frame the effective force given at the top of this sheet must be invoked to account for the dynamics. In the above given expression ω is the angular velocity of the turntable (rotating counterclockwise as seen from above) and v_r is the velocity measured by the observer on the rotating surface.

- (a) Show that the effective acceleration measured on the surface of the turntable is:

$$\ddot{x} = \omega^2 x + 2\omega \dot{y} \quad \text{and} \quad \ddot{y} = \omega^2 y - 2\omega \dot{x}$$

- (b) Integrate these coupled equations of motion numerically (using 4th-order Runge Kutta) to compute the trajectories (c) and (e) shown in Fig. 10-4 (using the numerical constants specified in example 10.2). Plot these trajectories.
- (c) Use trial and error to find a trajectory like that shown in Fig. 10-4e, but which passes through the starting point (see problem 10-5). Plot this trajectory.

Note: R code for this problem provided below.

```
x1 <- c()
y1 <- c()
w <- 1
R <- 1
x0 <- -0.5
y0 <- 0.0
v <- 0.47
theta <- pi*45/180
vx <- v*cos(theta)
vy <- v*sin(theta)
x <- x0
y <- y0
```

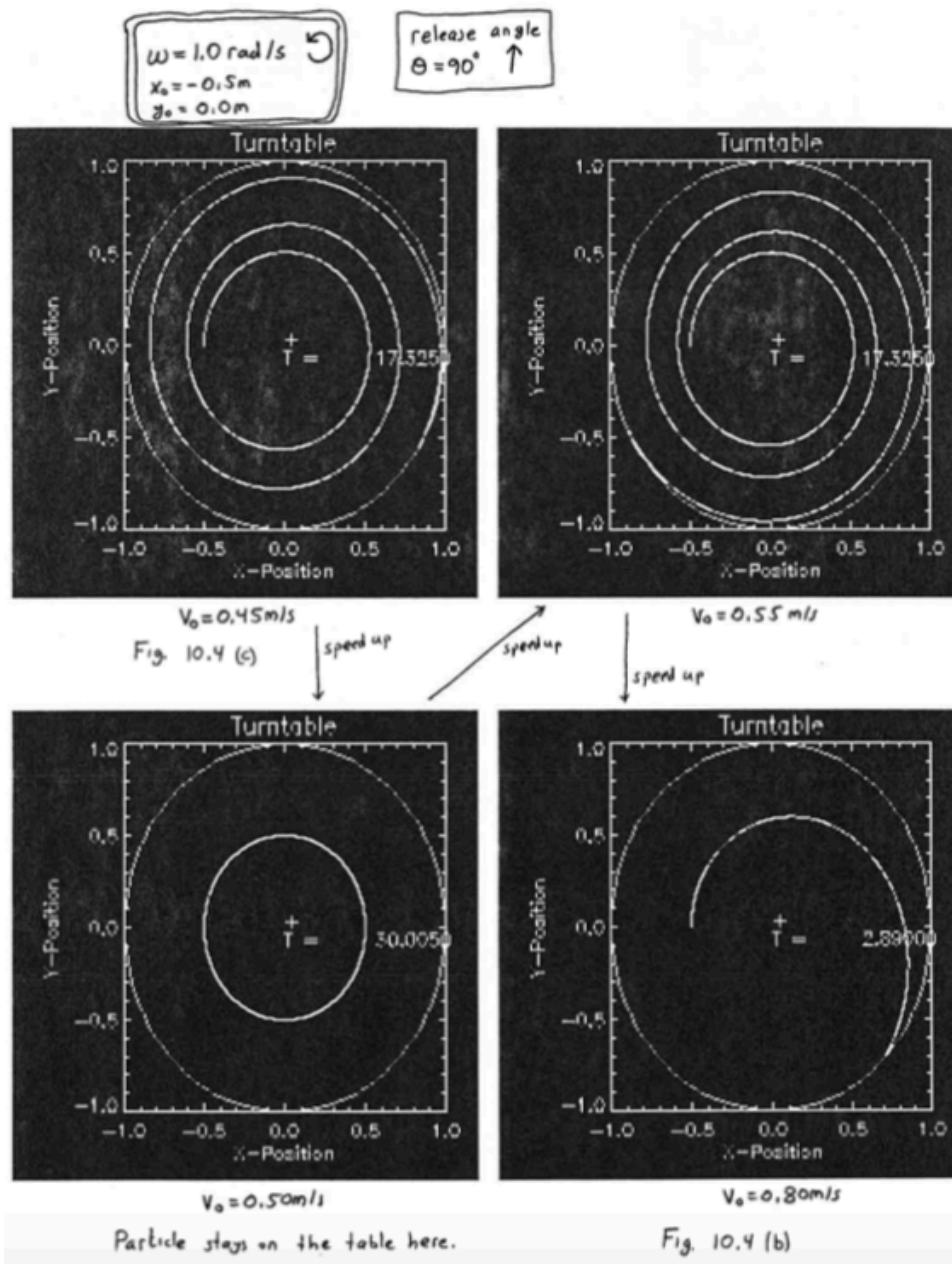
```

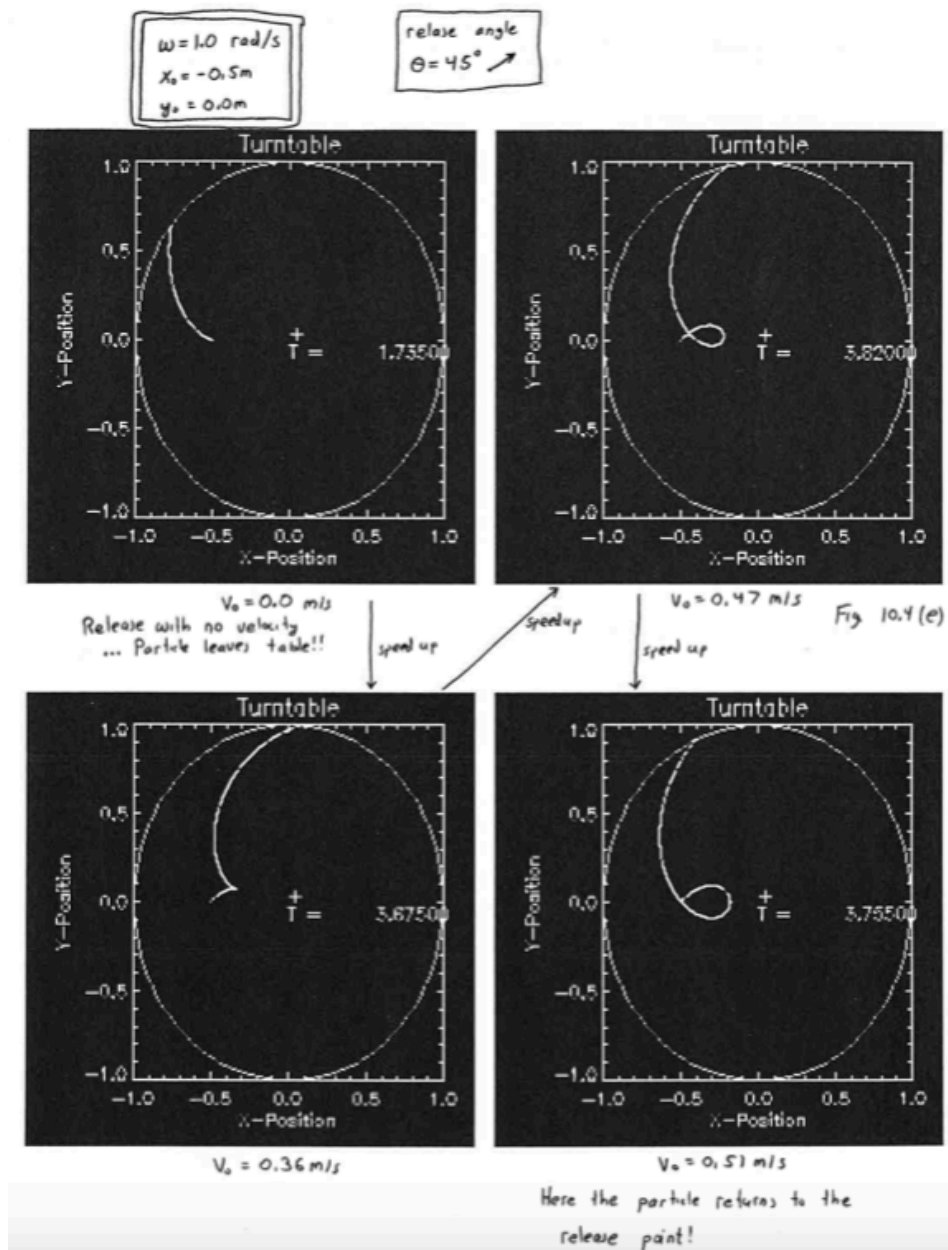
runtime <- 30
h=0.005
nstep <- runtime/h
accx <- function(x,vy,w){w*(w*x+2*vy)}
accy <- function(y,vx,w){w*(w*y-2*v x)}
count <- 0
th <- 2*pi*(0:100)/100
x2 <- R*cos(th)
y2 <- R*sin(th)
for (j in 0:nstep){
  count <- count+1
  fx1 <- vx
  gx1 <- accx(x,vy,w)
  fy1 <- vy
  gy1 <- accy(y,vx,w)
  fx2 <- vx+h*gx1/2
  gx2 <- accx(x+h*fx1/2,vy+h*gy1/2,w)
  fy2 <- vy+h*gy1/2
  gy2 <- accy(y+h*fy1/2.0,vx+h*gx1/2.0,w)
  fx3 <- vx+h*gx2/2
  gx3 <- accx(x+h*fx2/2.0,vy+h*gy2/2.0,w)
  fy3 <- vy+h*gy2/2
  gy3 <- accy(y+h*fy2/2.0,vx+h*gx2/2.0,w)
  fx4 <- vx+h*gx3
  gx4 <- accx(x+h*fx3,vy+h*gy3,w)
  fy4 <- vy+h*gy3
  gy4 <- accy(y+h*fy3,vx+h*gx3,w)
  x <- x+h*(fx1+2*fx2+2*fx3+fx4)/6
  vx <- vx+h*(gx1+2*gx2+2*gx3+gx4)/6
  y <- y+h*(fy1+2*fy2+2*fy3+fy4)/6
  vy <- vy +h*(gy1+2*gy2+2*gy3+gy4)/6
  x1 <- append(x1,x)
  y1 <- append(y1,y)
  r <- sqrt(x*x+y*y)
  if (r > R) break
}
par(pty="s")
plot(x1,y1,pch=".",col="red",xlim=c(-2,2),ylim=c(-2,2))

```

```
lines(x2,y2,col="green")
```

Sample runs:





Assignment 10 - Rigid-Body Dynamics II

$$[L_i = \sum_j I_{ij} \omega_j, T_{rot} = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j]$$

Reading Assignment: Marion & Thornton: Chap. 11. Sec. 2-4 and 7-9.

Overview: The first step in solving the dynamics of rigid bodies is to separate the center of mass (CM) motion from rotational motion about the CM. This rotational motion depends strongly on how the mass of the object is distributed. In particular, the second moments of the mass distribution (which make up the elements of the inertia tensor I_{ij}) contain all the information needed to completely describe the rotational motion. Thus two objects with totally different shapes, but the same inertia tensor, will display the same rotational dynamics. From the inertia tensor we can calculate the angular momentum $\mathbf{L}$ and rotational kinetic energy T_{rot} of a rigid body rotating with angular velocity ω (as shown in the equations above). Note that $\mathbf{L}$ and ω are vectors defined with respect to a “body” coordinate system with origin at the CM. It is usually most convenient to define this body coordinate system in terms of the “principle axes” of the body, in which case the inertia tensor is diagonal. For rotation about a principle axis, $\mathbf{L}$ and ω are collinear and we have the familiar results $\mathbf{L} = I\boldsymbol{\omega}$ and $T = I\omega^2/2$ (where I is the [scaler] moment of inertia with respect to the [principle] rotation axis). When the rotation axis does not coincide with a principle axis, $\mathbf{L}$ and ω will not be collinear and very interesting (and often complicated) rotational motion results, even in the absence of external forces. The wobbling of a football or frisbee and the so called “Chandler wobble” of the earth are examples of such motion. These wobbling effects occur in “force free” (i.e., torque free so $d\mathbf{L}/dt = 0$) motion, thus demonstrating that a constant $\mathbf{L}$ does not imply a constant ω . When external forces such as gravity and friction produce torques on our rotating body, both $\mathbf{L}$ and ω will change with time and the resulting motion can be quite bizarre (as we will see in “Toy-Fest 2000” in next seminar). Our goals for this seminar will be to understand how to calculate and manipulate the inertia tensor, how to determine principle axes for an object, how to use Euler’s equations to compute rotational dynamics relative to a body coordinate system and, finally, how to make that killer pool shot.

Problem Assignment:

Problem 11.5 - No English condition in billiards

(a) Find the height at which a billiard ball should be struck so that it will roll with no initial slipping. (b) Calculate the optimum height of the rail of a billiard table. On what basis is the calculation predicated?

Problem 11.11 - Toppled cube

A homogeneous cube, each edge of which has a length l , is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a small displacement and allowed to fall. Show that the angular velocity of the cube when one face strikes the plane is given by

$$\omega^2 = A \frac{g}{l} (\sqrt{2} - 1)$$

where $A = 3/2$ if the edge cannot slide on the plane and where $A = 12/5$ if sliding can occur without friction.

Problem 11.13 - Inertia tensor and principle axes for a 3 particle system

A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$\begin{aligned} m_1 &= 3m \quad , \quad (b, 0, b) \\ m_2 &= 4m \quad , \quad (b, b, -b) \\ m_3 &= 2m \quad , \quad (-b, b, 0) \end{aligned}$$

Find the inertia tensor, principal axes, and principal moments of inertia.

Problem 11.18 - Transformation of the inertia tensor for a thin plate

if, in the previous problem, the coordinate axes are rotated through an angle θ about the x_3 -axis, show that the new inertia tensor is

$$\{\mathbf{I}\} = \begin{pmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A' + B' \end{pmatrix}$$

where

$$\begin{aligned} A' &= A \cos^2 \theta - C \sin 2\theta + B \sin^2 \theta \\ B' &= A \sin^2 \theta + C \sin 2\theta + B \cos^2 \theta \\ C' &= C \cos 2\theta - \frac{1}{2}(B - A) \sin 2\theta \end{aligned}$$

and hence show that the x_1 - and x_2 -axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2C}{B-A} \right)$$

Problem 11.31 - Force free motion of a rotating plate

Consider a thin homogeneous plate with principal momenta of inertia

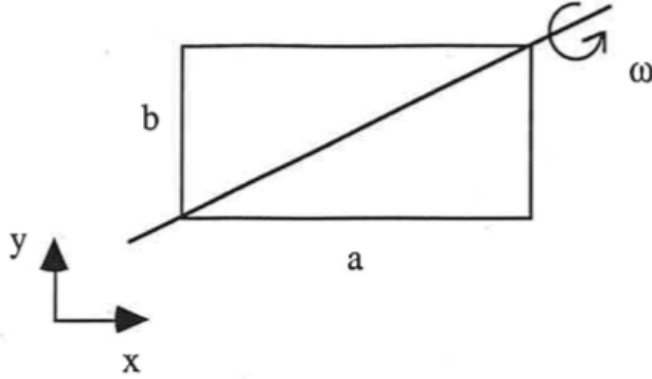
$$\begin{aligned} I_1 &\text{ along the principal axis } x_1 \\ I_2 &> I_1 \text{ along the principal axis } x_2 \\ I_3 &= I_1 + I_2 \text{ along the principal axis } x_3 \end{aligned}$$

Let the origins of the x_i and x'_i systems coincide and be located at the center of mass O of the plate. At time $t = 0$, the plate is set rotating in a force-free manner with an angular velocity Ω about an axis inclined at an angle α from the plane of the plate and perpendicular to the x_2 -axis. If $I_1/I_2 \equiv \cos 2\alpha$, show that at time t the angular velocity about the x_2 -axis is

$$\omega_2(t) = \Omega \cos \alpha \tanh(\Omega t \sin \alpha)$$

Problem A1 - Rectangular plate rotating about a diagonal

Consider a thin rectangular plate of mass M and side lengths a and b rotating with constant angular speed ω about a diagonal.



Assume the plate thickness is negligible and ignore gravity. As we saw earlier, the inertia tensor for this object relative to an $x - y$ body coordinate system with origin at the CM is

$$I = \frac{M}{12} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

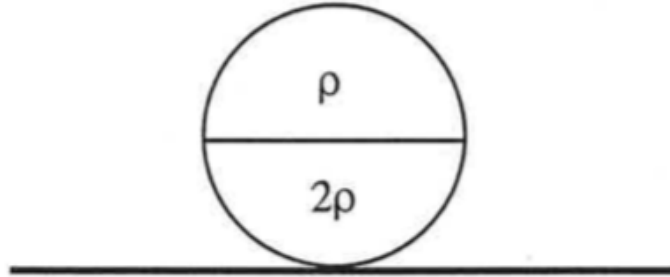
- (a) Compute the rotational kinetic energy of the plate.
- (b) Determine the angle between the L and ω vectors in the body coordinate system and determine under what conditions they will be collinear.
- (c) Use Euler's equations to show that a torque of magnitude:

$$\tau = \frac{Mab(a^2 - b^2)\omega^2}{12(a^2 + b^2)}$$

- (d) Determine the direction of this applied torque and explain how you could produce it.

Problem A2 - Lagrangian for a rolling composite disk

Consider a thin disk composed of two homogeneous halves connected along a diameter.



One half of the disk has density ρ and the other has density 2ρ . The disk rolls without slipping on a horizontal surface and all motion takes place within the plane of the disk. (See problem 1I-25).

- (a) Determine the Lagrangian for this system
- (b) Derive the equation of motion for the disk.
- (c) Show that the frequency of small oscillations of the disk about its equilibrium position is given by

$$\omega^2 = \frac{8}{27\pi - 16} \frac{g}{R}$$

where R is the disk radius.

Assignment 11 - Rigid-Body Dynamics III & Toy-Fest

Reading Assignment: Marion & Thornton: Chap. 11. Reread and concentrate on sections 7, 10, and 11.

Overview: This week we will study the rotational dynamics of rigid bodies when external torques act on the spinning object. The precession and nutation of a toy top or gyroscope provide dramatic illustrations of the complex motion resulting from an applied torque due to gravity. Add in friction and things get even more interesting (as will be seen for several of the featured toys). Last week we investigated rotational dynamics using Euler's equations, which describe the motion of a spinning body relative to the instantaneous

position of its principle axes. However, these equations don't tell us how the principle axes themselves move relative to a fixed lab frame. (Notice that in Marion's discussion of force free precession of a symmetric top [Sec 11.9] one must appeal to conservation of energy and angular momentum to ferret out the fixed frame "space-cone" dynamics). To describe, in general, rotational dynamics in the lab frame we need a means of specifying the orientation of our rigid body relative to a fixed inertial coordinate system. There is not a unique way to do this, but using the set of angles introduced by Euler (and called the Euler angles wouldn't you know) is the standard approach. The Euler angles provide a convenient set of generalized coordinates to use in constructing the Lagrangian for a rotating rigid body. Marion provides an illustration of this with the analysis of the symmetric top with fixed pivot point [Sec. 11-10]. The majority of this weeks seminar will be devoted to our own analysis (and demonstration) of some rotating (as well as wobbling, tipping, and rattling) rigid body toys. The equations of motion for these object are complicated, but hopefully, experimenting with the toys will aid in our understanding the dynamics hidden in the equations. One other important topic we need to cover this week is the stability of rigid-body rotations, i.e., about which principle axes can we actually have stable wobbling. Perhaps our frisbee expert can address this issue. We should also try to make sense of the claim that "flying saucers make better spacecraft than do flying cigars". Our stability arguments do not support this claim ... but, what if we allow for energy lose due to internal friction? The same logic is often used to explain why a person falling down a mountain tends to tumble head-over heels!

The toys we will discuss:

1. Rattle-back

Jearl Walker, Scientific American, Oct.1979, pg 172

Hermann Bondi, Proc. R. Soc. Lond. A **405**, 265 (1986)

2. Tippy-top

C.G. Gray and B.G. Nickel, Am. J. Phys. **68**, 821 (2000). Contains all references.

R.J. Cohen, Am. J. Phys. **45**,72 (1977). Best of references.

3. Big penny - When set upright and spun it displays rather curious and striking behavior as it loses energy to friction.
4. Frisbee and Boomerang

Classical Mechanics by Barger and Olsson, pgs. 195-202 – boomerang

M. Schulrnans, New Scientist, July 1990, pgs. 37-40 – frisbee

Problem Assignment:

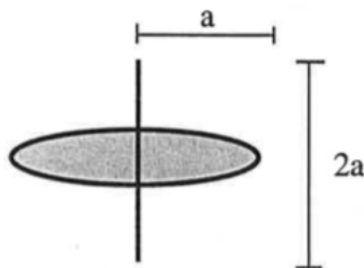
Problem A1 - Spinning pencil

A pencil is set spinning in an upright position (tip down). Treat the pencil as a uniform cylinder of length a and diameter b .

- (a) How fast must the pencil spin in order to remain upright?
- (b) Obtain a numerical result for the minimum spinning speed if $a = 20$ cm and $b = 1$ cm.

Problem A2 - Symmetric toy top: Sleeping, precession, and nutaton

Consider a toy top constructed from a heavy circular disk of mass $m = 100$ g and radius $a = 2.0$ cm mounted at the center of a thin rod of total length $2a$. The top is placed on a stand so the pivot point is fixed. (Ignore friction)



- (a) Find the minimum spinning speed of the top such that it can “sleep” in the vertical position.

Now suppose we start the top spinning at $\omega_0 = 35$ rev/s and initially its spin axis is held fixed at an angle of $\theta_0 = 60^\circ$ from the vertical. The axis is then released and the top starts to topple over (i.e., initial conditions are $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) = 0$). As it falls the top starts to precess as well as nutate between the limiting polar angles θ_1 and θ_2 .

- (b) Calculate the two limiting angles θ_1 and θ_2 (see Marion problem 11-30) and make a quantitative plot of the effective potential $V(\theta)$. Does $|\theta_1 - \theta_2|$ increase or decrease with increasing ω_0 ?
- (c) Estimate the period of nutation analytically.
- (d) Estimate the average period of precession analytically.
- (e) If we had started the top by pushing it “just so”, such that $\dot{\theta}(0) = 0$ and $\dot{\phi}(0) \neq 0$, we can have precession without nutation. What two initial values of $\dot{\phi}(0)$ will result in such motion (for the above given ω_0 and θ_0)? These two values correspond to so called fast and slow precession. Why do you think that it is the slow precession that is usually observed?

Toy Write-Ups from Swarthmore

Rattleback

$I_3 > I_2$, $I_3 \approx I_1$
 $I_1 > I_2$

Consider rocking about short axis

$$I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = N_1$$

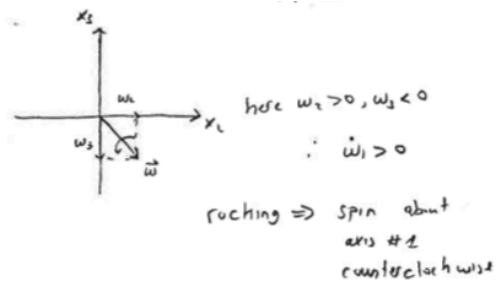
$$I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 = N_2$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = N_3$$

11/00
(MPT)

Euler says:
 $(I_2 - I_3) \omega_2 \omega_3 = I_1 \dot{\omega}_1$
 for $N_1 = 0$
 here $\omega_2 > 0$, $\omega_3 < 0$, $I_3 > I_2$
 $\therefore \dot{\omega}_1 > 0$

∴ This type of rucking builds spin counterclockwise or reduces existing clockwise spin.



Rucking about long axis results in opposite spin direction... Thus the possibility of two types of spin reversal.

(w_2 and w_3 will always have the same sign here)

$$\dot{w}_3 = \frac{N_3 + (I_1 - I_2)w_1w_2}{I_3}$$

due to surface constraint w_2 and w_3 have opposite signs. Generation of w_3 forces generation of w_2 . N_3 always acts as restoring torque (will have opposite sign of w_3) ← forget axis rucking

for $w_1 > 0$ (natural direct) $\dot{w}_3$ has same sign as $w_2 \rightarrow w_3$ decays

for $w_1 < 0$ (bad direct) $\dot{w}_3$ has opposite sign as $w_2 \rightarrow w_3$ grows

$$\left(\dot{w}_3 \approx \frac{N_3}{I_3} \leftarrow \text{restoring only, if } I_1 \approx I_2 \right)$$

$$\vec{\omega} = A(t) \cos \Omega t (-\sin \Theta \hat{w}_1 + \cos \Theta \hat{w}_3)$$

$$\omega_2 = -A(t) \sin \Theta \cos \Omega t$$

$$\omega_3 = A(t) \cos \Theta \cos \Omega t$$

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3$$

$$= \left(\frac{I_2 - I_3}{I_1} \right) A^2(t) \cos^2 \Omega t \sin \Theta \cos \Theta$$

$$E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$E_0 = \frac{1}{2} A_0^2 \cos^2 \Omega t \left\{ I_2 \sin^2 \Theta + I_3 \cos^2 \Theta \right\}$$

$$E(t) = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} A^2(t) \cos^2 \Omega t \left\{ I_2 \sin^2 \Theta + I_3 \cos^2 \Theta \right\}$$

$$\dot{A}^2(t) = \frac{A_0^2 \cos^2 \Omega t \left\{ \right\} - I_1 \omega_1^2}{\cos^2 \Omega t \left\{ \right\}}$$

$$= A_0^2 - \frac{I_1 \omega_1^2}{\cos^2 \Omega t \left\{ I_2 \sin^2 \Theta + I_3 \cos^2 \Theta \right\}}$$

$$\begin{aligned}\frac{d\omega_1}{dt} &= \frac{I_3 - I_1}{I_1} \sin\theta \cos\theta \left[A_0^2 - \frac{I_1 \omega_1^2}{\cos^2 \Omega t (I_1 \sin^2 \theta + I_3 \cos^2 \theta)} \right] \cos^2 \Omega t \\ &= \frac{I_3 - I_1}{2I_1} \sin\theta \cos\theta \left[A_0^2 \cos^2 \Omega t - \frac{I_1 \omega_1^2}{I_1 \sin^2 \theta + I_3 \cos^2 \theta} \right]\end{aligned}$$

$$\frac{d\omega_1}{dt} = B \{ A_0^2 \cos^2 \Omega t - C \omega_1^2 \}$$

$$\frac{d\omega_1}{dt} + BC \omega_1^2 = BA_0^2 \cos^2 \Omega t$$

$$\omega_1 = e^{i\alpha t}$$

The Tippy-Top

General Description

As seen in Figure (1), when a tippy-top is spun on a smooth surface, it turns itself upside-down to rotate on its stem. In the process, it raises its center of mass and reverses its direction of rotation. Similarly, when a glass ring is spun with the stone down, it lofts the stone in the air, and when a hard-boiled egg is spun flat, it rises to spin on one end.

This peculiar behavior has intrigued physicists since at least the 19th century, but it was not accurately described until the 1950's, and even then the description was limited by large approximations.

Friction is the key to understanding the behavior of the tippy-top. Since the CM is elevated during inversion (increasing the potential energy), the rotational kinetic energy must decrease. Thus, the total angular momentum $\mathbf{L}$ must decrease, which requires an external torque. $\mathbf{L}$ is almost entirely in the vertical $\hat{z}$ direction, so it cannot be changed by the normal or gravitational forces, since both point along the z axis. Friction is therefore necessary to explain the top's peculiar behavior.

A Simple Model¹

Because the CM of a tippy-top is near the center of curvature (above which the top spins), as seen in Figure (2), the gravitational torque can be neglected. Note that the frictional force is coming out of the page, opposing the direction of slipping. The frictional torque is nearly horizontal and time averages to zero, so by neglecting it as well, we have no external torques on the top. Assuming that the tippy-top is initially spun vertically, this means that the angular momentum remains a constant along the z -axis.

Now let $\mathbf{L}$ be the angular momentum described in the body frame. Using the definition of torque and the general relation between the time derivative of a vector in the fixed and rotating frames, we have

$$\mathbf{N} = \frac{d\mathbf{L}_{\text{fixed}}}{dt} = \frac{d\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} \quad (1)$$

Since the tippy-top is approximately spherical, we let $I_1 \cong I_2 \cong I_3$, so that $\mathbf{L} \cong I\boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \mathbf{L} \cong 0$. $\mathbf{L}$ remains vertical and $\mathbf{N}$ remains horizontal. Thus, by Eq. (1), $\frac{d\mathbf{L}}{dt}$ is always perpendicular to $\mathbf{L}$, causing $\mathbf{L}$ to precess uniformly in the body frame.

Taking the component of Eq. (1) along the symmetry axis of the top, $\hat{\mathbf{e}}_3$, we have

$$\hat{\mathbf{e}}_3 \cdot \mathbf{N} = \frac{d(\hat{\mathbf{e}}_3 \cdot \mathbf{L})}{dt} \quad (2)$$

Since $\hat{\mathbf{e}}_3 \cdot \mathbf{N} = N \cos(\theta + \frac{\pi}{2}) = -N \sin \theta$ and $\hat{\mathbf{e}}_3 \cdot \mathbf{L} = L \cos \theta$, Eq. (2) becomes

$$-N \sin \theta = \frac{d}{dt} L \cos \theta = -L \dot{\theta} \sin \theta + \dot{L} \cos \theta \quad (3)$$

We have assumed that $\dot{L} = 0$, and so we have

$$\dot{\theta} = \frac{N}{L} \cong \frac{\mu MgR}{I\omega} \quad (4)$$

We see that θ increases, causing the top to begin to flip.

¹Barger and Olsson, *Classical Mechanics: A Modern Perspective*, 1973.

Once the stem of the top touches the table, we have an ordinary rising top. $\mathbf{N}$ will still be perpendicular to $\mathbf{L}$, and $\mathbf{L}$ will still change in the direction of $\mathbf{N}$. However, since we now assume that $\mathbf{L}$ lies along the symmetry axis $\hat{\mathbf{e}}_3$, $\dot{\theta}$ is negative, and the tippy-top rights itself to the stable verticle position, where it will remain as long as

$$\omega_3 \geq \frac{(4MgRI)^{\frac{1}{2}}}{I_3} \quad (5)$$

as we saw in problem A1.

Other Considerations

While this analysis has given us a basic understanding of how the tippy-top works, we have made some large approximations. $\mathbf{L}$ is not really a constant during the first part of inversion, then suddenly switching from being along the fixed vertical axis to the body axis when the stem touches the table. Also, we assumed that the center of mass was on the axis of rotation, which allowed us to ignore the normal and gravitational forces.

A more rigorous analysis leads to six nonlinear differential equations (three translational and three angular), and a numerical solution allows you to solve for the motion of the top. The θ for a small coefficient of friction is seen in Figure (3).² Note that it is not monotonically increasing, but is nutating as the top flips.

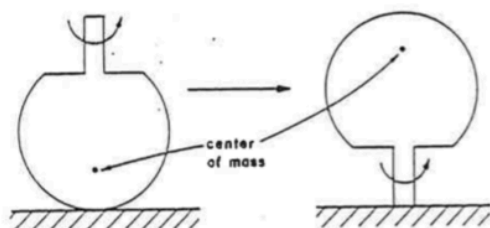
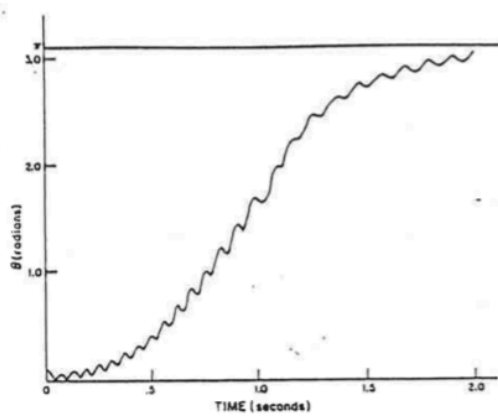


Fig. 1. Tipler top inverting.



²R.J. Cohen, *Am. J. Phys.* 45, 12 (1977).

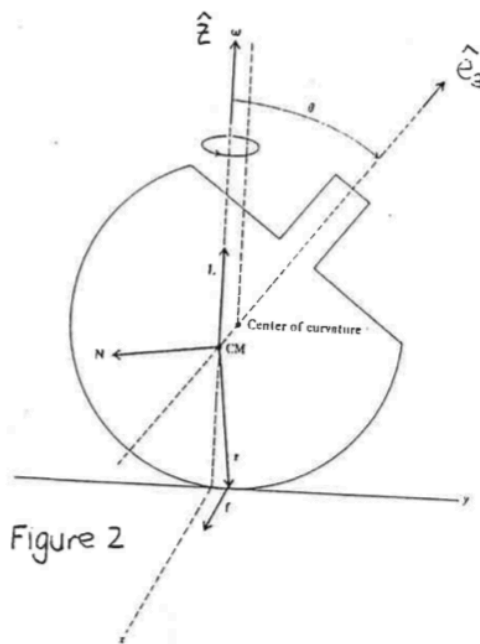


Figure 2

Figure 3

2.55

Boomerangs

Returning boomerangs are designed to be thrown great distances and to return to the thrower. Australian natives have thrown them as far as 100 yards and to heights of 150 feet with five complete circles.

The nonreturning type, which is more practical for hunting, can be thrown as far as 180 yards.

The ordinary boomerang is shaped like a bent banana. Is it essential that the boomerang have this particular shape? Can one make a returning boomerang in the shape of an X or a Y? Most

boomerangs are designed to be thrown with the right hand. What is the difference between left- and right-handed boomerangs? Why does a boomerang (of any shape) return? Why does it loop around in its path (see Figure 2.55)? Finally, how does the path depend on the boomerang's orientation as it leaves the thrower's hand?

26, pp. 153-154; 37, pp. 291-296; 65 through 67; 1564.

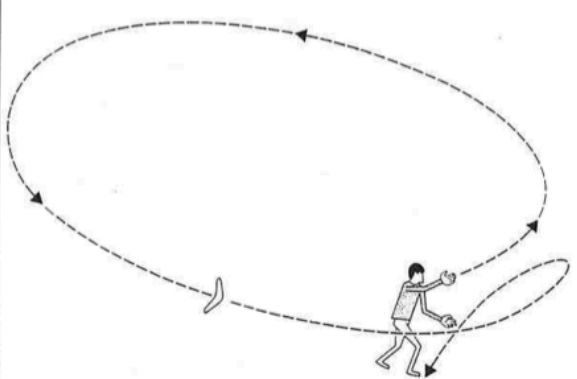


Figure 2.55
Boomerang path.

26 Routh, E. J., *Dynamics of a System of Rigid Bodies, Advanced Part*, Macmillan, London (1930).

37 Gray, A., *Treatise on Gyrostatics and Rotational Motion*, Dover, New York (1959).

65 Hess, F., "The Aerodynamics of Boomerangs," *Sci. Amer.*, 219, 124 (Nov. 1968).

66 Walker, G. T., "On Boomerangs," *Phil. Trans. Roy. Soc. Lond.*, A190, 23 (1897).

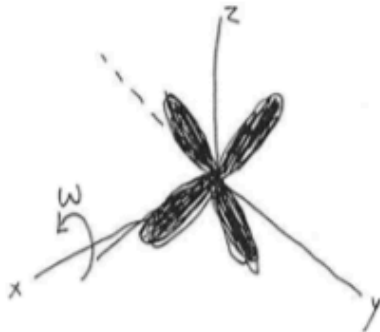
67 "Boomerang," *Encyclopaedia Britannica*, William Benton, Chicago (1970), Vol. 3, pp. 945-946.

2.55 The right-handed boomerang is thrown in a vertical plane so that it spins about a horizontal axis. Since it is an air-foil, there is a sideways "lift" on it, the lift being larger on the top half than on the bottom half, because the top half is turning in the same direction that the boomerang is traveling, whereas the bottom half is turning in the opposite direction. Therefore, there is a torque attempting to tilt the boomerang, but instead of tilting, the boomerang veers to the left and remains vertical. Sufficient veering causes the boomerang to turn full circle in its flight.

Boomerangs

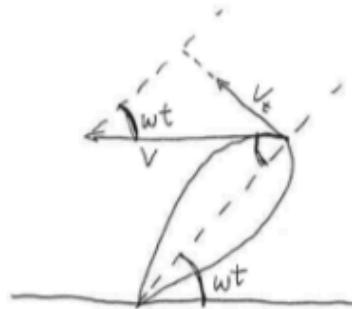
Why do they come back?

Two things: transverse acceleration and precession, result in circular motion. The air causes a net lift force perpendicular to the motion. This lift force also exerts a torque, since the upper blades are travelling faster than the lower blades. Therefore, in order to conserve angular momentum, the boomerang precesses. From this we get circular motion.



$$\vec{r} = r(\hat{y}\cos\omega t + \hat{z}\sin\omega t)$$

$$v_t = \omega r + V\sin\omega t$$



$$\vec{V}_\perp = -V\hat{y}$$

Lift force: varies with square of transverse velocity

$$d\vec{F} = \hat{x}c v_t^2 dr$$

$$\vec{F} = 4cl\left(\frac{\omega^2 l^2}{3} + V^2\right)\hat{x} \quad - \text{total inward acceleration}$$

Torque: $d\vec{N} = \vec{r} \times d\vec{F}$

$$\vec{N} = \frac{4c\omega V l^3}{3}\hat{y} \quad - \text{net torque (note: non-zero)}$$

This net torque causes the boomerang to precess in the counterclockwise from above, with frequency given by:

$$\omega_p = \frac{N}{L} = \frac{N}{\frac{M}{\omega}} = \frac{4cVl}{M}, \text{ which for circular motion will equal } V/R.$$

From this, and by setting the net lift force equal to

$$MV^2/R, \text{ we get:}$$

$R = \frac{\mu}{c}$ and $V = \omega l \sqrt{\frac{2}{3}}$, where μ is the mass density. Thus, the radius of flight only depends on the boomerang and the air, not on how hard it is thrown.

EXAMPLE 9.6.1

PRECESSION OF A FRISBEE

As an example of the above theory we consider the case of a thin disc, or any symmetric and fairly "flat" object, such as a china plate or a Frisbee. The perpendicular-axis theorem for principal axes is $I_1 + I_2 = I_3$, and, for a symmetric body $I_1 = I_2$, so that $2I_1 = I_3$. In our present notation this is $2I = I_z$, so the ratio

$$\frac{I_z}{I} = 2$$

to a good approximation. If our object is thrown into the air in such a way that the angular velocity ω is inclined to the symmetry axis by an angle α , then Equation 9.5.8 gives

$$\Omega = \omega \cos \alpha$$

for the rate of precession of the rotational axis about the symmetry axis.

For the precession of the symmetry axis about the invariable line, the wobble as seen from the outside, Equation 9.6.2 yields

$$\dot{\phi} = \omega(1 + 3 \cos^2 \alpha)^{1/2}$$

In particular, if α is quite small so that $\cos \alpha$ is very nearly unity, then we have approximately

$$\Omega \approx \omega$$

$$\dot{\phi} \approx 2\omega$$

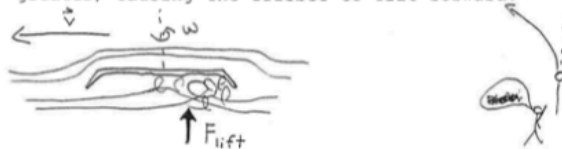
Thus, the wobble rate is very nearly twice the angular speed of rotation.

Frisbees

Frisbees are much simpler toys, yet they exhibit some interesting behaviors that can be explained using conservation of momentum and other concepts in Chapter 11.

They topple:

Toppling is what the experts call it when your frisbee veers off and turns over. This happens because the lift force doesn't act evenly on the disc. Towards the back of the disc the underneath air travels slower, so the lift is greater, causing the frisbee to tilt forward.

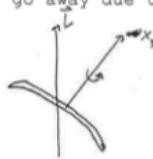


Because of conservation of momentum, and similar to the precession of the boomerang, the frisbee adjusts by being torqued to one side or the other. At this point the flying disc has essentially doomed itself.

Good throwers adjust for this effect when throwing the frisbee.

They wobble:

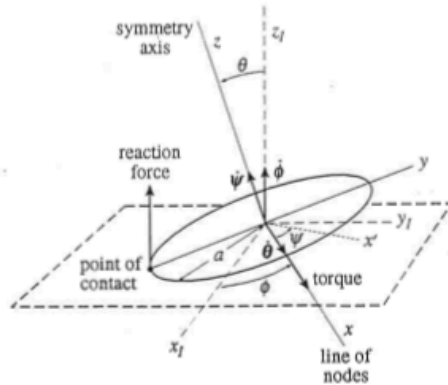
As discussed in my previous talk, symmetric tops given spin not quite along their symmetry axis will wobble (like the football). Interestingly, since our system is in fact not force-free, and there is air resistance, a frisbee's wobble will usually go away due to air friction.



Frisbee construction is an art. One must balance the desire for a high moment of inertia - in order to resist toppling - with a need for levity and low drag. One very successful version is the "Aerobie" ring, which can easily be thrown twice as far as a conventional frisbee, almost without toppling.

- 4 A coin is initially set spinning rapidly on its edge on a table. As the spin decreases due to frictional forces, a point is reached where the coin begins to precess (wobble). With further loss of energy the coin will gradually "lie down" with an increase of the precession rate while at the same time the actual turning rate, as seen from above, decreases. Explain the curious behavior of a spinning coin just described, and show your calculations in detail. (Assume frictional torque is negligible in comparison with the gravitational torque.)

A disk is spun about a vertical diameter. As it loses energy through friction it begins to wobble with slowly decreasing angle θ . Assume that the only motion of the CM is to fall slowly and that the disk rolls without slipping.



- Show that the rolling condition is $\omega_3 = 0$.
- Show that the precession of the line of nodes $\dot{\phi}$ is given by $\dot{\phi} = \omega_0 / \sqrt{\sin \theta}$, where $\omega_0 = 2\sqrt{g/a}$. The wobble rate thus increases as the disk lies down. *Hint: Use part a) to establish that $\mathbf{L}$ lies along the $\hat{y}$ axis. The torque lies along the $\hat{x}$ axis. Use $\mathbf{N} = \dot{\mathbf{L}} = \dot{\phi} \times \mathbf{L}$.*
- Show that the component of spin along the vertical x -axis is given by $\omega_{\text{vert}} = \omega_0 (\sin \theta)^{3/2}$. Thus even though the wobble rate increases without limit as the disk lies down, the spin as seen from above actually comes to a stop. This effect can be impressively demonstrated using a heavy disk, with a mark on top, spinning on a cement floor.

The Big Penny

The Big Penny is simply a large heavy disk. If we spin it about its diameter, it loses energy and eventually begins to wobble (precess) faster and faster until it suddenly stops. At the same time, however, the rotation as seen from above comes to a stop. We'll see in a moment why these effects occur. And, in case you were wondering, the Big Penny does indeed count as a toy - there is a commercial version called Euler's Disk that consists of a very heavy disk with a rounded edge on a mirrored surface that, due to the fact that it has very little friction, can wobble for upwards of a minute.

In order to analyze the penny, we need to make a few assumptions. First, we assume that the penny rolls on its edge without slipping. Secondly, we assume that the center of mass falls slowly down and that the frictional torque is much less than the torque due to gravity. With these assumptions, we can analyze the penny analytically.

First we examine what it means to have the penny roll without slipping. If the penny were to slip as it rolls, then at any instant in time there would be some nonzero ω_3 about the symmetry axis (z) of the penny. Thus, the condition that the penny rolls without slipping amounts to specifying that

$$\omega_3 = 0$$

Given this, we can find the angular momentum of the system. At any instant in time, we know that both the center of mass and the contact point must be at rest, and so the instantaneous motion of the penny is a rotation about the y -axis with an angular speed of $\dot{\phi} \sin \theta$. The angular momentum is in the y direction and has a magnitude of

$$L = \frac{1}{4} M a^2 \dot{\phi} \sin \theta,$$

where M is the mass of the disk, a is the radius of the disk, and using the fact that the moment of inertia of the disk is $\frac{1}{4} M a^2$.

As this angular momentum changes, it creates a torque. Looking at the geometry of the system, we find that

$$\mathbf{N} = \dot{\mathbf{L}} = \dot{\phi} \times \mathbf{L}$$

This torque is due wholly to gravity (based on our assumption that the frictional torque is negligible) and has magnitude

$$N = M g a \cos \theta$$

We note that the angle between $\dot{\phi}$ and $\mathbf{L}$ is $90^\circ - \theta$. Given this information, then, we can then take the magnitude of the cross product and arrive at an equation soluble for $\dot{\phi}$:

$$M g a \cos \theta = \dot{\phi} \frac{1}{4} M a^2 \dot{\phi} \cos \theta$$

Solving for $\dot{\phi}$ gives us Solving for $\dot{\phi}$, we have

$$\dot{\phi} = \frac{\omega_0}{\sqrt{\sin \theta}}, \quad (1)$$

where $\omega_0 = 2\sqrt{g/a}$. This shows that $\dot{\phi}$ (the wobble rate, or the precession of the line of nodes) increases without bound as the disk lies down.

What remains is to demonstrate that the spin rate as seen from above goes to zero as the penny lies down. We need to find the total spin in the vertical direction. Examining the Euler angles, we can see that this spin ω_{vert} is given by

$$\omega_{\text{vert}} = \dot{\phi} + \dot{\psi} \cos \theta$$

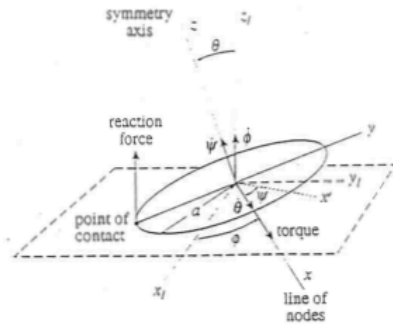
We showed previously that $\omega_3 = 0$; since $\omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$, we have that

$$\dot{\psi} = -\dot{\phi} \cos \theta$$

Plugging this into our expression for ω_{vert} and using a trigonometric identity, we have

$$\omega_{\text{vert}} = \dot{\phi} \sin^2 \theta = \omega_0 (\sin \theta)^{\frac{2}{3}} \quad (2)$$

We can see from this expression that the rotation rate about the vertical goes to zero as the penny lies down. Equations 1 and 2, then, explain the funny behavior we observe for the Big Penny.



Assignment 12 - Coupled Oscillations ($|A_{jk}-\omega^2 m_{jk}|$ and Waves $\left(\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau} \frac{\partial^2 q}{\partial t^2}\right)$)

Reading Assignment: Marion & Thornton: Chap. 12 (Secs. 1-4, 6, 8-9) and Chap. 13 (Secs. 1-4, 6-7).

Overview: As the name implies, the motion of a simple harmonic oscillator is ... well ... simple. But couple two SHO's together such that they can exchange energy and now the general motion is not so simple! The energy flow between the two oscillators can lead to complex behavior. Surprisingly, it is possible to describe such complicated coupled motion as a superposition of the motion of a set of uncoupled oscillators or normal modes. This remarkable “normal mode decomposition” amounts to finding a coordinate transformation which maps our system of n -coupled oscillators onto a system of n -uncoupled oscillators (but n.b., the frequencies or modes of these new uncoupled oscillators are NOT simply those of the original oscillators). Such a transformation is always possible for a conservative system if the potential energy is quadratic in the position variables (which is the usual case for small amplitude oscillations). For very simple systems, one can often identify the normal modes from symmetry considerations. In the general case, one has a set of n -coupled differential equations which can be reduced to n -coupled algebraic equations and thus, our problem becomes an exercise in linear algebra. The set of normal mode frequencies ω can immediately be determined from the secular determinant given above (where the A_{jk} and m_{jk} are coupling constants for the potential and kinetic energies, respectively). Indeed, finding the normal modes of a system amounts to the simultaneous diagonalization of the coupling constant matrices (tensors actually) $\{\mathbf{A}\}$ and $\{\mathbf{m}\}$. *This type of normal mode analysis is widely used in physics and is the origin of quasi-particle descriptions of “many-body” systems* (e.g., phonons in solids are the normal modes of the coupled lattice vibrations). Our analysis of coupled oscillations will begin with two coupled harmonic oscillators which can be described in terms of two normal modes, one symmetric (or in phase) and one anti-symmetric (or out of phase). More coupled oscillators lead to more normal modes (and more difficult analysis), however, the rather daunting problem of the “loaded string” (i.e., n masses equally spaced on an elastic string) turns out to be quite tractable. Taking the continuous

limit of the loaded string leads to the 1D-wave equation given above (where q is the string displacement and ρ and τ are the mass density and tension, respectively). Although this same equation is also easily obtained via force analysis, the coupled oscillator approach provides a natural description of wave motion in terms of normal modes (or Fourier components).

Problem Assignment:

Problem 12.16 - Swinging hoop with sliding bead

A thin hoop of radius R and mass M oscillates in its own plane hanging from a single fixed point. Attached to the hoop is a small mass M constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations, and show the the eigenfrequencies are

$$\omega_1 = \sqrt{2}\sqrt{\frac{g}{R}} \quad , \quad \omega_2 = \frac{\sqrt{2}}{2}\sqrt{\frac{g}{R}}$$

Find the two sets of initial conditions that allow the system to oscillate in its normal modes. Describe the physical situation for each mode.

Problem 12.17 - 3 mass,4 spring coupled system

Find the eigenfrequencies and describe the normal modes for a system such as the one discussed in Section 12.2 but with three equal masses m and four springs (all with equal spring constants) with the system fixed at the ends.

Problem 12.18 - Pendulum attached to a sliding mass

A mass M moves horizontally along a smooth rail. A pendulum is hung from M with a weightless rod and mass m at the end. Find the eigenfrequencies and describe the normal modes.

Problem 12.25 - 3 particle loaded string

Rework the problem in Example 12.7, assuming that all three particles are displaced a distance a and released from rest.

Problem 13.6 - Harmonic response of a hammered string

A string is set into motion by being struck at a point $L/4$ from one

end by a triangular hammer. The initial velocity is greatest at $x = L/4$ and decreases linearly to zero at $x = 0$ and $x = L/2$. The region $L/2 \leq x \leq L$ is initially undisturbed. Determine the subsequent motion of the string. Why are the fourth, eighth, and related harmonics absent? How many decibels down from the fundamental are the second and third harmonics?

Problem 13.20 - Reflection and transmission along a string with attached mass

Consider an infinitely long continuous string with tension τ . A mass M is attached to the string at $x = 0$. If a wave train with velocity ω/k is incident from the left, show that the reflection and transmission occur at $x = 0$ and that the coefficients R and T are given by

$$R = \sin^2 \theta \quad , \quad T = \cos^2 \theta$$

where

$$\tan \theta = \frac{M\omega^2}{2k\tau}$$

Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$. What are the phase changes for the reflected and transmitted waves?

Assignment 13 - Nonlinear Dynamics ($F(x) = -k(x + \alpha x^3)$) and Chaos ($d_i(t) = d_i(0)e^{\lambda_i t}$)

Reading Assignment: Marion & Thornton: Chap. 4

Also, explore the R programs: DDPENDULUM, LOG-MAP, LOG-B-DIAGRAM, and IIENON.

Overview: Overview: As we have seen throughout this course, the equations of motion for dynamical systems are, more often than not, nonlinear. Nonlinear equations are hard to solve so we have generally looked for linear approximations which lead to analytic solutions and nice properties like superposition. But in throwing away all those pesky nonlinear terms we have been in fact throwing away a lot of very interesting physics! The subject of

nonlinear systems is an old one but the use of computers has really revolutionized the field and led to a number of new discoveries. This week we'll just be able to get an introduction to this broad area and this will require our own use of computers. (Luckily we have some nice code to work with). We'll need to remind ourselves about phase space diagrams and the concept of an attractor in such a diagram. We'll also need to understand how we can get from these diagrams to Poincare sections and bifurcation diagrams (which are very helpful in sorting out complex dynamics). We'll start by taking a closer look at the occurrence of hysteresis and jumps in the dynamics of a nonlinear driven system, the Duffing oscillator. For weak driving this hysteresis is the primary new effect due to the nonlinearity. With increased driving we begin to observe a phenomena known as period doubling, a characteristic precursor to the chaotic behavior which eventually develops. Many simple dynamical systems, both dissipative and energy conserving, can exhibit such a transition to chaos. **And just what is chaos you may ask.** Here we are speaking of chaos in a deterministic system, which means that given perfect information about the initial state we can completely predict the time evolution of the system. How can we speak of chaotic motion in such a system? Well, the key is that bit about perfect information. Chaotic systems exhibit extreme sensitivity to initial conditions. The sensitivity is so extreme that unless that information is really perfect (which is practically and fundamentally impossible) long time prediction becomes impossible. A quantitative measure of such sensitivity is provided by the Lyapunov exponents λ defined in the equation at the top of the page. Here $d(t)$ measures the distance between two trajectories of a system at time t . If the exponent λ is positive the trajectories diverge no matter how close they were at time zero ... this means unpredictability and thus chaos. The study of chaotic dynamics is a computationally intensive exercise since one must integrate equations of motion for very long times to elucidate the phenomena. Simple discrete nonlinear maps provide a much "cheaper" way to explore some of the characteristics of chaotic behavior and the transition to chaos. In some cases one can motivate these maps from a real dynamical system but the relationships are often a bit tenuous. However, many of the features of chaos are thought to be "universal" and thus independent of the underlying model. So these simple maps can provide us with some valuable results (such as the Feigenbaum number) and, of course, some pretty pictures!

Problem Assignment:

Problem for everyone: - The Duffing Oscillator

The equation of motion for a driven damped oscillator with the non-linear restoring force given at the top of this sheet can be written as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x_\varepsilon x^3 = F \cos \omega t$$

In the following we will take $\omega_0 = 1$ and $\varepsilon = 1$ (which can always be achieved by an appropriate rescaling of x and t). To a first approximation we can assume a trial solution of the form

$$x(t) = A(\omega) \cos(\omega t - \theta)$$

Show that this trial solution leads to the following approximate relation between driving force F and response amplitude A [Hint: coefficients of $\sin \omega t$, $\cos \omega t$, and $\cos 3\omega t$ must vanish separately]:

$$F^2 = A^2[(1 - \omega^2 + 3A^2/4)^2 + (2\gamma\omega)^2]$$

Plot $A(\omega)$ for $0 \leq \omega \leq 2$ with $F = 0.5$ and $\gamma = 0.1$ noting that multiple solutions are possible for a certain range of ω . [Try solving for ω as a function of A]. Your plot should resemble Marion's Fig. 4-11.

Computational Projects: Lots of R code for some now classic chaotic systems at end of assignment. This code is easily modified to study other systems and maps.

- (1) **More fun with the Duffing oscillator:** - By solving the Duffing equation via numerical integration, construct an amplitude response curve (i.e., maximum amplitude vs. driving frequency) and demonstrate hysteresis by showing the existence of two steady state solutions (depending on starting conditions) for a certain range of driving frequencies. Note that you must wait for transients to die out before determining the steady state maximum amplitude. Use the same parameters as in the previous problem and compare with those analytic results. Modify the R pendulum code to construct phase space plots and Poincare sections for the Duffing oscillator. Look for period doubling at $F = 20$, $\gamma = 0.1$, $\omega = 1.4$ and for signs of chaos at $F = 25$, $\gamma = 0.1$, $\omega = 1.3$.

- (2) **The damped pendulum with vertically driven pivot:** - The equation of motion for this system is:

$$\ddot{\theta} = -\gamma\dot{\theta} - [\omega_0^2 + 2F \cos \omega t] \sin \theta$$

where θ is the angle the pendulum makes with the vertical, γ is the damping coefficient, $\omega_0^2 = g/\ell$, and, ω and $F = \omega^2 A/2\ell$ are the frequency and reduced amplitude of the driving force. (Note that the driver is equivalent to a time varying gravitational field). Modify the R pendulum code to study the behavior of this system as a function of reduced driving amplitude F for fixed $\omega_0 = 1$, $\omega = 2$ and, $\gamma = 0.2$. You should observe a first period doubling at $F = 0.79$. Find the approximate value of F for subsequent period doublings and use these results to estimate Feigenbaum's number.

- (3) **Lyapunov exponent for the logistic map [see Marion FIGURE 4-24]:** (a) Compute the Lyapunov exponent λ for the logistic map directly from the definition $|\Delta x_n| = |\Delta x_0| \exp(\lambda n)$. Consider the cases of $a = 3.4, 3.64, 3.88, 4.0$. Take $x_0 = 0.5$ and $\Delta x_0 = 10^{-6}$. (b) Now compute λ for the logistic map using Eq. 4.52. Do this for the range $3 \leq a \leq 4$ in steps of $a = 0.05$. Discard the first 1000 iterations and use 10^4 to 10^5 values of $\ln |f'(x_n)|$ for the averages.
- (4) **The tent and Henon maps [Marion 4-17 through 4-21]:** - Use the Logistic and Henon R codes as a starting point to studying these maps and working the Marion problems.
- (5) **The bouncing ball on vibrating table (Chirikov map) [Marion 4-23]:** - Modify the Henon R code to work the Marion problem. Be prepared to explain what your results mean.
- (6) **Rings of Saturn:** - The structure in Saturn's rings is due to perturbations caused Mimas, one of Saturn's moons. Here we will study a very simple discrete model of the dynamics. Consider an object orbiting Saturn which, at some particular time, has orbital radius r_n and relative angle to Mimas θ_n . One orbit later the new relative angle will be approximately (using Kepler's 3rd law)

$$\theta_{n+1} = \theta_n + 2\pi(\sigma/r_n)^{3/2}$$

where $\sigma = 185.7 \times 10^3$ km is the radius of Mimas' orbit. The radial acceleration experienced by the orbiting object can be expressed

in discrete form as follows

$$\begin{aligned}\frac{\Delta v_r}{\Delta t} &\approx \frac{v_r(t + \Delta t) - v_r(t)}{\Delta t} \approx \frac{r(t + \Delta t) - r(t)}{(\Delta t)^2} - \frac{r(t) - r(t - \Delta t)}{(\Delta t)^2} \\ &= \frac{r(t + \Delta t) - 2r(t) + r(t - \Delta t)}{(\Delta t)^2}\end{aligned}$$

Assuming a periodic inverse-square perturbing force leads (with handwaving) to the radial equation:

$$r_{n+1} = 2r_n - r_{n-1} - a \cos \theta_n / (r_n - \sigma)^2$$

where $a = 2 \times 10^{12} \text{ km}^3$. Investigate this two-dimensional map for r_n and θ_n using initial r values in the range $R_S < r < \sigma$ where $R_S = 60.4 \times 10^3 \text{ km}$ is the radius of Saturn. Take 10^3 km as your unit of length (in which case $a = 2000$). Plot points $(r_n \cos \theta_n, r_n \sin \theta_n)$ and find the bands of r_n where stable trajectories are found. Examine the effects of changing the parameter a . Try $a = 200$ and $a = 20000$.

R Codes:

```
# PENDULUMS.R - Damped driven pendulum
# This is the model studied extensively in Chapter 4 of Marion (see Fig 4-18)
# This program draws the full phase space diagram.
#
osc1 <- function(t,x,v,F,k,w) {F*cos (w*t)-k*v-sin(x)}
#
# define parameters
# F = driving amplitude, k = damping parameter, w = driving frequency
F <- 1.50
k <- 0.5
w <- 0.6666666
# define size for main loop
num <- 50000
# integration time step
h <- 0.05
# arbitrary starting point
x <- 0.0
```

```

v <- 0.0
# plot functions require vectors
x1 <- c()
y1 <- c()
x1 <- append(x1,x)
y1 <- append(y1,v)
# use loop to generate successive points in time
# and overplot points in real time
for (j in 0:num){
  # Runge-Kutta method
  f1 <- v
  g1 <- osc1(j*h, x, v, F, k, w)
  f2 <- v+h*g1/2
  g2 <- osc1((j+0.5)*h,x+h*f1/2,v+h*g1/2,F,k,w)
  f3 <- v+h*g2/2
  g3 <- osc1((j+0.5)*h,x+h*f2/2,v+h*g2/2,F,k,w)
  f4 <- v+h*g3
  g4 <- osc1((j+1)*h,x+h*f3,v+h*g3,F,k,w)
  x <- x+h*(f1+2*f2+2*f3+f4)/6
  # keep x = angular variable between -2*pi and +2*pi
  if (abs(x) > pi) {
    x <- x-2.0*pi*x/abs(x)
  }
  v <- v+h*(g1+2*g2+2*g3+g4)/6
  # Run for num/10 rycles before plotting to get rid of transients
  if (j > num/10) {
    x1 <- append(x1,x)
    y1 <- append(y1,v)
  }
  plot(x1,y1,pch=".",col="red",xlim=c(-pi,pi),ylim=c(-4,4),xlab="position",ylab="vel
}
}

# PENDULUMPO.R - Damped driven pendulum
# This is the model studied extensively in Chapter 4 of Marion (see Fig 4-18)
# This program constructs the Poincare section and is easily modified
# to draw the full phase space diagram.
#
osc1 <- function(t,x,v,F,k,w) {F*cos (w*t)-k*v-sin(x)}

```

```

#
# define parameters
# F = driving amplitude, k = damping parameter, w = driving frequency
F <- 1.15
k <- 0.5
w <- 0.6666666
# define size for main loop
num <- 200000
# integration time step chosen so that 1000 steps = 1 cycle
t <- 2.0*pi/w
h <- 0.001*t
# arbitrary starting point
x <- 0.0
v <- 0.0
x1 <- c()
v1 <- c()
count <- 0
for (j in 0:num){
  # Runge-Kutta method
  f1 <- v
  g1 <- osc1(j*h, x, v, F, k, w)
  f2 <- v+h*g1/2
  g2 <- osc1((j+0.5)*h, x+h*f1/2, v+h*g1/2, F, k, w)
  f3 <- v+h*g2/2
  g3 <- osc1((j+0.5)*h, x+h*f2/2, v+h*g2/2, F, k, w)
  f4 <- v+h*g3
  g4 <- osc1((j+1)*h, x+h*f3, v+h*g3, F, k, w)
  x <- x+h*(f1+2*f2+2*f3+f4)/6
  # keep x = angular variable between -2*pi and +2*pi
  if (abs(x) > pi) {
    x <- x-2.0*pi*x/abs(x)
  }
  v <- v+h*(g1+2*g2+2*g3+g4)/6
  if (j > num/10) {
    count <- count+1
    if (count == 1000){
      x1 <- append(x1,x)
      v1 <- append(v1,v)
    }
  }
}

```

```

plot(x1,v1,pch=".",col="red",xlim=c(-pi,pi),ylim=c(-4,4),xlab="position",ylab="vel
count <- 0
  }
}
}

```

```

# HENON.R
# Calculates and draws the Henon map
# This is a 2D map with a scale invariant strange attractor
x <- 0.0
x1 <- c()
x1 <- append(x1,x)
y <- 0.0
y1 <- c()
y1 <- append(y1,y)
a <- 1.4
b <- 0.3
for (n in 1:5000) {
  xnew <- y+ 1.0-a*x*x
  ynew <- b*x
  x1 <- append(x1,x)
  y1 <- append(y1,y)
  x <- xnew
  y <- ynew
  plot(x1,y1,pch=".",col="red",xlim=c(-1.5,1.5),ylim=c(-0.45,0.45),xlab="q_n",ylab="
}

```

```

# LOG_B_DIAGRAM.R
# Computes and draws a bifurcation diagram for the logistic map
# (See Marion Fig. 4-22) Note that the variable r is Marion's alpha
kk <- 10000
y <- c(rep(0,kk))
y[1] <- 0.5678
xpl <- c(rep(1,kk/2))
xp <- c(rep(0,kk/2))
ypl <- c(rep(0,kk/2))

for (r in (seq(1,4,by=0.005))) {

```

```

for (j in 2:kk) {
  y[j] <- r*y[j-1]*(1-y[j-1])
}
xp <- r* xpl
ypl <- y[(kk/2+1):kk]
plot(xp,ypl,pch=".",col="red",xlim=c(1.0,4.0),ylim=c(0.0,1.0),xlab="r_value",ylab="y")
par(new=TRUE)
}

```

RETURN_MAP.R

Computes and draws the return map Here the variable r is Marion's alpha
For each run a random starting xo is chosen

```

r <- 2.5
x <- runif(1)
y <- r*x*(1-x)
x1 <- c(x,x)
y1 <- c(0,y)
plot(x1,y1,type="l",xlab="x(n)",ylab="x(n+1)",
     main= "Return Map" , col="red", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
par(new=TRUE)
for (j in 1:1500) {
  xlast <- x
  x <- y
  ylast <- y
  x1 <- c(xlast,x)
  y1 <- c(ylast,ylast)
  plot(x1,y1,type="l",xlab="x(n)",ylab="x(n+1)",
       main= "Return Map" , col="green", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
  par(new=TRUE)
  ylast <- y
  y <- r*x* (1-x)
  x1 <- c(x,x)
  y1 <- c(ylast,y)
  plot(x1,y1,type="l",xlab="x(n)",ylab="x(n+1)",
       main= "Return Map" , col="green", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
  par(new=TRUE)
}

```

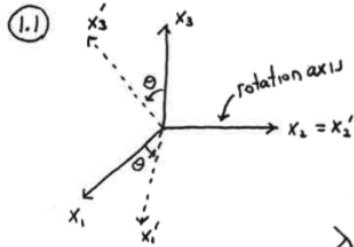
```

xa <- 0.01*(0:100)
ya <- r*xa*(1.0-xa)
plot(xa, ya, type="l", xlab="x(n)", ylab="x(n+1)",
     main= "Return Map" , col="purple", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
par(new=TRUE)
if (r < 1.0) {
  xb <- 0.0
} else {
  xb <- 1.0-1.0/r
}
xb1 <- c(xb,xb)
yb <- r*xb*(1.0-xb)
yb1 <- c(yb,yb)
plot(xb1, yb1, pch=22, xlab="x(n)", ylab="x(n+1)",
     main= "Return Map" , col="magenta", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
par(new=TRUE)
xc <- c(0.0,1.0)
yc <- xc
plot(xc, yc, type="l", xlab="x(n)", ylab="x(n+1)",
     main= "Return Map" , col="blue", xlim= c(0.0,1.0), ylim= c(0.0,1.0))
par(new=TRUE)

```


Assignment 1 - Solutions

Problem 1-1



Elements of the rotation matrix are given by direction cosines:

$\lambda_{ij} = \cos \theta_{ij}$ where $\theta_{ij} \equiv$ angle between x'_i and x_j

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} = \begin{pmatrix} \cos \theta & \cos 90^\circ & \cos (90^\circ + \theta) \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos (90^\circ - \theta) & \cos 90^\circ & \cos \theta \end{pmatrix}$$

Rotation:

$$x'_i = \sum_j \lambda_{ij} x_j$$

or

$$\vec{x}' = \lambda \vec{x}$$

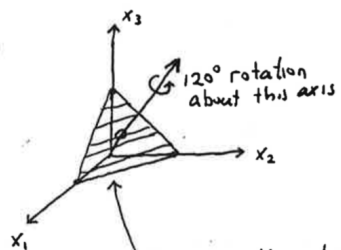
$$= \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

for $\theta = 45^\circ$

$$\lambda = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

Problem 1-3

(1.3)



Rotation axis given by:

$$\hat{u}_R = a\hat{u}_1 + a\hat{u}_2 + a\hat{u}_3$$

$$\text{where } \sqrt{3a^2} = 1 \Rightarrow a = 1/\sqrt{3}$$

... but we don't need this!

Consider this triangle... a 360° rotation about $\hat{u}_R$ returns coordinates to original state... a 120° rotation is $1/3$ of a full revolution which rotates the triangle such

$$\text{that: } \left. \begin{array}{l} \hat{x}_3 \rightarrow \hat{x}_1 \\ \hat{x}_1 \rightarrow \hat{x}_2 \\ \hat{x}_2 \rightarrow \hat{x}_3 \end{array} \right\} \Rightarrow \begin{array}{l} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1 \end{array} \quad \vec{x}' = \lambda \vec{x}$$

Rotation Matrix:

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} = \begin{pmatrix} \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \\ \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}}$$

$$\left[\text{For above two problems note that } \sum_i \lambda_{ij}^2 = \sum_j \lambda_{ij}^2 = 1 \right]$$

Problem 1-5

(1.5) Two-dimensional transformation matrix $\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$

$$|\lambda|^2 = (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})^2 = \lambda_{11}^2\lambda_{22}^2 - 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21} + \lambda_{12}^2\lambda_{21}^2$$

We want to show that $|\lambda|^2 = 1$. We know that for an orthogonal

matrix: $\lambda\lambda^T = 1$

$$\Rightarrow \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11}^2 + \lambda_{12}^2 & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} & \lambda_{21}^2 + \lambda_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives us the following 3 relations:

$$\begin{aligned} \text{(I)} \quad & \lambda_{11}^2 + \lambda_{12}^2 = 1 \\ \text{(II)} \quad & \lambda_{21}^2 + \lambda_{22}^2 = 1 \\ \text{(III)} \quad & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0 \end{aligned}$$

If mult. (III) by $\lambda_{11}\lambda_{21}$ and add this to (III) mult. by $\lambda_{12}\lambda_{22}$ we get

$$\lambda_{11}^2\lambda_{21}^2 + 2\lambda_{11}\lambda_{21}\lambda_{12}\lambda_{22} + \lambda_{12}^2\lambda_{22}^2 = 0$$

$\uparrow \lambda_{21}^2 = 1 - \lambda_{22}^2$
 $\uparrow \lambda_{22}^2 = 1 - \lambda_{21}^2$

← Now make these clever substitutions using (II) to get all the terms appearing in our expansion of $|\lambda|^2$

$$\Rightarrow \lambda_{11}^2 - \lambda_{11}^2\lambda_{22}^2 + 2\lambda_{11}\lambda_{21}\lambda_{12}\lambda_{22} + \lambda_{12}^2 - \lambda_{22}^2\lambda_{21}^2 = 0$$

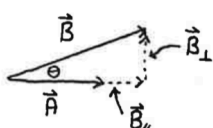
$$\underbrace{\lambda_{11}^2\lambda_{22}^2 - 2\lambda_{11}\lambda_{21}\lambda_{12}\lambda_{22} + \lambda_{22}^2\lambda_{21}^2}_{=|\lambda|^2 \text{ as shown above}} = \underbrace{\lambda_{11}^2 + \lambda_{12}^2}_{=1 \text{ from (I)}}$$

$$\therefore \underline{\underline{|\lambda|^2 = 1}}$$

Problem 1-9

(1-9) Vector Manipulations: $\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$ $\vec{B} = -2\hat{i} + 3\hat{j} + \hat{k}$

a) $\vec{A} - \vec{B} = \underline{3\hat{i} - \hat{j} - 2\hat{k}}$ $|\vec{A} - \vec{B}| = \sqrt{9+1+4} = \underline{\underline{\sqrt{14}}}$

b)  $\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$
 $\vec{B}_{\parallel}$ component of $\vec{B}$ parallel to $\vec{A}$

$\vec{B}_{\parallel} = |\vec{B}_{\parallel}| \hat{u}_A$ where $|\vec{B}_{\parallel}| = |\vec{B}| \cos \theta$ and $\hat{u}_A$ is a unit vector along $\vec{A}$: $\hat{u}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{1}{\sqrt{6}}(\hat{i} + 2\hat{j} - \hat{k})$

Note that $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$

$\therefore \vec{B}_{\parallel} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} \hat{u}_A = \left(\frac{3}{\sqrt{6}}\right) \frac{1}{\sqrt{6}} (\hat{i} + 2\hat{j} - \hat{k}) = \underline{\underline{\frac{1}{2}(\hat{i} + 2\hat{j} - \hat{k})}}$

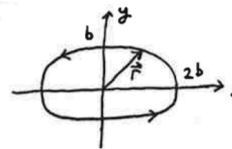
c) $\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{14}}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \sim \underline{\underline{71^\circ}}$

d) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{vmatrix} = (2+3)\hat{i} + (2-1)\hat{j} + (3+4)\hat{k} = \underline{\underline{5\hat{i} + \hat{j} + 7\hat{k}}}$

e) $(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) = \vec{A} \times \vec{A} + \vec{A} \times \vec{B} - \vec{B} \times \vec{A} - \vec{B} \times \vec{B}$
 $= \vec{A} \times \vec{B} - \vec{B} \times \vec{A} = 2(\vec{A} \times \vec{B}) = \underline{\underline{10\hat{i} + 2\hat{j} + 14\hat{k}}}$

Problem 1-10

(1-10) Elliptical Trajectory given by $\vec{r}(t) = 2b \sin \omega t \hat{x} + b \cos \omega t \hat{y}$



$$a) \quad \vec{v} = \frac{d\vec{r}}{dt} = \underline{2b\omega \cos \omega t \hat{x} - b\omega \sin \omega t \hat{y}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -2b\omega^2 \sin \omega t \hat{x} - b\omega^2 \cos \omega t \hat{y} = \underline{-\omega^2 \vec{r}}$$

$$\begin{aligned} \text{speed} = |\vec{v}| &= [4b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t]^{1/2} = b\omega [4\cos^2 \omega t + \sin^2 \omega t]^{1/2} \\ &= \underline{b\omega \sqrt{1 + 3\cos^2 \omega t}} \end{aligned}$$

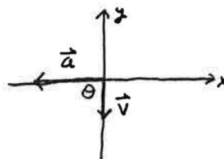
$$b) \quad @ t = \pi/2\omega \quad \omega t = \pi/2$$

$$\vec{v} = -b\omega \hat{y}$$

$$\vec{a} = -2b\omega^2 \hat{x}$$

$$\underline{\text{angle } \theta = 90^\circ}$$

$$(\vec{v} \cdot \vec{a} = 0)$$



Problem 1-11

(1-11) Triple scalar product $(\vec{A} \times \vec{B}) \cdot \vec{C}$

definition of cross product: $(\vec{A} \times \vec{B})_i = \sum_{j,k} \epsilon_{ijk} A_j B_k \Rightarrow$

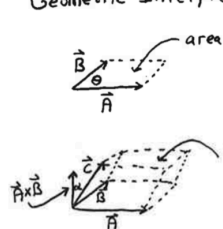
$$\begin{aligned} (\vec{A} \times \vec{B})_1 &= A_2 B_3 - A_3 B_2 \\ (\vec{A} \times \vec{B})_2 &= A_3 B_1 - A_1 B_3 \\ (\vec{A} \times \vec{B})_3 &= A_1 B_2 - A_2 B_1 \end{aligned}$$

$$\begin{aligned} (\vec{A} \times \vec{B}) \cdot \vec{C} &= \sum_i (\vec{A} \times \vec{B})_i C_i \\ &= \sum_i C_i (\vec{A} \times \vec{B})_i = C_1 (A_2 B_3 - A_3 B_2) + C_2 (A_3 B_1 - A_1 B_3) + C_3 (A_1 B_2 - A_2 B_1) \\ &= C_1 \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} + C_2 \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix} + C_3 \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \\ &= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = - \begin{vmatrix} A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ &\quad \uparrow \\ &\quad \text{[Note: row (or column) exchange simply gives a factor of } -1 \text{]} \end{aligned}$$

Similarly we find:

$$\begin{aligned} \vec{A} \cdot (\vec{B} \times \vec{C}) &= (\vec{B} \times \vec{C}) \cdot \vec{A} = \sum_i A_i (\vec{B} \times \vec{C})_i = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ \vec{B} \cdot (\vec{C} \times \vec{A}) &= (\vec{C} \times \vec{A}) \cdot \vec{B} = \sum_i B_i (\vec{C} \times \vec{A})_i = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} \Rightarrow \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ &\quad \uparrow \\ &\quad \text{2 row exchanges} \end{aligned}$$

Geometric Interpretation:



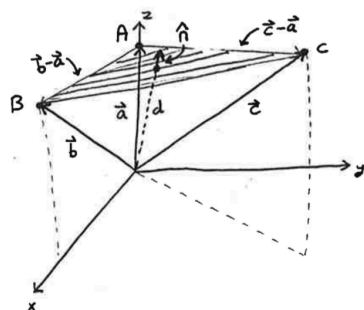
area of parallelogram = $|\vec{A}||\vec{B}|\sin\theta = |\vec{A} \times \vec{B}|$

Volume of parallelepiped = (area of base) (height)

$$\begin{aligned} &= (AB \sin\theta) (\text{projection of } \vec{C} \text{ onto } \vec{A} \times \vec{B}) \\ &= (AB \sin\theta) (C \cos\alpha) \\ &= |\vec{A} \times \vec{B}| |\vec{C}| \cos\alpha \\ &= \underline{(\vec{A} \times \vec{B}) \cdot \vec{C}} \equiv \vec{A} \cdot \vec{B} \cdot \vec{C} \end{aligned}$$

Problem 1-12

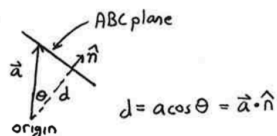
(1-12) Analytic Geometry



$\hat{n} \equiv$ normal unit vector to ABC plane

$d =$ distance from origin to ABC plane

side view:



[similarly: $d = \vec{b} \cdot \hat{n} = \vec{c} \cdot \hat{n}$]

normal vector given by $\hat{n} = \frac{(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})}{|(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})|}$

[note: $\hat{n}$ can be defined in terms of any 2 sides of the ABC triangle]

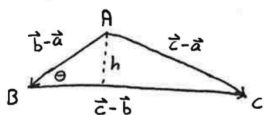
The distance is $d = \vec{a} \cdot \hat{n} = \frac{\vec{a} \cdot ((\vec{b}-\vec{a}) \times (\vec{c}-\vec{a}))}{|(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})|}$

note: $(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})$
 $= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$
 $= \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$

$$= \frac{\vec{a} \cdot [\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}]}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

$$= \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

Area of the ABC triangle:



$$\text{Area} = \frac{1}{2} |\vec{c}-\vec{b}| h$$

$$= \frac{1}{2} |\vec{c}-\vec{b}| |\vec{b}-\vec{a}| \sin \theta$$

$$= \frac{1}{2} |(\vec{c}-\vec{b}) \times (\vec{a}-\vec{b})|$$

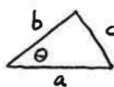
$$= \frac{1}{2} |\vec{c} \times \vec{a} - \vec{c} \times \vec{b} - \vec{b} \times \vec{a} + \vec{b} \times \vec{b}|$$

$$= \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

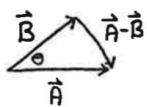
can use any 2 sides in this construction.

Problem 1-17

(1-17) Law of cosines



$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



$$(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A} - \vec{B}|^2 = c^2$$

or expanding ...

$$\begin{aligned} (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) &= \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B} \\ &= |\vec{A}|^2 + |\vec{B}|^2 - 2 \vec{A} \cdot \vec{B} \\ &= a^2 + b^2 - 2ab \cos \theta \end{aligned}$$

let $a = |\vec{A}|$

$b = |\vec{B}|$

$c = |\vec{A} - \vec{B}|$

equating these two $\Rightarrow \underline{\underline{c^2 = a^2 + b^2 - 2ab \cos \theta}} \checkmark$

Problem 1-20

(1-20)

$$a) \sum_{i \neq j} \epsilon_{ijk} \delta_{ij} = \sum_i \epsilon_{iik} = 0$$

↑ use δ_{ij} to do j -summation

$$b) \sum_{j \neq k} \epsilon_{ijk} \epsilon_{ljk} = \epsilon_{i12} \epsilon_{l12} + \epsilon_{i21} \epsilon_{l21} + \epsilon_{i13} \epsilon_{l13} + \epsilon_{i31} \epsilon_{l31} + \epsilon_{i23} \epsilon_{l23} + \epsilon_{i32} \epsilon_{l32}$$

only $j \neq k$ terms survive

$$= 2 \epsilon_{i12} \epsilon_{l12} + 2 \epsilon_{i13} \epsilon_{l13} + 2 \epsilon_{i23} \epsilon_{l23}$$

← only one can be nonzero!

$$= \begin{cases} 0 & i \neq l \\ 2 \epsilon_{123} \epsilon_{123} & i=l=1 \\ 2 \epsilon_{213} \epsilon_{213} & i=l=2 \\ 2 \epsilon_{312} \epsilon_{312} & i=l=3 \end{cases}$$

$$= \underline{\underline{2 \delta_{il}}}$$

$$c) \sum_{i \neq j \neq k} \epsilon_{ijk} \epsilon_{ijk} = \epsilon_{123} \epsilon_{123} + \epsilon_{231} \epsilon_{231} + \epsilon_{312} \epsilon_{312} \quad \leftarrow \text{even permutations}$$

$$+ \epsilon_{132} \epsilon_{132} + \epsilon_{213} \epsilon_{213} + \epsilon_{321} \epsilon_{321} \quad \leftarrow \text{odd permutations}$$

$$= \underline{\underline{6}}$$

Problem 1-21

(1-21) Triple scalar product via ϵ_{ijk} : $(\vec{A} \times \vec{B}) \cdot \vec{C}$

def. of cross product : $(\vec{A} \times \vec{B})_i = \sum_{jk} \epsilon_{ijk} A_j B_k$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \sum_i (\vec{A} \times \vec{B})_i C_i = \sum_i \left\{ \sum_{jk} \epsilon_{ijk} A_j B_k \right\} C_i$$

$$= \sum_{ijk} \epsilon_{ijk} A_j B_k C_i$$

$$\epsilon_{jik} = -\epsilon_{ijk}$$

$$= - \sum_{ijk} \epsilon_{jik} A_j B_k C_i$$

$$= \sum_{ijk} \epsilon_{jki} A_j B_k C_i$$

now relabel
indices : $j \rightarrow l$
 $k \rightarrow m$
 $i \rightarrow n$

$$\rightarrow \sum_{lmn} \epsilon_{lmn} A_l B_m C_n$$

Problem 1-22a

(1-22) Compute $\sum_k \epsilon_{ijk} \epsilon_{lmk}$

a) $i=j$ $\sum_k \epsilon_{iik} \epsilon_{lmk} = 0$

b) $i=l$ $\sum_k \epsilon_{ijl} \epsilon_{lmk} = \epsilon_{ij1} \epsilon_{im1} + \epsilon_{ij2} \epsilon_{im2} + \epsilon_{ij3} \epsilon_{im3}$ only nonzero if $i \neq j, i \neq m$

only one of these terms can contribute ... each term involves i, j, m, k but there are only 3 allowed indices, $\{1, 2, 3\}$ $\therefore$ if i and k are set, we require $j=m$!

$= \delta_{jm}$

c) $i=m$ $\sum_k \epsilon_{ijl} \epsilon_{lmk} = -\sum_k \epsilon_{ijl} \epsilon_{ilk} = -\delta_{jl}$ as in (b)

d) $j=l$ $\sum_k \epsilon_{ijl} \epsilon_{lmk} = -\sum_k \epsilon_{jlk} \epsilon_{lmk} = -\delta_{im}$

e) $j=m$ $\sum_k \epsilon_{ijl} \epsilon_{lmk} = \sum_k \delta_{ilk} \delta_{jem} = \delta_{il}$

f) $l=m$ $\sum_k \epsilon_{ijl} \epsilon_{lmk} = 0$

g) $i \neq l$ or m $\sum_k \rightarrow 0$ since only the ϵ_{lmk} w/ $l \neq m \neq k$ term survives which forces $i=k$ so $\epsilon_{ijk} \rightarrow 0$

h) $j \neq l$ or m $\sum_k \rightarrow 0$ for same reason as in (g)

$\therefore$ We get $\sum_k \epsilon_{ijk} \epsilon_{lmk} = \begin{cases} 1 & i=l \text{ and } j=m \\ -1 & i=m \text{ and } j=l \\ 0 & \text{otherwise} \end{cases} = \underline{\underline{\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}}}$

Problem 1-22b

$$\begin{aligned}
 \underline{1-22 \text{ cont.}} \quad [\vec{A} \times (\vec{B} \times \vec{C})]_i &= \sum_{jk} \epsilon_{ijk} A_j [\vec{B} \times \vec{C}]_k \\
 &= \sum_{jk} \epsilon_{ijk} A_j \sum_{lm} \epsilon_{klm} B_l C_m \\
 &= \sum_{jklm} \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m \quad [\epsilon_{klm} = -\epsilon_{lkm} = \epsilon_{mlk}] \\
 &= \sum_{jklm} \epsilon_{ijk} \epsilon_{lmk} A_j B_l C_m \\
 &= \sum_{jlm} [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}] A_j B_l C_m \quad \left. \begin{array}{l} \text{do } k\text{-summation using} \\ \text{derived identity} \end{array} \right\} \\
 &= \sum_{jl} \{ \delta_{il} A_j B_l C_j - \delta_{jl} A_j B_l C_i \} \quad \left. \begin{array}{l} \text{do } m\text{-summation} \\ \text{using } \delta\text{-func.} \end{array} \right\} \\
 &= \sum_j \{ A_j C_j B_i - A_j B_j C_i \} \quad \left. \begin{array}{l} \text{do } l\text{-summation} \\ \text{using } \delta\text{-func.} \end{array} \right\} \\
 &= B_i \sum_j A_j C_j - C_i \sum_j A_j B_j \\
 &= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B}) \quad \leftarrow \text{the } i\text{-th component}
 \end{aligned}$$

full vector: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

Problem 1-25a

(1-25) Spherical coordinates given by:

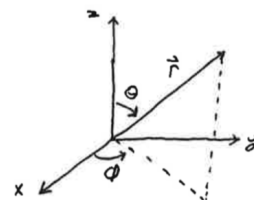
$$x = r \cos \phi \sin \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \theta$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$= r \cos \phi \sin \theta \hat{i} + r \sin \phi \sin \theta \hat{j} + r \cos \theta \hat{k}$$



for unit vectors:

$$\vec{u}_r = \frac{\partial \vec{r}}{\partial r} = \cos \phi \sin \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \theta \hat{k} \quad h_r = |\vec{u}_r| = 1$$

$$\vec{u}_\theta = \frac{\partial \vec{r}}{\partial \theta} = r \cos \phi \cos \theta \hat{i} + r \sin \phi \cos \theta \hat{j} - r \sin \theta \hat{k} \quad h_\theta = |\vec{u}_\theta| = r$$

$$\vec{u}_\phi = \frac{\partial \vec{r}}{\partial \phi} = -r \sin \phi \sin \theta \hat{i} + r \cos \phi \sin \theta \hat{j} \quad h_\phi = |\vec{u}_\phi| = r \sin \theta$$

$$\therefore \begin{aligned} \hat{r} &= \frac{\vec{u}_r}{h_r} = \cos \phi \sin \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \theta \hat{k} \\ \hat{\theta} &= \frac{\vec{u}_\theta}{h_\theta} = \cos \phi \cos \theta \hat{i} + \sin \phi \cos \theta \hat{j} - \sin \theta \hat{k} \\ \hat{\phi} &= \frac{\vec{u}_\phi}{h_\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \end{aligned}$$

Problem 1-25b

1-25 cont.

$$\text{Velocity: } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) \\ = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$\dot{\hat{r}} = [-\dot{\phi} \sin\phi \sin\theta + \dot{\theta} \cos\phi \cos\theta]\hat{x} \\ + [\dot{\phi} \cos\phi \sin\theta + \dot{\theta} \sin\phi \cos\theta]\hat{y} \\ + [-\dot{\theta} \sin\theta]\hat{z} \\ = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}$$

$$\Rightarrow \underline{\underline{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}}}$$

Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\phi}\dot{\hat{\phi}} + r\sin\theta\ddot{\phi}\hat{\phi} + r\sin\theta\dot{\phi}\dot{\hat{\phi}} \\ = [\ddot{r}]\hat{r} + [\dot{r}\dot{\theta} + r\ddot{\theta}]\hat{\theta} + [\dot{r}\sin\theta\dot{\phi} + r\cos\theta\dot{\phi}\dot{\hat{\phi}} + r\sin\theta\ddot{\phi}]\hat{\phi} \\ + \dot{r}\dot{\hat{r}} + r\dot{\theta}\dot{\hat{\theta}} + r\sin\theta\dot{\phi}\dot{\hat{\phi}} \quad \text{we have } \dot{\hat{r}} \text{ above, need } \dot{\hat{\theta}}, \dot{\hat{\phi}}$$

$$\dot{\hat{\theta}} = [-\dot{\phi} \sin\phi \cos\theta - \dot{\theta} \cos\phi \sin\theta]\hat{x} + [\dot{\phi} \cos\phi \cos\theta - \dot{\theta} \sin\phi \sin\theta]\hat{y} + [-\dot{\theta} \cos\theta]\hat{z} \\ = \dot{\phi}\{-\sin\phi \cos\theta\hat{x} + \cos\phi \cos\theta\hat{y}\} + \dot{\theta}\{-\cos\phi \sin\theta\hat{x} - \sin\phi \sin\theta\hat{y} - \cos\theta\hat{z}\} \\ = \dot{\phi} \cos\theta \hat{\phi} - \dot{\theta} \hat{r} \\ \dot{\hat{\phi}} = -\dot{\phi} \cos\phi \hat{x} - \dot{\phi} \sin\phi \hat{y} = -\dot{\phi}\{\cos\phi \hat{x} + \sin\phi \hat{y}\} \\ = -\dot{\phi}[\sin\theta \hat{r} + \cos\theta \hat{\theta}] \\ = -\dot{\phi} \sin\theta \hat{r} - \dot{\phi} \cos\theta \hat{\theta}$$

Assemble the

required pieces:

$$\dot{r}\dot{\hat{r}} = \dot{r}\dot{\phi} \sin\theta \hat{\phi} + \dot{r}\dot{\theta}\hat{\theta}$$

$$\dot{r}\dot{\theta}\hat{\theta} = r\dot{\theta}\dot{\phi} \cos\theta \hat{\phi} - r\dot{\theta}^2 \hat{r}$$

$$r\sin\theta\dot{\phi}\dot{\hat{\phi}} = -r\sin^2\theta\dot{\phi}^2 \hat{r} - r\sin\theta\cos\theta\dot{\phi}^2 \hat{\theta}$$

Put it all together:

$$\vec{a} = [\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2]\hat{r} \\ + [2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2]\hat{\theta} \\ + [2\dot{r}\sin\theta\dot{\phi} + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}]\hat{\phi} \quad (\text{Pheew!})$$

Problem 1-26a

(1-26) Motion with constant speed v along path $r = k(1 + \cos \theta)$

In polar coordinates: $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ and

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

[see previous problem for the general procedure]
to get this type of result

We need to determine $\dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta}$

From the path equation we know: $r = k(1 + \cos \theta)$

$$\dot{r} = -k \sin \theta \dot{\theta}$$

$$\ddot{r} = -k \cos \theta (\dot{\theta})^2 - k \sin \theta \ddot{\theta}$$

From the constant speed condition we know: $v^2 = |\vec{v}|^2 = (\dot{r})^2 + r^2(\dot{\theta})^2$

this is
all the info!!

We can determine $\dot{\theta}$ as follows:

$$\begin{aligned} v^2 &= (\dot{r})^2 + r^2(\dot{\theta})^2 = k^2 \sin^2 \theta (\dot{\theta})^2 + r^2 (\dot{\theta})^2 \\ &= (r^2 + k^2 \sin^2 \theta) (\dot{\theta})^2 \\ &= k^2 (1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta) (\dot{\theta})^2 \\ &= k^2 (2 + 2 \cos \theta) (\dot{\theta})^2 \\ &= k^2 \left(\frac{2r}{k} \right) (\dot{\theta})^2 = 2rk (\dot{\theta})^2 \Rightarrow \underline{\underline{\dot{\theta} = \frac{v}{\sqrt{2rk}}}} \end{aligned}$$

We can use the $\dot{\theta}$ result to find $\ddot{\theta}$ as follows:

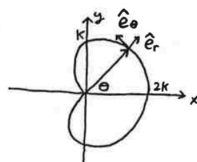
$$\text{since } v^2 = 2rk (\dot{\theta})^2 \text{ and } \frac{d}{dt}(v^2) = 0$$

$$\begin{aligned} \Rightarrow 0 &= \frac{d}{dt} [2rk (\dot{\theta})^2] = 2\dot{r}k (\dot{\theta})^2 + 4rk \dot{\theta} \ddot{\theta} \\ &= 2k \dot{\theta} (\dot{r} \dot{\theta} + 2r \ddot{\theta}) \end{aligned}$$

from above result for $\dot{\theta}$ we see that $\dot{\theta} \neq 0$ so we can divide it out,

$$\therefore \text{ we have } \dot{r} \dot{\theta} + 2r \ddot{\theta} = 0$$

$$\begin{aligned} \Rightarrow \ddot{\theta} &= -\frac{\dot{r} \dot{\theta}}{2r} \\ &= -\frac{1}{2r} (-k \sin \theta \dot{\theta}) (\dot{\theta}) \quad \left. \begin{array}{l} \text{use above result} \\ \text{for } \dot{\theta} \end{array} \right\} \\ &= \frac{k}{2r} \sin \theta \left(\frac{v^2}{2rk} \right) \\ &= \underline{\underline{\frac{v^2}{4r^2} \sin \theta}} \end{aligned}$$



Problem 1-26b

1-26 cont.

Now that we have expressions for $\dot{\theta}$ and $\ddot{\theta}$ we can insert these into our original formula for $\ddot{\mathbf{r}}$:

$$\begin{aligned}\ddot{\mathbf{r}} &= -k \cos \theta (\dot{\theta})^2 - k \sin \theta \ddot{\theta} \\ &= -k \cos \theta \frac{v^2}{2kr} - k \sin \theta \frac{v^2}{4r^2} \sin \theta \\ &= -\frac{v^2}{2r} \left\{ \cos \theta + \frac{k}{2r} \sin^2 \theta \right\} \quad \text{note that } \cos \theta = \frac{r}{k} - 1 \\ &= -\frac{v^2}{2r} \left\{ \left(\frac{r}{k} - 1 \right) + \frac{k}{2r} \cdot \frac{r}{k} \left(2 - \frac{r}{k} \right) \right\} \quad \cos^2 \theta = \left(\frac{r}{k} - 1 \right)^2 = \frac{r^2}{k^2} - 2 \frac{r}{k} + 1 \\ &= -\frac{v^2}{2r} \left\{ \frac{r}{k} - 1 + 1 - \frac{r}{2k} \right\} = -\frac{v^2}{4k} \quad \sin^2 \theta = 1 - \cos^2 \theta = 2 \frac{r}{k} - \frac{r^2}{k^2} \\ & \quad \quad \quad = \frac{r}{k} \left(2 - \frac{r}{k} \right)\end{aligned}$$

Now we have all the pieces:

$$\begin{aligned}\dot{\theta} &= \frac{v}{\sqrt{2kr}} & \dot{r} &= -v \sqrt{\frac{k}{2r}} \sin \theta \\ \ddot{\theta} &= \frac{v^2}{4r^2} \sin \theta & \ddot{r} &= -\frac{v^2}{4k}\end{aligned}$$

$$\begin{aligned}a_r &= \ddot{\mathbf{a}} \cdot \hat{\mathbf{e}}_r = \ddot{r} - r(\dot{\theta})^2 \\ &= -\frac{v^2}{4k} - \frac{v^2}{2k} \\ &= -\frac{3}{4} \frac{v^2}{k}\end{aligned}$$

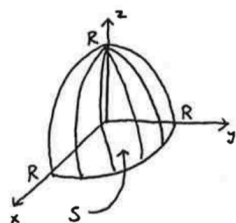
$$\begin{aligned}\hat{a}_\theta &= \ddot{\mathbf{a}} \cdot \hat{\mathbf{e}}_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} \\ &= \frac{v^2}{4r} \sin \theta - 2v \sqrt{\frac{k}{2r}} \sin \theta \frac{v}{\sqrt{2kr}} \\ &= \frac{v^2}{4r} \sin \theta - \frac{v^2}{r} \sin \theta \\ &= -\frac{3v^2}{4r} \sin \theta\end{aligned}$$

$$\begin{aligned}|\ddot{\mathbf{a}}| &= \sqrt{a_r^2 + a_\theta^2} = \frac{3}{4} \frac{v^2}{k} \sqrt{1 + \frac{k^2}{r^2} \sin^2 \theta} \\ &= \frac{3}{4} \frac{v^2}{k} \left[1 + \frac{\sin^2 \theta}{(1 + \cos \theta)^2} \right]^{1/2} \\ &= \frac{3}{4} \frac{v^2}{k} \left[1 + \frac{1 - \cos^2 \theta}{(1 + \cos \theta)^2} \right]^{1/2} \\ &= \frac{3}{4} \frac{v^2}{k} \left[1 + \frac{1 - \cos \theta}{1 + \cos \theta} \right]^{1/2} \\ &= \frac{3}{4} \frac{v^2}{k} \sqrt{\frac{2}{1 + \cos \theta}}\end{aligned}$$

$$\frac{k}{r} = \frac{1}{1 + \cos \theta}$$

Problem 1-37

(1-37) Surface integral
via Gauss



$$\int_S \vec{A} \cdot d\vec{a} \quad \text{for } \vec{A} = (x^2 + y^2 + z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (R^2)(\hat{r})$$

$$= R^3 \hat{r} \quad \text{in spherical coordinates}$$

S is a sphere w/ $R^2 = x^2 + y^2 + z^2$

in spherical coord. $d\vec{a} = \hat{r} da$ for this surface

$$\therefore \vec{A} \cdot d\vec{a} = (R^3 \hat{r}) \cdot \hat{r} da = R^3 da$$

$$\int_S \vec{A} \cdot d\vec{a} = R^3 \int_S da = R^3 (4\pi R^2) = \underline{\underline{4\pi R^5}}$$

} direct
calculation

For Gauss's law we need $\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3x^2 + y^2 + z^2 + 3y^2 + x^2 + z^2 + 3z^2 + x^2 + y^2$

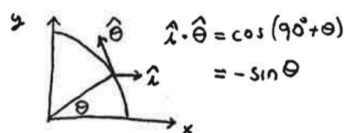
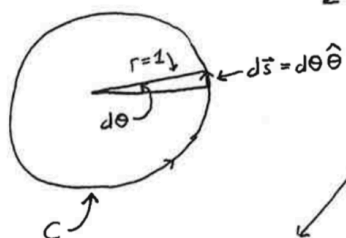
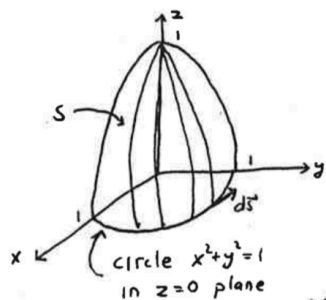
$$= 5(x^2 + y^2 + z^2) = 5r^2$$

$$\int_S \vec{A} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{A} dv = \int_V 5r^2 dv = 4\pi \int_0^R (5r^2) r^2 dr = 20\pi \left. \frac{r^5}{5} \right|_0^R = \underline{\underline{4\pi R^5}}$$

Problem 1-38

(1-38) Surface integral
via Stokes

$\int_S (\nabla \times \vec{A}) \cdot d\vec{a}$ for $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$
with S the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$



$$\text{Stokes: } \int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \int_C \vec{A} \cdot d\vec{s}$$

$$d\vec{s} = d\theta \hat{\theta}$$

$$\text{on path } C (z=0) \quad \vec{A} = y\hat{i} + x\hat{k} \\ = \sin\theta \hat{i} + \cos\theta \hat{k}$$

$$\hat{i} \cdot \hat{\theta} = -\sin\theta$$

$$\hat{k} \cdot \hat{\theta} = 0$$

$$\begin{aligned} \therefore \int_C \vec{A} \cdot d\vec{s} &= \int_C (\sin\theta \hat{i} + \cos\theta \hat{k}) \cdot \hat{\theta} d\theta \\ &= \int_C (-\sin^2\theta) d\theta \\ &= -\int_0^{2\pi} \sin^2\theta d\theta \\ &= -\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \underline{\underline{-\pi}} \end{aligned}$$

Assignment 2 - Solutions

Numerical Methods: Integrating the Equations of Motion

The first step in solving a mechanics problem using the method of Newtonian mechanics is to identify all the forces contributing to the $\mathbf{F}$ in Newton's second law $\mathbf{a}=\mathbf{F}/m$. This is a vector equation and each component will be of the following form:

$$a_i = \frac{dv_i}{dt} = \frac{d^2x_i}{dt^2} = F_i(\mathbf{x}, \mathbf{v}, t)/m$$

where, as indicated, the force $\mathbf{F}$ may depend on position $\mathbf{x}$, velocity $\mathbf{v}$, and time t . (The force will also probably depend on one or more constants such as the acceleration due to gravity g , a spring constant, or a drag coefficient). These are the equations of motion for the system and in one sense we have solved the problem once we've written down these equations. In another sense our problems have only begun as we usually want to know the velocity and position as functions of time and this requires solution of the above set of differential equations. For simple cases we can explicitly solve this set of equations. However, the "real" world is a complicated place and the equations of motion for most systems are analytically intractable. Thus numerical solutions are required. In either case, a set of initial conditions $(\mathbf{x}_0, \mathbf{v}_0, t_0)$ needs to be specified.

Numerical Techniques:

Let's consider the simple case of 1D motion in the x -direction. Given a force $F(x, v, t)$, and thus an acceleration $a(x, v, t)$, our goal is to determine the functions $v(t)$ and $x(t)$ by numerically integrating the equations of motion:

$$\frac{dv}{dt} = a(x, v, t) \quad \text{and} \quad \frac{dx}{dt} = v(x, t).$$

We do this by starting at x_0, v_0, t_0 and solving the equations step by step as we increment time by an amount Δt at each step. (Note that in many discussions the time step is called h).

Euler's Method: The simplest method is the Euler method which proceeds as follows:

$$v_{n+1} = v_n + a(x_n, v_n, t_n) \Delta t$$

$$x_{n+1} = x_n + v_n \Delta t$$

This method is easy to understand and easy to implement but it is never actually used in practice as it is neither very accurate or very stable.

Runge-Kutta Method: An only slightly more complicated, but much more accurate and stable algorithm is the fourth-order Runge-Kutta formula:

$$v_{n+1} = v_n + \frac{1}{6} (k_1 + 2 k_2 + 2 k_3 + k_4)$$

$$x_{n+1} = x_n + \frac{1}{6} (f_1 + 2 f_2 + 2 f_3 + f_4)$$

where

$$k_1 = a(x_n, v_n, t_n) \Delta t$$

$$f_1 = v_n \Delta t$$

$$k_2 = a(x_n + f_1/2, v_n + k_1/2, t_n + \Delta t/2) \Delta t$$

$$f_2 = (v_n + k_1/2) \Delta t$$

$$k_3 = a(x_n + f_2/2, v_n + k_2/2, t_n + \Delta t/2) \Delta t$$

$$f_3 = (v_n + k_2/2) \Delta t$$

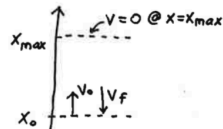
$$k_4 = a(x_n + f_3, v_n + k_3, t_n + \Delta t) \Delta t$$

$$f_4 = (v_n + k_3) \Delta t$$

Note that this Runge-Kutta algorithm requires four evaluations of the function $a(x,v,t)$ to advance the solution from time t_n to time $t_{n+1}=t_n+\Delta t$. This algorithm is the accepted "work horse" method for solving ODE's.

To implement the above algorithms you will need to specify the time step Δt . In principle a short time step means a more accurate solution, but you will need many more iterations to solve the equations out to some final time t_f . Also, at each iteration there will be some error introduced due to the approximate nature of the algorithm (and also possibly due the finite precision of computer calculations). This error tends to accumulate so the more iterations, the greater the final error. The Euler method is accurate only to first order in Δt (which means you will potentially be accumulating an error of order $(\Delta t)^2$ at each iteration). The above Runge-Kutta method is accurate to fourth order in Δt (so an error of order $(\Delta t)^5$ may be accumulated at each iteration).

(2-12)



$$\text{Drag force: } F_D = -k|v| = \begin{cases} -kv^2 & \text{for up motion} \\ kv^2 & \text{for down motion} \end{cases}$$

$$F_{\text{total}} = -mg - k|v| \quad (\text{where up is positive})$$

We want v in terms of x , so easiest to eliminate t from the start:

$$\left. \begin{aligned} \frac{dv}{dt} &= \frac{F}{m} \\ \frac{dx}{dt} &= v \end{aligned} \right\} \Rightarrow dx = \frac{v dv}{F/m}$$

$$\begin{aligned} \text{Up motion: } dx &= \frac{v dv}{-g - \frac{k}{m} v^2} \Rightarrow x_{\text{max}} - x_0 = -\frac{m}{k} \int_{v_0}^0 \frac{v dv}{\frac{mg}{k} + v^2} \\ &= -\frac{m}{2k} \ln \left(\frac{mg}{k} + v^2 \right) \Big|_{v_0}^0 \\ &= \frac{m}{2k} \ln \left(1 + \frac{kv_0^2}{mg} \right) \end{aligned}$$

$$\begin{aligned} \text{Down motion: } dx &= \frac{v dv}{-g + \frac{k}{m} v^2} \Rightarrow x - x_{\text{max}} = -\frac{m}{k} \int_0^v \frac{v dv}{\frac{mg}{k} - v^2} \\ &= \frac{m}{2k} \ln \left(\frac{mg}{k} - v^2 \right) \Big|_0^v \\ &= \frac{m}{2k} \ln \left(1 - \frac{kv^2}{mg} \right) \end{aligned}$$

$$\therefore \text{ on way down } v(x) = -\sqrt{\frac{mg}{k} \left[1 - \exp \left\{ \frac{2k}{m} (x - x_{\text{max}}) \right\} \right]}$$

$$\text{terminal } v \text{ given by } v(x \rightarrow -\infty) = -\sqrt{\frac{mg}{k}}$$

$$\begin{aligned} v @ x_0 \text{ given by } v(x_0) &= v_t \sqrt{1 - \exp \left\{ \frac{2k}{m} (x_0 - x_{\text{max}}) \right\}} \\ &= v_t \sqrt{1 - \exp \left\{ -\ln \left(1 + \frac{kv_0^2}{mg} \right) \right\}} \\ &= v_t \sqrt{1 - \frac{1}{\left(1 + \frac{v_0^2}{v_t^2} \right)}} = v_t \sqrt{\frac{v_0^2/v_t^2}{1 + v_0^2/v_t^2}} = \underline{\underline{\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}}} \end{aligned}$$

(2-13)

Drag force: $F = -mk(v^2 + a^2v)$ for $k > 0$ force opposes motion

To determine max. possible displacement we determine $x(v)$:

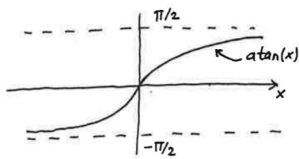
$$\left. \begin{aligned} \frac{dv}{dt} &= \frac{F}{m} \\ \frac{dx}{dt} &= v \end{aligned} \right\}$$

$$dx = \frac{v dv}{F/m} \Rightarrow x - x_0 = \int_{v_0}^v \frac{-v dv}{k(v^2 + a^2v)}$$

$$= -\frac{1}{k} \int_{v_0}^v \frac{dv}{a^2 + v^2}$$

$$= -\frac{1}{ka} \tan^{-1}\left(\frac{v}{a}\right) \Big|_{v_0}^v$$

$$= \frac{1}{ka} \left\{ \tan^{-1}\left(\frac{v_0}{a}\right) - \tan^{-1}\left(\frac{v}{a}\right) \right\}$$



max displacement @ $v \rightarrow 0$

$$\therefore \Delta x_{\max} = \frac{1}{ka} \tan^{-1}\left(\frac{v_0}{a}\right)$$

max value of $\tan^{-1}$ is $\pi/2$
for $v_0 \rightarrow \infty$

$$\therefore \Delta x_{\max} = \frac{\pi}{2ka}$$

To determine time to stop we need $v(t)$:

$$dt = \frac{dv}{F/m}$$

$$\Delta t = -\frac{1}{k} \int_{v_0}^{v_f} \frac{dv}{(v^2 + a^2v)} = -\frac{1}{k} \int_{v_0}^{v_f} \frac{dv}{v(a^2 + v^2)} \quad [\text{Integral E.3, pg 604}]$$

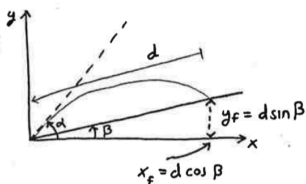
$$= -\frac{1}{2ka^2} \ln\left(\frac{v^2}{a^2 + v^2}\right) \Big|_{v_0}^{v_f}$$

$$= \frac{1}{2ka^2} \left\{ \ln\left(\frac{a^2 + v_f^2}{v_f^2}\right) - \ln\left(\frac{a^2 + v_0^2}{v_0^2}\right) \right\}$$

$$= \frac{1}{2ka^2} \ln\left[\frac{v_0^2}{v_f^2} \left(\frac{a^2 + v_f^2}{a^2 + v_0^2}\right)\right]$$

$\rightarrow \infty$ as $v_f \rightarrow 0$ (even if $v_0 \approx \infty$!)

(2-14)

 $V_0 = \text{initial speed}$ let $x_0 = y_0 = 0$ $d = \text{distance up the hill}$

$$\begin{aligned} \text{a) } V_{0x} &= V_0 \cos \alpha & x_f &= V_{0x} t & \Rightarrow t &= \frac{x_f}{V_{0x}} = \frac{d \cos \beta}{V_0 \cos \alpha} \\ V_{0y} &= V_0 \sin \alpha & y_f &= V_{0y} t - \frac{1}{2} g t^2 \end{aligned}$$

← plug this expression for t into y_f eq.

$$\Rightarrow d \sin \beta = V_0 \sin \alpha \frac{d \cos \beta}{V_0 \cos \alpha} - \frac{1}{2} g \frac{d^2 \cos^2 \beta}{V_0^2 \cos^2 \alpha}$$

$$\tan \beta = \tan \alpha - \frac{g}{2 V_0^2} \frac{\cos \beta}{\cos^2 \alpha} d$$

$$\begin{aligned} d &= \frac{\tan \alpha - \tan \beta}{\left(\frac{g}{2 V_0^2} \frac{\cos \beta}{\cos^2 \alpha} \right)} = \frac{2 V_0^2}{g} \frac{\cos^2 \alpha}{\cos \beta} [\tan \alpha - \tan \beta] \\ &= \frac{2 V_0^2}{g} \frac{\cos^2 \alpha}{\cos^2 \beta} \{ \tan \alpha \cos \beta - \sin \beta \} \\ &= \frac{2 V_0^2}{g} \frac{\cos \alpha}{\cos^2 \beta} \{ \sin \alpha \cos \beta - \sin \beta \cos \alpha \} \\ &= \frac{2 V_0^2}{g} \frac{\cos \alpha \sin(\alpha - \beta)}{\cos^2 \beta} \end{aligned}$$

check: for $\alpha = \beta$ $d = 0 \checkmark$ for $\beta = 0$

$$d = \frac{2 V_0^2}{g} \sin \alpha \cos \alpha \checkmark$$

b) α for max range given by:

$$\frac{\partial d}{\partial \alpha} = \frac{2 V_0^2}{g \cos^2 \beta} \{ -\sin \alpha \sin(\alpha - \beta) + \cos \alpha \cos(\alpha - \beta) \} = 0$$

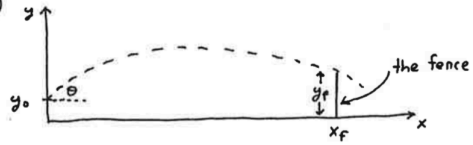
$$\cos(2\alpha - \beta) = 0$$

$$2\alpha - \beta = \pi/2 \Rightarrow \alpha = \frac{1}{2}(\beta + \pi/2) \quad \text{for } \beta = 0 \quad \alpha_{\max} = \pi/4 \checkmark$$

$$\begin{aligned} \text{c) } d_{\max} &= \frac{2 V_0^2}{g} \frac{\cos(\frac{\pi}{4} + \beta/2) \sin(\pi/4 - \beta/2)}{\cos^2 \beta} \\ &= \frac{2 V_0^2}{g} \frac{\cos[\frac{1}{2}(\frac{\pi}{2} + \beta)] \sin[\frac{1}{2}(\frac{\pi}{2} - \beta)]}{\cos^2 \beta} \\ &= \frac{V_0^2}{g} \frac{\sin \pi/2 - \sin \beta}{\cos^2 \beta} = \frac{V_0^2}{g} \frac{1 - \sin \beta}{1 - \sin^2 \beta} = \frac{V_0^2/g}{1 + \sin \beta} \end{aligned}$$

for b and c see
trig identities
D.11 and D.12 on pg. 600

(2-17)



$$y_0 = 0.7\text{m}$$

$$y_f = 2.0\text{m}$$

$$x_f = 60.0\text{m}$$

$$\theta = 35^\circ$$

$$V_{0x} = V_0 \cos \theta$$

$$V_{0y} = V_0 \sin \theta$$

equations of motion
for constant acceleration:

$$x = V_{0x} t$$

$$= V_0 \cos \theta t$$

$$y = y_0 + V_{0y} t - \frac{1}{2} g t^2$$

$$= y_0 + V_0 \sin \theta t - \frac{1}{2} g t^2$$

time to reach fence: $t_f = \frac{x_f}{V_0 \cos \theta}$ ← plug this into y equation

$$\Rightarrow y_f = y_0 + V_0 \sin \theta \frac{x_f}{V_0 \cos \theta} - \frac{1}{2} g \frac{x_f^2}{V_0^2 \cos^2 \theta}$$

$$= y_0 + x_f \tan \theta - \frac{g x_f^2}{2 V_0^2 \cos^2 \theta}$$

solve for V_0 :

$$V_0^2 = \frac{g x_f^2}{2 \cos^2 \theta} \left[y_0 + x_f \tan \theta - y_f \right]^{-1}$$

$$V_0 = \frac{x_f}{\cos \theta} \left[\frac{g}{2 (x_f \tan \theta + y_0 - y_f)} \right]^{1/2}$$

$$= \frac{60.0\text{m}}{\cos 35^\circ} \left[\frac{9.8\text{m/s}^2}{2 (60.0\text{m} \tan 35^\circ + 0.7\text{m} - 2.0\text{m})} \right]^{1/2}$$

$$= \underline{\underline{25.4\text{ m/s}}}$$

(2-18) Baseball w/ v^2 drag

$$C_w = 0.5, r = 0.05 \text{ m}, m = 0.2 \text{ kg}$$

$$A = \pi r^2, \rho = 1.3 \text{ kg/m}^3$$

$\nwarrow$ cross-sectional area of baseball
 $\nearrow$ density of air

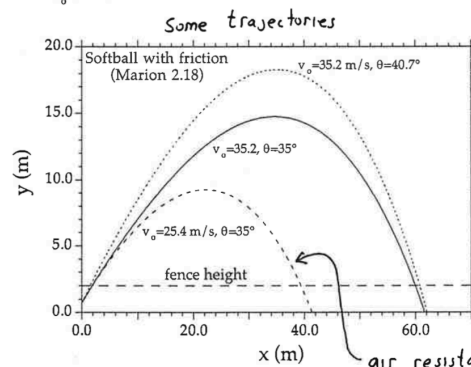
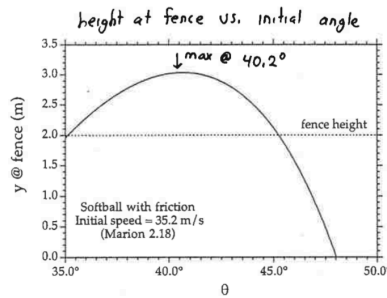
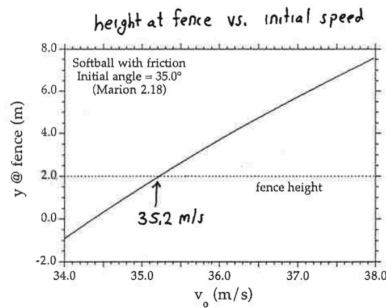
$$\vec{F}_D = \frac{1}{2} C_w \rho A |\vec{v}| \vec{v} \quad \text{where } v = \sqrt{v_x^2 + v_y^2}$$

$$\begin{aligned} \therefore (F_D)_x &= \frac{1}{2} C_w \rho A \sqrt{v_x^2 + v_y^2} v_x \\ (F_D)_y &= \frac{1}{2} C_w \rho A \sqrt{v_x^2 + v_y^2} v_y \end{aligned} \quad \left. \vphantom{\begin{aligned} (F_D)_x \\ (F_D)_y \end{aligned}} \right\} \begin{array}{l} x \text{ and } y \text{ motion are coupled} \\ \text{through the drag force!!} \end{array}$$

The equations of motion are:

$$\begin{aligned} \dot{v}_x &= -\frac{k}{m} \sqrt{v_x^2 + v_y^2} v_x \\ \dot{v}_y &= -g - \frac{k}{m} \sqrt{v_x^2 + v_y^2} v_y \end{aligned} \quad \text{where } k = \frac{1}{2} C_w \rho A$$

These must be solved numerically - I used the Euler algorithm and got the following results:



(2-22) Lorentz Force: $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ ($v \ll c$)

a) $\vec{E} = 0, \vec{v}_0 \perp \vec{B} \Rightarrow |\vec{F}| = qv_0 B$

The B-field does no work on the particle since it produces a force $\perp$ to the particle's path $\Rightarrow \vec{F} \cdot d\vec{r} = 0$.

$\therefore$ The particle experiences a constant force $\perp$ to its path

$\Rightarrow$ uniform circular motion w/ $a = v^2/R (= F/m)$

$$R = \frac{mv^2}{F} = \frac{mv^2}{qvB} = \frac{mv}{qB}$$

[This can all be done more rigorously by following the analysis given below w/ $\vec{E} = 0$ and $\vec{v}_0 = v_x \hat{x} + v_y \hat{y}$]

b) Now let $\vec{B} = B_z \hat{k}, \vec{E} = E_y \hat{j} + E_z \hat{k}$

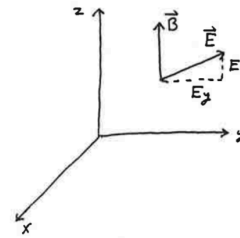
For $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{k}$ we have:

$$\begin{aligned} \vec{F} &= q\vec{E} + q\vec{v} \times \vec{B} \\ &= qE_y \hat{j} + qE_z \hat{k} + q(v_y B_z \hat{x} - v_x B_z \hat{y}) \end{aligned}$$

$$\therefore F_x = qv_y B_z$$

$$F_y = qE_y - qv_x B_z$$

$$F_z = qE_z$$



z-motion: $\frac{dv_z}{dt} = \frac{F_z}{m} = \frac{q}{m} E_z \Rightarrow v_z(t) = v_z(0) + \frac{q}{m} E_z t$

$$v_z = \frac{dz}{dt} \Rightarrow \underline{\underline{z(t) = z(0) + v_z(0)t + \frac{1}{2} \frac{q}{m} E_z t^2}}$$

c) x and y motion are coupled via:

$$\frac{dv_x}{dt} = \frac{q}{m} v_y B_z$$

$$\frac{dv_y}{dt} = \frac{q}{m} E_y - \frac{q}{m} v_x B_z$$

we can decouple these by taking a derivative of the 1st equation:

$$\frac{d^2 v_x}{dt^2} = \frac{q B_z}{m} \frac{dv_y}{dt}$$

and now substituting in the 2nd equation:

$$\frac{d^2 v_x}{dt^2} = \frac{q B_z}{m} \left\{ \frac{q}{m} E_y - \frac{q}{m} v_x B_z \right\}$$

We rewrite this as: $\ddot{v}_x + \omega^2 v_x = \omega^2 \frac{E_y}{B_z}$ where $\omega = \frac{q B_z}{m}$

2-22 cont.

Solution: $v_x(t) = \overbrace{A_1 \sin \omega t + A_2 \cos \omega t}^{\text{solution to homogeneous equation}} + \underbrace{E_y/B_z}_{\text{particular solution (here we take } v_x = \text{constant solution)}}$

We can compute $\langle v_x \rangle$ w/o determining the constants A_1 and A_2 :

$$\langle v_x \rangle \equiv \frac{1}{T} \int_0^T v_x(t) dt = A_1 \langle \sin \omega t \rangle_T + A_2 \langle \cos \omega t \rangle_T + \langle E_y/B_z \rangle_T = \frac{E_y}{B_z}$$

$(T = 2\pi/\omega)$ average of sin or cos over a period = 0! this is a constant

For the y-motion we substitute our $v_x(t)$ result into the equation for $\frac{dv_y}{dt}$ which gives:

$$v_y(t) = \frac{m}{qB_z} \frac{dv_x}{dt} = \frac{m}{qB_z} \{ A_1 \omega \cos \omega t - A_2 \omega \sin \omega t \} = A_1 \cos \omega t - A_2 \sin \omega t$$

$\omega = \frac{qB_z}{m}$

$$\langle v_y \rangle = A_1 \langle \cos \omega t \rangle_T - A_2 \langle \sin \omega t \rangle_T = 0$$

d) From (c) we have:

$$\dot{x} = A_1 \sin \omega t + A_2 \cos \omega t + E_y/B_z$$

$$\dot{y} = A_1 \cos \omega t - A_2 \sin \omega t$$

initial conditions: $\begin{cases} \dot{x}(0) = E_y/B_z \Rightarrow A_2 = 0 \\ \dot{y}(0) = A_1 \Rightarrow A_1 = A \end{cases}$

$$\therefore \dot{x} = A \sin \omega t + E_y/B_z$$

$$\dot{y} = A \cos \omega t$$

Integrating these gives:

$$x(t) - x(0) = -\frac{A}{\omega} \cos \omega t + \frac{A}{\omega} + \frac{E_y}{B_z} t$$

$$y(t) - y(0) = \frac{A}{\omega} \sin \omega t$$

Now insert initial condition, $x(0) = -\frac{A}{\omega}$, $y(0) = 0$:

$$\boxed{\begin{aligned} x(t) &= -\frac{A}{\omega} \cos \omega t + \frac{E_y}{B_z} t \\ y(t) &= \frac{A}{\omega} \sin \omega t \end{aligned}}$$

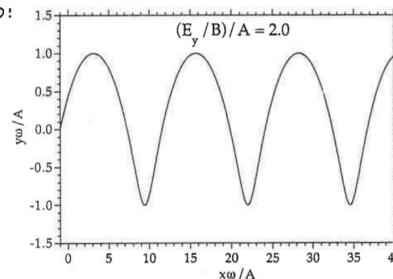
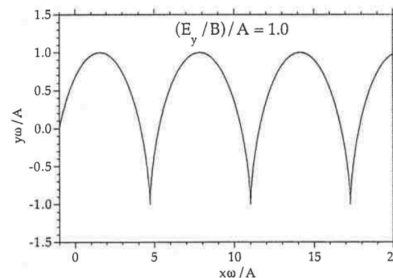
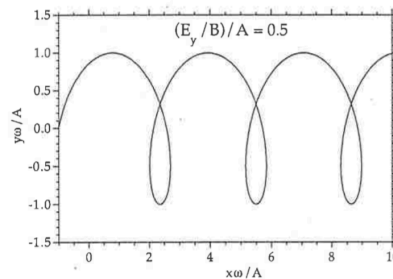
For plotting we rewrite these equations in terms of dimensionless quantities:

$$\frac{x\omega}{A} = -\cos \omega t + \frac{(E_y/B_z)}{A} \omega t$$

$$\frac{y\omega}{A} = \sin \omega t$$

*Note the A and E/B have units of velocity and ω has units of $1/\text{time}$.

$\therefore \frac{(E/B)}{A}$, $\frac{x\omega}{A}$, $\frac{y\omega}{A}$, and ωt are all dimensionless



(2-23) 1D motion w/ time dependent force: $F(t) = kt e^{-\alpha t}$ where $k = 1 \text{ N/s}$
 $\alpha = 0.5 \text{ s}^{-1}$
 [Note that the force starts out linear but then falls off exponentially for large time.] and mass of particle is $m = 1 \text{ kg}$

$$a(t) = \frac{F(t)}{m} = \frac{k}{m} t e^{-\alpha t} \quad \left(\text{in the following we let } v(0)=0 \text{ and } x(0)=0 \right)$$

$$\begin{aligned} v(t) &= \int_0^t a(t) dt = \frac{k}{m} \int_0^t t e^{-\alpha t} dt = \frac{k}{m} \left[\frac{e^{-\alpha t}}{\alpha^2} (-\alpha t - 1) \right]_0^t \\ &= \frac{k}{m \alpha^2} \left[-e^{-\alpha t} (\alpha t + 1) + 1 \right] \\ &= \frac{k}{m \alpha^2} \left\{ 1 - e^{-\alpha t} (1 + \alpha t) \right\} \end{aligned}$$

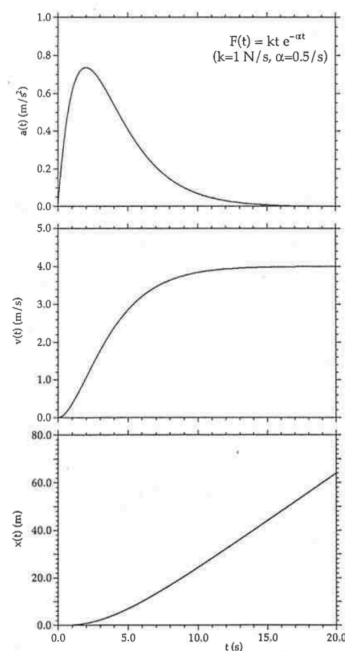
$$\begin{aligned} x(t) &= \int_0^t v(t) dt \\ &= \frac{k}{m \alpha^2} \int_0^t (1 - e^{-\alpha t} - \alpha t e^{-\alpha t}) dt \\ &= \frac{k}{m \alpha^2} \left[t + \frac{1}{\alpha} e^{-\alpha t} - \alpha \frac{e^{-\alpha t}}{\alpha^2} (-\alpha t - 1) \right]_0^t \\ &= \frac{k}{m \alpha^2} \left\{ t + \frac{1}{\alpha} e^{-\alpha t} - \frac{1}{\alpha} + \frac{e^{-\alpha t}}{\alpha} (\alpha t + 1) - \frac{1}{\alpha} \right\} \\ &= \frac{k}{m \alpha^3} \left\{ \alpha t - 2 + (\alpha t + 2) e^{-\alpha t} \right\} \end{aligned}$$

for plotting w/ t in seconds:

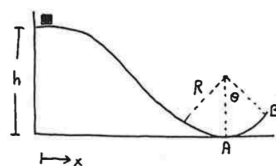
$$a(t) = t e^{-t/2}$$

$$v(t) = 4 \left[1 - e^{-t/2} (1 + t/2) \right]$$

$$x(t) = 8 \left\{ \frac{t}{2} - 2 + \left(\frac{t}{2} + 2 \right) e^{-t/2} \right\}$$

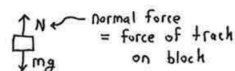


(2-25)



$$\theta = 45^\circ$$

a) @ point A:



Normal force
= force of track
on block

$$\text{acceleration @ A: } \frac{mv^2}{R} \hat{y}$$

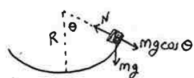
$$\text{speed @ A given by: } mgh = \frac{1}{2}mv^2$$

$$\text{or } v = \sqrt{2gh}$$

$$\therefore F_y = N - mg = \frac{mv^2}{R}$$

$$N = mg + \frac{mv^2}{R} = mg \left(1 + \frac{2h}{R} \right)$$

b) @ point B:



$$\text{circular motion: } a = \frac{v^2}{R} \Rightarrow N - mg \cos \theta = \frac{mv^2}{R}$$

$$\text{height of block @ B} = R - R \cos \theta$$

$$\text{speed @ B given by: } \frac{1}{2}mv^2 = mg \left\{ h - R(1 - \cos \theta) \right\}$$

$$\therefore N = mg \cos \theta + \frac{mv^2}{R}$$

$$= mg \cos \theta + \frac{2mg}{R} \left\{ h - R(1 - \cos \theta) \right\}$$

$$= mg \left\{ 3 \cos \theta - 2 + \frac{2h}{R} \right\} = mg \left\{ \frac{3\sqrt{2}}{2} - 2 + \frac{2h}{R} \right\}$$

$$\text{c) speed @ B from part b: } v_B = \sqrt{2g \left[h - R \left(1 - \frac{\sqrt{2}}{2} \right) \right]}$$

d) Projectile motion:

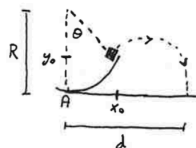
$$d = x_0 + v_{x0} t$$

$$y = y_0 + v_{y0} t - \frac{1}{2} g t^2$$

$$v_{x0} = v_{y0} = \frac{\sqrt{2}}{2} v_B = \sqrt{g(h - y_0)}$$

$$y_0 = R(1 - \cos \theta) = R \left(1 - \frac{\sqrt{2}}{2} \right)$$

$$x_0 = R \sin \theta = R \frac{\sqrt{2}}{2}$$



$$\text{time for } y \rightarrow 0 \text{ given by } \frac{1}{2} g t^2 - v_{y0} t - y_0 = 0$$

$$\Rightarrow t = \frac{v_{y0} \pm \sqrt{v_{y0}^2 + 2gy_0}}{g} \quad [\text{take "+" root since "-" root gives } t < 0]$$

$$\therefore d = \frac{\sqrt{2}}{2} R + v_{x0} \left\{ \frac{v_{y0} + \sqrt{v_{y0}^2 + 2gy_0}}{g} \right\} = \frac{\sqrt{2}}{2} R + \frac{1}{g} v_B^2 + \frac{\sqrt{2}}{2g} v_B \left[\frac{2}{g} v_B^2 + 2gR \left(1 - \frac{\sqrt{2}}{2} \right) \right]^{1/2}$$

$$= \frac{\sqrt{2}}{2} R + (h - y_0) + \left\{ \frac{1}{g} g(h - y_0) \left[g(h - y_0) + 2gy_0 \right] \right\}^{1/2}$$

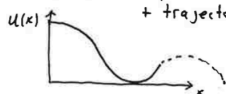
$$= \frac{\sqrt{2}}{2} R + (h - y_0) + \sqrt{(h - y_0)(h + y_0)}$$

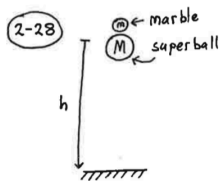
$$= h + (\sqrt{2} - 1)R + \sqrt{h^2 - y_0^2}$$

$$= h + (\sqrt{2} - 1)R + \sqrt{h^2 - R^2 \left(\frac{3}{2} - \sqrt{2} \right)}$$

e) Potential Energy: $U(x) = mgy(x)$ $\Rightarrow U(x) \equiv \text{shape of track!}$

+ trajectory



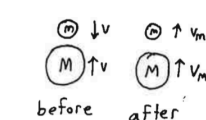


Assume h is large w/ respect to size of superball and marble.

Speed after falling distance h given by: $v = \sqrt{2gh}$

Superball rebounds elastically w/ floor (thus its speed is unchanged) and then collides with the marble.

The superball-marble collision:



[Note: our calculation will allow for $V_M < 0$ and $v_m < 0$]

Momentum Conservation: $Mv - mv = MV_M + mv_m$
 $\Rightarrow M(v - V_M) = m(v + v_m)$ (1)

Energy Conservation: $\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 = \frac{1}{2}MV_M^2 + \frac{1}{2}mv_m^2$
 $\Rightarrow M(v^2 - V_M^2) = m(v_m^2 - v^2)$ (2)

Results for the marble:

Divide Eq. (2) by (1): $v + V_M = v_m - v \Rightarrow V_M = v_m - 2v$

Now eliminate V_M from (1): $M(3v - v_m) = m(v + v_m) \Rightarrow v_m = \frac{3M - m}{M + m} v$
 requires $m < 3M$ for marble to bounce up!

$H_m = \frac{1}{2g} v_m^2 = \frac{1}{2g} \left(\frac{3M - m}{M + m} \right)^2 v^2 = \left(\frac{3 - m/M}{1 + m/M} \right)^2 h$ ($m < 3M$)

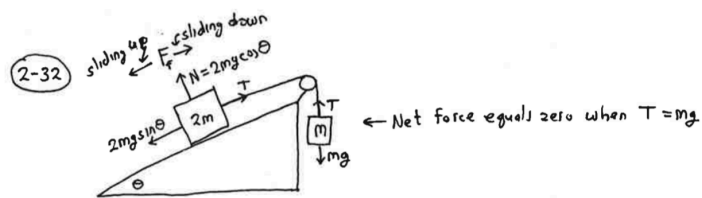
Results for the superball:

Eliminate $v_m = V_M + 2v$ from (1): $M(v - V_M) = m(3v + V_M) \Rightarrow V_M = \frac{M - 3m}{M + m} v$
 requires $M > 3m$ for superball to continue moving up

$H_M = \frac{1}{2g} V_M^2 = \frac{1}{2g} \left(\frac{M - 3m}{M + m} \right)^2 v^2 = \left(\frac{1 - 3m/M}{1 + m/M} \right)^2 h$ ($M > 3m$)

Note: for $m \ll M$ $H_m \sim 9h$ (marble)
 $H_M \sim h$ (superball)

if $m = M$ $H_m = H_M = h$



Friction force: $F_f = \pm \mu_k N$

Force balance for sliding block: $T - 2mg \sin \theta \pm F_f = 0$

$$mg - 2mg \sin \theta \pm \mu_k 2mg \cos \theta = 0$$

↑ $\begin{cases} + \text{ for sliding down} \\ - \text{ for sliding up} \end{cases}$

$\therefore \theta$ given by: $1 - 2 \sin \theta \pm 2 \mu_k \cos \theta = 0$

$$(1 - 2 \sin \theta)^2 = 4 \mu_k^2 (1 - \sin^2 \theta)$$

$$1 - 4 \sin \theta + 4 \sin^2 \theta - 4 \mu_k^2 + 4 \mu_k^2 \sin^2 \theta = 0$$

$$\sin^2 \theta - \frac{1}{1 + \mu_k^2} \sin \theta + \frac{1 - 4 \mu_k^2}{4(1 + \mu_k^2)} = 0$$

$$\sin \theta = \frac{1}{2(1 + \mu_k^2)} \pm \sqrt{\left(\frac{1}{1 + \mu_k^2}\right)^2 - \left(\frac{1 - 4 \mu_k^2}{1 + \mu_k^2}\right) \left(\frac{1}{2}\right)}$$

$$= \frac{1}{2(1 + \mu_k^2)} \left\{ 1 \pm \sqrt{1 - (1 - 4 \mu_k^2)(1 + \mu_k^2)} \right\}$$

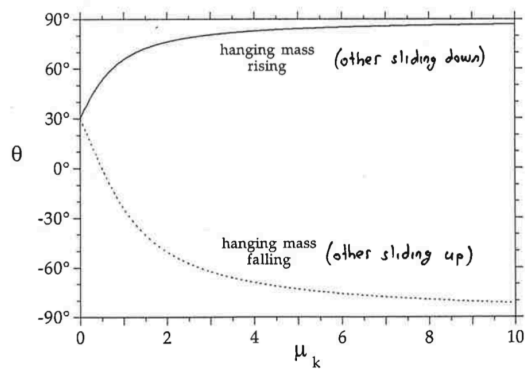
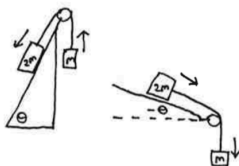
$$= \frac{1}{2(1 + \mu_k^2)} \left\{ 1 \pm \mu_k \sqrt{4 \mu_k^2 + 3} \right\} \quad \begin{cases} + \text{ for sliding up} \\ - \text{ for sliding down} \end{cases}$$

Note that for:

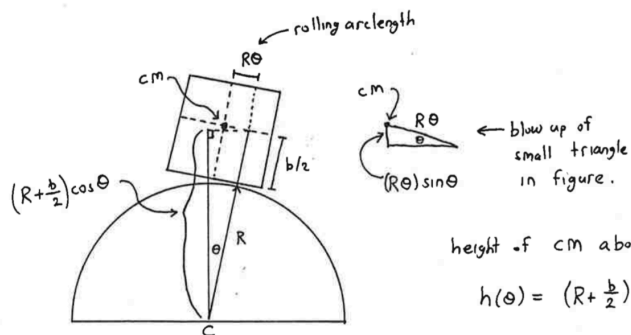
$$\mu_k = 0 \Rightarrow \sin \theta = \frac{1}{2}$$

$$\mu_k \rightarrow \infty \Rightarrow \sin \theta \rightarrow \pm 1$$

In the case of large μ_k the sliding up and sliding down config. are quite different:



2-42 stability of cube on cylinder



height of cm above point C:

$$h(\theta) = (R + \frac{b}{2}) \cos \theta + R\theta \sin \theta$$

Potential energy: $U(\theta) = mgh(\theta)$

$$\frac{1}{m_g} \frac{\partial U}{\partial \theta} = - (R + \frac{b}{2}) \sin \theta + R \sin \theta + R\theta \cos \theta$$

$$= - \frac{b}{2} \sin \theta + R\theta \cos \theta$$

$$\frac{\partial U}{\partial \theta} = 0 \text{ @ } \theta = 0 \Rightarrow \text{equil. position } \checkmark$$

Stability given by condition $\left. \frac{\partial^2 U}{\partial \theta^2} \right|_{\theta_{\text{equil}}} > 0$

$$\text{We have } \frac{1}{m_g} \frac{\partial^2 U}{\partial \theta^2} = - \frac{b}{2} \cos \theta + R \cos \theta - R\theta \sin \theta$$

$$= (R - \frac{b}{2}) \cos \theta - R\theta \sin \theta$$

$$\left. \frac{1}{m_g} \frac{\partial^2 U}{\partial \theta^2} \right|_{\theta=0} = R - \frac{b}{2} \quad \begin{cases} > 0 & \text{for } R > b/2 & \text{stable} \\ < 0 & \text{for } R < b/2 & \text{unstable} \end{cases}$$

small block is stable ($b < 2R$)

large block is unstable ($b > 2R$)

(2-43)

Force given by: $F = -kx + \frac{kx^3}{\alpha^2}$ with $k > 0$

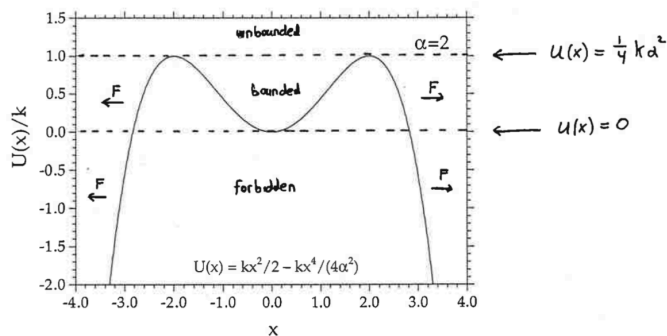
for small x , the linear term dominates ... a spring force
for large x , the cubic term dominates $\rightarrow$ force pushes things away from the origin

$F=0$ @ $x=0$ and $\pm\alpha$... these are the equilibrium points.

Potential given by:

$$U(x) = - \int F(x) dx = \frac{1}{2} kx^2 - \frac{1}{4} \frac{k}{\alpha^2} x^4 + C$$

Here we let the constant $C=0$, which sets our zero of energy.



- $E > \frac{1}{4} k\alpha^2$ unbounded, always moving in same direction
- $0 < E < \frac{1}{4} k\alpha^2$ bound if $|x| < \alpha$, reflects off wall @ $E = U(x)$
if $|x| > \alpha$, pushed away from wall } $E = U(x) \rightarrow$ turning point
- $E < 0$ pushed away from wall
- $E = \frac{1}{4} k\alpha^2$ gets trapped in unstable equilibrium @ $x = \pm\alpha$?!

(2-50) Falling water droplet

$\begin{array}{c} r \\ \bigcirc \\ \downarrow mg \end{array}$
 $A = \pi r^2$ Here r, A, m are all changing with time.

a) $\frac{dm}{dt} = kA$ $m = \rho \frac{4\pi}{3} r^3$ where $\rho = \text{density of water}$
 $= k\pi r^2$ $\downarrow$
 $\frac{dm}{dt} = 4\pi\rho r^2 \frac{dr}{dt}$

Equate these two expressions for $\frac{dm}{dt}$: $4\rho \frac{dr}{dt} = k$

$$\int_{r_0}^r dr = \frac{k}{4\rho} \int_0^t dt \Rightarrow \underline{\underline{r(t) = r_0 + \frac{k}{4\rho} t}}$$

b) $F = mg \Rightarrow$ produces change in momentum $\frac{dp}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt}$

$$\begin{aligned} \therefore \frac{dv}{dt} &= g - \frac{v}{m} \frac{dm}{dt} \\ &= g - v \left(\frac{k\pi r^2}{\rho \frac{4\pi}{3} r^3} \right) = g - v \left(\frac{3k}{4\rho r} \right) = g - v \left(\frac{3k}{4\rho r_0 + kt} \right) \end{aligned}$$

Here we have a differential equation of the form: $\dot{v} + A(t)v = g$ where $A(t) = \frac{3k}{4\rho r_0 + kt}$

The general solution is given by: $v e^{\int A(t) dt} = \int g e^{\int A(t) dt} dt + c$

Here $\int A(t) dt = 3 \ln(4\rho r_0 + kt)$

$e^{\int A(t) dt} = (4\rho r_0 + kt)^3$ (this is the "integrating factor")

$$\begin{aligned} \therefore (4\rho r_0 + kt)^3 v(t) &= \int g (4\rho r_0 + kt)^3 dt + c \\ &= \frac{g}{4k} (4\rho r_0 + kt)^4 + c \end{aligned}$$

constant c given by:
 $v(t) = \frac{g}{4k} (4\rho r_0 + kt) + \frac{c}{(4\rho r_0 + kt)^3}$ $v(0) = v_0 = \frac{2\rho r_0}{k} + \frac{c}{(4\rho r_0)^3}$

$$= \frac{g}{4k} (4\rho r_0 + kt) + (4\rho r_0)^3 \frac{(v_0 - \frac{2\rho r_0}{k})}{(4\rho r_0 + kt)^3} \Rightarrow c = (4\rho r_0)^3 \left[v_0 - \frac{2\rho r_0}{k} \right]$$

$$= \underline{\underline{\frac{g}{4k} (4\rho r_0 + kt) + \mathcal{O}(r_0^3)}} \left\{ \begin{array}{l} \text{we can neglect the second} \\ \text{term for small } r_0 \end{array} \right.$$

$\therefore \underline{\underline{v(t) \propto t}}$

(2-52) Launch of a single stage rocket in uniform gravity

let: m_0 = initial mass u = exhaust speed (relative to rocket), $u > 0$
 m_b = mass after burnout $\alpha = -\frac{dm}{dt}$ fuel burn rate ($\alpha > 0$)

external force: $F_{ext} = mg \Rightarrow$ equals $\frac{dp}{dt}$ where $dp = m dv + u dm$

$\therefore dv = -g dt - u \frac{dm}{m}$ note that $m = m_0 - \alpha t$ and $dm = -\alpha dt$

$= \left[-g + \frac{\alpha u}{m_0 - \alpha t} \right] dt$ integrate this from $0 \rightarrow v$ and $0 \rightarrow t$

$\Rightarrow v(t) = -gt - u \ln \left(\frac{m_0 - \alpha t}{m_0} \right) = -gt + u \ln \left(\frac{m_0}{m} \right)$ for $0 \leq t \leq t_b$
 where $t_b = \frac{m_0 - m_b}{\alpha}$
 velocity during burn... the "rocket" equation burnout time.

$y(t)$ during burn given by: $\frac{dy}{dt} = -gt + u \ln \left(\frac{m_0}{m_0 - \alpha t} \right)$

$$\int_0^y dy = -g \int_0^t t dt + u \ln m_0 \int_0^t \frac{1}{m_0 - \alpha t} dt - u \int_0^t \ln(m_0 - \alpha t) dt \quad \left[\int \ln(ax+b) = \frac{a+b}{b} \ln(ax+b) - x \right]$$

$$y(t) = -\frac{1}{2} g t^2 + [u \ln m_0] t - u \left[\left(\frac{m_0 - \alpha t}{-\alpha} \right) \ln(m_0 - \alpha t) - t \right]_0^t$$

$$= -\frac{1}{2} g t^2 + [u \ln m_0] t + \frac{u}{\alpha} (m_0 - \alpha t) \ln(m_0 - \alpha t) - u \frac{m_0}{\alpha} \ln m_0 + u t \quad 0 \leq t \leq t_b$$

@ burnout $y = y_b$, $t = t_b = \frac{m_0 - m_b}{\alpha}$ (note: $m_0 - \alpha t_b = m_b$)

$$y_b = -\frac{1}{2} g t_b^2 + u t_b + (u \ln m_0) \frac{m_0 - m_b}{\alpha} + u \frac{m_b}{\alpha} \ln m_b - u \frac{m_0}{\alpha} \ln m_0$$

$$= u t_b - \frac{1}{2} g t_b^2 - \frac{u m_b}{\alpha} \ln m_0 + \frac{u m_b}{\alpha} \ln m_b$$

$$= \underline{\underline{u t_b - \frac{1}{2} g t_b^2 - \frac{u m_b}{\alpha} \ln \left(\frac{m_0}{m_b} \right)}}$$

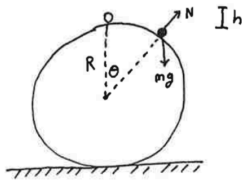
Additional height after burnout:

$$\Delta y = v_b t - \frac{1}{2} g t^2 \quad v(t) = v_b - g t$$

$$= v_b \left(\frac{v_b}{g} \right) - \frac{1}{2} g \left(\frac{v_b}{g} \right)^2 \quad \text{time after burnout to } v(t) = 0 \Rightarrow t = \frac{v_b}{g}$$

$$= \underline{\underline{\frac{1}{2} \frac{v_b^2}{g}}} \quad \text{where } v_b = -g \frac{m_0 - m_b}{\alpha} + u \ln \left(\frac{m_0}{m_b} \right)$$

(A1) Sliding off a sphere



let h = vertical distance fallen : $h = R(1 - \cos \theta)$

speed given by $\frac{1}{2}mv^2 = mgh \Rightarrow v^2 = 2gR(1 - \cos \theta)$

Net radial force: $F_r = N - mg \cos \theta$

Acceleration: $-\frac{v^2}{R}$ (circular motion)

$$\therefore N - mg \cos \theta = -\frac{mv^2}{R}$$

$$= -\frac{m}{R}(2gR[1 - \cos \theta])$$

loses contact when $N = 0 \Rightarrow g \cos \theta = 2g(1 - \cos \theta)$

$$\cos \theta = \frac{2}{3} \Rightarrow \theta \approx 48.2^\circ$$

Once particle loses contact, projectile motion with

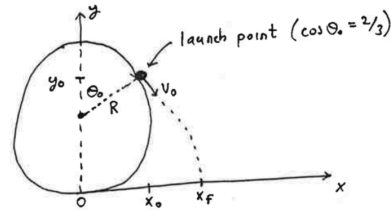
$$x_0 = R \sin \theta_0 = \frac{\sqrt{5}}{3} R$$

$$y_0 = R(1 - \cos \theta_0) = \frac{5}{3} R$$

$$v_0 = \sqrt{\frac{2}{3}gR}$$

$$v_{0x} = v_0 \cos \theta_0 = \frac{2}{3} \sqrt{\frac{2}{3}gR}$$

$$v_{0y} = -v_0 \sin \theta_0 = -\frac{\sqrt{5}}{3} \sqrt{\frac{2}{3}gR}$$



Time to hit ground given by: $y_f = 0 = y_0 + v_{0y}t - \frac{1}{2}gt^2$

$$\Rightarrow t = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gy_0}}{g} = \frac{-\frac{\sqrt{5}}{3} \sqrt{\frac{2}{3}gR} + \sqrt{\frac{5}{9} \frac{2gR}{3} + 2g \frac{5R}{3}}}{g}$$

$$= \frac{\sqrt{10} - 1}{3g} \sqrt{\frac{10gR}{3}}$$

Total x -distance:

$$x_f = x_0 + v_{0x}t = \frac{\sqrt{5}}{3} R + \frac{2}{3} \sqrt{\frac{2}{3}gR} \left(\frac{\sqrt{10} - 1}{3g} \right) \sqrt{\frac{10gR}{3}}$$

$$= \frac{\sqrt{5}}{3} R + \frac{2}{27} R \sqrt{20} (\sqrt{10} - 1) = \frac{\sqrt{5}}{27} R (5 + 4\sqrt{10})$$

Velocity @ Impact:

$$v_{xf} = v_{x0} = \frac{2}{3} \sqrt{\frac{2}{3}gR}$$

$$v_{yf} = v_{y0} - gt$$

$$= -\frac{\sqrt{5}}{3} \sqrt{\frac{2}{3}gR} - \left(\frac{\sqrt{10} - 1}{3} \right) \sqrt{\frac{10gR}{3}} = \frac{-10}{3\sqrt{3}} \sqrt{gR}$$

$$\vec{v}_f = v_{xf} \hat{i} + v_{yf} \hat{j}$$

$$\text{Note: } |\vec{v}_f|^2 = \frac{4}{9} \frac{2}{3} gR + \frac{100}{27} gR$$

$$= \frac{108}{27} gR = 4gR$$

from energy conservation:

$$\frac{1}{2}mv_f^2 = mg2R$$

$$\Rightarrow v_f^2 = 4gR \checkmark$$

(A2) Peg and the Pendulum



condition to just go around circle will

be $T=0$ @ top $\Rightarrow mg = \frac{mv^2}{R}$ where $R=L-d$
 $\nwarrow$ string tension

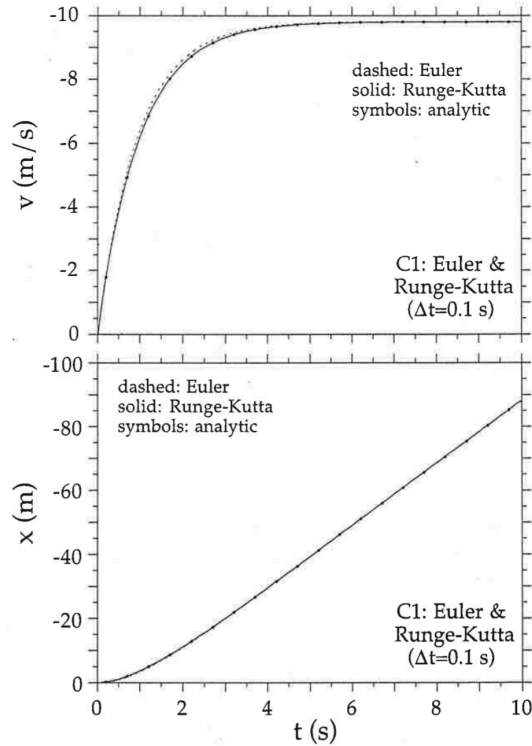
speed @ top of circle given by energy conservation: $\frac{1}{2}mv^2 = mg[L - 2(L-d)]$

$$v^2 = 2g(L-2R)$$

$\therefore$ condition on R: $\frac{2}{R}(L-2R) = 1$
 $R = \frac{2}{5}L$

Peg distance: $d = L - R = \frac{3}{5}L$

① Test the Algorithms : Falling in gravity w/ linear drag $\Rightarrow a = -g - kv$



solve $\frac{dv}{dt} = -g - kv$

$\frac{dx}{dt} = v$

using both the Euler and Runge-Kutta algorithms

Analytic solution:

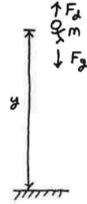
$$v(t) = -\frac{g}{k} + \frac{kv_0 + g}{k} e^{-kt}$$

$$x(t) = x_0 - \frac{g}{k}t + \frac{kv_0 + g}{k^2}(1 - e^{-kt})$$

Let $x_0 = 0$, $v_0 = 0$, $k = 1 \text{ s}^{-1}$

if $v_0 = 9.8 \text{ m/s}$ there will be no change in speed as this is the terminal velocity for the given k .

(C2) Plummeting Parachutist



$$y_0 = 32 \times 10^3 \text{ m}$$

$$v_0 = 0$$

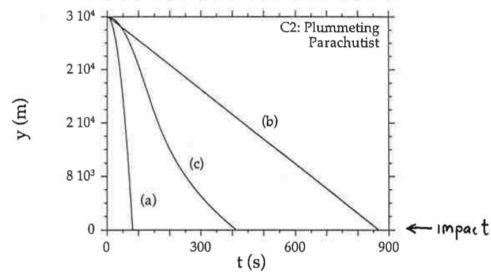
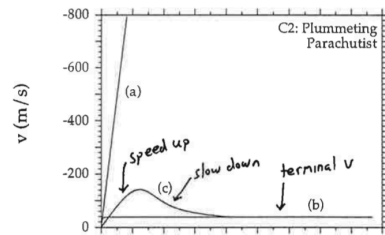
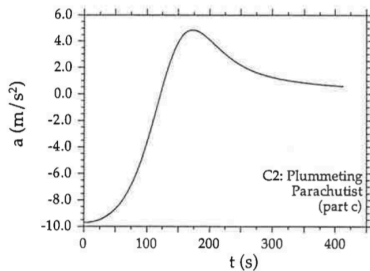
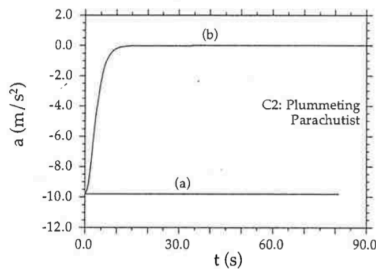
$$g = 9.8 \text{ m/s}^2$$

$$\left. \begin{aligned} \frac{dv}{dt} &= -\frac{F_g}{m} + \frac{F_d}{m} \\ \frac{dy}{dt} &= v \end{aligned} \right\} \begin{array}{l} \text{Solve numerically} \\ \text{via Runge-Kutta} \end{array}$$

a) $\frac{dv}{dt} = -g$ (no drag) time to impact $\approx 81 \text{ s}$ (analytic solution $t = \sqrt{\frac{2y_0}{g}}$)

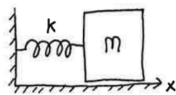
b) $\frac{dv}{dt} = -g - \frac{a}{m} v|v|$ with $a = 0.5 \text{ N s}^2/\text{m}^2$ time to impact $\approx 867 \text{ s}$
 $m = 70 \text{ kg}$ terminal velocity $\sim 37 \text{ m/s}$
time to reach $v_t < 10 \text{ s}!!$

c) $\frac{dv}{dt} = \frac{-g}{(1 + y/R_e)^2} - \frac{1}{2} \frac{C}{m} v|v|$ with $R_e = 6370 \times 10^3 \text{ m}$ time to impact = 412 s
 $H = 8 \times 10^3 \text{ m}$ max speed $\sim 141 \text{ m/s}$ @ $t \sim 120 \text{ s}$
speed drops for $t > 120 \text{ s}$ as you have exceeded terminal velocity for that altitude.



Assignment 3 - Solutions

3-1



$$k = 10^4 \text{ dyne/cm} \quad x_0 = 3 \text{ cm (initial displacement from equilibrium position)}$$

$$m = 100 \text{ g}$$

$$a) \quad \omega_0 = \sqrt{\frac{k}{m}} = 10 \text{ rad/s} \quad \nu_0 = \frac{\omega_0}{2\pi} = \frac{10}{2\pi} \text{ Hz} \sim \underline{\underline{1.6 \text{ Hz}}}$$

$$\tau_0 = \frac{1}{\nu_0} = \frac{2\pi}{\omega_0} = \frac{\pi}{5} \text{ s} \sim \underline{\underline{0.63 \text{ s}}}$$

$$b) \quad E_{\text{total}} = \frac{1}{2} k x_0^2 = \underline{\underline{4.5 \times 10^4 \text{ erg}}}$$

$$c) \quad V_{\text{max}} \text{ when } \frac{1}{2} m v^2 = E_{\text{total}} \Rightarrow V_{\text{max}} = \sqrt{\frac{2E_{\text{total}}}{m}} = \sqrt{\frac{k x_0^2}{m}} = \omega_0 x_0 = \underline{\underline{30 \text{ cm/s}}}$$

3-2

$$\text{for damped motion: } \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x(t) = A(t) \cos \omega_1 t \quad \text{where } A(t) = x_0 e^{-\beta t}$$

$$\omega_1^2 = \omega_0^2 - \beta^2$$

$$a) \quad @ \quad t = 10 \text{ s} \quad \frac{A(t)}{A(0)} = \frac{1}{2} \Rightarrow e^{-\beta t} = \frac{1}{2} \Rightarrow \beta = \frac{1}{t} \ln 2 = \frac{\ln 2}{10 \text{ s}} = \underline{\underline{6.9 \times 10^{-2} \text{ s}^{-1}}}$$

$$b) \quad \nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi} \sqrt{\omega_0^2 - \beta^2} = \frac{\omega_0}{2\pi} \sqrt{1 - \left(\frac{\beta}{\omega_0}\right)^2} = \nu_0 \left\{ 1 - (6.9 \times 10^{-3})^2 \right\}^{1/2}$$

$$\approx \nu_0 \left\{ 1 - \frac{1}{2} (6.9 \times 10^{-3})^2 \right\}$$

$$= \nu_0 \left\{ 1 - 2.4 \times 10^{-5} \right\}$$

$$c) \quad \text{decrement of the motion} = e^{\beta \tau_1} = e^{\beta/\nu_1} = \exp \left\{ \frac{6.9 \times 10^{-2} \text{ s}^{-1}}{1.6 \text{ s}^{-1}} \right\} = \underline{\underline{1.044}}$$

3-3

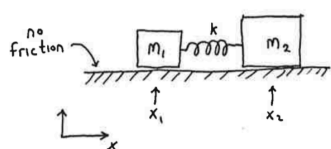
$$V_0 = 1 \text{ cm/s for problem 3-1 oscillator}$$

$$a) \quad x(t) = A \sin \omega_0 t$$

$$v(t) = \omega_0 A \cos \omega_0 t \Rightarrow v_0 = \omega_0 A \Rightarrow A = \frac{v_0}{\omega_0} = \frac{1 \text{ cm/s}}{10 \text{ /s}} = \underline{\underline{0.1 \text{ cm}}}$$

$$b) \quad U_{\text{max}} = \frac{1}{2} k A^2 = \frac{1}{2} (10^4 \text{ dyne/cm}) (0.1 \text{ cm})^2 = \underline{\underline{50 \text{ erg}}}$$

3-6



let $l \equiv$ equilibrium length of spring

$$\begin{aligned} m_1 &= 0.10 \text{ kg} \\ m_2 &= 0.20 \text{ kg} \\ k &= 0.5 \text{ N/m} \end{aligned}$$

$$\begin{aligned} \text{force on } m_2: F_2 &= k(l - (x_2 - x_1)) = m_2 \ddot{x}_2 \\ \text{" " } m_1: F_1 &= -k(l - (x_2 - x_1)) = m_1 \ddot{x}_1 \end{aligned}$$

We can write a differential equation for $y = x_2 - x_1$:

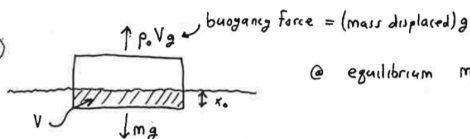
$$\ddot{y} = \frac{k}{m_1} (l - y) + \frac{k}{m_2} (l - y)$$

or in terms of $z = y - l$:

$$\ddot{z} + \left(\frac{k}{m_1} + \frac{k}{m_2} \right) z = 0$$

$$\begin{aligned} \therefore \text{ we have SHM w/ } \omega^2 &= \frac{k}{m_1} + \frac{k}{m_2} = k \left(\frac{m_1 + m_2}{m_1 m_2} \right) = \frac{k}{\mu} \quad \left. \begin{aligned} \mu &= 0.067 \text{ kg} \\ \omega &= 2.74 \text{ rad/s} \\ \gamma &= 0.44 \text{ Hz} \end{aligned} \right\} \\ \text{where } \mu &= \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass} \end{aligned}$$

3-7



$$\text{@ equilibrium } mg = \rho_0 V g$$

$$V = x_0 A \Rightarrow x_0 = 0.8 \text{ cm}$$

$$\begin{aligned} A &= 1 \text{ cm}^2 && \text{cross-sectional area of block} \\ \rho &= 0.8 \text{ g/cm}^3 && \text{density of block} \\ \rho_0 &= 1.0 \text{ g/cm}^3 && \text{density of liquid} \\ V &= 0.8 \text{ cm}^3 && \text{volume displaced @ equilibrium} \end{aligned}$$

If pushed down $(-x)$ net force is up.

$$\begin{aligned} F &= \rho_0 A (x_0 + \Delta x) g - mg \\ &= \rho_0 A g \Delta x + \underbrace{\rho_0 A x_0 g - mg}_{\text{these cancel from equilibrium condition}} \end{aligned}$$

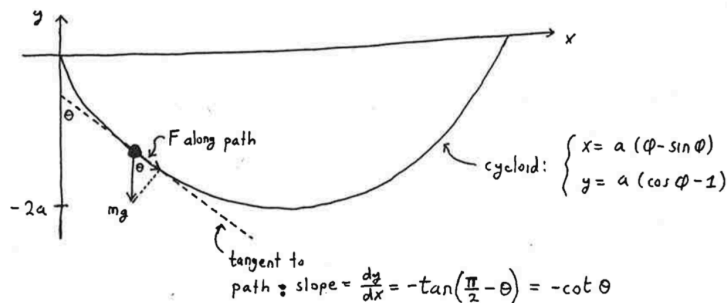
$$= \rho_0 A g \Delta x$$

this is a restoring force since $\Delta x < 0$; $F = k \Delta x$ w/ $k = \rho_0 A g$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\rho_0 A g}{m}} = \sqrt{\frac{\rho_0 A g}{\rho_0 V}} = \sqrt{\frac{A g}{V}}$$

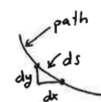
$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{V}{g A}} = 2\pi \sqrt{\frac{0.8 \text{ cm}^3}{(980 \text{ cm/s}^2)(1 \text{ cm}^2)}} = 0.18 \text{ s}$$

(3-8) Isochronous Pendulum (we map the $l=4a$ pendulum to the problem of a particle sliding in a cycloidal bowl!):



Force along path: $F = mg \cos \theta$
 $= m \ddot{v}$

Velocity along path: $v = \frac{ds}{dt}$
 where $ds = \sqrt{dx^2 + dy^2}$



We can write both θ and v in terms of the parameter ϕ :

$$dx = a(1 - \cos \phi) d\phi$$

$$dy = -a \sin \phi d\phi$$

$$\frac{dy}{dx} = \frac{-\sin \phi}{1 - \cos \phi} = \frac{-2 \sin \phi/2 \cos \phi/2}{2 \sin^2 \phi/2} = -\cot \phi/2$$

Comparing w/ the tangent line construction we have: $\theta = \phi/2$

$$\begin{aligned} ds &= a \sqrt{(1 - \cos \phi)^2 + \sin^2 \phi} d\phi \\ &= a \sqrt{2 - 2 \cos \phi} d\phi \\ &= 2a \sin \phi/2 d\phi \\ v &= \frac{ds}{dt} = 2a \sin \phi/2 \frac{d\phi}{dt} \\ &= \frac{d}{dt} (-4a \cos \phi/2) \end{aligned}$$

Our force equation states:

$$\ddot{v} = g \cos \theta$$

From the above results

for θ and v we have: $\frac{d^2}{dt^2} (-4a \cos \phi/2) = g \cos \phi/2$

$$\frac{d^2}{dt^2} (\cos \phi/2) + \frac{g}{4a} (\cos \phi/2) = 0$$

Letting $z \equiv \cos \phi/2 \Rightarrow \ddot{z} + \frac{g}{4a} z = 0$ SHM w/ $\omega^2 = g/4a = g/l$

No Approximations Here!
 Period is independent of amplitude.

(3-10) For a damped oscillator: $x(t) = A(t) \cos \omega_1 t$ where $A(t) = x_0 e^{-\beta t}$
 $\omega_1^2 = \omega_0^2 - \beta^2$

$$\frac{A(t)}{A(0)} = \frac{1}{e} \text{ @ } t = 1/\beta \quad (\text{the } 1/e \text{ time})$$

$$\text{Here } t = \frac{1}{\beta} = nT = n \frac{2\pi}{\omega_1} \Rightarrow \omega_1 = 2\pi n \beta \text{ or } \beta = \frac{\omega_1}{2\pi n}$$

$$\therefore \omega_1^2 = \omega_0^2 - \left(\frac{\omega_1}{2\pi n}\right)^2 \Rightarrow \omega_1^2 = \frac{\omega_0^2}{1 + \left(\frac{1}{2\pi n}\right)^2}$$

$$\omega_1 = \omega_0 \left[1 + \frac{1}{4\pi^2 n^2}\right]^{-1/2} \approx \omega_0 \left\{1 - \frac{1}{8\pi^2 n^2}\right\} \quad \left(\text{see D.6, pg. 600 for Taylor expansion of } (1+x)^{-1/2}\right)$$

(3-18) For a driven oscillator the steady state solution is $x(t) = D(\omega) \cos(\omega t - \delta)$

$$\text{Total energy} = \frac{1}{2} m (\dot{x}_{\max})^2 \quad \text{where } \dot{x}_{\max} = \omega D$$

$$= \frac{1}{2} m \omega^2 D^2$$

$$\text{Energy loss / cycle} = - \int_{\text{cycle}} F_D dx \quad \text{where } F_D = -b\dot{x} = -2m\beta\dot{x} \quad (\beta \equiv b/2m)$$

$$= - \int_0^T F_D (\dot{x} dt) \quad \leftarrow \text{rewrite as integral over time}$$

$$= 2m\beta \int_0^T (\dot{x})^2 dt$$

$$= 2m\beta \omega^2 D^2 \int_0^T \cos^2(\omega t - \delta) dt$$

$$= 2m\beta \omega^2 D^2 T \langle \cos^2(\omega t - \delta) \rangle_T \quad \text{note: } T = \frac{2\pi}{\omega}$$

$$= 2m\beta \pi \omega D^2 \quad \langle \cos^2 \theta \rangle_T = \frac{1}{2}$$

$$\langle f(t) \rangle \equiv \frac{1}{T} \int_0^T f(t) dt$$

$$\therefore \frac{E_{\text{total}}}{\Delta E_{\text{cycle}}} = \frac{\frac{1}{2} m \omega^2 D^2}{2m\beta \pi \omega D^2} = \frac{\omega}{4\pi\beta} \approx \frac{1}{2\pi} \left(\frac{\omega_R}{2\beta}\right) = \frac{1}{2\pi} Q$$

[driving near resonance]
so $\omega \approx \omega_R$]

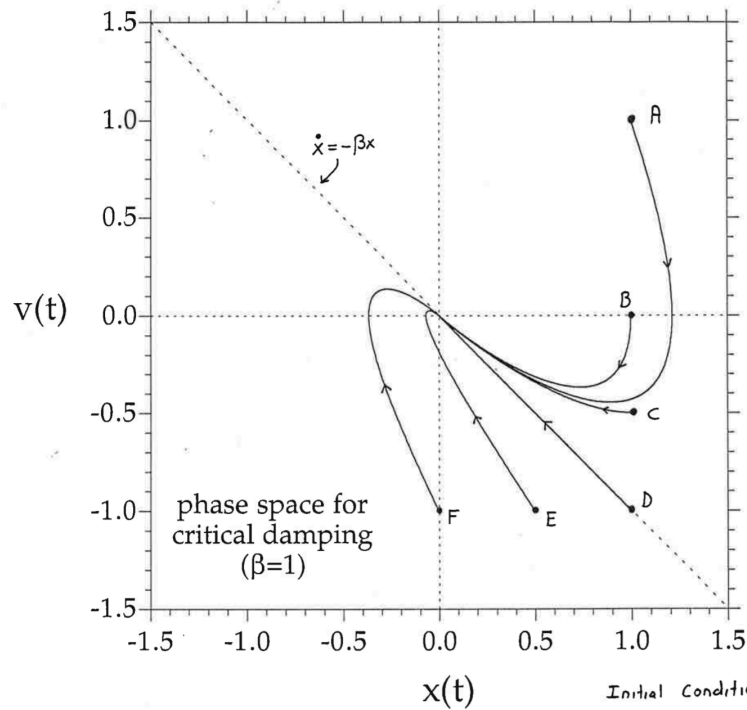
(3-21) For critical damping $\beta^2 = \omega_n^2$

$$x(t) = (a + bt)e^{-\beta t}$$

$$\dot{x}(t) = -\beta(a + bt)e^{-\beta t} + be^{-\beta t}$$

$$= -\beta x(t) + be^{-\beta t} \quad \leftarrow \text{for large } t \quad \dot{x}(t) \rightarrow -\beta x(t)$$

$$a \text{ and } b \text{ given in terms of } x_0 \text{ and } \dot{x}_0 \text{ as: } \begin{cases} a = x_0 \\ b = \beta x_0 + \dot{x}_0 \end{cases}$$



$$x(t) = [x_0 + (\beta x_0 + v_0)t]e^{-\beta t}$$

$$v(t) = -\beta x(t) + (\beta x_0 + v_0)e^{-\beta t}$$

Initial Conditions:

	x_0	v_0	$x=0$
A	1.0	1.0	
B	1.0	0.0	
C	1.0	-0.5	
D	1.0	-1.0	
E	0.5	-1.0	
F	0.0	-1.0	

3-24 Driven Damped Oscillator : Transient solution: $x_c(t) = A_c e^{-\beta t} \cos \omega_1 t$
 Steady State : $x_p(t) = D_p(\omega) \cos(\omega t - \delta_p)$

where $D_p = \frac{A_r}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$

$\delta_p = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$

$x(t) = x_c(t) + x_p(t)$

Here $\beta = 0.2 \text{ s}^{-1}$

$k = 1 \text{ kg/s}^2$

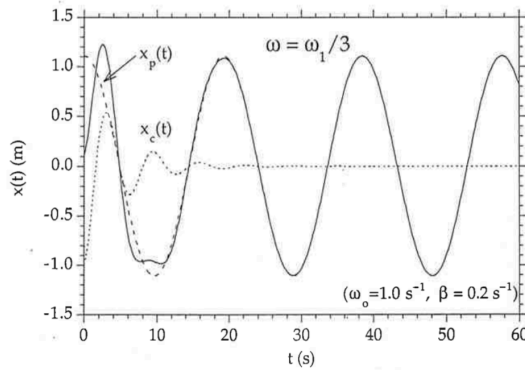
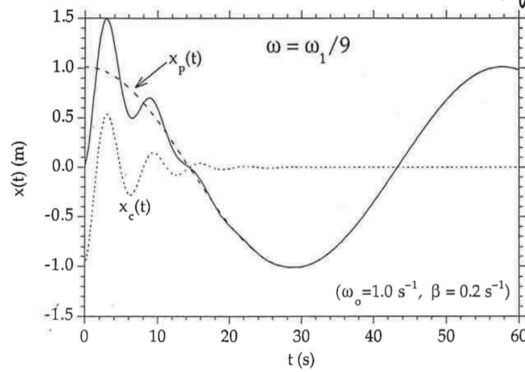
$m = 1 \text{ kg}$

$\omega_0 = 1 \text{ s}^{-1}$

$A_c = -1 \text{ m}$

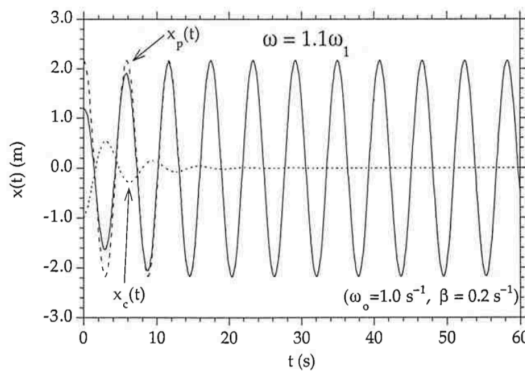
$A_p = 1 \text{ m/s}^2$

$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \approx 0.979 \text{ s}^{-1}$

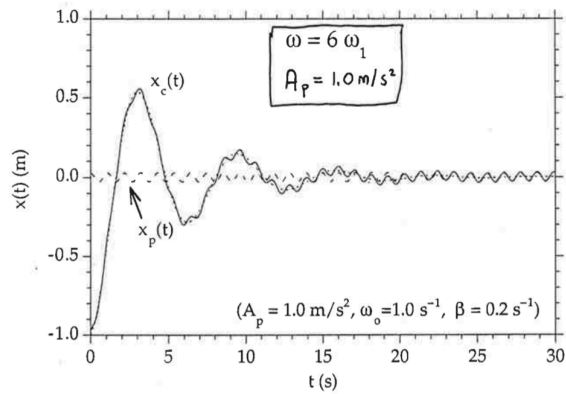
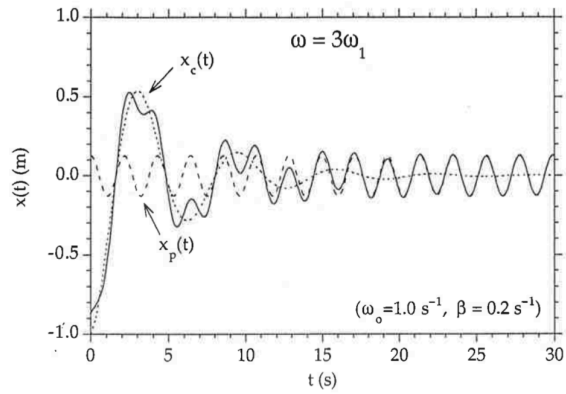


In these plots:

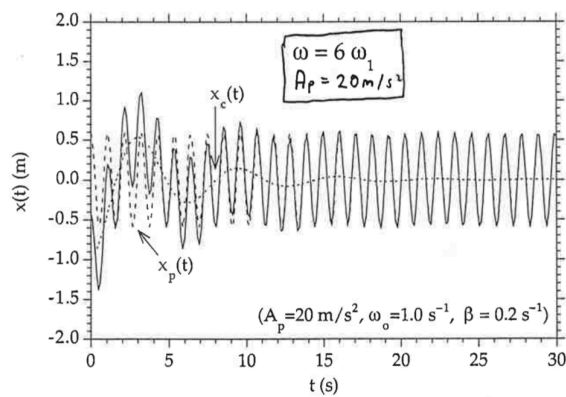
- full solution
- transient
- steady state



Driving near resonance minimizes distortions due to the transient solution.

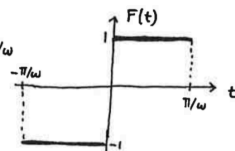


Steady state
amplitude decreases
as driving frequency
increases... the
system can not
respond fast
enough to see
any effect.



More response
if you drive
harder

(3-32) Fourier expansion of $F(t) = \begin{cases} -1 & -\pi/\omega < t < 0 \\ +1 & 0 < t < \pi/\omega \end{cases}$



$$\text{Let } F(t) = \sum_{n=0}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) = \sum_{n=1}^{\infty} B_n \sin n\omega t$$

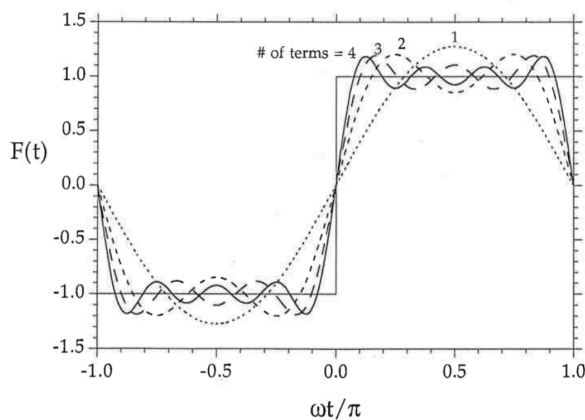
[from symmetry we have $A_n = 0$
 $\forall n$ since $F(t)$ is an odd function]

Use orthogonality condition to determine B_n coefficients: $\int_{-\pi/\omega}^{\pi/\omega} \sin n\omega t \sin m\omega t dt = \frac{\pi}{\omega} \delta_{nm}$

$$\Rightarrow \int_{-\pi/\omega}^{\pi/\omega} F(t) \sin m\omega t dt = \int_{-\pi/\omega}^{\pi/\omega} \left(\sum_{n=1}^{\infty} B_n \sin n\omega t \right) \sin m\omega t dt = \sum_{n=1}^{\infty} B_n \frac{\pi}{\omega} \delta_{nm} = B_m \frac{\pi}{\omega}$$

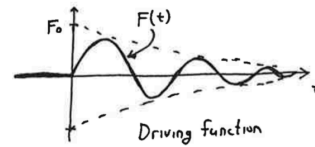
$$\begin{aligned} \therefore B_n &= \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} F(t) \sin n\omega t dt \\ &= \frac{\omega}{\pi} \int_{-\pi/\omega}^0 -\sin n\omega t dt + \frac{\omega}{\pi} \int_0^{\pi/\omega} \sin n\omega t dt \\ &= \frac{\omega}{\pi} \left[\frac{1}{n\omega} \cos n\omega t \right]_{-\pi/\omega}^0 - \frac{\omega}{\pi} \left[\frac{1}{n\omega} \cos n\omega t \right]_0^{\pi/\omega} \\ &= \frac{1}{n\pi} (2 - 2 \cos n\pi) \\ &= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

$$\therefore F(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin n\omega t}{n} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\omega t]}{(2k+1)}$$



Damped Oscillator with

Driving Force: $F(t) = \begin{cases} 0 & t < 0 \\ F_0 e^{-\gamma t} \sin \omega t & t > 0 \end{cases}$

Oscillator described by: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m} F(t)$ let $x(0)=0$ and $\dot{x}(0)=0$,

then the Green's function is: $G(t, t') = \begin{cases} 0 & t < t' \\ \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1(t-t') & t \geq t' \end{cases}$

where $\omega_1^2 = \omega_0^2 - \beta^2$

Oscillator response given by:

$$\begin{aligned}
 x(t) &= \int_{-\infty}^t F(t') G(t, t') dt' \\
 &= \int_0^t [F_0 e^{-\gamma t'} \sin \omega t'] \left[\frac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1(t-t') \right] dt' \quad [\sin(a-b) = \sin a \cos b - \cos a \sin b] \\
 &= \frac{F_0}{m\omega_1} e^{-\beta t} \int_0^t e^{(\beta-\gamma)t'} \sin \omega t' [\sin \omega_1 t \cos \omega_1 t' - \cos \omega_1 t \sin \omega_1 t'] dt' \\
 &= \frac{F_0}{m\omega_1} e^{-\beta t} \left\{ \sin \omega_1 t \int_0^t e^{(\beta-\gamma)t'} \sin \omega t' \cos \omega_1 t' dt' \right. \\
 &\quad \left. - \cos \omega_1 t \int_0^t e^{(\beta-\gamma)t'} \sin \omega t' \sin \omega_1 t' dt' \right\} \\
 &= \frac{F_0}{m\omega_1} e^{-\beta t} \left\{ \sin \omega_1 t \left[\frac{e^{(\beta-\gamma)t'} [(\beta-\gamma) \sin(\omega-\omega_1)t' - (\omega-\omega_1) \cos(\omega-\omega_1)t']}{2[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \right. \right. \\
 &\quad \left. + \frac{e^{(\beta-\gamma)t'} [(\beta-\gamma) \sin(\omega+\omega_1)t' - (\omega+\omega_1) \cos(\omega+\omega_1)t']}{2[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \right]_0^t \\
 &\quad \left. - \cos \omega_1 t \left[\frac{e^{(\beta-\gamma)t'} [(\omega-\omega_1) \sin(\omega-\omega_1)t' + (\beta-\gamma) \cos(\omega-\omega_1)t']}{2[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \right. \right. \\
 &\quad \left. - \frac{e^{(\beta-\gamma)t'} [(\omega+\omega_1) \sin(\omega+\omega_1)t' + (\beta-\gamma) \cos(\omega+\omega_1)t']}{2[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \right]_0^t \right\}
 \end{aligned}$$

3-38 cont.

$$\begin{aligned}
 &= \frac{F_0}{m\omega_1} e^{-\beta t} \left\{ \sin \omega_1 t \frac{e^{(\beta-\gamma)t} [(\beta-\gamma) \sin(\omega-\omega_1)t - (\omega-\omega_1) \cos(\omega-\omega_1)t] + (\omega-\omega_1)}{2[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \right. \\
 &\quad + \sin \omega_1 t \frac{e^{(\beta-\gamma)t} [(\beta-\gamma) \sin(\omega+\omega_1)t - (\omega+\omega_1) \cos(\omega+\omega_1)t] + (\omega+\omega_1)}{2[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \\
 &\quad - \cos \omega_1 t \frac{e^{(\beta-\gamma)t} [(\beta-\gamma) \cos(\omega-\omega_1)t + (\omega-\omega_1) \sin(\omega-\omega_1)t] - (\beta-\gamma)}{2[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \\
 &\quad \left. + \cos \omega_1 t \frac{e^{(\beta-\gamma)t} [(\beta-\gamma) \cos(\omega+\omega_1)t + (\omega+\omega_1) \sin(\omega+\omega_1)t] - (\beta-\gamma)}{2[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \right\} \\
 &= \frac{F_0}{2m\omega_1} \left\{ e^{-\gamma t} \left[\frac{(\beta-\gamma) [\sin \omega_1 t \sin(\omega-\omega_1)t - \cos \omega_1 t \cos(\omega-\omega_1)t]}{[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \right. \right. \\
 &\quad - \frac{(\omega-\omega_1) [\sin \omega_1 t \cos(\omega-\omega_1)t + \cos \omega_1 t \sin(\omega-\omega_1)t]}{[(\beta-\gamma)^2 + (\omega-\omega_1)^2]} \\
 &\quad + \frac{(\beta-\gamma) [\sin \omega_1 t \sin(\omega+\omega_1)t + \cos \omega_1 t \cos(\omega+\omega_1)t]}{[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \\
 &\quad \left. - \frac{(\omega+\omega_1) [\sin \omega_1 t \cos(\omega+\omega_1)t - \cos \omega_1 t \sin(\omega+\omega_1)t]}{[(\beta-\gamma)^2 + (\omega+\omega_1)^2]} \right] \\
 &\quad + e^{-\beta t} \left[\frac{(\omega-\omega_1) \sin \omega_1 t + (\beta-\gamma) \cos \omega_1 t}{(\beta-\gamma)^2 + (\omega-\omega_1)^2} + \frac{(\omega+\omega_1) \sin \omega_1 t - (\beta-\gamma) \cos \omega_1 t}{(\beta-\gamma)^2 + (\omega+\omega_1)^2} \right] \Big\} \\
 &= \frac{F_0}{2m\omega_1} \left\{ e^{-\gamma t} \left[\frac{-(\beta-\gamma) \cos \omega t - (\omega-\omega_1) \sin \omega t}{(\beta-\gamma)^2 + (\omega-\omega_1)^2} + \frac{(\beta-\gamma) \cos \omega t + (\omega+\omega_1) \sin \omega t}{(\beta-\gamma)^2 + (\omega+\omega_1)^2} \right] \right. \\
 &\quad \left. + e^{-\beta t} \left[\frac{(\beta-\gamma) \cos \omega_1 t + (\omega-\omega_1) \sin \omega_1 t}{(\beta-\gamma)^2 + (\omega-\omega_1)^2} - \frac{(\beta-\gamma) \cos \omega_1 t - (\omega+\omega_1) \sin \omega_1 t}{(\beta-\gamma)^2 + (\omega+\omega_1)^2} \right] \right\}
 \end{aligned}$$

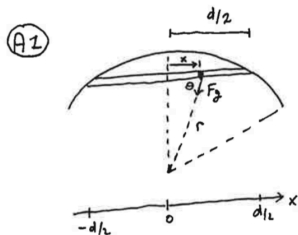
Using the identities: $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
 $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

Now define
$$A(\omega) = \frac{1}{[(\beta-\gamma)^2 + (\omega+\omega_1)^2][(\beta-\gamma)^2 + (\omega-\omega_1)^2]}$$

3-38 cont.

$$\begin{aligned}
 &= \frac{F_0}{2m\omega_1} A(\omega) \left\{ e^{-\gamma t} \left([-(\beta-\gamma) \cos \omega t - (\omega-\omega_1) \sin \omega t] [(\beta-\gamma)^2 + (\omega+\omega_1)^2] \right. \right. \\
 &\quad \left. \left. + [(\beta-\gamma) \cos \omega t + (\omega+\omega_1) \sin \omega t] [(\beta-\gamma)^2 + (\omega-\omega_1)^2] \right) \right. \\
 &\quad \left. + e^{-\beta t} \left([(\beta-\gamma) \cos \omega_1 t + (\omega-\omega_1) \sin \omega_1 t] [(\beta-\gamma)^2 + (\omega+\omega_1)^2] \right. \right. \\
 &\quad \left. \left. - [(\beta-\gamma) \cos \omega_1 t - (\omega+\omega_1) \sin \omega_1 t] [(\beta-\gamma)^2 + (\omega-\omega_1)^2] \right) \right\} \\
 &= \frac{F_0}{2m\omega_1} A(\omega) \left\{ e^{-\gamma t} \left(-(\beta-\gamma)(4\omega\omega_1) \cos \omega t - \omega(4\omega\omega_1) \sin \omega t + 2\omega [(\beta-\gamma)^2 + \omega^2 + \omega_1^2] \sin \omega t \right) \right. \\
 &\quad \left. + e^{-\beta t} \left((\beta-\gamma)(4\omega\omega_1) \cos \omega_1 t - \omega_1(4\omega\omega_1) \sin \omega_1 t + 2\omega [(\beta-\gamma)^2 + \omega^2 + \omega_1^2] \sin \omega_1 t \right) \right\} \\
 &= \frac{F_0}{m} \omega A(\omega) \left\{ e^{-\gamma t} \left(2(\gamma-\beta) \overset{\cos \omega t}{\lambda} - 2\omega \sin \omega t + [(\beta-\gamma)^2 + \omega^2 + \omega_1^2] \frac{\sin \omega t}{\omega} \right) \right. \\
 &\quad \left. + e^{-\beta t} \left(2(\beta-\gamma) \cos \omega_1 t - 2\omega_1 \sin \omega_1 t + [(\beta-\gamma)^2 + \omega^2 + \omega_1^2] \frac{\sin \omega_1 t}{\omega_1} \right) \right\} \\
 &= \frac{F_0}{m} \omega A(\omega) \left\{ e^{-\gamma t} \left(2(\gamma-\beta) \cos \omega t + [(\beta-\gamma)^2 + \omega^2 - \omega_1^2] \frac{\sin \omega t}{\omega} \right) \right. \\
 &\quad \left. + e^{-\beta t} \left(2(\beta-\gamma) \cos \omega_1 t + [(\beta-\gamma)^2 + \omega^2 - \omega_1^2] \frac{\sin \omega_1 t}{\omega_1} \right) \right\}
 \end{aligned}$$

$\nearrow$ strong driving, strong damping
 Note that for $\gamma \ll \beta$ system responds @ driving frequency ω
 $\beta \ll \gamma$ system responds @ natural frequency ω_1
 $\nwarrow$ weak driving, weak damping



$$F_g = G \frac{m}{r^2} \left(\frac{\frac{4\pi}{3} r^3}{\frac{4\pi}{3} R_e^3} \right) m_e = \frac{G m_e}{R_e^3} m r \quad \left[\begin{array}{l} \text{towards} \\ \text{center} \end{array} \right]$$

$$\text{fraction of earth contributing to gravitational force} = \frac{m_g}{R_e} r$$

$$F_x = -F_g \cos \theta = -F_g \frac{x}{r} = -\frac{m_g}{R_e} x \quad \text{linear restoring force!}$$

$$F_x = m \ddot{x}$$

$$\ddot{x} + \frac{g}{R_e} x = 0$$

$$\text{SHM w/ } \omega = \sqrt{\frac{g}{R_e}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R_e}{g}} = 5.08 \times 10^3 \text{ s} \sim 84 \text{ min}$$

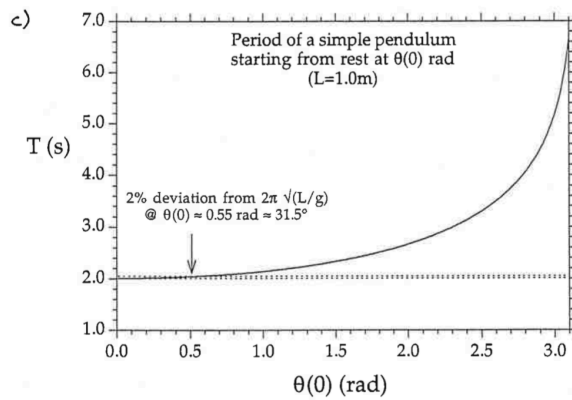
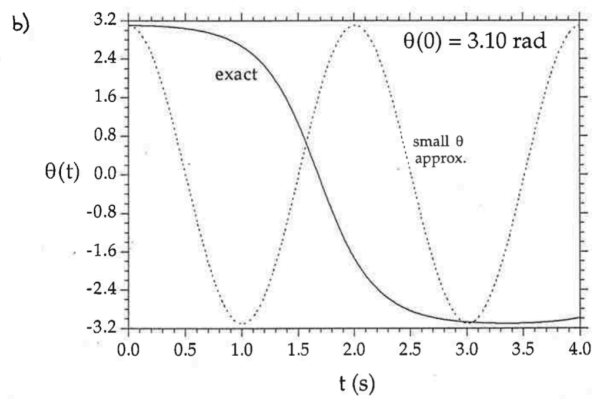
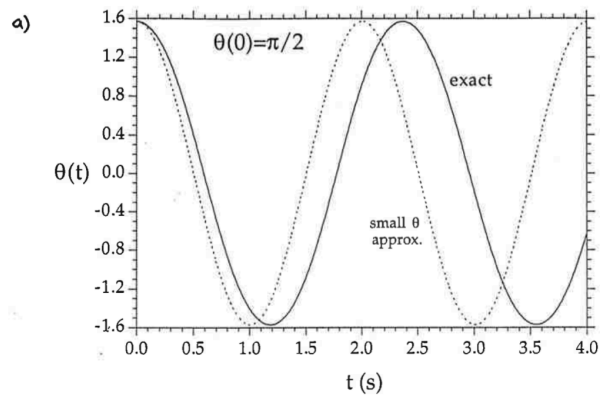
$$\begin{cases} x(t) = \frac{d}{2} \cos \omega t & (\text{start @ } x = d/2) \\ v(t) = -\omega \frac{d}{2} \sin \omega t \end{cases}$$

$$v_{\max} = \frac{1}{2} \omega d = \frac{d}{2} \sqrt{\frac{g}{R_e}} \sim 185 \text{ m/s}$$

$$t_{\text{rip time}} = \frac{1}{2} T \sim 42 \text{ min}$$

$$\begin{aligned} R_e &= 6400 \times 10^3 \text{ m} \\ d &= 300 \times 10^3 \text{ m} \\ g &= 9.8 \text{ m/s}^2 \end{aligned}$$

(C1) Numerical Solution for $\ddot{\theta} + \omega^2 \sin \theta = 0$ $\omega^2 = g/L$, $L = 1.0\text{m}$
 (4th-order Runge-Kutta)
 We start from rest in all cases (i.e., $\dot{\theta}(0) = 0$).



C1 - cont.

d) exact expression for period of a simple pendulum ... first we eliminate time from our D.E.:

$$\left. \begin{aligned} \frac{d\dot{\theta}}{dt} + \omega^2 \sin \theta &= 0 \\ \frac{d\theta}{dt} &= \dot{\theta} \end{aligned} \right\} \Rightarrow \dot{\theta} \frac{d\dot{\theta}}{d\theta} + \omega^2 \sin \theta = 0$$

$$\dot{\theta} d\dot{\theta} = -\omega^2 \sin \theta d\theta \quad \leftarrow \text{Now we integrate this equation twice.}$$

Solve with $\dot{\theta}(0)=0$, $\theta(0)=\theta_0$:

$$\int_0^{\dot{\theta}} \dot{\theta} d\dot{\theta} = -\omega^2 \int_{\theta_0}^{\theta} \sin \theta d\theta \quad \leftarrow 1^{st} \text{ integration}$$

$$\frac{1}{2} \dot{\theta}^2 = \omega^2 [\cos \theta - \cos \theta_0]$$

$$\therefore \dot{\theta} = \pm \sqrt{2\omega^2 (\cos \theta - \cos \theta_0)}$$

$\leftarrow 2^{nd} \text{ integration}$
 $\leftarrow$ we can integrate this over half a period to get an integral expression for T . We let θ vary from $-\theta_0 \rightarrow \theta_0$ for $t: 0 \rightarrow T/2$ (and thus $\dot{\theta} > 0$).

$$\int_{-\theta_0}^{\theta_0} \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}} = \sqrt{2} \omega \int_0^{T/2} dt = \frac{\sqrt{2}}{2} \omega T$$

$$\Rightarrow T = \frac{\sqrt{2}}{\omega} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}}$$

This result is an elliptic integral ... to get it into "standard" form requires a bit of manipulation.
 $\rightarrow$ We will be motivated by the desired result.

First we use the identity $\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1 \Rightarrow \cos \theta - \cos \theta_0 = 2 \cos^2 \frac{\theta}{2} - 2 \cos^2 \frac{\theta_0}{2}$
 $= 2 \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)$

$$\Rightarrow T = \frac{1}{\omega} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{[\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}]^{1/2}} = \frac{1}{\omega K} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{[1 - \frac{1}{K^2} \sin^2 \frac{\theta}{2}]^{1/2}} \quad \text{where } K = \sin \frac{\theta_0}{2}$$

Now we change integration variables: let $\sin u = \frac{1}{K} \sin \frac{\theta}{2}$ for $\theta = \theta_0$ $\sin u = 1$ $u = \pi/2$
 (this is the tricky step!!) for $\theta = -\theta_0$ $\sin u = -1$ $u = -\pi/2$

$$\cos u du = \frac{1}{2K} \cos \frac{\theta}{2} d\theta$$

$$\therefore d\theta = \frac{\cos u du}{\frac{1}{2K} \cos \frac{\theta}{2}} = \frac{\sqrt{1 - \sin^2 u} du}{\frac{1}{2K} \sqrt{1 - \sin^2 \frac{\theta}{2}}} = 2K \frac{\sqrt{1 - \sin^2 u} du}{\sqrt{1 - K^2 \sin^2 u}} \quad \leftarrow \text{integrand is an even function}$$

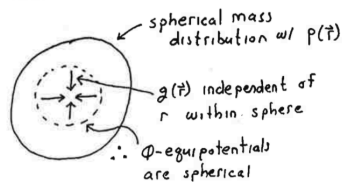
$$\therefore T = \frac{1}{\omega K} \int_{-\pi/2}^{\pi/2} 2K \frac{\sqrt{1 - \sin^2 u}}{\sqrt{1 - K^2 \sin^2 u}} \frac{du}{\sqrt{1 - \sin^2 u}} = \frac{2}{\omega} \int_{-\pi/2}^{\pi/2} \frac{du}{\sqrt{1 - K^2 \sin^2 u}} = \frac{4}{\omega} \int_0^{\pi/2} \frac{du}{\sqrt{1 - K^2 \sin^2 u}}$$

Notice that the integrand in our final expression is periodic w/ period $\pi/2$. The integration range $0 \rightarrow \pi/2$ represents one period... to get the desired result we simply extend the range to cover four periods of the integrand:

$$T = \frac{1}{\omega} \int_0^{2\pi} \frac{du}{\sqrt{1 - K^2 \sin^2 u}} \quad \text{where } \omega = \sqrt{\frac{g}{L}}$$

Assignment 4 - Solutions

(S-2) Mass distribution for constant radial g-field



From the figure we have $\phi(\vec{r}) = \phi(r)$
and $\vec{g} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} = -C\hat{r}$
constant

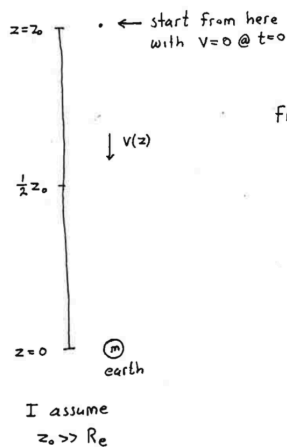
We will use the Poisson equation:

$$\nabla^2\phi = 4\pi G \rho(\vec{r})$$

$$\nabla^2\phi = \vec{\nabla} \cdot \vec{\nabla}\phi = \vec{\nabla} \cdot (-C\hat{r}) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 C) = \frac{2}{r} C$$

$$\therefore \rho(r) = \frac{\nabla^2\phi}{4\pi G} = \frac{C}{2\pi G r} \quad \text{where } C = \frac{\partial\phi}{\partial r}$$

(S-5) Time to fall



• ← start from here
with $v=0$ at $t=0$

We can determine $v(z)$ from $\frac{1}{2}mv^2 = -\Delta U = GmM\left(\frac{1}{z} - \frac{1}{z_0}\right)$

$$\therefore v(z) = -\sqrt{2Gm\left(\frac{1}{z} - \frac{1}{z_0}\right)} = -\sqrt{2Gm \frac{z_0 - z}{z_0 z}} \quad \text{negative sign since } z \text{ is decreasing.}$$

from $v(z) = \frac{dz}{dt}$ we have:

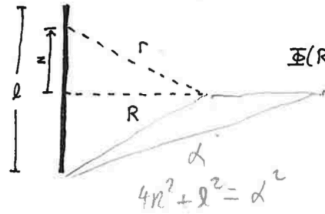
$$\begin{aligned} t(z) &= \int_{z_0}^z \frac{dz}{v(z)} = \frac{-1}{\sqrt{2Gm}} \int_{z_0}^z \frac{z_0}{z_0 - z} dz \quad \text{let } y^2 = z \\ & \quad 2y dy = dz \\ &= \frac{\sqrt{2z_0}}{Gm} \int_{\sqrt{z_0}}^{\sqrt{z}} \frac{y^2}{\sqrt{z_0 - y^2}} dy \quad \leftarrow (\text{see E.7, pg. 605}) \\ &= \frac{\sqrt{2z_0}}{Gm} \left[-\frac{y}{2} \sqrt{z_0 - y^2} + \frac{z_0}{2} \arcsin \frac{y}{\sqrt{z_0}} \right]_{\sqrt{z_0}}^{\sqrt{z}} \\ &= \frac{\sqrt{2z_0}}{Gm} \left\{ \frac{\sqrt{z}}{2} \sqrt{z_0 - z} + \frac{z_0}{2} \arcsin(1) - \frac{z_0}{2} \arcsin \sqrt{\frac{z}{z_0}} \right\} \end{aligned}$$

$$\text{Time to fall to } z_0/2 : t(z_0/2) = \frac{\sqrt{2z_0}}{Gm} \left\{ \frac{1}{2} \sqrt{\frac{z_0}{2}} \sqrt{\frac{z_0}{2}} + \frac{z_0}{2} \frac{\pi}{2} - \frac{z_0}{2} \arcsin \frac{1}{\sqrt{2}} \right\} = \frac{\sqrt{2z_0}}{Gm} \left(\frac{z_0}{4} + \frac{\pi z_0}{8} \right)$$

$$\text{Total fall time : } t(z=0) = \frac{\sqrt{2z_0}}{Gm} \left(\frac{z_0}{2} \frac{\pi}{2} \right) \quad \text{Note: for } z_0 \gg R_e \quad t(0) \approx t(R_e)$$

$$\text{Fraction of time to fall } z_0/2 : \frac{t(z_0/2)}{t(0)} = \frac{\frac{1}{4}(1 + \pi/2)}{\pi/4} = \frac{1}{\pi} + \frac{1}{2} = \frac{11/\pi + 5.5}{11} \approx \frac{3.5 + 5.5}{11} = \underline{\underline{\frac{9}{11}}}$$

(5-7) Gravitational potential for a thin rod (along the $\perp$ -bisector)



$$\begin{aligned}\Phi(R) &= -G \int \frac{\rho(z)}{r} dz & \rho(z) &= \begin{cases} m/l & -\frac{l}{2} \leq z \leq \frac{l}{2} \\ 0 & |z| > \frac{l}{2} \end{cases} \\ &= -G \int_{-l/2}^{l/2} \frac{(m/l)}{\sqrt{R^2 + z^2}} dz & r &= \sqrt{R^2 + z^2} \\ &= -\frac{Gm}{l} \left[\ln(z + \sqrt{R^2 + z^2}) \right]_{-l/2}^{l/2} & & \leftarrow (\text{see E.G., pg 605}) \\ &= -\frac{Gm}{l} \ln \left\{ \frac{l/2 + \sqrt{R^2 + l^2/4}}{-l/2 + \sqrt{R^2 + l^2/4}} \right\} \\ &= -\frac{Gm}{l} \ln \left\{ \frac{\sqrt{4R^2 + l^2} + l}{\sqrt{4R^2 + l^2} - l} \right\}\end{aligned}$$

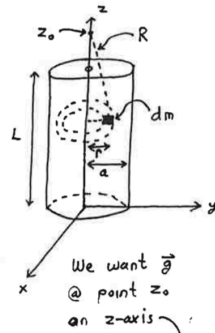
this can also be written as:

$$\Phi(R) = -\frac{2Gm}{l} \ln \left\{ \frac{\sqrt{4R^2 + l^2} + l}{2R} \right\}$$

we can check our result by considering the limit $R \gg l$:

$$\begin{aligned}\Phi(R) &= -\frac{2Gm}{l} \ln \left\{ \frac{2R\sqrt{1 + \frac{l^2}{4R^2}} + l}{2R} \right\} \\ &\approx -\frac{2Gm}{l} \ln \left(\frac{2R+l}{2R} \right) & \text{for } l \ll R \\ &= -\frac{2Gm}{l} \ln \left(1 + \frac{l}{2R} \right) \\ &\approx -\frac{Gm}{R} & \text{since } \ln(1+x) \approx x \text{ for } x \ll 1.\end{aligned}$$

5-8 Gravitational field along symmetry axis outside a finite cylinder



a) Field due to mass element dm :

$$d\vec{g} = -G \frac{dm}{R^2} \hat{e}_R \quad \text{where } \hat{e}_R \text{ points from } dm \text{ to } z_0$$

$$R^2 = (z_0 - z)^2 + r^2$$

$$dm = \rho r dr d\theta dz \quad \text{where } \rho = \frac{M}{\pi a^2 L}$$

On the symmetry axis we will only have a z -component:

$$dg_z = -|d\vec{g}| \cos \theta \quad \text{where } \cos \theta = \frac{z_0 - z}{R}$$

$$\therefore g_{z_0} = \int dg_z = - \int G \rho \frac{r dr d\theta dz}{R^2} \frac{z_0 - z}{R}$$

integrate over volume of cylinder

$$= -2\pi G \rho \int_0^L \int_0^a \frac{(z_0 - z) r}{[(z_0 - z)^2 + r^2]^{3/2}} dr dz$$

let $u = z_0 - z$

$$= 2\pi G \rho \int_{z_0}^{z_0-L} \int_0^a \frac{u r}{(u^2 + r^2)^{3/2}} dr du$$

$du = -dz$

@ $z=0$, $u=z_0$

@ $z=L$, $u=z_0-L$

$$= 2\pi G \rho \int_{z_0}^{z_0-L} \left[\frac{-u}{\sqrt{u^2 + r^2}} \right]_{r=0}^{r=a} du$$

$$= 2\pi G \rho \int_{z_0}^{z_0-L} \left(1 - \frac{u}{\sqrt{u^2 + a^2}} \right) du$$

$$= 2\pi G \rho \left[u - \sqrt{u^2 + a^2} \right]_{z_0}^{z_0-L}$$

$$= 2\pi G \rho \left\{ -L - \sqrt{(z_0-L)^2 + a^2} + \sqrt{z_0^2 + a^2} \right\}$$

$$= -2\pi G \rho \left\{ \sqrt{(z_0-L)^2 + a^2} - \sqrt{z_0^2 + a^2} + L \right\}$$

$$\text{where } \rho = \frac{M}{\pi a^2 L}$$

S-8 cont.

b) Same as part (a), but determine g_{z_0} via potential Φ .

$$\begin{aligned}
 d\Phi &= -G \frac{dm}{R} & dm &= \rho r dr d\theta dz & R &= \sqrt{(z_0 - z)^2 + r^2} \\
 & & &= 2\pi \rho r dr dz \\
 \Phi &= \int_V d\Phi = -2\pi G \rho \int_0^L \int_0^a \frac{r dr dz}{\sqrt{(z_0 - z)^2 + r^2}} \\
 &= -2\pi G \rho \int_0^L \left[\sqrt{(z_0 - z)^2 + r^2} \right]_{r=0}^{r=a} dz \\
 &= -2\pi G \rho \int_0^L \left(\sqrt{(z_0 - z)^2 + a^2} - (z_0 - z) \right) dz & \text{let } u = z_0 - z \quad du = -dz \\
 &= 2\pi G \rho \int_{z_0}^{z_0-L} \left(\sqrt{u^2 + a^2} - u \right) du \\
 &= 2\pi G \rho \left[\frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln \left[u + \sqrt{u^2 + a^2} \right] - \frac{u^2}{2} \right]_{z_0}^{z_0-L} \\
 &= \pi G \rho \left\{ (z_0 - L) \sqrt{(z_0 - L)^2 + a^2} - z_0 \sqrt{z_0^2 + a^2} \right. \\
 &\quad \left. + a^2 \ln \left[\frac{(z_0 - L) + \sqrt{(z_0 - L)^2 + a^2}}{z_0 + \sqrt{z_0^2 + a^2}} \right] - (z_0 - L)^2 + z_0^2 \right\}
 \end{aligned}$$

← Not so pretty!!

$$\vec{g} = -\vec{\nabla} \Phi$$

$$\begin{aligned}
 \therefore g_{z_0} &= -\frac{\partial}{\partial z_0} \Phi \quad (z_0 > L) \\
 &= -\pi G \rho \left\{ \frac{(z_0 - L)^2}{\sqrt{(z_0 - L)^2 + a^2}} + \frac{z_0 - L}{\sqrt{(z_0 - L)^2 + a^2}} - \sqrt{z_0^2 + a^2} - \frac{z_0}{\sqrt{z_0^2 + a^2}} \right. \\
 &\quad \left. + a^2 \frac{1 + \frac{z_0 - L}{\sqrt{(z_0 - L)^2 + a^2}}}{(z_0 - L) + \sqrt{(z_0 - L)^2 + a^2}} - a^2 \frac{1 + \frac{z_0}{\sqrt{z_0^2 + a^2}}}{z_0 + \sqrt{z_0^2 + a^2}} - 2(z_0 - L) + 2z_0 \right\} \\
 &\quad \left[\begin{array}{l} \text{this term} \\ = \frac{a^2}{\sqrt{(z_0 - L)^2 + a^2}} \end{array} \right] \quad \left[\begin{array}{l} \text{this term} \\ = \frac{a^2}{\sqrt{z_0^2 + a^2}} \end{array} \right] \\
 &= -\pi G \rho \left\{ \sqrt{(z_0 - L)^2 + a^2} + \frac{(z_0 - L)^2 + a^2}{\sqrt{(z_0 - L)^2 + a^2}} - \sqrt{z_0^2 + a^2} - \frac{z_0^2 + a^2}{\sqrt{z_0^2 + a^2}} + 2L \right\} \\
 &= -2\pi G \rho \left\{ \sqrt{(z_0 - L)^2 + a^2} - \sqrt{z_0^2 + a^2} + L \right\} \quad \checkmark \text{ as in part (a)} \\
 &\quad \dots \text{ but more work!}
 \end{aligned}$$

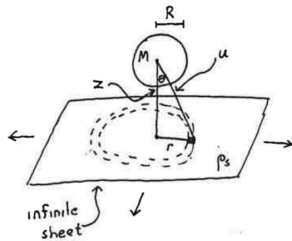
(S-14) Gravitational self-energy of a sphere of radius R , mass M



assemble sphere
shell-by-shell

$$\begin{aligned}
 dU_{\text{shell}} &= \overset{\substack{\text{potential due to sphere of radius } r @ r \\ \text{mass of shell @ radius } r}}{\Phi(r)} dm_{\text{shell}} \\
 &= -G \frac{\rho \left(\frac{4\pi}{3} r^3\right)}{r} \left[\rho (4\pi r^2) dr \right] \quad \text{where } \rho = \frac{M}{\frac{4\pi}{3} R^3} \\
 U_{\text{total}} &= \int_0^R dU_{\text{shell}} = -G \rho^2 \frac{16\pi^2}{3} \int_0^R r^4 dr \\
 &= -G \rho^2 \frac{16\pi^2}{15} R^5 \\
 &= -G \left(\frac{3M}{4\pi R^3}\right)^2 \frac{16\pi^2}{15} R^5 = \underline{\underline{-\frac{3}{5} \frac{GM^2}{R}}}
 \end{aligned}$$

(S-16) Force on sphere due to infinite sheet. Center of sphere @ z above sheet.

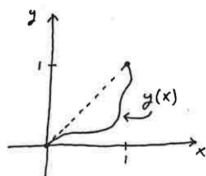


$$\begin{aligned}
 \text{Force due to ring of radius } r: & \quad \text{from symmetry we will only have a } z\text{-component.} \\
 d\vec{F} &= -GM \frac{\rho_s (2\pi r dr)}{u^2} \cos \theta \hat{z} \quad \text{where } \cos \theta = \frac{z}{u} \\
 &= -GM \rho_s \frac{2\pi r z}{(r^2 + z^2)^{3/2}} dr \quad u = \sqrt{r^2 + z^2} \\
 \vec{F} &= \int d\vec{F} = \int_0^\infty -GM \rho_s \frac{2\pi r z}{(r^2 + z^2)^{3/2}} dr \hat{z} \\
 &= -GM \rho_s 2\pi z \int_0^\infty \frac{r dr}{(r^2 + z^2)^{3/2}} \hat{z} \\
 &= -2\pi GM \rho_s z \left[\frac{-1}{(r^2 + z^2)^{1/2}} \right]_{r=0}^{r=\infty} \hat{z} \\
 &= \underline{\underline{-2\pi GM \rho_s \hat{z}}}
 \end{aligned}$$

Note: result is independent of sphere height z !

The infinite sheet produces a uniform gravitational field of $\vec{g} = -2\pi G \rho_s \hat{z}$

⑥-1 Variations on a straight line



$$D_{12} = \int_{(x_1, y_1)}^{(x_2, y_2)} ds \quad \text{where } ds = \sqrt{(dx)^2 + (dy)^2} \\ = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$D_{12} = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$$

Consider trial function: $y(\alpha, x) = x + \alpha \sin[\pi(1-x)]$

note that $y(\alpha, x)$ satisfies the
end point conditions: @ $x=0 \rightarrow y=0$
@ $x=1 \rightarrow y=1$

$$\frac{dy}{dx} = 1 - \alpha \pi \cos[\pi(1-x)]$$

$$D[y(\alpha, x)] = \int_0^1 \sqrt{1 + (1 - \alpha \pi \cos[\pi(1-x)])^2} dx$$

this integral cannot be evaluated
in terms of elementary functions
... we will expand the integrand

$$= \sqrt{2} \int_0^1 \sqrt{1 - \alpha \pi \cos[\pi(1-x)] + \frac{\alpha^2 \pi^2 \cos^2[\pi(1-x)]}{2}} dx \quad \text{for small } \alpha$$

$$= \sqrt{2} \int_0^1 [1 - \alpha \beta(\alpha, x)]^{1/2} dx \quad \text{where } \beta(\alpha, x) = \pi \cos[\pi(1-x)] - \frac{\alpha \pi^2 \cos^2[\pi(1-x)]}{2}$$

$$\approx \sqrt{2} \int_0^1 \left\{ 1 - \frac{1}{2} \alpha \beta(\alpha, x) - \frac{1}{8} \alpha^2 \beta^2(\alpha, x) + \dots \right\} dx \quad [\text{see Eg. D.4, pg 599}]$$

Let's take a derivative w/ respect to α before integrating ...

$$\frac{\partial D(y, \alpha)}{\partial \alpha} = \sqrt{2} \int_0^1 \left\{ -\frac{1}{2} \beta(\alpha, x) - \frac{1}{2} \alpha \frac{\partial \beta}{\partial \alpha} - \frac{1}{4} \alpha \beta^2 - \frac{1}{4} \alpha^2 \beta \frac{\partial \beta}{\partial \alpha} + \dots \right\} dx$$

$$\left. \frac{\partial D}{\partial \alpha} \right|_{\alpha=0} = \sqrt{2} \int_0^1 \left\{ -\frac{1}{2} \beta(0, x) \right\} dx$$

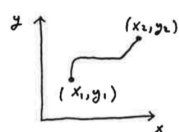
$$= \sqrt{2} \int_0^1 \left\{ -\frac{\pi}{2} \cos[\pi(1-x)] \right\} dx = \sqrt{2} \left[-\frac{1}{2} \sin \pi(1-x) \right]_0^1 = 0 \quad \checkmark$$

If I had integrated ~~first~~ first I would find:

$$D[y(\alpha, x)] \approx \sqrt{2} + \frac{\sqrt{2}}{16} \pi^2 \alpha^2 + O(\alpha^3)$$

all real values of α give $D > \sqrt{2}$!

(6-2) Shortest path - 2D



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (\text{distance measure})$$

$$D_{12} = \int_{(x_1, y_1)}^{(x_2, y_2)} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$$

Problem: Find path $y(x)$ which minimizes D_{12}

Condition for extreme value of D_{12} given by

Euler equation: $\frac{\partial \ell}{\partial y} - \frac{d}{dx} \frac{\partial \ell}{\partial y'} = 0$ where $\ell = \sqrt{1 + (y')^2}$

We have $\frac{\partial \ell}{\partial y} = 0$ and $\frac{\partial \ell}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}$

$$\therefore -\frac{d}{dx} \left[\frac{y'}{\sqrt{1 + (y')^2}} \right] = 0 \Rightarrow \frac{y'}{\sqrt{1 + (y')^2}} = c \quad (= \text{constant})$$

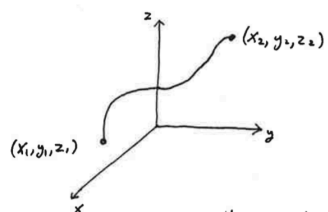
solve for y' : $y' = \frac{c}{\sqrt{1 - c^2}} = m \quad (= \text{constant})$

solve equation $\frac{dy}{dx} = m \Rightarrow \underline{y = mx + b}$ straight line!

constants m and b are given by endpoint values:

$$\left. \begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \end{array} \right\} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}, \quad b = \frac{-y_2 x_1 + y_1 x_2}{x_2 - x_1}$$

(6-3) Shortest path - 3D



$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx$$

$$= f(y, z'; x) dx$$

$$D_{12} = \int ds = \int_{x_1}^{x_2} f(y, z'; x) dx$$

Here we have a functional depending on two functions $y(x)$ and $z(x)$.

$\therefore$ We have 2 Euler equations: $\frac{\partial \ell}{\partial y} - \frac{d}{dx} \frac{\partial \ell}{\partial y'} = 0$

$$\frac{\partial \ell}{\partial z} - \frac{d}{dx} \frac{\partial \ell}{\partial z'} = 0$$

6-3 cont.

As for the 2-D case we have:

$$\frac{\partial \ell}{\partial y} = 0, \quad \frac{\partial \ell}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2 + (z')^2}} \Rightarrow \frac{d}{dx} \left(\frac{\partial \ell}{\partial y'} \right) = 0 \quad \text{or} \quad \frac{\partial \ell}{\partial y'} = C_1$$

$$\frac{\partial \ell}{\partial z} = 0, \quad \frac{\partial \ell}{\partial z'} = \frac{z'}{\sqrt{1 + (y')^2 + (z')^2}} \Rightarrow \frac{d}{dx} \left(\frac{\partial \ell}{\partial z'} \right) = 0 \quad \text{or} \quad \frac{\partial \ell}{\partial z'} = C_2$$

constants

We rewrite the final equations as:

$$\begin{aligned} (y')^2 &= C_1^2 [1 + (y')^2 + (z')^2] \\ (z')^2 &= C_2^2 [1 + (y')^2 + (z')^2] \end{aligned} \quad \Rightarrow \quad \underbrace{C_2^2 (y')^2 = C_1^2 (z')^2}_{\substack{\text{substituting this back} \\ \text{into the } y' \text{ and } z' \text{ equations} \\ \text{leads to explicit expressions} \\ \text{for } y' \text{ and } z'}}$$

For example:

$$\begin{aligned} (y')^2 &= C_1^2 \left[1 + (y')^2 + \frac{C_2^2}{C_1^2} (y')^2 \right] \\ &= C_1^2 + (C_1^2 + C_2^2) (y')^2 \end{aligned} \quad \Rightarrow \quad y' = \frac{C_1}{\sqrt{1 - C_1^2 - C_2^2}} = m_1$$

a constant

$$\text{similarly we find: } z' = \frac{C_2}{\sqrt{1 - C_1^2 - C_2^2}} = m_2$$

$$\begin{aligned} y &= m_1 x + b_1 \\ z &= m_2 x + b_2 \end{aligned}$$

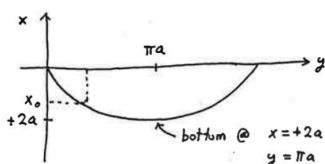
the constants m_1, m_2, b_1, b_2 can be determined from the endpoint values (x_1, y_1, z_1) and (x_2, y_2, z_2) .

$$\begin{aligned} \text{you get } m_1 &= \frac{y_2 - y_1}{x_2 - x_1} & b_1 &= \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \\ m_2 &= \frac{z_2 - z_1}{x_2 - x_1} & b_2 &= \frac{x_2 z_1 - x_1 z_2}{x_2 - x_1} \end{aligned}$$

Using these it is easily shown that:

$$\underline{\underline{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}} \quad \leftarrow \text{standard form for a straight line in 3D.}$$

6-6 Time to travel to the bottom of a cycloid



(axes labelled as in text, Fig. 6.4)

start @ x_0 ... speed @ x given by $v = \sqrt{2g(x-x_0)}$

$$\begin{aligned} \text{cycloid: } x &= a(1 - \cos \theta) & dx &= a \sin \theta d\theta \\ y &= a(\theta - \sin \theta) & dy &= a(1 - \cos \theta) d\theta \end{aligned}$$

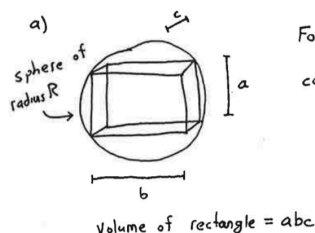
$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 - \cos \theta}{\sin \theta} & \text{note that } 1 - \cos \theta &= x/a \\ & & \cos \theta &= 1 - x/a \\ &= \frac{x/a}{\sqrt{1 - \cos^2 \theta}} &= \frac{x/a}{\sqrt{2 \frac{x}{a} - \frac{x^2}{a^2}}} &= \frac{x}{\sqrt{2ax - x^2}} \end{aligned}$$

$$\begin{aligned} t &= \int_{x=x_0}^{x=2a} \frac{ds}{v} = \int_{x_0}^{2a} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(x-x_0)}} dx \\ &= \int_{x_0}^{2a} \left[\frac{1 + \frac{x^2}{2ax - x^2}}{2g(x-x_0)} \right]^{1/2} dx \\ &= \int_{x_0}^{2a} \left(\frac{2ax}{(2ax - x^2) 2g(x-x_0)} \right)^{1/2} dx \\ &= \sqrt{\frac{a}{g}} \int_{x_0}^{2a} \frac{dx}{\sqrt{(2a-x)(x-x_0)}} \\ &= \sqrt{\frac{a}{g}} \int_{x_0}^{2a} \frac{dx}{\sqrt{-x^2 + (x_0+2a)x - 2ax_0}} \\ &= \sqrt{\frac{a}{g}} \left[-\sin^{-1} \frac{-2x + x_0 + 2a}{\sqrt{(x_0+2a)^2 - 8ax_0}} \right]_{x_0}^{2a} \\ &= -\sqrt{\frac{a}{g}} \sin^{-1} \left(\frac{x_0+2a-2x}{x_0-2a} \right) \Big|_{x=x_0}^{x=2a} \\ &= -\sqrt{\frac{a}{g}} [\sin^{-1}(1) - \sin^{-1}(-1)] \\ &= \underline{\underline{\pi \sqrt{\frac{a}{g}}}} \quad \text{independent of starting point } x_0! \end{aligned}$$

(see integral E.8c, pg 605)

- 6-8) Maximum volume rectangular box inscribed within a sphere and ellipsoid.
[via Lagrange multiplier method]



For sphere centered @ origin: $x^2 + y^2 + z^2 = R^2$
corners of rectangle given by $\pm a/2, \pm b/2, \pm c/2$

$\therefore$ constraint equation is $a^2 + b^2 + c^2 = 4R^2$

or $g = a^2 + b^2 + c^2 - 4R^2 = 0$

Extreme value
on volume V given by:

$$\frac{\partial V}{\partial a} + \lambda \frac{\partial g}{\partial a} = 0 \Rightarrow bc + \lambda 2a = 0$$

$$\frac{\partial V}{\partial b} + \lambda \frac{\partial g}{\partial b} = 0 \Rightarrow ac + \lambda 2b = 0$$

$$\frac{\partial V}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0 \Rightarrow ab + \lambda 2c = 0$$

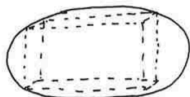
from above 3 equations we have $abc = -2a\lambda = -2b\lambda = -2c\lambda$

$\therefore a^2 = b^2 = c^2$

combining this w/ constraint equation gives:

$$a = b = c = \frac{2R}{\sqrt{3}}$$

b) ellipsoid w/ semiaxes A, B, C



For ellipsoid centered @ origin: $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$

corners of rectangle given by $\pm a/2, \pm b/2, \pm c/2$

Volume of rectangle: $V = abc$

$\therefore$ constraint equation: $g = \frac{a^2}{4A^2} + \frac{b^2}{4B^2} + \frac{c^2}{4C^2} - 1 = 0$

As in part (a) we have:

$$\frac{\partial V}{\partial a} + \lambda \frac{\partial g}{\partial a} = 0 \Rightarrow bc + \lambda \frac{a}{2A^2} = 0$$

$$\frac{\partial V}{\partial b} + \lambda \frac{\partial g}{\partial b} = 0 \Rightarrow ac + \lambda \frac{b}{2B^2} = 0$$

$$\frac{\partial V}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0 \Rightarrow ab + \lambda \frac{c}{2C^2} = 0$$

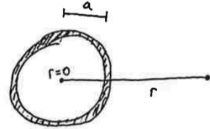
$$\Rightarrow abc = -\lambda \frac{a^2}{2A^2} = -\lambda \frac{b^2}{2B^2} = -\lambda \frac{c^2}{2C^2}$$

$$\Rightarrow \frac{a^2}{A^2} = \frac{b^2}{B^2} = \frac{c^2}{C^2}$$

combining w/ constraint equation:

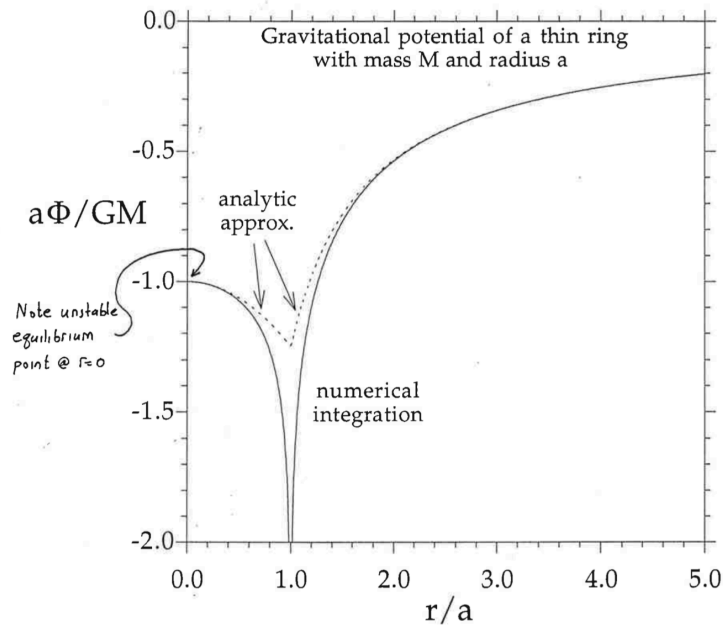
$$a = \frac{2A}{\sqrt{3}} \quad b = \frac{2B}{\sqrt{3}} \quad c = \frac{2C}{\sqrt{3}}$$

(C1) In plane gravitational potential due to circular ring of mass.



$$\Phi(r) = -\frac{GM}{\pi} \int_0^\pi \frac{d\phi}{(a^2 + r^2 - 2ar \cos \phi)^{1/2}}$$

b) @ $r=a$ $\Phi \rightarrow -\infty$ since $\Phi \sim \frac{-GM}{|r-a|}$



a) Analytic approximation for $r > a$:

$$\Phi(r) = -\frac{GM}{\pi r} \int_0^\pi \frac{d\phi}{\left[1 + \left(\frac{a}{r}\right)^2 - 2\frac{a}{r} \cos \phi\right]^{1/2}}$$

note: $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \mathcal{O}(x^3)$ for $x \ll 1$

here: $x = \frac{a}{r} \left(\frac{a}{r} - 2 \cos \phi \right)$

$$\approx -\frac{GM}{\pi r} \int_0^\pi \left\{ 1 - \frac{1}{2} \left(\frac{a}{r} \right)^2 + \frac{a}{r} \cos \phi + \frac{3}{2} \left(\frac{a}{r} \right)^2 \cos^2 \phi \right\} d\phi$$

$x^2 = \left(\frac{a}{r} \right)^2 \left[\left(\frac{a}{r} \right)^2 - 4 \frac{a}{r} \cos \phi + 4 \cos^2 \phi \right]$

where we have only kept terms to order $\left(\frac{a}{r}\right)^2$

$$= -\frac{GM}{\pi r} \int_0^\pi \left[1 + \frac{a}{r} \cos \phi - \frac{1}{2} \left(\frac{a}{r} \right)^2 (1 - 3 \cos^2 \phi) \right] d\phi$$

note $\cos^2 \phi = \frac{1}{2} + \frac{1}{2} \cos 2\phi$

$$= -\frac{GM}{\pi r} \int_0^\pi \left[1 + \frac{a}{r} \cos \phi - \frac{1}{2} \left(\frac{a}{r} \right)^2 \left(-\frac{1}{2} - \frac{3}{2} \cos 2\phi \right) \right] d\phi = -\frac{GM}{\pi r} \left[1 + \frac{1}{4} \left(\frac{a}{r} \right)^2 \right] \pi$$

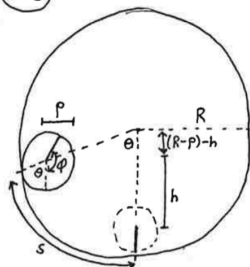
* $\Phi(r) = -\frac{GM}{r} \left\{ 1 + \frac{1}{4} \frac{a^2}{r^2} + \mathcal{O}\left(\left(\frac{a}{r}\right)^3\right) \right\}$

[Integrals over $\cos \phi$, $\cos 2\phi$ Vanish]

Assignment 5 - Solutions

7-3

Sphere of radius p rolling within a cylinder of radius R .



equation of constraint
from rolling condition:

$$s = R\theta = p(\theta + \phi)$$

$$\text{or} \quad \phi = \left(\frac{R}{p} - 1\right)\theta$$

θ = angular position of CM of sphere

ϕ = rotation of sphere about its CM

Must be very careful here with the
constraint relation which connects these two angles!

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \omega^2$$

$$v_{cm} = (R-p) \dot{\theta}$$

$$= \frac{1}{2} m (R-p)^2 \dot{\theta}^2 + \frac{1}{5} m p^2 \dot{\phi}^2$$

$$\omega = \dot{\phi}$$

$$I = \frac{2}{5} m p^2$$

$$= \frac{1}{2} m (R-p)^2 \dot{\theta}^2 + \frac{1}{5} m (R-p)^2 \dot{\theta}^2$$

$$= \frac{7}{10} m (R-p)^2 \dot{\theta}^2$$

$$U = mgh \quad \text{where } (R-p) - h = (R-p) \cos \theta$$

$$= mg(R-p)[1 - \cos \theta]$$

$$\begin{array}{c} R-p \\ \swarrow \quad \searrow \\ \theta \end{array} \quad (R-p) - h$$

$$L = T - U = \frac{7}{10} m (R-p)^2 \dot{\theta}^2 - mg(R-p)[1 - \cos \theta]$$

$$\frac{\partial L}{\partial \theta} = -mg(R-p) \sin \theta \quad \frac{\partial L}{\partial \dot{\theta}} = \frac{7}{5} m (R-p)^2 \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{7}{5} m (R-p)^2 \ddot{\theta} + mg(R-p) \sin \theta = 0$$

$$\ddot{\theta} + \frac{5g}{7(R-p)} \sin \theta = 0$$

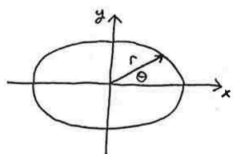
For small oscillations: $\sin \theta \approx \theta$

$$\Rightarrow \ddot{\theta} + \omega^2 \theta = 0$$

SHM w/

$$\omega = \sqrt{\frac{5g}{7(R-p)}}$$

7-4 Particle in a 2D central potential



Use polar coordinates $\{r, \theta\}$ since $U = U(r)$

$$\vec{f}(r) = -A r^{\alpha-1} \quad [\text{central force ... i.e., } r\text{-dependent only}]$$

$$\text{In general } \vec{f} = -\vec{\nabla} U \Rightarrow -\frac{\partial}{\partial r} U(r) \hat{r}$$

$$U(r) = -\int \vec{f}(r) \cdot d\vec{r} = \frac{A}{\alpha} r^{\alpha} \quad \text{where } U(0) = 0.$$

$$\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \Rightarrow v^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{A}{\alpha} r^{\alpha}$$

$$\text{r-motion: } \left. \begin{aligned} \frac{\partial L}{\partial r} &= m r \dot{\theta}^2 - A r^{\alpha-1} \\ \frac{\partial L}{\partial \dot{r}} &= m \dot{r} \end{aligned} \right\} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \underline{m \ddot{r} - m r \dot{\theta}^2 + A r^{\alpha-1} = 0}$$

$$\text{\theta-motion: } \left. \begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= m r^2 \dot{\theta} \end{aligned} \right\} \frac{d}{dt} (m r^2 \dot{\theta}) = 0 \Rightarrow m r^2 \dot{\theta} = l \equiv \text{angular momentum about origin}$$

where $l = \text{constant}$
 $\rightarrow$ angular momentum conserved!

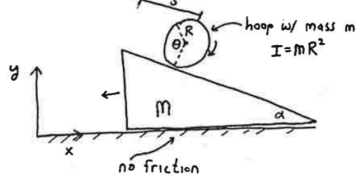
$$[l = |\vec{r} \times \vec{p}| = r m v_{\theta} = r m (r \dot{\theta}) = m r^2 \dot{\theta}]$$

$$\text{Total energy: } E = T + U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{A}{\alpha} r^{\alpha}$$

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} m (2 \dot{r} \ddot{r} + 2 r \dot{r} \dot{\theta}^2 + 2 r^2 \dot{\theta} \ddot{\theta}) + A r^{\alpha-1} \dot{r} \\ &= \dot{r} (m \ddot{r} - m r \dot{\theta}^2 + A r^{\alpha-1}) + 2 \dot{r} (m r \dot{\theta}^2) + m r^2 \dot{\theta} \ddot{\theta} \\ &= \dot{r} (m \ddot{r} - m r \dot{\theta}^2 + A r^{\alpha-1}) + \frac{d}{dt} (m r^2 \dot{\theta}) \\ &\quad \begin{matrix} \nwarrow \text{vanishes from} \\ \text{r-equation} \end{matrix} \quad \begin{matrix} \nwarrow \text{vanishes from} \\ \text{\theta-equation} \end{matrix} \\ &= 0 \end{aligned}$$

$\therefore$ total energy conserved

7-6 Hoop rolling down sliding incline



generalized coordinates:

$x \equiv$ position of left end of incline

$s \equiv$ linear distance rolled by hoop down incline

constraint: $s = R\theta$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m v_{cm}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$U = mgy_0 = mg(y_0 - s \sin \alpha)$$

$\uparrow$ initial height of hoop CM

$$v_{cm}^2 = v_x^2 + v_y^2 \quad \leftarrow \text{speed of hoop CM in fixed xy-frame}$$

$$= (\dot{x} + \dot{s} \cos \alpha)^2 + (\dot{s} \sin \alpha)^2$$

$$= \dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \alpha$$

$$L = T - U$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \alpha) + \frac{1}{2}mR^2\left(\frac{\dot{s}}{R}\right)^2 - mg(y_0 - s \sin \alpha)$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + m\dot{s}^2 + m\dot{x}\dot{s} \cos \alpha - mg(y_0 - s \sin \alpha)$$

$$\left. \begin{aligned} x\text{-motion: } \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial \dot{x}} &= (M+m)\dot{x} + m\dot{s} \cos \alpha \end{aligned} \right\} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \Rightarrow \underline{(M+m)\dot{x} + m\dot{s} \cos \alpha = \text{constant}}$$

conservation of linear momentum

$$\left. \begin{aligned} s\text{-motion: } \frac{\partial L}{\partial s} &= mg \sin \alpha \\ \frac{\partial L}{\partial \dot{s}} &= 2m\dot{s} + m\dot{x} \cos \alpha \end{aligned} \right\} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \frac{\partial L}{\partial s} = 0$$

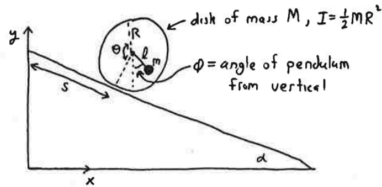
$$\Rightarrow 2m\ddot{s} + m\ddot{x} \cos \alpha - mg \sin \alpha = 0$$

or

$$\underline{2\ddot{s} + \ddot{x} \cos \alpha - g \sin \alpha = 0}$$

These differential equations are easily solved, but we'll leave it here.

7-9 Pendulum on axle of wheel rolling downhill



generalized coordinates:

$s \equiv$ linear distance rolled by disk

$\phi \equiv$ angle of pendulum

$\theta \equiv$ angle through which disk has rolled: $s = R\theta$

Position of disk CM: $x_{cm} = x_0 + s \cos \alpha$
 $y_{cm} = y_0 - s \sin \alpha$

Position of pendulum bob: $x_{bob} = x_{cm} + l \sin \phi$
 $y_{bob} = y_{cm} - l \cos \phi$

$U_{disk} = Mgy_{cm} = Mg(y_0 - s \sin \alpha)$

$U_{bob} = mgy_{bob} = mg(y_0 - s \sin \alpha - l \cos \phi)$

$T = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mv_{bob}^2$
 $= \frac{1}{2}M\dot{s}^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{\dot{s}}{R})^2 + \frac{1}{2}m(\dot{s}^2 + l^2\dot{\phi}^2 + 2\dot{s}l\dot{\phi}[\cos \alpha \cos \phi - \sin \alpha \sin \phi])$
 $= \frac{3}{4}M\dot{s}^2 + \frac{1}{2}m(\dot{s}^2 + l^2\dot{\phi}^2 + 2\dot{s}l\dot{\phi} \cos(\alpha + \phi))$

$v_{cm}^2 = \dot{x}_{cm}^2 + \dot{y}_{cm}^2$
 $= (\dot{s} \cos \alpha)^2 + (-\dot{s} \sin \alpha)^2 = \dot{s}^2$

$v_{bob}^2 = \dot{x}_{bob}^2 + \dot{y}_{bob}^2$
 $= (\dot{x}_{cm} + l\dot{\phi} \cos \phi)^2 + (\dot{y}_{cm} + l\dot{\phi} \sin \phi)^2$
 $= (\dot{s} \cos \alpha + l\dot{\phi} \cos \phi)^2 + (-\dot{s} \sin \alpha + l\dot{\phi} \sin \phi)^2$
 $= \dot{s}^2 + l^2\dot{\phi}^2 + 2\dot{s}l\dot{\phi} \cos \alpha \cos \phi - 2\dot{s}l\dot{\phi} \sin \alpha \sin \phi$

$L = T - U = \left(\frac{3}{4}M + \frac{1}{2}m\right)\dot{s}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + m\dot{s}l\dot{\phi} \cos(\alpha + \phi) - (M+m)g(y_0 - s \sin \alpha) + mgl \cos \phi$

s-motion: $\frac{\partial L}{\partial s} = (M+m)g \sin \alpha$

$\frac{\partial L}{\partial \dot{s}} = \left(\frac{3}{2}M + m\right)\dot{s} + ml\dot{\phi} \cos(\alpha + \phi)$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \left(\frac{3}{2}M + m\right)\ddot{s} + ml\ddot{\phi} \cos(\alpha + \phi) - ml(\dot{\phi})^2 \sin(\alpha + \phi)$
 $\Rightarrow \left(\frac{3}{2}M + m\right)\ddot{s} + ml\ddot{\phi} \cos(\alpha + \phi) - ml(\dot{\phi})^2 \sin(\alpha + \phi) - (M+m)g \sin \alpha = 0$

Note: if $\dot{\phi} = 0, \ddot{\phi} = 0 \Rightarrow \ddot{s} - \frac{(M+m)g}{(\frac{3}{2}M + m)} \sin \alpha = 0 \leftarrow$ constant acceleration down incline if pendulum not swinging $\checkmark$

ϕ -motion: $\frac{\partial L}{\partial \phi} = -m\dot{s}l\dot{\phi} \sin(\alpha + \phi) - mgl \sin \phi$

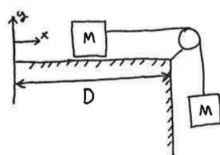
$\frac{\partial L}{\partial \dot{\phi}} = ml^2\ddot{\phi} + m\dot{s}l \cos(\alpha + \phi)$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi} + m\ddot{s}l \cos(\alpha + \phi) - m\dot{s}l\dot{\phi} \sin(\alpha + \phi)$

$\Rightarrow ml^2\ddot{\phi} + m\dot{s}l \cos(\alpha + \phi) + mgl \sin \phi = 0$

or
 $\ddot{\phi} + \frac{1}{l}\ddot{s} \cos(\alpha + \phi) + \frac{g}{l} \sin \phi = 0$

Note: if $\ddot{s} = 0$ we have the equation for a simple pendulum $\checkmark$

7-10 Sliding blocks



$l = \text{string length} \quad \therefore (D-x) - y = 0$
 $\dot{x} = -\dot{y}$

a) Ignoring string mass: $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 = M\dot{y}^2$
 $U = mgy$

$L = T - U = M\dot{y}^2 - Mgy$

$\frac{\partial L}{\partial y} = -Mg \quad \frac{\partial L}{\partial \dot{y}} = 2M\dot{y} \Rightarrow 2M\ddot{y} + Mg = 0$

$\ddot{y} = -g/2$

$\therefore y(t) = -\frac{g}{4}t^2 + C_1t + C_2$ where $C_1 = \dot{y}(0), C_2 = y(0)$

b) String w/ mass m

$T = M\dot{y}^2 + \frac{1}{2}m\dot{y}^2$
 kinetic energy of string

$U = Mgy + g \int_{-y}^0 y dm = Mgy + g \int_{-y}^0 y \left(\frac{m}{l}\right) dy$
 $= Mgy + \frac{gm}{2l} y^2$
 $= Mgy - \frac{gm}{2l} y^2$

$L = T - U = M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 - Mgy + \frac{gm}{2l} y^2$

$\frac{\partial L}{\partial y} = -Mg + \frac{gm}{l} y$
 $\frac{\partial L}{\partial \dot{y}} = 2(M + \frac{1}{2}m)\dot{y}$

$(2M+m)\ddot{y} - \frac{mg}{l} y + Mg = 0$

or $\ddot{y} - \frac{mg}{(2M+m)l} y = -Mg/(2M+m)$

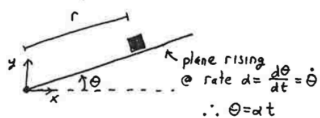
$y(t) = Ae^{\tau t} + Be^{-\tau t} + y_p$ where $y_p = \left(\frac{+Mg}{2M+m}\right) \frac{(2M+m)l}{mg} = +\frac{M}{m}l$
 $\dot{y}(t) = \tau Ae^{\tau t} - \tau Be^{-\tau t}$ $\nwarrow$ particular solution ($y = \text{constant}$)

let $\dot{y}(0) = 0 \Rightarrow A = B$

let $y(0) = 0 \Rightarrow 2A + y_p = 0 \Rightarrow A = -\frac{Ml}{2m}$

$\therefore y(t) = \frac{-Ml}{m} \cosh(\tau t) + \frac{M}{m}l$ where $\tau = \sqrt{\frac{mg}{(2M+m)l}}$
 $= \frac{M}{m}l \{1 - \cosh(\tau t)\}$

7-12 Sliding down a tilting plane



Motion has 1-degree of freedom: generalized coordinate r

let $r(0) = r_0$

Position of particle: $x = r \cos \theta$
 $y = r \sin \theta$

$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta = \dot{r} \cos \theta - r \alpha \sin \theta$
 $\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta = \dot{r} \sin \theta + r \alpha \cos \theta$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$
 $= \frac{1}{2} m [\dot{r}^2 + r^2 \alpha^2 - 2 r \dot{r} \alpha \cos \theta \sin \theta + 2 r \dot{r} \alpha \sin \theta \cos \theta]$
 $= \frac{1}{2} m (\dot{r}^2 + r^2 \alpha^2)$

$U = mgy = mgr \sin \theta = mgr \sin \alpha t$

$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \alpha^2) - mgr \sin \alpha t$

$\frac{\partial L}{\partial r} = m r \alpha^2 - m g \sin \alpha t$
 $\frac{\partial L}{\partial \dot{r}} = m \dot{r}$ Euler-Lagrange gives:
 $m \ddot{r} - m \alpha^2 r + m g \sin \alpha t = 0$
 $\ddot{r} - \alpha^2 r = -g \sin \alpha t$

General solution: $r(t) = A e^{\alpha t} + B e^{-\alpha t} + r_p$ let $r_p = C \sin \alpha t$ ← try this for a particular solution.
 $\ddot{r}_p = -\alpha^2 C \sin \alpha t$
 $\therefore -\alpha^2 C - \alpha^2 C = -g$
 $C = g/2\alpha^2$

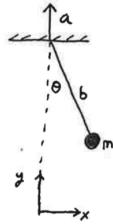
Initial $r(0) = A + B = r_0$

Conditions $\dot{r}(0) = \alpha A - \alpha B + \frac{g \alpha}{2 \alpha^2} = 0$

$\therefore \begin{cases} A + B = r_0 \\ A - B = -g/2\alpha^2 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} (r_0 - g/2\alpha^2) \\ B = r_0 - A = \frac{1}{2} (r_0 + g/2\alpha^2) \end{cases}$

$r(t) = \frac{1}{2} (r_0 - \frac{g}{2\alpha^2}) e^{\alpha t} + \frac{1}{2} (r_0 + \frac{g}{2\alpha^2}) e^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$
 $= \frac{1}{2} r_0 (e^{\alpha t} + e^{-\alpha t}) - \frac{1}{2} \frac{g}{2\alpha^2} (e^{\alpha t} - e^{-\alpha t}) + \frac{g}{2\alpha^2} \sin \alpha t$
 $= r_0 \cosh \alpha t - \frac{g}{2\alpha^2} [\sinh \alpha t - \sin \alpha t]$

(7-14) Vertically accelerated simple pendulum



Position of bob: $y = y_0 + \frac{1}{2}at^2 - b \cos \theta$

$$x = b \sin \theta$$

$$\dot{y} = at + b \dot{\theta} \sin \theta$$

$$\dot{x} = b \dot{\theta} \cos \theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(b^2 \dot{\theta}^2 + a^2 t^2 + 2atb \dot{\theta} \sin \theta)$$

$$U = mgy = mg(y_0 + \frac{1}{2}at^2 - b \cos \theta)$$

$$L = T - U = \frac{1}{2}m[b^2 \dot{\theta}^2 + a^2 t^2 + 2atb \dot{\theta} \sin \theta] - mg[y_0 + \frac{1}{2}at^2 - b \cos \theta]$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m(2abt \dot{\theta} \cos \theta) - mgb \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mb^2 \dot{\theta} + mabt \sin \theta \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mb^2 \ddot{\theta} + mab \sin \theta + mabt \dot{\theta} \cos \theta$$

Euler-Lagrange

gives: $\Rightarrow mb^2 \ddot{\theta} + mab \sin \theta + mgb \sin \theta = 0$

$$\ddot{\theta} + \frac{a}{b} \sin \theta + \frac{g}{b} \sin \theta = 0$$

$$\ddot{\theta} + \left(\frac{a+g}{b}\right) \sin \theta = 0$$

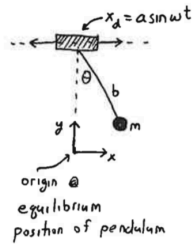
For small oscillations: $\sin \theta \approx \theta$

$$\ddot{\theta} + \omega^2 \theta = 0 \Rightarrow \text{SHM w/}$$

$$\omega = \sqrt{\frac{a+g}{b}}$$

(as expected)

7-16 Driven simple pendulum



Position of bob:

$$y = b - b \cos \theta$$

$$x = x_d + b \sin \theta$$

$$= a \sin \omega t + b \sin \theta$$

$$\dot{y} = b \dot{\theta} \sin \theta$$

$$\dot{x} = a \omega \cos \omega t + b \dot{\theta} \cos \theta$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (b^2 \dot{\theta}^2 + a^2 \omega^2 \cos^2 \omega t + 2ab\omega \dot{\theta} \cos \omega t \cos \theta)$$

$$U = mgy = mg(b - b \cos \theta)$$

$$\mathcal{L} = T - U = \frac{1}{2} m [b^2 \dot{\theta}^2 + 2ab\omega \dot{\theta} \cos \theta \cos \omega t + a^2 \omega^2 \cos^2 \omega t] - mgb(1 - \cos \theta)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mab\omega \dot{\theta} \sin \theta \cos \omega t - mgb \sin \theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mb^2 \dot{\theta} + mab\omega \cos \theta \cos \omega t \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = mb^2 \ddot{\theta} - mab\omega \dot{\theta} \sin \theta \cos \omega t - mab\omega^2 \cos \theta \sin \omega t$$

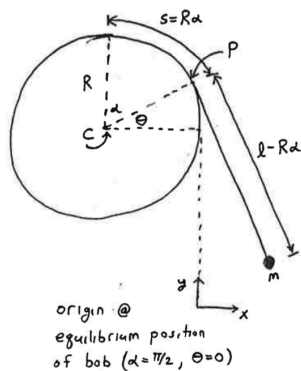
Euler-Lagrange gives: $\Rightarrow mb^2 \ddot{\theta} - mab\omega^2 \cos \theta \sin \omega t + mgb \sin \theta = 0$

or

$$\ddot{\theta} + \frac{g}{b} \sin \theta - \frac{a\omega^2}{b} \sin \omega t \cos \theta = 0$$

for small θ : $\ddot{\theta} + \frac{g}{b} \theta = \frac{a\omega^2}{b} \sin \omega t \leftarrow \text{driven simple pendulum (which we can solve).}$

7-18 Pendulum on a disk



l = total string length

Note that θ is the angle the bob is displaced from the vertical and $\alpha + \theta = \pi/2$

In our x - y coordinate system points C and P are located by:

$$\begin{aligned} x_C &= -R & x_P &= x_C + R \sin \alpha \\ y_C &= l - R \pi/2 & y_P &= y_C + R \cos \alpha \end{aligned}$$

The bob is located @:

$$\begin{aligned} x_B &= x_P + (l - R\alpha) \sin \theta = -R + R \sin \alpha + (l - R\alpha) \sin \theta \\ y_B &= y_P - (l - R\alpha) \cos \theta = l - \frac{\pi}{2} R + R \cos \alpha - (l - R\alpha) \cos \theta \end{aligned}$$

To study small oscillations we will work in terms of the angle θ :

$$x_B = -R + R \cos \theta + l \sin \theta - R \left(\frac{\pi}{2} - \theta \right) \sin \theta \quad (\text{Note: } \sin \alpha = \cos \theta, \cos \alpha = \sin \theta)$$

$$\begin{aligned} \dot{x}_B &= -R \sin \theta \dot{\theta} + l \cos \theta \dot{\theta} + R \dot{\theta} \sin \theta - R \left(\frac{\pi}{2} - \theta \right) \cos \theta \dot{\theta} \\ &= [l - R \left(\frac{\pi}{2} - \theta \right)] \cos \theta \dot{\theta} \end{aligned}$$

$$\begin{aligned} y_B &= l - \frac{\pi}{2} R + R \sin \theta - l \cos \theta + R \left(\frac{\pi}{2} - \theta \right) \cos \theta \\ \dot{y}_B &= R \cos \theta \dot{\theta} + l \sin \theta \dot{\theta} - R \dot{\theta} \cos \theta - R \left(\frac{\pi}{2} - \theta \right) \sin \theta \dot{\theta} \\ &= [l - R \left(\frac{\pi}{2} - \theta \right)] \sin \theta \dot{\theta} \end{aligned}$$

$$T = \frac{1}{2} m (\dot{x}_B^2 + \dot{y}_B^2) = \frac{1}{2} m [l - R \left(\frac{\pi}{2} - \theta \right)]^2 \dot{\theta}^2$$

$$U = mgy_B = mg \left\{ l - \frac{\pi}{2} R + R \sin \theta - l \cos \theta + R \left(\frac{\pi}{2} - \theta \right) \cos \theta \right\}$$

$$L = T - U$$

$$= \frac{1}{2} m \left\{ [l - R \left(\frac{\pi}{2} - \theta \right)]^2 \dot{\theta}^2 \right\} - mg \left\{ l - \frac{\pi}{2} R + R \sin \theta - [l - R \left(\frac{\pi}{2} - \theta \right)] \cos \theta \right\}$$

7-18 cont.

$$\frac{\partial L}{\partial \theta} = mR \left[l - R \left(\frac{\pi}{2} - \theta \right) \right] \dot{\theta}^2 - mgR \cos \theta + mgR \cos \theta - mg \left[l - R \left(\frac{\pi}{2} - \theta \right) \right] \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m \left[l - R \left(\frac{\pi}{2} - \theta \right) \right]^2 \dot{\theta} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 2mR \left[l - R \left(\frac{\pi}{2} - \theta \right) \right] \dot{\theta}^2 + m \left[l - R \left(\frac{\pi}{2} - \theta \right) \right]^2 \ddot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow m \left[l - R \left(\frac{\pi}{2} - \theta \right) \right]^2 \ddot{\theta} + mg \left[l - R \left(\frac{\pi}{2} - \theta \right) \right] \sin \theta + mR \left[l - R \left(\frac{\pi}{2} - \theta \right) \right] \dot{\theta}^2 = 0$$

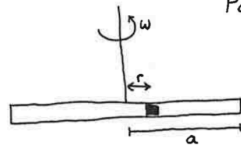
or

$$\ddot{\theta} + \frac{g}{l - R \left(\frac{\pi}{2} - \theta \right)} \sin \theta + \frac{R}{l - R \left(\frac{\pi}{2} - \theta \right)} \dot{\theta}^2 = 0$$

We have already identified the equilibrium point as $\theta = 0$. For small oscillations we have $\theta \ll 1$, $\sin \theta \approx \theta$ and ignore terms of $\mathcal{O}(\theta^2)$.

$$\Rightarrow \ddot{\theta} + \omega^2 \theta = 0 \quad \text{with} \quad \omega = \sqrt{\frac{g}{l - \frac{\pi}{2}R}} \quad \text{motion is simple harmonic about } \theta = 0$$

(A1)



Particle in rotating tube ... in-plane motion, no friction.

This is a 1D problem. We use the generalized coordinate r .Given initial condition: $r(0) = b$
 $\dot{r}(0) = 0$ a) In polar coordinates: $\vec{V} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$, here $\dot{\theta} = \omega$ is fixed

$$T = \frac{1}{2} m |\vec{V}|^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$

$$U = 0$$

$$L = T - U$$

$$\left. \begin{array}{l} \frac{\partial L}{\partial r} = m r \omega^2 \\ \frac{\partial L}{\partial \dot{r}} = m \dot{r} \end{array} \right\} \text{Euler-Lagrange} \Rightarrow \underline{\underline{\ddot{r} - \omega^2 r = 0}}$$

b) General solution to equation of motion:

$$r(t) = A e^{\omega t} + B e^{-\omega t}$$

we use initial conditions to get A and B:

$$r(0) = A + B = b$$

$$\dot{r}(0) = \omega A - \omega B = 0$$

$$\therefore A = B = b/2$$

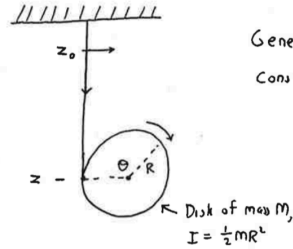
$$\therefore r(t) = \frac{b}{2} (e^{\omega t} + e^{-\omega t})$$

$$= \underline{\underline{b \cosh \omega t}}$$

c) Mass exits tube when $r(T) = a$ or $a = b \cosh \omega T$

$$\therefore \underline{\underline{T = \frac{1}{\omega} \cosh^{-1} \left(\frac{a}{b} \right)}}$$

(A2) Constraint forces for falling yo-yo



Generalized coordinates z and θ .

Constraint equation: $z = R\theta$ or $\dot{z} - R\dot{\theta} = 0$

$$T = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$U = -mgz$$

$$L = T - U = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 + mgz$$

a)

$$z\text{-equation: } \frac{\partial L}{\partial z} = mg$$

$$\frac{\partial L}{\partial \dot{z}} = M\dot{z}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 1$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) + \lambda \frac{\partial \mathcal{L}}{\partial z} = 0$$

$$\Rightarrow mg - m\ddot{z} + \lambda = 0 \Rightarrow \ddot{z} = g + \frac{\lambda}{m}$$

$$\theta\text{-equation: } \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -R$$

$$\Rightarrow -I\ddot{\theta} - \lambda R = 0 \Rightarrow \ddot{\theta} = -\frac{\lambda R}{I}$$

b)

$$\text{From constraint we have } \ddot{z} = R\ddot{\theta} \Rightarrow g + \frac{\lambda}{m} = -\frac{\lambda R^2}{I}$$

$$\lambda \left(\frac{1}{m} + \frac{R^2}{I} \right) = -g$$

$$\lambda = \frac{-g}{\left(\frac{1}{m} + \frac{R^2}{I} \right)} = -mg/3$$

$$\therefore \ddot{z} = \frac{2}{3}g$$

$$\underline{\underline{z(t) = z_0 + \frac{1}{3}gt^2}} \quad (\text{for } \dot{z}(0)=0)$$

$$\ddot{\theta} = \frac{mg}{3} \frac{2}{mR} = \frac{2}{3} \frac{g}{R}$$

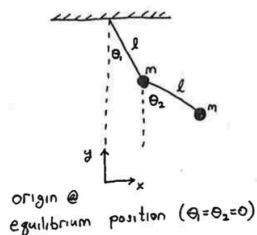
$$\underline{\underline{\theta(t) = \frac{1}{3} \frac{g}{R} t^2}} \quad (\text{for } \theta(0)=0, \dot{\theta}(0)=0)$$

Constraint forces:

$$f_z = \lambda \frac{\partial \mathcal{L}}{\partial z} = -\frac{mg}{3} \leftarrow \text{string tension (in "up" direction, i.e., } -z\text{-direction)}$$

$$f_\theta = \lambda \frac{\partial \mathcal{L}}{\partial \theta} = \left(-\frac{mg}{3} \right) (-R) = \underline{\underline{\frac{1}{3}mgR}} \leftarrow \text{torque about disk center}$$

(A3) Double Pendulum ... a classic



a) We'll label the upper bob "1" and the lower bob "2".

$$\begin{aligned} x_1 &= l \sin \theta_1 & \dot{x}_1 &= l \cos \theta_1 \dot{\theta}_1 \\ y_1 &= 2l - l \cos \theta_1 & \dot{y}_1 &= l \sin \theta_1 \dot{\theta}_1 \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 + l \sin \theta_2 & \dot{x}_2 &= l (\cos \theta_1 \dot{\theta}_1 + \cos \theta_2 \dot{\theta}_2) \\ &= l (\sin \theta_1 + \sin \theta_2) \\ y_2 &= y_1 - l \cos \theta_2 & \dot{y}_2 &= l (\sin \theta_1 \dot{\theta}_1 + \sin \theta_2 \dot{\theta}_2) \\ &= 2l - l (\cos \theta_1 + \cos \theta_2) \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m l^2 \{ \dot{\theta}_1^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \} \\ &= \frac{1}{2} m l^2 \{ 2 \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \} \\ U &= m g (y_1 + y_2) = m g l \{ 4 - 2 \cos \theta_1 - \cos \theta_2 \} \\ L = T - U &= \frac{1}{2} m l^2 \{ 2 \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \} - m g l \{ 4 - 2 \cos \theta_1 - \cos \theta_2 \} \end{aligned}$$

b) θ_1 - equation of motion:

$$\frac{\partial L}{\partial \theta_1} = -m l^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - 2 m g l \sin \theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m l^2 (2 \dot{\theta}_1 + \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = m l^2 \{ 2 \ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_1 - \theta_2) \}$$

Euler-Lagrange gives

$$\Rightarrow m l^2 \{ 2 \ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + (\dot{\theta}_2)^2 \sin(\theta_1 - \theta_2) \} + 2 m g l \sin \theta_1 = 0$$

or

$$\underline{\underline{2 \ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + \frac{2g}{l} \sin \theta_1 = 0}}$$

Assignment 6 - Solutions

Legendre Transforms (a special type of variable change)

Math first: Consider a function of several variables $\beta(x, y, t)$

Total differential: $d\beta = u dx + v dy + w dt$

where $u \equiv \left(\frac{\partial \beta}{\partial x}\right)_{y,t}$

$v \equiv \left(\frac{\partial \beta}{\partial y}\right)_{x,t}$

$w \equiv \left(\frac{\partial \beta}{\partial t}\right)_{x,y}$

The sets (u, x)
 (v, y)
 (w, t) are pairs of
"conjugate"
variables

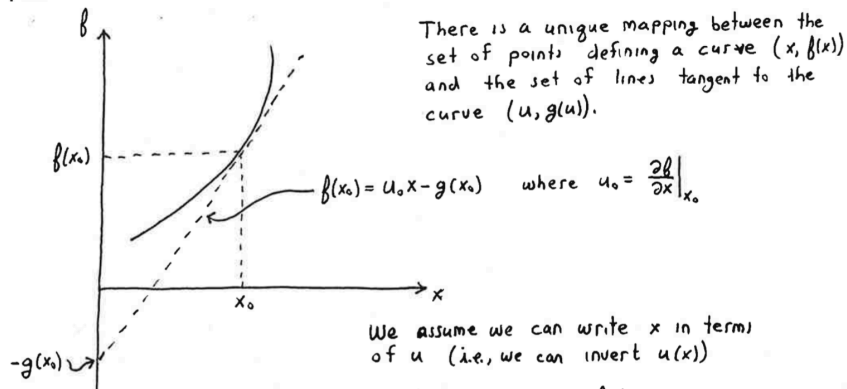
It is often desirable to make changes of variable of the form:

$\beta(x, y, t) \rightarrow g(u, y, t)$ where $\beta \leftrightarrow g$ is unique + invertible

This is a Legendre Transformation

[Note: there are 7 possible Legendre transformations of $\beta(x, y, t)$]

This transformation can be illustrated graphically:



We define the Legendre transform of $\beta(x)$ as:

$$g(u) = ux - \beta(x)$$

But is it true that g depends only on u ?

$$\rightarrow dg = u dx + x du - d\beta \quad \text{and} \quad d\beta = \frac{\partial \beta}{\partial x} dx = u dx$$

$$\Rightarrow dg = x du \Rightarrow g = g(u)$$

Examples: In thermo we have many conjugate pairs: $(T, S), (P, V), (\mu, N)$, etc.
 All of the "thermodynamic potentials" are simply Legendre transforms of the energy: $E(S, V, N) \rightarrow A(T, V, N) \rightarrow G(T, P, N) \rightarrow \dots$

In mechanics we have two pairs $(q, \dot{q})$ and (q, p) and thus 3 possible transforms of $L(q, \dot{q})$:

$$L(q, \dot{q}) \rightarrow H(q, p) \rightarrow K(\dot{p}, p) \rightarrow I(\dot{q}, q)$$

never seen this one defined.

Hamiltonian Mechanics

The Hamiltonian is simply the $\dot{q} \rightarrow p$ Legendre transform of the Lagrangian

$$H(q, p) \equiv \dot{q}p - L(q, \dot{q}) \quad \left[\begin{array}{l} \text{We must be able to solve for } \dot{q} \\ \text{in terms of } p \text{ and } q \text{ to construct } H \end{array} \right]$$

$$\begin{aligned} dH &= \dot{q}dp + p d\dot{q} - dL \\ &= \dot{q}dp + p d\dot{q} - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q} \end{aligned}$$

defined as $p \equiv \frac{\partial L}{\partial \dot{q}}$

$\downarrow$
 $\dot{p} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$
 Euler-Lagrange

$$= \dot{q}dp - \dot{p}dq$$

We see that $H = H(q, p)$ and

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

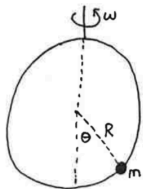
← Hamilton's Equations

We haven't added anything new here... just changed variables.

But p and q are treated more equally and symmetrically than $\dot{q}$ and q ; this can be advantageous in some cases.

[It just so happens that in many cases $H = E = T + U$ but this is NOT the definition of H and is not always true.]

(7-21) Whirling hoop w/ sliding bead



$$T = \frac{1}{2} m (R^2 \sin^2 \theta \omega^2 + R^2 \dot{\theta}^2)$$

$$U = mg(R - R \cos \theta)$$

$$L = T - U = \frac{1}{2} m R^2 (\omega^2 \sin^2 \theta + \dot{\theta}^2) - mgR(1 - \cos \theta)$$

$$\left. \begin{aligned} \frac{\partial L}{\partial \theta} &= m R^2 \omega^2 \sin \theta \cos \theta - mgR \sin \theta \\ \frac{\partial L}{\partial \dot{\theta}} &= m R^2 \dot{\theta} \end{aligned} \right\} \text{ Euler-Lagrange gives: } \begin{aligned} m R^2 \ddot{\theta} - m R^2 \omega^2 \sin \theta \cos \theta + mgR \sin \theta &= 0 \end{aligned}$$

or

$$\ddot{\theta} - \omega^2 \sin \theta \cos \theta + \frac{g}{R} \sin \theta = 0$$

$$\text{Equilibrium points given by } \ddot{\theta} = 0 \Rightarrow \left(\frac{g}{R} - \omega^2 \cos \theta \right) \sin \theta = 0$$

$$\therefore \text{ equilibrium @ } \theta_0 = 0, \pi, \arccos\left(\frac{g}{\omega^2 R}\right)$$

In order to examine the stability of these equilibrium points we consider solutions to our equation of motion of the form $\theta(t) = \theta_0 + n(t)$

where $n(t) \ll 1$. Note that $\ddot{\theta} = \ddot{n}$. Inserting into the equation of motion:

$$\Rightarrow \ddot{n} + \left[\frac{g}{R} - \omega^2 \cos(\theta_0 + n) \right] \sin(\theta_0 + n) = 0$$

$$\Rightarrow \ddot{n} + \left[\frac{g}{R} - \omega^2 \cos \theta_0 + \omega^2 n \sin \theta_0 \right] (\sin \theta_0 + n \cos \theta_0) = 0$$

$$\left[\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b \end{aligned} \right]$$

$$\begin{aligned} \sin n &\approx n \\ \cos n &\approx 1 \end{aligned}$$

For $\theta_0 = \pi$:

$$\ddot{n} + \left(\omega^2 + \frac{g}{R} \right) (-n) = 0$$

or

$$\ddot{n} - \left(\frac{g}{R} + \omega^2 \right) n = 0 \leftarrow \text{has exponential solutions} \dots \text{no oscillations} \dots \text{unstable equilibrium point}$$

For $\theta_0 = 0$:

$$\ddot{n} + \left(\frac{g}{R} - \omega^2 \right) n = 0 \leftarrow \text{oscillatory solutions for } \frac{g}{R} - \omega^2 > 0$$

$$\therefore \text{ stable equilibrium for } \omega^2 < \frac{g}{R}$$

$$\text{frequency of small oscillations} = \sqrt{\frac{g}{R} - \omega^2}$$

7-21 conti

For $\Theta_0 = \arccos\left(\frac{g}{\omega^2 R}\right)$: For this to be possible we require $\omega^2 > \frac{g}{R}$!!

$$\ddot{n} + \left[\frac{g}{R} - \omega^2 \left(\frac{g}{\omega^2 R} \right) + \omega^2 n \sqrt{1 - \left(\frac{g}{\omega^2 R} \right)^2} \right] \left[\sqrt{1 - \left(\frac{g}{\omega^2 R} \right)^2} + n \left(\frac{g}{\omega^2 R} \right) \right] = 0$$

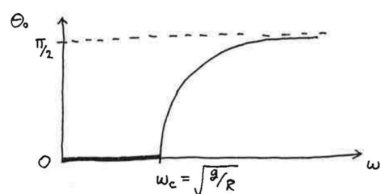
Ignoring terms of $O(n^2)$ gives:

$$\ddot{n} + \omega^2 n \left[1 - \left(\frac{g}{\omega^2 R} \right)^2 \right] = 0 \Rightarrow \ddot{n} + \left[\omega^2 - \left(\frac{g}{R} \right)^2 \right] n = 0$$

← oscillatory solutions for $\omega^2 > \left(\frac{g}{R} \right)^2$

$$\text{frequency of small oscillations} = \sqrt{\omega^2 - \left(\frac{g}{R} \right)^2}$$

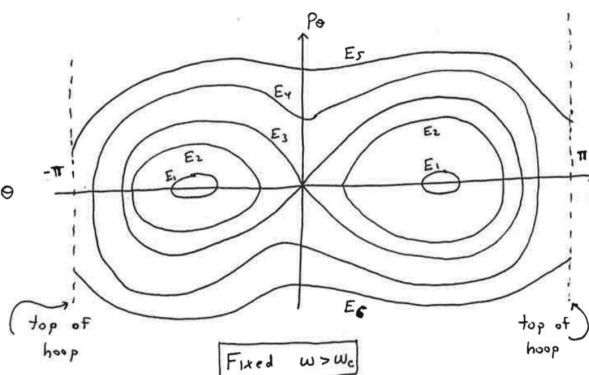
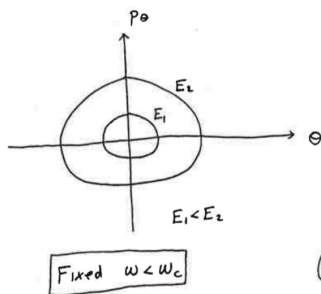
The equilibrium position vs. hoop spinning speed looks like this:



frequency of small oscillations about Θ_0 :

for $\omega < \omega_c$	$\sqrt{\frac{g}{R} - \omega^2}$
for $\omega > \omega_c$	$\sqrt{\omega^2 - \left(\frac{g}{R} \right)^2}$
@ $\omega = \omega_c$	no oscillations

Phase Diagrams:



- E_1, E_2 - oscillates about one of two possible equilibrium points $\pm \Theta_0$
- E_3 - just sneaks through bottom of hoop
- E_4 - Now oscillates back and forth about bottom of hoop
- E_5, E_6 - Now goes all the way around hoop! (2 possible directions)
- $E_1 < E_2 < E_3 < E_4 < E_5, E_6$

(7-22) Time dependent force $F(x,t) = \frac{k}{x^2} e^{-t/\tau}$ $k, \tau > 0$

1-D motion. $T = \frac{1}{2} m \dot{x}^2$

$$U = - \int F dx = -k e^{-t/\tau} \int \frac{dx}{x^2} = \frac{k e^{-t/\tau}}{x} + C$$

let $U(x \rightarrow \infty) = 0$, $\therefore C = 0$

$$L = T - U = \underline{\underline{\frac{1}{2} m \dot{x}^2 - \frac{k}{x} e^{-t/\tau}}}$$

$$H = p \dot{x} - L, \quad p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

$$= \frac{p^2}{m} - \frac{1}{2} m \left(\frac{p}{m} \right)^2 + \frac{k}{x} e^{-t/\tau}$$

$$= \underline{\underline{\frac{p^2}{2m} + \frac{k}{x} e^{-t/\tau}}}$$

$$E_{\text{total}} = T + U = \frac{1}{2} m \dot{x}^2 + \frac{k}{x} e^{-t/\tau}$$

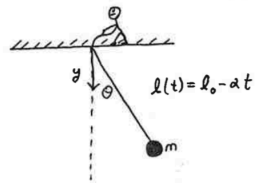
$$= \frac{p^2}{2m} + \frac{k}{x} e^{-t/\tau}$$

$$= H$$

Total energy equals the Hamiltonian...

But $\frac{\partial E}{\partial t} = -\frac{k}{x\tau} e^{-t/\tau} \neq 0$ Energy is NOT conserved.

7-24 Simple pendulum w/ changing string length.



$$\begin{aligned} y &= l(t) \cos \theta & \dot{y} &= \dot{l} \cos \theta - l \sin \theta \dot{\theta} \\ x &= l(t) \sin \theta & \dot{x} &= \dot{l} \sin \theta + l \cos \theta \dot{\theta} \\ \dot{l} &= -\alpha \end{aligned}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \{ \dot{l}^2 + l^2 \dot{\theta}^2 \} = \frac{m}{2} \{ \alpha^2 + l^2 \dot{\theta}^2 \}$$

$$U = -mgy = -mgl \cos \theta$$

$$L = T - U = \frac{1}{2} m \{ \alpha^2 + l^2 \dot{\theta}^2 \} + mgl \cos \theta$$

$$\left. \begin{aligned} \frac{\partial L}{\partial \theta} &= -mgl \sin \theta \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2 \dot{\theta} \end{aligned} \right\} \text{Euler-Lagrange gives:}$$

$$ml^2 \ddot{\theta} + 2ml\dot{l}\dot{\theta} + mgl \sin \theta \Rightarrow \ddot{\theta} + \frac{\dot{l}}{l} \sin \theta - \frac{2\dot{l}}{l} \dot{\theta} = 0$$

$$H = p_{\theta} \dot{\theta} - L \quad p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_{\theta}}{ml^2}$$

$$= p_{\theta} \left(\frac{p_{\theta}}{ml^2} \right) - \frac{1}{2} m \alpha^2 - \frac{1}{2} ml^2 \left(\frac{p_{\theta}}{ml^2} \right)^2 - mgl \cos \theta$$

$$= \frac{p_{\theta}^2}{2ml^2} - \frac{1}{2} m \alpha^2 - mgl \cos \theta$$

$$E = T + U = \frac{1}{2} m \alpha^2 + \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos \theta$$

$$= \frac{p_{\theta}^2}{2ml^2} + \frac{1}{2} m \alpha^2 - mgl \cos \theta$$

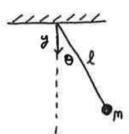
$$\underline{E \neq H}$$

$$\frac{\partial E}{\partial t} \neq 0 \text{ since } l=l(t)$$

energy is not conserved
as work is done on the system.

7-26

a) Simple Pendulum



$$x = l \sin \theta \quad \dot{x} = l \cos \theta \dot{\theta}$$

$$y = l \cos \theta \quad \dot{y} = -l \sin \theta \dot{\theta}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = -mgy = -mgl \cos \theta$$

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$

$$H = p_{\theta} \dot{\theta} - L$$

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \Rightarrow \dot{\theta} = p_{\theta} / m l^2$$

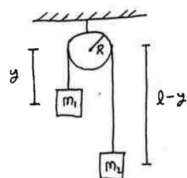
$$= \frac{p_{\theta}^2}{m l^2} - \frac{1}{2} m l^2 \left(\frac{p_{\theta}}{m l^2} \right)^2 - mgl \cos \theta$$

$$= \frac{p_{\theta}^2}{2 m l^2} - mgl \cos \theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p_{\theta}} = \frac{p_{\theta}}{m l^2}$$

$$\dot{p}_{\theta} = -\frac{\partial H}{\partial \theta} = -mgl \sin \theta$$

b) Atwood Machine



$$T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$\Delta y = R \theta \Rightarrow \dot{y} = R \dot{\theta}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}^2 + \frac{I}{2 R^2} \dot{y}^2$$

$$U = -m_1 g y - m_2 g (l - y)$$

$$L = T - U = \frac{1}{2} (m_1 + m_2 + I/R^2) \dot{y}^2 + m_1 g y + m_2 g (l - y)$$

$$p = \frac{\partial L}{\partial \dot{y}} = (m_1 + m_2 + I/R^2) \dot{y} \Rightarrow \dot{y} = \frac{p}{(m_1 + m_2 + I/R^2)}$$

$$\left(\begin{array}{l} \text{total string length} \\ = l + \pi R \end{array} \right)$$

$$H(p, y) = p \dot{y} - L = \frac{p^2}{(m_1 + m_2 + I/R^2)} - \frac{1}{2} \frac{p^2}{(m_1 + m_2 + I/R^2)} - m_1 g y - m_2 g (l - y)$$

$$= \frac{p^2}{2 (m_1 + m_2 + I/R^2)} - m_1 g y - m_2 g (l - y)$$

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{(m_1 + m_2 + I/R^2)}$$

$$\dot{p} = -\frac{\partial H}{\partial y} = m_1 g - m_2 g = (m_1 - m_2) g$$

7-30 Poisson Brackets : $[g, h] \equiv \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)$

where $g = g(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n, t)$

$$dg = \sum_k \left[\left(\frac{\partial g}{\partial q_k} \right) dq_k + \left(\frac{\partial g}{\partial p_k} \right) dp_k \right] + \left(\frac{\partial g}{\partial t} \right) dt \quad \leftarrow \text{total differential of } g$$

$$\begin{aligned} \text{a) } [g, H] &= \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right) = \sum_k \left(\frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial p_k} \dot{p}_k \right) \quad \text{where we use Hamilton's equations} \\ &= \sum_k \frac{d}{dt} \left(\frac{\partial g}{\partial q_k} dq_k + \frac{\partial g}{\partial p_k} dp_k \right) \\ &= \frac{dg}{dt} - \left(\frac{\partial g}{\partial t} \right) \end{aligned}$$

these are defined for fixed t
and therefore are treated as constants
w/ respect to d/dt .

$$\therefore \underline{\underline{\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}}}$$

$$\text{b) } [q_i, H] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) = \sum_k \delta_{ik} \dot{q}_k = \underline{\underline{\dot{q}_i}}$$

$\nwarrow$ equals δ_{ik} $\nwarrow$ equals zero

$$[p_i, H] = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right) = \sum_k \delta_{ik} \dot{p}_k = \underline{\underline{\dot{p}_i}}$$

$\nwarrow$ equals zero $\nwarrow$ equals δ_{ik}

$$\text{c) } [p_i, p_j] = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_k [(0)\delta_{jk} - \delta_{ik}(0)] = \underline{\underline{0}}$$

$$[q_i, q_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = \sum_k [\delta_{ik}(0) - (0)\delta_{jk}] = \underline{\underline{0}}$$

$$\text{d) } [q_i, p_j] = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_k [\delta_{ik} \delta_{jk} - 0] = \underline{\underline{\delta_{ij}}}$$

e) For f with $\frac{\partial f}{\partial t} = 0$ and $[f, H] = 0$

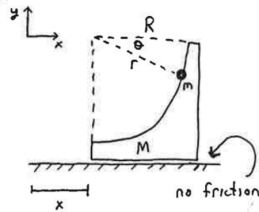
we have from part (a):

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}$$

$$= 0$$

$\therefore f$ is a constant of the motion
(i.e., a conserved quantity)

(7-34) Normal force via Lagrange multipliers



Two degrees of freedom ... use coordinates x, θ

position of particle:

$$\begin{aligned} x_m &= x + R \cos \theta \\ y_m &= -R \sin \theta \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_m &= \dot{x} - R \sin \theta \dot{\theta} \\ \dot{y}_m &= -R \cos \theta \dot{\theta} \end{aligned}$$

$$\begin{aligned} a) \quad T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2) \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (x^2 + R^2 \dot{\theta}^2 - 2 \dot{x} R \dot{\theta} \sin \theta) \end{aligned}$$

$$U = mgy_m = -mgR \sin \theta$$

$$L = T - U = \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} m R^2 \dot{\theta}^2 - m R \dot{x} \dot{\theta} \sin \theta + mgR \sin \theta$$

θ -motion:

$$\frac{\partial L}{\partial \theta} = -m R \dot{x} \dot{\theta} \cos \theta + mgR \cos \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m R^2 \ddot{\theta} - m R \ddot{x} \sin \theta - m R \dot{x} \dot{\theta} \cos \theta \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = m R^2 \ddot{\theta} - m R \ddot{x} \sin \theta - m R \dot{x} \dot{\theta} \cos \theta$$

$$\text{Euler-Lagrange: } m R^2 \ddot{\theta} - m R \ddot{x} \sin \theta - m R \dot{x} \dot{\theta} \cos \theta = 0$$

x -motion:

$$\frac{\partial L}{\partial x} = 0 \quad (x \text{ is a cyclic variable ... there will be a conserved quantity } x = \frac{\partial L}{\partial \dot{x}})$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m) \dot{x} - m R \dot{\theta} \sin \theta \quad (= \text{constant})$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = (M+m) \ddot{x} - m R \ddot{\theta} \sin \theta - m R \dot{\theta}^2 \cos \theta = 0$$

Equations of motion:

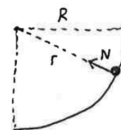
$$\begin{aligned} \ddot{x} &= a R (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) & a &= \frac{m}{m+M} \\ \ddot{\theta} &= \frac{1}{R} (\ddot{x} \sin \theta + g \cos \theta) \end{aligned}$$

$$\text{Also Note that: } \ddot{x} = \frac{a \cos \theta}{1 - a \sin^2 \theta} (g \sin \theta + R \dot{\theta}^2)$$

$$\ddot{\theta} = \frac{\cos \theta}{R(1 - a \sin^2 \theta)} (g + a R \dot{\theta}^2 \sin \theta)$$

7-34 cont.

b) Now we want to determine the normal force of the wedge on the particle. In order to use Lagrange multipliers we must introduce a new coordinate in the direction of $\vec{N}$. This will be r as shown and the constraint is $r=R$.



$$\therefore g = r - R = 0 \quad \frac{\partial g}{\partial r} = 1 \quad \therefore N = \lambda \quad (\text{the magnitude of the normal force is } \lambda)$$

Now we must reformulate L treating the distance r as a variable:

$$\left. \begin{aligned} x_m &= x + r \cos \theta \\ y_m &= -r \sin \theta \end{aligned} \right\} \Rightarrow \begin{aligned} \dot{x}_m &= \dot{x} + \dot{r} \cos \theta - r \sin \theta \dot{\theta} \\ \dot{y}_m &= -\dot{r} \sin \theta - r \cos \theta \dot{\theta} \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2) \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \{ \dot{x}^2 + \dot{r}^2 + r^2 \dot{\theta}^2 + 2 \dot{x} \dot{r} \cos \theta - 2 \dot{x} r \sin \theta \dot{\theta} - 2 \dot{r} \dot{r} \cos \theta \sin \theta + 2 \dot{r} r \cos \theta \sin \theta \dot{\theta} \} \\ &= \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} m \{ \dot{r}^2 + r^2 \dot{\theta}^2 + 2 \dot{x} \dot{r} \cos \theta - 2 \dot{x} r \sin \theta \dot{\theta} \} \end{aligned}$$

$$U = m g y_m = -m g r \sin \theta$$

$$L = T - U$$

r -motion (this equation will be the only one involving λ):

$$\frac{\partial L}{\partial r} = \frac{1}{2} m \{ 2 \dot{\theta}^2 - 2 \dot{x} \sin \theta \dot{\theta} \} + m g \sin \theta$$

$$\frac{\partial L}{\partial \dot{r}} = m \dot{r} + m \dot{x} \cos \theta \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = m \ddot{r} + m \ddot{x} \cos \theta - m \dot{x} \sin \theta \dot{\theta}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) + \lambda = 0 \Rightarrow m r \dot{\theta}^2 + m g \sin \theta - m \ddot{r} - m \ddot{x} \cos \theta + \lambda = 0$$

$$\begin{aligned} \text{or} \\ \lambda &= m \ddot{r} + m \ddot{x} \cos \theta - m r \dot{\theta}^2 - m g \sin \theta \\ &= m \ddot{x} \cos \theta - m R \dot{\theta}^2 - m g \sin \theta \end{aligned} \quad \left. \begin{array}{l} \text{use constraint} \\ r = R \\ \dot{r} = 0 \end{array} \right\}$$

We would like to find $\lambda(\theta)$, therefore we need to eliminate $\ddot{x}$ and $\dot{\theta}$ from our expression. From part (a) we have $\ddot{x}$ in terms of θ and $\dot{\theta}$:

$$\begin{aligned} \lambda &= \frac{m a \cos^2 \theta}{1 - a \sin^2 \theta} (g \sin \theta + R \dot{\theta}^2) - m R \dot{\theta}^2 - m g \sin \theta \\ &= \frac{m a \cos^2 \theta (g \sin \theta + R \dot{\theta}^2) - m (g \sin \theta + R \dot{\theta}^2) (1 - a \sin^2 \theta)}{1 - a \sin^2 \theta} \\ &= \frac{m a (g \sin \theta + R \dot{\theta}^2) - m (g \sin \theta + R \dot{\theta}^2)}{1 - a \sin^2 \theta} = \frac{m(a-1)(g \sin \theta + R \dot{\theta}^2)}{1 - a \sin^2 \theta} \end{aligned}$$

7-34 cont'd

We can use conservation of linear momentum and energy to determine $\dot{\theta}$.

Assume initial conditions $\dot{x}(0) = 0$, $\dot{\theta}(0) = 0$. Then we have zero total linear momentum $\Rightarrow (M+m)\dot{x} - mR\dot{\theta}\sin\theta = 0$ (from part a)

$$\text{or} \\ \dot{x} = aR\dot{\theta}\sin\theta$$

Initial energy: $T_0 = 0$

$$U_0 = -mgR\sin\theta_0 \quad (\text{where } \theta_0 = \text{initial angle})$$

$$\therefore T + U = -mgR\sin\theta_0$$

$$\Rightarrow \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}mR^2\dot{\theta}^2 - mR\dot{x}\dot{\theta}\sin\theta - mgR\sin\theta = -mgR\sin\theta_0$$

$$(aR\dot{\theta}\sin\theta)^2 + aR^2\dot{\theta}^2 - 2aR(aR\dot{\theta}\sin\theta)\dot{\theta}\sin\theta = 2agR(\sin\theta - \sin\theta_0)$$

$$(aR^2 - a^2R^2\sin^2\theta)\dot{\theta}^2 = 2agR(\sin\theta - \sin\theta_0)$$

$$\dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R(1 - a\sin^2\theta)}$$

Inserting this into our equation for λ :

$$\lambda = \frac{m(a-1)}{1-a\sin^2\theta} \left[g\sin\theta + \frac{2g(\sin\theta - \sin\theta_0)}{1-a\sin^2\theta} \right]$$

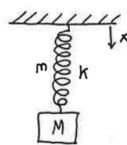
$$= \frac{-Ma}{1-a\sin^2\theta} \left[\frac{g\sin\theta - ag\sin^3\theta + 2g\sin\theta - 2g\sin\theta_0}{1-a\sin^2\theta} \right]$$

$$= -\frac{mM}{m+M} g \left[\frac{3\sin\theta - a\sin^3\theta - 2\sin\theta_0}{(1-a\sin^2\theta)^2} \right]$$

$$a-1 = \frac{m}{m+M} - 1 = \frac{-M}{m+M}$$

Note: if $\theta = \theta_0 = 0 \Rightarrow \lambda = 0 \checkmark$

(A1) Massive Spring



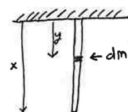
Let down be the positive x -direction and let x be the position of hanging mass M .

$$T_M = \frac{1}{2} M \dot{x}^2$$

$$U_M = -Mgx$$

Spring mass will not effect the stored elastic energy: $U_{\text{elastic}} = \frac{1}{2} k (l_0 - x)^2$
where l_0 = equilibrium length of the spring

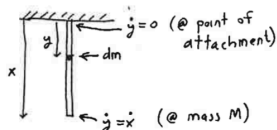
For the gravitational potential energy of the spring we have:



$$U_s = - \int_0^x g y dm = - \frac{m g}{x} \int_0^x y dy = - \frac{m g}{x} \frac{x^2}{2} = - \frac{1}{2} m g x$$

We could have guessed this by considering the CM of the spring.

Spring kinetic energy:



for $\dot{y}$ @ y we interpolate between values @ $x=0$ and $x=x$: $\dot{y} = \frac{y}{x} \dot{x}$

$$T_s = \int_0^x \frac{1}{2} dm (\dot{y})^2 = \frac{m}{2} \int_0^x \frac{dy}{x} \left(\frac{y}{x} \dot{x} \right)^2 = \frac{m \dot{x}^2}{2x^3} \int_0^x y^2 dy = \frac{1}{6} m \dot{x}^2$$

The Lagrangian is:

$$L = T_M + T_s - U_M - U_{\text{elastic}} - U_s$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{6} m \dot{x}^2 + Mgx + \frac{1}{2} mgx - \frac{1}{2} k (l_0 - x)^2$$

$$\frac{\partial L}{\partial x} = Mg + \frac{1}{2} mg + k(l_0 - x)$$

$$\frac{\partial L}{\partial \dot{x}} = M \dot{x} + \frac{1}{3} m \dot{x}$$

Euler-Lagrange gives:

$$(M + \frac{1}{3} m) \ddot{x} + kx = (M + \frac{1}{2} m)g + k l_0 = k x_{\text{equil.}}$$

define variable: $z = x - x_{\text{equil}}$ (z = displacement from equilibrium)

$$\ddot{z} = \ddot{x}$$

$$\Rightarrow \ddot{z} + \omega^2 z = 0$$

$$\text{where } \omega = \sqrt{\frac{k}{M + m/3}}, \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{M + m/3}{k}}$$

(A2) Momentum Space Lagrangian: $k(p, \dot{p}, t) = p\dot{q} + q\dot{p} - L(q, \dot{q}, t)$

a) Hamiltonian for 1D SHO: $H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2)$

$$\text{We know } H = p\dot{q} - L \Rightarrow L(q, \dot{q}) = p\dot{q} - H = p\dot{q} - \frac{1}{2}p^2 - \frac{1}{2}\omega^2 q^2$$

we need p in terms of q and $\dot{q}$: From Hamilton's eq, $\dot{q} = \frac{\partial H}{\partial p} = p$

$$\therefore L(q, \dot{q}) = \dot{q}^2 - \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \\ = \underline{\underline{\frac{1}{2}(\dot{q}^2 - \omega^2 q^2)}}$$

$$k(p, \dot{p}, t) = p\dot{q} + q\dot{p} - L$$

$$= p\dot{q} + q\dot{p} - \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2$$

need q in terms of p and $\dot{p}$:

$$= p^2 - \frac{\dot{p}^2}{\omega^2} - \frac{1}{2}p^2 + \frac{1}{2}\omega^2 \frac{\dot{p}^2}{\omega^4}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q \Rightarrow q = -\frac{\dot{p}}{\omega^2}$$

From above: $\dot{q} = p$

$$= \underline{\underline{\frac{1}{2}(p^2 - \frac{1}{\omega^2}\dot{p}^2)}}$$

b)

$$\frac{\partial L}{\partial q} = -\omega^2 q$$

$$\frac{\partial L}{\partial \dot{q}} = \dot{q}$$

$$\left. \begin{array}{l} \frac{\partial L}{\partial q} = -\omega^2 q \\ \frac{\partial L}{\partial \dot{q}} = \dot{q} \end{array} \right\} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \Rightarrow \underline{\underline{\ddot{q} + \omega^2 q = 0}}$$

$$\frac{\partial k}{\partial p} = p$$

$$\frac{\partial k}{\partial \dot{p}} = -\frac{1}{\omega^2} \dot{p}$$

$$\left. \begin{array}{l} \frac{\partial k}{\partial p} = p \\ \frac{\partial k}{\partial \dot{p}} = -\frac{1}{\omega^2} \dot{p} \end{array} \right\} \frac{d}{dt} \left(\frac{\partial k}{\partial \dot{p}} \right) - \frac{\partial k}{\partial p} = 0 \Rightarrow \begin{array}{l} -\frac{1}{\omega^2} \ddot{p} - p = 0 \\ \text{or} \\ \underline{\underline{\ddot{p} + \omega^2 p = 0}} \end{array}$$

(A3) Poisson brackets for angular momentum: $\vec{L} = \vec{r} \times \vec{p}$ In Cartesian coordinates:

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

$$\begin{aligned} a) [L_x, L_y] &= \left(\frac{\partial L_x}{\partial x} \frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x} \frac{\partial L_y}{\partial x} \right) \\ &\quad + \left(\frac{\partial L_x}{\partial y} \frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y} \frac{\partial L_y}{\partial y} \right) + \left(\frac{\partial L_x}{\partial z} \frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z} \frac{\partial L_y}{\partial z} \right) \\ &= (-p_y)(-x) - y p_x \\ &= x p_y - y p_x = \underline{L_z} \end{aligned}$$

← the z-term is the only one that contributes!

$$\begin{aligned} [L_y, L_z] &= \frac{\partial L_y}{\partial x} \frac{\partial L_z}{\partial p_x} - \frac{\partial L_y}{\partial p_x} \frac{\partial L_z}{\partial x} \quad \leftarrow \text{only x-term contributes} \\ &= (-p_z)(-y) - z p_y = y p_z - z p_y = \underline{L_x} \end{aligned}$$

$$\begin{aligned} [L_z, L_x] &= \frac{\partial L_z}{\partial y} \frac{\partial L_x}{\partial p_y} - \frac{\partial L_z}{\partial p_y} \frac{\partial L_x}{\partial y} \quad \leftarrow \text{only y-term contributes} \\ &= (-p_x)(-z) - x p_z = z p_x - x p_z = \underline{L_y} \end{aligned}$$

$$\begin{aligned} b) [x, L_x] &= [x, y p_z - z p_y] = 0 & [p_x, L_x] &= [p_x, y p_z - z p_y] = 0 \\ [x, L_y] &= [x, z p_x - x p_z] = z & [p_x, L_y] &= [p_x, z p_x - x p_z] = p_z \\ [x, L_z] &= [x, x p_y - y p_x] = -y & [p_x, L_z] &= [p_x, x p_y - y p_x] = -p_y \end{aligned}$$

The pattern is clear:

$$[x_i, L_j] = \epsilon_{ijk} x_k$$

$$[p_i, L_j] = \epsilon_{ijk} p_k$$

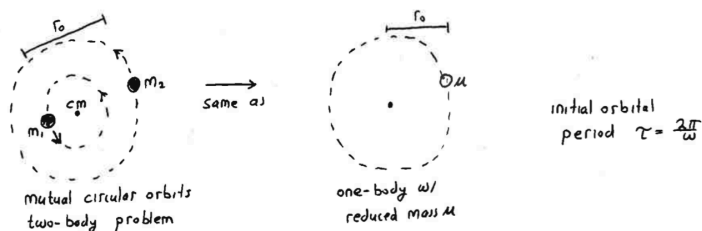
$$c) H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2$$

$$\begin{aligned} c) \frac{dL_z}{dt} &= [L_z, H] = \left[x p_y - y p_x, \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2 \right] \\ &= \left(p_y \frac{p_x^2}{m} - (-y) k_x x \right) + \left((-p_x) \left(\frac{p_y^2}{m} \right) - (x) (k_y y) \right) + 0 \\ &= \frac{1}{m} p_x p_y^2 + k_x x y - \frac{1}{m} p_x^2 p_y - k_y x y \quad \leftarrow \begin{matrix} \text{x-term} & \text{y-term} & \text{z-term} \end{matrix} \\ &= \underline{(k_x - k_y) x y} \end{aligned}$$

$$\underline{\frac{dL_z}{dt} = 0 \text{ for } k_x = k_y} \quad \leftarrow \begin{matrix} \text{angular momentum conserved} \\ \text{for isotropic oscillator} \end{matrix}$$

Assignment 7 - Solutions

8-5 Time to collide



For circular orbit we have $F_c = \mu \omega^2 r_0$ where F_c is provided by the gravitational attraction: $F_g = -G \frac{m_1 m_2}{r_0^2}$. We get the initial r_0 from

$$F_c = F_g \Rightarrow r_0 = \left(\frac{G m_1 m_2}{\mu \omega^2} \right)^{1/3} = \left(\frac{G m_1 m_2}{4\pi^2 \mu} \tau^2 \right)^{1/3}$$

When motion is stopped, total energy is: $E_{tot} = -\frac{G m_1 m_2}{r_0}$

While masses fall towards each other: $E_{tot} = \frac{1}{2} \mu \dot{r}^2 - \frac{G m_1 m_2}{r} = \text{constant}$
(i.e., reduced mass falls towards force center)

$$\therefore \frac{dr}{dt} = - \left[\frac{2}{\mu} \left(\frac{G m_1 m_2}{r} - \frac{G m_1 m_2}{r_0} \right) \right]^{1/2} = - \sqrt{\frac{2 G m_1 m_2}{\mu}} \left(\frac{1}{r} - \frac{1}{r_0} \right)^{1/2}$$

negative sign since r gets smaller w/ increasing time

Time to collide given by:

$$\begin{aligned} t &= \int_0^t dt = - \int_{r_0}^0 \frac{dr}{\sqrt{\frac{2 G m_1 m_2}{\mu} \left(\frac{1}{r} - \frac{1}{r_0} \right)}} = - \sqrt{\frac{\mu r_0}{2 G m_1 m_2}} \int_{r_0}^0 \frac{dr}{\sqrt{\frac{r}{r_0} - 1}} \quad \text{let } r = x^2 \\ &= - 2 \sqrt{\frac{\mu r_0}{2 G m_1 m_2}} \int_{\sqrt{r_0}}^0 \frac{x^2}{\sqrt{r_0 - x^2}} dx \quad \left(\begin{array}{l} dr = 2x dx \\ r=0 \rightarrow x=0 \\ r=r_0 \rightarrow x=\sqrt{r_0} \end{array} \right) \\ &= - 2 \sqrt{\frac{\mu r_0}{2 G m_1 m_2}} \left[- \frac{x}{2} \sqrt{r_0 - x^2} + \frac{r_0}{2} \sin^{-1} \frac{x}{\sqrt{r_0}} \right]_{\sqrt{r_0}}^0 \\ &= - \sqrt{\frac{\mu r_0}{2 G m_1 m_2}} \left(r_0 \sin^{-1}(0) - r_0 \sin^{-1}(1) \right) \\ &= r_0 \frac{\pi}{2} \sqrt{\frac{\mu r_0}{2 G m_1 m_2}} \\ &= \frac{\pi}{2\sqrt{2}} \sqrt{\frac{\mu r_0^3}{G m_1 m_2}} \\ &= \frac{\pi}{2\sqrt{2}} \frac{\tau}{2\pi} = \frac{\tau}{4\sqrt{2}} \end{aligned}$$

from above we have:

$$\tau = 2\pi \sqrt{\frac{\mu r_0^3}{G m_1 m_2}}$$

8.8 Orbit for a repulsive force $F = kr$

The orbit is given by:

$$\begin{aligned}\Theta(r) &= \int \frac{(l/r^2) dr}{\sqrt{2\mu(E - U - l^2/2\mu r^2)}} \quad \text{with } U = -\int F dr = -\frac{1}{2}kr^2 + C \quad \leftarrow \begin{array}{l} \text{let this} \\ \text{be zero} \\ \text{so } U(0) = 0. \end{array} \\ &= \frac{l}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{E + \frac{1}{2}kr^2 - l^2/2\mu r^2}} \quad \text{let } x = r^2 \\ &= \frac{l}{\sqrt{2\mu}} \int \frac{dr}{r \sqrt{\frac{1}{2}kr^4 + Er^2 - l^2/2\mu}} \quad dx = 2r dr \Rightarrow dr = \frac{dx}{2\sqrt{x}} \\ &= \frac{l}{2\sqrt{2\mu}} \int \frac{dx}{x \sqrt{\frac{1}{2}kx^2 + Ex - l^2/2\mu}} \quad \text{this integral is of the form E.10b:} \\ &= \frac{1}{2} \sin^{-1} \left(\frac{\frac{\mu E}{l^2} x - 1}{x \sqrt{\frac{\mu^2 E^2}{l^4} + \frac{\mu k}{l^2}}} \right) + \Theta_0 \quad \int \frac{dx}{x \sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \sin^{-1} \left(\frac{bx + 2c}{x \sqrt{b^2 - 4ac}} \right)\end{aligned}$$

or

$$\begin{aligned}\sin[2(\Theta - \Theta_0)] &= \frac{1}{\sqrt{1 + \frac{l^2 k}{\mu E^2}}} - \frac{1}{r^2} \frac{l^2/\mu E}{\sqrt{1 + \frac{l^2 k}{\mu E^2}}} \\ &= \frac{1}{\epsilon} - \frac{\alpha}{\epsilon r^2} \quad \text{where } \epsilon = \sqrt{1 + \frac{l^2 k}{\mu E^2}}, \quad \alpha = \frac{l^2}{\mu E}\end{aligned}$$

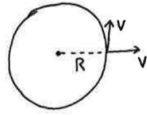
Orbit equation: $\boxed{\frac{\alpha}{r^2} = 1 - \epsilon \sin[2(\Theta - \Theta_0)]}$

To show this is a hyperbola we let $\Theta_0 = \pi/4 \Rightarrow \frac{\alpha}{r^2} = 1 + \epsilon \cos 2\Theta$

and change to Cartesian variables: $x = r \cos \Theta$
 $y = r \sin \Theta$

$$\begin{aligned}\alpha &= r^2 + \epsilon r^2 \cos 2\Theta \\ &= x^2 + y^2 + \epsilon r^2 (\cos^2 \Theta - \sin^2 \Theta) \\ &= (1 + \epsilon)x^2 + (1 - \epsilon)y^2 \quad \text{but } \epsilon > 1 \quad \text{since } l^2, k, \mu, E^2 \text{ are all positive quantities} \\ &= (1 + \epsilon)x^2 - (\epsilon - 1)y^2 \\ \Rightarrow \quad &\boxed{\frac{x^2}{\left(\frac{\alpha}{\epsilon + 1}\right)} - \frac{y^2}{\left(\frac{\alpha}{\epsilon - 1}\right)} = 1} \quad \text{which is a hyperbola } \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)\end{aligned}$$

(8-9) Radial kick to circular orbit



a) radial kick doesn't change angular momentum ($\vec{L} = \vec{r} \times m\vec{v}$)

$$\therefore L_{\text{final}} = L_{\text{initial}}$$

$$\frac{L_f}{L_i} = 1$$

Initial energy: $E_i = T_0 + U$ where $T_0 = \frac{1}{2}mv^2$

Final energy: $E_f = 2T_0 + U$ $U = -\frac{GMm}{R}$

for a circular orbit: $\frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow T_0 = -\frac{1}{2}U$

$$\therefore E_i = -\frac{1}{2}U + U = \frac{1}{2}U$$

$$E_f = -U + U = 0$$

$$\frac{E_f}{E_i} = 0$$

b) since $E_f = 0$, orbit is parabolic ... satellite is lost!

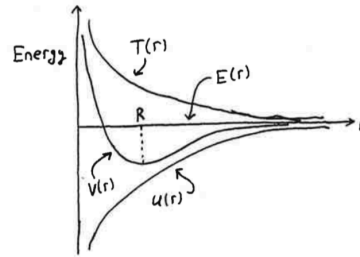
After rocket fires we have:

$$E(r) = 0$$

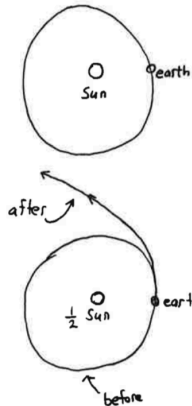
$$U(r) = -\frac{GMm}{r}$$

$$T(r) = E - U = \frac{GMm}{r}$$

$$V(r) = \frac{L^2}{2\mu r^2} + U(r) = \frac{1}{2}\mu v^2 \frac{R^2}{r^2} - \frac{GMm}{r}$$



(8-10) Sun loses $\frac{1}{2}$ its mass



before we have $T_0 = \frac{1}{2}mv^2$ $U_0 = -\frac{GM_s m_e}{R}$

after we have $T_f = T_0$ $U_f = \frac{1}{2}U_0$

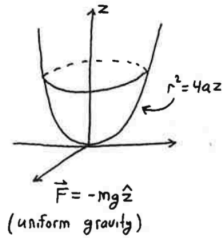
initially in circular orbit: $\frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow T_0 = -\frac{1}{2}U_0$

$$\therefore E_0 = T_0 + U_0 = -\frac{1}{2}U_0$$

$$E_f = T_f + U_f = -\frac{1}{2}U_0 + \frac{1}{2}U_0 = 0$$

final orbit is parabolic
... earth escapes.

8-33 Circular orbit inside a paraboloid



velocity in cylindrical coordinates:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

$$T = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m\{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\} \quad U = mgz$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

we eliminate \$z\$ using
the constraint \$r^2 = 4az\$

$$= \frac{1}{2}m\left(\dot{r}^2 + \frac{r^2\dot{r}^2}{4a^2} + r^2\dot{\theta}^2\right) - \frac{mgr^2}{4a}$$

$$z = \frac{r^2}{4a}, \quad \dot{z} = \frac{r\dot{r}}{2a}$$

$$\frac{\partial L}{\partial r} = m\frac{r\dot{r}^2}{4a^2} + mr\dot{\theta}^2 - \frac{mg}{2a}r$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\frac{r^2\dot{r}}{4a^2}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \ell = mr^2\dot{\theta} = \text{constant}$$

Euler-Lagrange

for \$r\$-motion: $m\ddot{r} + m\frac{r(\dot{r})^2}{2a^2} + m\frac{r^2\ddot{r}}{4a^2} - m\frac{r(\dot{r})^2}{4a^2} - mr\dot{\theta}^2 + \frac{mg}{2a}r = 0$

or

$$\left(1 + \frac{r^2}{4a^2}\right)\ddot{r} + \frac{1}{4a^2}r(\dot{r})^2 - \frac{\ell^2}{m^2r^3} + \frac{g}{2a}r = 0 \quad \leftarrow \text{full equation of motion}$$

For a circular orbit we require \$r = p = \text{constant}\$, \$\dot{r} = 0, \ddot{r} = 0 \Rightarrow \ell = mp^2\sqrt{\frac{g}{2a}}\$
angular momentum in circular orbit

For a nearly circular orbit we let \$r(t) = p + x(t)\$ where \$\frac{x}{p} \ll 1\$

Inserting into full \$r\$-equation (noting \$\dot{r} = \dot{x}, \ddot{r} = \ddot{x}\$):

$$\left(1 + \frac{(p+x)^2}{4a^2}\right)\ddot{x} + \frac{p+x}{4a^2}(\dot{x})^2 - \frac{\ell^2}{m^2(p+x)^3} + \frac{g}{2a}(p+x) = 0$$

note that $\frac{1}{(p+x)^3} = \frac{1}{p^3} \frac{1}{(1+x/p)^3} \approx \frac{1}{p^3} \left(1 - \frac{3x}{p}\right)$

Now linearize \$x\$-equation (i.e., discard all terms of \$O(x^2)\$ and greater):

$$\left(1 + \frac{p^2}{4a^2}\right)\ddot{x} - \frac{\ell^2}{m^2} \frac{1}{p^3} \left(1 - \frac{3x}{p}\right) + \frac{g}{2a}(p+x) = 0$$

\$\rightarrow\$ Insert expression for \$\ell\$ from circular orbit:

$$\left(1 + \frac{p^2}{4a^2}\right)\ddot{x} - \frac{pg}{2a} \left(1 - \frac{3x}{p}\right) + \frac{pg}{2a} + \frac{g}{2a}x = 0$$

$$\Rightarrow \ddot{x} + \left(\frac{2g/a}{1 + p^2/4a^2}\right)x = 0$$

SHM \$\omega\$ $\omega^2 = \frac{2g}{a + p^2/4a}$ let \$p = \sqrt{4az_0}\$

$$= \frac{2g}{a + z_0}$$

8-35

Condition for closed orbit

Consider a nearly circular orbit in the central force field $F(r) = -\frac{k}{r^n}$ described by $r(t) = \rho + x(t)$ where $x = x_0 \cos \omega_0 t$, $x_0/\rho \ll 1$.

In section 8.10 it is shown that in general, for a central force, the frequency of the perturbation is given by:

$$\omega_0^2 = \frac{3g(\rho)}{\rho} + \left. \frac{dg}{dr} \right|_{r=\rho} \leftarrow \text{Eq. 8.89}$$

where $g(r) \equiv -\frac{1}{\mu} F(r)$.

For our $\frac{1}{r^n}$ law we have: $g(r) = \frac{k}{\mu r^n}$ and $\left. \frac{dg}{dr} \right|_{r=\rho} = -\frac{nk}{\mu \rho^{n+1}}$

$$\therefore \omega_0^2 = \frac{3k}{\mu \rho^{n+1}} - \frac{nk}{\mu \rho^{n+1}} = (3-n) \frac{k}{\mu \rho^{n+1}}$$

The frequency of the circular orbit is given by $\mu \omega_c^2 \rho = \frac{k}{\rho^n}$
↑
centripetal
acceleration ← force

$$\therefore \omega_c^2 = \frac{k}{\mu \rho^{n+1}}$$

So the perturbing frequency and orbital frequency are related by:

$$\omega_0 = \sqrt{3-n} \omega_c$$

In the circular orbit we have $\Theta = \omega_c t$, thus we can rewrite our original orbit equation as:

$$\begin{aligned} r(\Theta) &= \rho + x(\Theta) = \rho + x_0 \cos\left(\omega_0 \frac{\Theta}{\omega_c}\right) \\ &= \rho + x_0 \cos(\sqrt{3-n} \Theta) \end{aligned}$$

The apsidal angle is given by $\sqrt{3-n} \Theta = \pi$ (angle between $r_{\min} = \rho - x_0$ and $r_{\max} = \rho + x_0$)

$$\text{or } \Theta_{\text{apsidal}} = \frac{\pi}{\sqrt{3-n}}$$

For closed orbit we require $\Theta_{\text{apsidal}} = \frac{n}{m} 2\pi$ where $n, m \in \text{integers}$

$\therefore$ for closed orbit we need $\sqrt{3-n} = \text{integer}$

for $n > -6$ there are only two possibilities:

$n = -1$	$F = -kr$	$\sqrt{3-n} = 2$
$n = 2$	$F = -k/r^2$	$\sqrt{3-n} = 1$

(8-36) Precession in a $F(r) = -\left(\frac{k}{r^2}\right)e^{-r/a}$ field.

The orbit $r(\theta)$ for any central force $F(r)$ is given by Eq. 8.211

$$\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} = -\frac{mr^2}{l^2} F(r) = \frac{mk}{l^2} e^{-r/a} \quad \text{substituting in our } F(r)$$

We will assume a nearly circular orbit given by $r(\theta) = p[1 + \delta(\theta)]$

where $p = \text{constant}$ and $\delta(\theta) \ll 1$ for all θ .

We insert $r(\theta)$ expression into the above orbit equation, noting that

$\frac{1}{r} \approx \frac{1}{p}(1 - \delta)$, to get an equation for $\delta(\theta)$:

$$\begin{aligned} -\frac{1}{p} \frac{d^2 \delta}{d\theta^2} + \frac{1}{p}(1 - \delta) &= \frac{mk}{l^2} e^{-p/a} e^{-p\delta/a} \\ &\approx \frac{mk}{l^2} e^{-p/a} \left(1 - \frac{p}{a} \delta \right) \quad \text{where we have expanded the exponential to linearize the equation.} \end{aligned}$$

We get the angular momentum l from the unperturbed circular orbit via:

$$\frac{mv^2}{p} = F(p) \quad \text{and} \quad l = mvr \quad \Rightarrow \quad l^2 = mkp e^{-p/a}$$

Inserting this into the δ -equation gives:

$$-\frac{d^2 \delta}{d\theta^2} + (1 - \delta) = 1 - \frac{p}{a} \delta \quad \Rightarrow \quad \frac{d^2 \delta}{d\theta^2} + \left(1 - \frac{p}{a} \right) \delta = 0$$

Since $p < a$ (as given in the problem) we have:

$$\delta(\theta) = \delta_0 \cos \omega \theta \quad \text{where} \quad \omega = \sqrt{1 - p/a}$$

$$\text{and} \quad r(\theta) = p[1 + \delta_0 \cos \omega \theta].$$

The apses are $r_{\min} = p(1 - \delta_0)$ and $r_{\max} = p(1 + \delta_0)$ and the

apsidal angle is given by $\omega \theta_{\text{aps}} = \pi$ or $\theta_{\text{aps}} = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{1 - p/a}}$

In one complete circular orbit ($\theta = 2\pi$) $r_{\max}$ will precess

by the amount

$$\Delta = 2\theta_{\text{aps}} - 2\pi$$

$$= \frac{2\pi}{(1 - p/a)^{1/2}} - 2\pi \quad \leftarrow p/a \ll 1 \text{ so expand denominator}$$

$$\approx 2\pi \left(1 + \frac{1}{2} \frac{p}{a} \right) - 2\pi = \boxed{\frac{\pi p}{a}}$$

(A1) Return of the Comets

- a) Hyakutake: $\epsilon = 0.999846$ ← eccentricity
 $r_{\min} = 0.23012 \text{ AU}$ ← perihelion

Semi-major axis: $a = \frac{r_{\min}}{1-\epsilon} = \frac{(0.23012 \text{ AU})(1.495 \times 10^{11} \text{ m/AU})}{0.000154} = 2.23 \times 10^{14} \text{ m}$

Period from Kepler's 3rd law: $\tau = \left[\frac{4\pi^2 a^3}{GM_{\text{sun}}} \right]^{1/2} = 1.82 \times 10^{12} \text{ s} = 2.10 \times 10^7 \text{ days} = 5.76 \times 10^4 \text{ year} \Rightarrow \underline{\underline{58,000 \text{ year}}}$

$M_{\text{sun}} = 1.99 \times 10^{30} \text{ kg}$
 $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$

Aphelion given by: $r_{\max} = \left(\frac{1+\epsilon}{1-\epsilon} \right) r_{\min} = \underline{\underline{2,990 \text{ AU}}}$

Pluto data: $a = 39.5 \text{ AU}$, $\epsilon = 0.2484$

$\Rightarrow$ Pluto aphelion: $r_{\max} = a(1+\epsilon) = \underline{\underline{49.3 \text{ AU}}}$

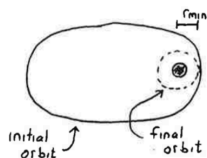
- b) Hale-Bopp: $\epsilon = 0.995075$
 $r_{\min} = 0.913959 \text{ AU}$

Semi-major axis: $a = \frac{r_{\min}}{1-\epsilon} = 2.77 \times 10^{13} \text{ m}$

Period: $\tau = \left[\frac{4\pi^2 a^3}{GM_{\text{sun}}} \right]^{1/2} = 7.95 \times 10^{10} \text{ s} = 9.20 \times 10^5 \text{ days} = \underline{\underline{2,500 \text{ year}}}$

Aphelion: $r_{\max} = \left(\frac{1+\epsilon}{1-\epsilon} \right) r_{\min} = 370 \text{ AU}$

(A2) Circularization of a satellite orbit



- a) With pass through perigee we have a speed change of $v_n = (1-\delta) v_{n-1} = (1-\delta)^n v_0 \approx (1-n\delta) v_0$

$\therefore n = \frac{1}{\delta} \left(1 - \frac{v_n}{v_0} \right)$

We can relate the speed @ perigee to the orbit eccentricity using $\alpha = r_{\min} (1+\epsilon)$ and

$\alpha = \frac{L^2}{\mu k} = \frac{\mu v^2 r_{\min}}{k}$

$\therefore \frac{\alpha_n}{\alpha_0} = \frac{v_n^2}{v_0^2} = \frac{1+\epsilon_n}{1+\epsilon_0} \Rightarrow \boxed{n = \frac{1}{\delta} \left(1 - \sqrt{\frac{1+\epsilon_n}{1+\epsilon_0}} \right)}$

- b) for $\delta = 0.01$, $\epsilon_0 = 0.9656$, $\epsilon_n = 1/7$

$\Rightarrow \boxed{n \approx 24}$

- c) Speed @ apses given by:

$v = \sqrt{\frac{\alpha k}{\mu r_{\text{apside}}}}$ for apogee $r_{\max} = \left(\frac{1+\epsilon}{1-\epsilon} \right) r_{\min}$ and $\alpha = r_{\min} (1+\epsilon)$ } $\frac{\alpha}{r_{\max}} = \frac{(1-\epsilon)^2}{1+\epsilon} \frac{1}{r_{\min}}$ $\frac{k}{\mu} = GM_e$

Apogee speed: $v_{\text{apo}} = \frac{1-\epsilon}{\sqrt{1+\epsilon}} \sqrt{\frac{GM_e}{r_{\min}}}$ Note that $\frac{dv}{d\epsilon} = \frac{-(3+\epsilon)}{2(1+\epsilon)^{3/2}} \sqrt{\frac{GM_e}{r_{\min}}} < 0$ becomes more circular here.
 $\therefore \underline{\underline{V \text{ increases as } \epsilon \text{ decreases}}}$

(C1) Numerical solution via orbits2d.pro

b) Marion 8-41: Hohmann transfer from $h=200\text{ km}$ earth orbit to moon orbit

$$r_0 = R_E + h = 6.57 \times 10^6 \text{ m} \quad \text{speed in circular orbit given by } v = \sqrt{\frac{GM_E}{r}}$$

$$r_f = r_{\text{moon}} = 3.8 \times 10^8 \text{ m}$$

Transfer orbit is an ellipse with $a = \frac{1}{2}(r_0 + r_f)$

$$\text{Transfer orbit speed @ perigee: } v_{t0} = \sqrt{\frac{2GM_E}{r_0} \left(\frac{r_f}{r_0 + r_f} \right)} \quad \Delta v_0 = v_{t0} - v_0$$

$$\text{Transfer orbit speed @ apogee: } v_{tf} = \sqrt{\frac{2GM_E}{r_f} \left(\frac{r_0}{r_0 + r_f} \right)} \quad \Delta v_f = v_f - v_{tf}$$

$$\text{Total speed change: } \Delta v = \Delta v_0 + \Delta v_f = \underline{\underline{3966 \text{ m/s}}}$$

$$\text{Transfer time} = \frac{1}{2} \tau_{\text{transfer orbit}} = \pi \sqrt{\frac{a^3}{GM_E}} = 4.3 \times 10^5 \text{ sec} = \underline{\underline{119 \text{ hr} \sim 5 \text{ days}}}$$

$$\text{Speed for orbit calculation: } v_{t0} = 1.092 \times 10^4 \text{ m/s} = \underline{\underline{10.92 \text{ km/s}}}$$

c) Marion 8-42: Hohmann transfer from $h=200\text{ km}$ to geosync. orbit.

$$r_0 = R_E + h = 6.57 \times 10^6 \text{ m}$$

get r_f from Kepler 3rd law with $\tau = 24 \text{ hr}$

$$\Rightarrow r_f = \left(\frac{GM_E \tau^2}{4\pi^2} \right)^{1/3} = 4.225 \times 10^7 \text{ m}$$

$$\text{For circular orbit: } E = -\frac{GM_E m}{2r}$$

$$\text{for } m = 1 \times 10^4 \text{ kg: } \left. \begin{array}{l} E(r_0) = -3.04 \times 10^{10} \text{ J} \\ E(r_f) = -4.72 \times 10^{10} \text{ J} \end{array} \right\} \Delta E = E_f - E_0 = 2.57 \times 10^{10} \text{ J}$$

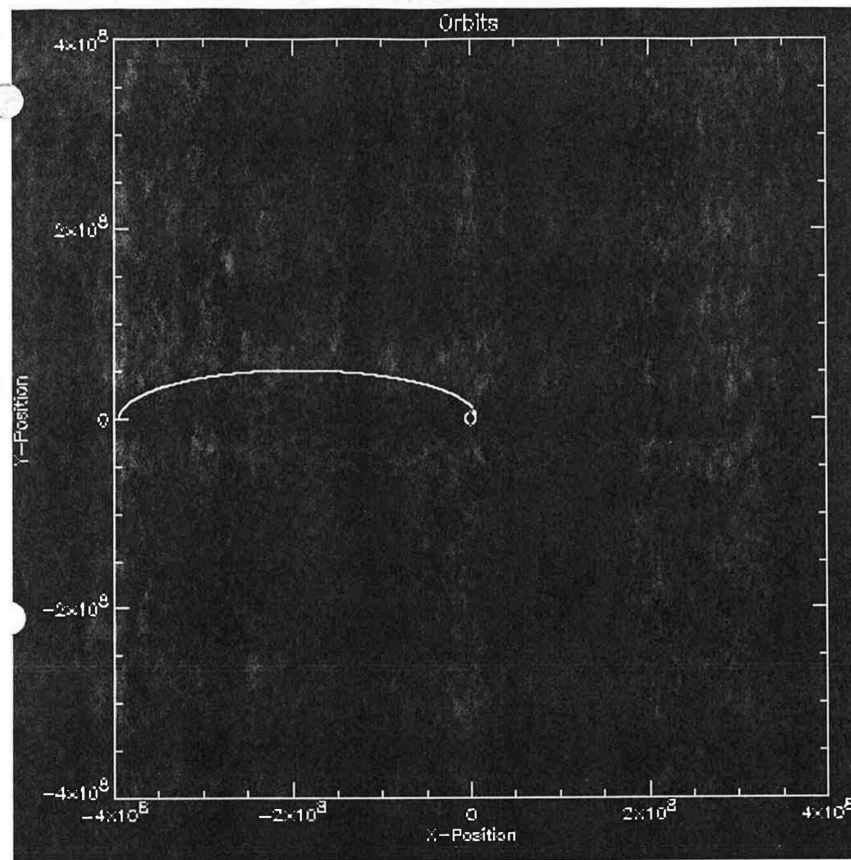
$$\text{speed for transfer orbit: } v_{t0} = \sqrt{\frac{2GM_E}{r_0} \left(\frac{r_f}{r_0 + r_f} \right)} = 1.024 \times 10^4 \text{ m/s}$$

$$= \underline{\underline{10.24 \text{ km/s}}}$$

$$\text{Transfer time} = \frac{1}{2} \tau_{\text{transfer orbit}} = \pi \sqrt{\frac{a^3}{GM_E}} = 1.898 \times 10^4 \text{ s} = \underline{\underline{5.27 \text{ hr}}}$$

$$a = \frac{1}{2}(r_0 + r_f)$$

$$\text{d) I used } r_{\text{min}} = 3R_E, \quad v_{\text{perigee}} = 3.8 \text{ km/s}, \quad F = -\frac{k}{r^{2.05}}$$

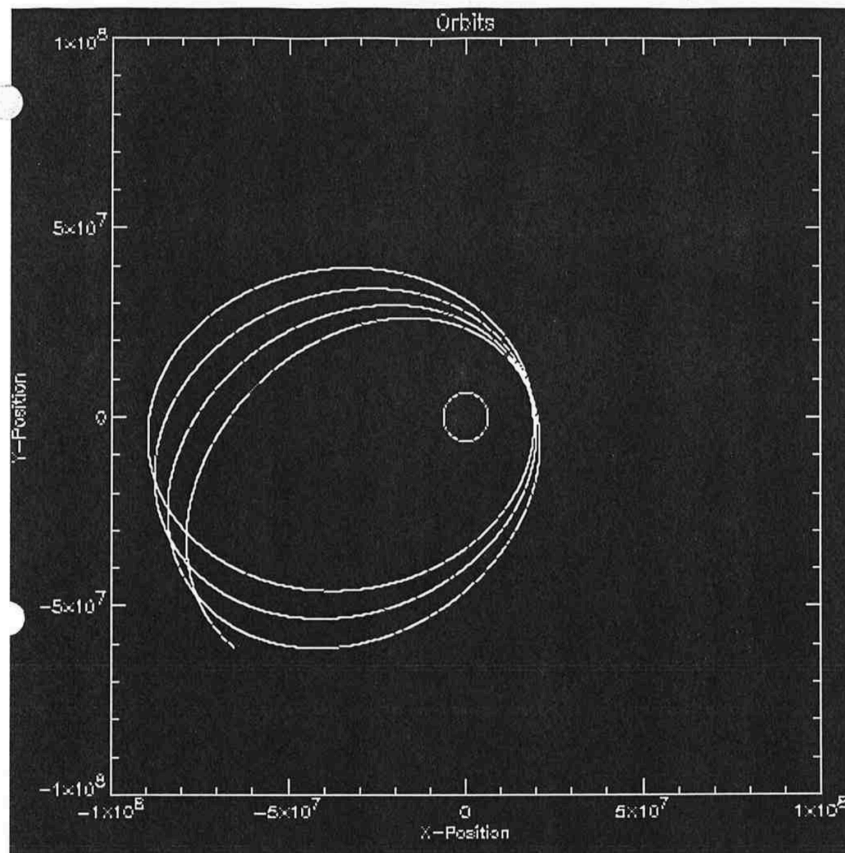


b. Earth $\rightarrow$ Moon transfer orbit (120hr run w/ $\Delta t = 30s$)

$V_{0t} = \underline{10.92 \text{ km/s}}$ @ $h_0 = 200\text{km}$ above surface of earth

$R_{\text{moon orbit}} = 3.80 \times 10^8 \text{ m}$

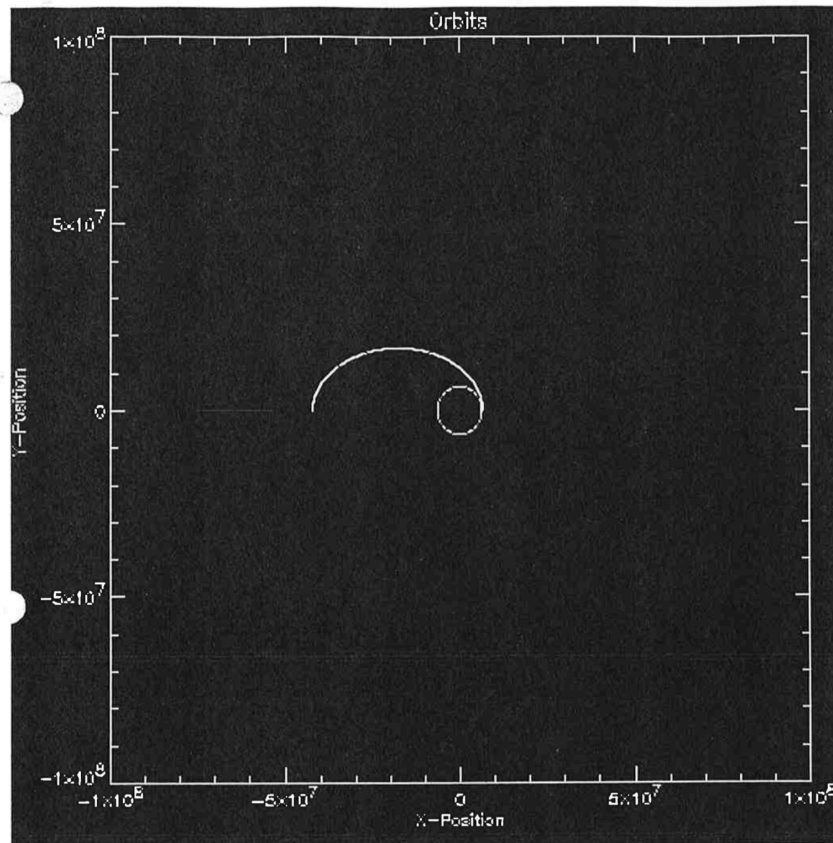
Transfer time = 119 hr $\sim$ (5 days)



d. Non-inverse square force
 $V_y^0 = 3.8 \text{ km/s}$ $x_0 = 3 R_e$
 $V_x^0 = 0$ $y_0 = 0$

total time = 200 hr ($\Delta t = 30 \text{ sec}$)

$$F \approx \frac{-k}{r^{2.05}}$$



c. transfer to synchronous orbit (5.27 hr run w/ $\Delta t = 2s$)

$v_{\text{ot}} = \underline{10,24 \text{ km/s}}$ @ $h_0 = 200 \text{ km}$ above surface of earth

$R_{\text{sync}} = 4,225 \times 10^7 \text{ m}$

Transfer time = 5.27 hr

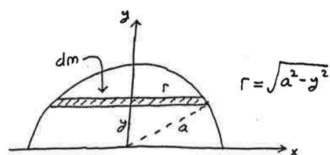
Assignment 8 - Solutions

9.3 CM of an ice-cream cone:

I'll find the CM of the "ice-cream" and the cone separately and then combine the results. Assume each part has the same mass density ρ .



CM for a hemisphere:



$$dm = \rho \pi r^2 dy = \rho \pi (a^2 - y^2) dy$$

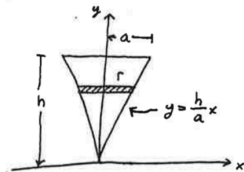
$$\text{total mass: } M_H = \int_0^a dm = \rho \pi \int_0^a (a^2 - y^2) dy$$

$$= \rho \pi \left(a^3 - \frac{1}{3} a^3 \right) = \frac{2\pi}{3} \rho a^3$$

$$y_{cm}^H = \frac{1}{M_H} \int_0^a y dm$$

$$= \frac{\rho \pi}{\left(\frac{2\pi}{3} \rho a^3 \right)} \int_0^a y (a^2 - y^2) dy = \frac{3}{2a^3} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \underline{\underline{\frac{3a}{8}}}$$

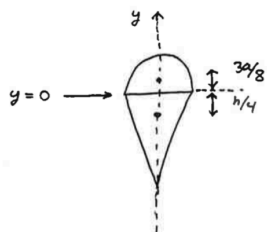
CM for a cone:



$$dm = \rho \pi r^2 dy = \rho \pi \left(\frac{a}{h} y \right)^2 dy$$

$$\text{total mass: } M_C = \int_0^h dm = \frac{\rho \pi a^2}{h^2} \int_0^h y^2 dy = \rho \frac{\pi a^2 h}{3}$$

$$y_{cm}^C = \frac{1}{M_C} \int_0^h y dm = \frac{\rho \pi \frac{a^2}{h^2}}{\rho \pi \frac{a^2 h}{3}} \int_0^h y^3 dy = \left(\frac{3}{h^3} \right) \frac{h^4}{4} = \underline{\underline{\frac{3}{4} h}}$$

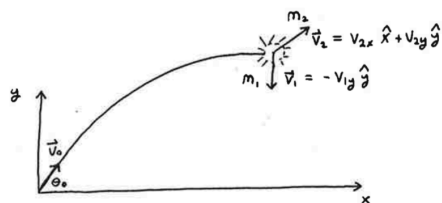


$$y_{cm} = \frac{M_H \frac{3a}{8} + M_C \left(-\frac{h}{4} \right)}{M_H + M_C}$$

$$= \frac{\left(\frac{2\pi}{3} \rho a^3 \right) \left(\frac{3a}{8} \right) - \left(\frac{\pi}{3} \rho a^2 h \right) \left(\frac{h}{4} \right)}{\frac{2\pi}{3} \rho a^3 + \frac{\pi}{3} \rho a^2 h} = \underline{\underline{\frac{3a^2 - h^2}{4(2a+h)}}}$$

Note: CM can be up in the ice-cream or down in the cone depending on the relative values of a and h .

9.9 Projectile Breakup



$$\theta_0 = 45^\circ, E_0 = \frac{1}{2}(m_1 + m_2)v_0^2$$

$$\vec{v}_0 = \frac{\sqrt{2}}{2}v_0 \hat{x} + \frac{\sqrt{2}}{2}v_0 \hat{y}$$

At breakup point:

$$\text{Initial momentum: } \vec{p}_i = (m_1 + m_2) \frac{\sqrt{2}}{2} v_0 \hat{x}$$

Final momentum:

$$\vec{p}_f = m_2 v_{2x} \hat{x} + (m_2 v_{2y} - m_1 v_{1y}) \hat{y}$$

$$\text{Initial energy: } E_i = \frac{1}{2} E_0 + E_0$$

kinetic due to explosion

Final energy:

$$E_f = \frac{1}{2} m_1 v_{1y}^2 + \frac{1}{2} m_2 (v_{2x}^2 + v_{2y}^2)$$

Momentum Conservation:

Energy Conservation:

$$v_{2x} = \frac{\sqrt{2}}{2} \left(\frac{m_1 + m_2}{m_2} \right) v_0$$

$$3E_0 = m_1 v_{1y}^2 + m_2 (v_{2x}^2 + v_{2y}^2)$$

$$v_{2y} = \frac{m_1}{m_2} v_{1y}$$

← 3 equations, 3 unknowns

$$\text{Solve for } v_{1y}: 3E_0 = m_1 v_{1y}^2 + m_2 \left(\frac{m_1 + m_2}{m_2} \right)^2 v_0^2 + m_2 \left(\frac{m_1}{m_2} \right)^2 v_{1y}^2$$

$$= m_1 \left(1 + \frac{m_1}{m_2} \right) v_{1y}^2 + \frac{m_1 + m_2}{m_2} E_0$$

$$v_{1y} = \left(\frac{[3 - (1 + \frac{m_1}{m_2})] E_0}{m_1 (1 + \frac{m_1}{m_2})} \right)^{1/2} = \sqrt{\frac{(2m_2 - m_1) E_0}{m_1 (m_1 + m_2)}}$$

For fragment #2 we have:

$$v_{2x} = \frac{\sqrt{(m_1 + m_2) E_0}}{m_2} \quad v_{2y} = \frac{m_1}{m_2} \sqrt{\frac{(2m_2 - m_1) E_0}{m_1 (m_1 + m_2)}}$$

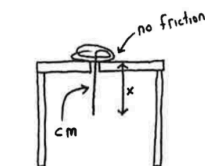
$$|\vec{v}_2| = \sqrt{v_{2x}^2 + v_{2y}^2} = \left\{ \left[(m_1 + m_2) + \frac{m_1^2 (2m_2 - m_1)}{m_1 (m_1 + m_2)} \right] \frac{E_0}{m_2^2} \right\}^{1/2} = \left[\frac{(m_1 + m_2)^2 + 2m_1 m_2 - m_1^2}{m_1 + m_2} \left(\frac{E_0}{m_2^2} \right) \right]^{1/2}$$

$$= \sqrt{\frac{4m_1 + m_2}{m_2 (m_1 + m_2)}} E_0$$

$$\theta_2 = \tan^{-1} \left(\frac{v_{2y}}{v_{2x}} \right) = \tan^{-1} \left(\frac{m_1 \sqrt{\frac{2m_2 - m_1}{m_1 (m_1 + m_2)}}}{\sqrt{m_1 + m_2}} \right) = \tan^{-1} \left(\frac{\sqrt{m_1 (2m_2 - m_1)}}{m_1 + m_2} \right)$$

$$m_1 \text{ will be max when } v_{1y} = 0 \Rightarrow \frac{m_1}{m_2} = 2$$

9-15 Rope slipping off frictionless table



mass of "system" changes w/ time.
linear density of rope: $\rho = m/L$

1st Method - Assume energy conserved:

let $U(0) = T(0) = 0$ so total energy = $U + T = 0$

$$U(t) = -M(t)g \frac{x}{2} \quad T(t) = \frac{1}{2} M(t) \dot{x}^2$$

where $M(t) = \rho x(t)$

$$T + U = 0 \Rightarrow \frac{1}{2} \rho x \dot{x}^2 - \frac{1}{2} \rho g x^2 = 0$$

$$\text{or} \quad \dot{x} = \sqrt{gx}$$

$$a = \frac{dv}{dt} = v \frac{dv}{dx} = \sqrt{gx} \cdot \frac{1}{2} \sqrt{\frac{g}{x}} = \underline{\underline{g/2}}$$

2nd Method - Assume rope is in free fall (no tension in rope... gravity is the only ^{external} force acting)

$$F_g = \frac{dP}{dt} \quad \text{where } F_g = M(t)g \quad P = M(t)v(t) = \rho x v$$

$$\therefore \rho x g = \rho \dot{x} v + \rho x \dot{v}$$

or

$$xg = v^2 + x \dot{v} = v^2 + x v \frac{dv}{dx} \quad \text{since } \dot{v} = \frac{dv}{dt} = \frac{dx}{dt} \frac{dv}{dx} = v \frac{dv}{dx}$$

We have the differential equation: $xv \frac{dv}{dx} + v^2 = xg \Rightarrow \frac{dv}{dx} + \frac{v}{x} = \frac{g}{v}$

We try the solution: $v = cx^n, \frac{dv}{dx} = ncx^{n-1}$

Substituting into the D.E. gives: $x(cx^n)(ncx^{n-1}) + c^2 x^{2n} = xg$

$$c^2(n+1)x^{2n} = xg \leftarrow \text{this can only be true if } n = 1/2 \text{ and } \therefore c = \sqrt{2g/3}$$

This D.E. is of the Bernoulli form:
 $\frac{dv}{dx} + P(x)v = Q(x)v^n$

$$\therefore v = \sqrt{\frac{2g}{3}x}$$

$$a = \frac{dv}{dt} = v \frac{dv}{dx} = \sqrt{\frac{2g}{3}x} \cdot \frac{1}{2} \sqrt{\frac{2g}{3x}} = \underline{\underline{g/3}}$$

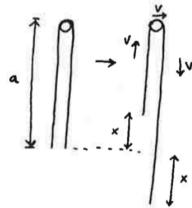
When $x=L$: $U_f = -mg \frac{L}{2}, T_f = \frac{1}{2} m \left(\frac{2g}{3} L \right)$

$$\therefore E_f = T_f + U_f = \frac{1}{3} mgL - \frac{1}{2} mgL = \underline{\underline{-\frac{1}{6} mgL}}$$

but $E_0 = 0 \therefore \Delta E = -\frac{1}{6} mgL \leftarrow \text{energy lost ... but what is the dissipative mechanism?}$

Or should we simply interpret this as the energy required to change the mass of the system?

9-20 Rope sliding off peg



Total external force acting on rope: $\vec{F}_{\text{ext}} = \vec{F}_g + \vec{F}_{\text{peg}}$

but $\vec{F}_{\text{peg}}$ is $\perp$ the rope's velocity and thus will do no work here.

$$F_g = (a+x)\rho g - (a-x)\rho g \quad \text{where } \rho = m/2a$$

$$= 2x\rho g$$

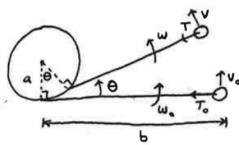
$$F = \frac{dP}{dt} = m \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = \frac{2x\rho g}{m} = \frac{g}{a}x$$

noting that $\frac{dv}{dt} = v \frac{dv}{dx}$ we have: $v dv = \frac{g}{a}x dx$

Integrating: $\frac{1}{2}v(t)^2 = \frac{g}{2a}x(t)^2$ (since $x(0)=0, v(0)=0$)

When $x(t)=a \Rightarrow \boxed{v = \sqrt{ga}}$

9-22 Particle on a string



Energy is conserved here but angular momentum is not (since an external torque must be supplied by the cylinder... the associated force does no work since it is $\perp$ to $\vec{v}$).

Energy conservation: $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2$

Also note that: $v_0 = \omega_0 b$

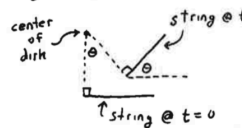
$$v = \omega(b-a\theta)$$

$$\therefore \omega_0^2 b^2 = \omega^2 (b-a\theta)^2$$

or

$$\boxed{\omega = \frac{\omega_0 b}{b-a\theta}} \quad (\text{thus } \omega > \omega_0 \text{ as expected})$$

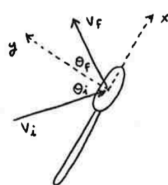
Note geometry:



String tension gives rise to centripetal acceleration

$$\therefore T = \frac{mv^2}{b-a\theta} = \frac{m\omega^2(b-a\theta)^2}{b-a\theta} = \underline{m\omega^2(b-a\theta)} = \boxed{m\omega\omega_0 b}$$

9-28 Tennis Racket Impulse



$$\begin{aligned}\theta_i &= 45^\circ & v_i &= 8 \text{ m/s} & m &= 0.06 \text{ kg} \\ \theta_f &= 15^\circ & v_f &= 16 \text{ m/s}\end{aligned}$$

a) Impulse $= \vec{J} = \vec{p}_f - \vec{p}_i$

$$\vec{p}_i = m v_i (\sin \theta_i \hat{x} - \cos \theta_i \hat{y})$$

$$\vec{p}_f = m v_f (\sin \theta_f \hat{x} + \cos \theta_f \hat{y})$$

$$\sin \theta_i = \cos \theta_i = \frac{\sqrt{2}}{2}$$

$$\sin \theta_f = \frac{\sqrt{3}}{4} (\sqrt{3} - 1)$$

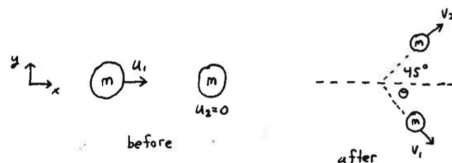
$$\cos \theta_f = \frac{\sqrt{3}}{4} (\sqrt{3} + 1)$$

$$\vec{J} = m v_i [(2 \sin \theta_f - \sin \theta_i) \hat{x} + (2 \cos \theta_f + \cos \theta_i) \hat{y}] \quad (v_f = 2v_i)$$

$$= m v_i \frac{\sqrt{2}}{2} [(\sqrt{3}-2) \hat{x} + (\sqrt{3}+2) \hat{y}] = \underline{\underline{[-0.091 \hat{x} + 1.27 \hat{y}] \text{ kg m/s}}}$$

b) $\vec{F} = \frac{\vec{J}}{\Delta t} = \frac{\vec{J}}{0.01 \text{ s}} = \underline{\underline{(-9.1 \hat{x} + 127 \hat{y}) \text{ N}}}$

9-32 Billiard ball collision



$$\left. \begin{array}{ll} \text{Conservation of } p_x: & m u_1 = m v_1 \cos \theta + m v_2 \cos 45^\circ \\ \text{" " } p_y: & 0 = -m v_1 \sin \theta + m v_2 \sin 45^\circ \\ \text{" " } E: & \frac{1}{2} m u_1^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 \end{array} \right\} \begin{array}{l} 3 \text{ equations, 3 unknowns} \\ (v_1, v_2, \theta) \end{array}$$

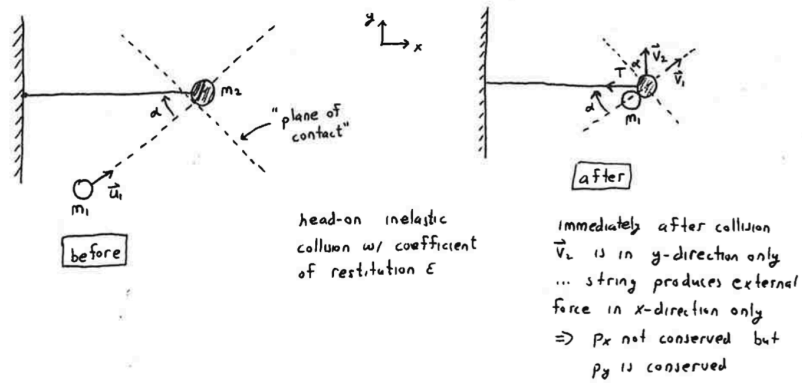
From symmetry we'd guess that $\theta = 45^\circ$, $v_1 = v_2$... but we'll verify this:

$$\left. \begin{array}{l} v_1 \cos \theta = -\frac{\sqrt{2}}{2} v_2 + u_1 \\ v_1 \sin \theta = \frac{\sqrt{2}}{2} v_2 \\ v_1^2 = u_1^2 - v_2^2 \end{array} \right\} \begin{aligned} u_1^2 - v_2^2 &= (u_1 - \frac{\sqrt{2}}{2} v_2)^2 + (\frac{\sqrt{2}}{2} v_2)^2 \\ &= u_1^2 - \sqrt{2} u_1 v_2 + \frac{1}{2} v_2^2 + \frac{1}{2} v_2^2 \\ &= u_1^2 + v_2^2 - \sqrt{2} u_1 v_2 \\ \Rightarrow \boxed{v_2} &= \boxed{\frac{\sqrt{3}}{2} u_1} \end{aligned}$$

$$v_1 = \sqrt{u_1^2 - v_2^2} = \sqrt{u_1^2 - \frac{1}{2} u_1^2} = \frac{\sqrt{2}}{2} u_1 \Rightarrow \boxed{v_1 = v_2} \quad \text{as expected.}$$

$$\sin \theta = \frac{\frac{\sqrt{2}}{2} u_1}{\frac{\sqrt{3}}{2} u_1} = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\theta = 45^\circ}$$

9-38 Inelastic collision w/ a tethered particle



m_1 will continue moving in original direction: $\vec{v}$ component parallel to plane of contact is unchanged in a collision

p_y conservation: $m_1 u_1 \sin \alpha = m_2 v_2 + m_1 v_1 \sin \alpha$

energy change described by ϵ : $\epsilon = \frac{|\Delta v_n|}{|\Delta u_n|}$ where subscript "n" means component normal to plane of contact

$$= \frac{v_2 \sin \alpha - v_1}{u_1} \Rightarrow v_1 = v_2 \sin \alpha - \epsilon u_1$$

We have 2 equations and 2 unknowns (v_1 and v_2)

Solving for v_2 : $m_1 u_1 \sin \alpha = m_2 v_2 + m_1 (v_2 \sin \alpha - \epsilon u_1) \sin \alpha$

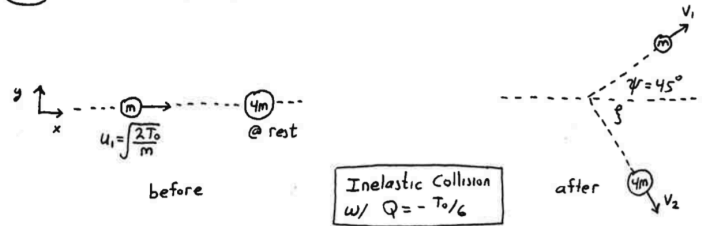
$$= v_2 (m_2 + m_1 \sin^2 \alpha) - m_1 \epsilon u_1 \sin \alpha$$

$$\therefore v_2 = \frac{(1 + \epsilon) m_1 u_1 \sin \alpha}{m_2 + m_1 \sin^2 \alpha} \quad \text{in y-direction}$$

$$v_1 = \frac{(1 + \epsilon) m_1 u_1 \sin^2 \alpha}{m_2 + m_1 \sin^2 \alpha} - \epsilon u_1$$

or $v_1 = \frac{u_1 (m_1 \sin^2 \alpha - m_2 \epsilon)}{m_2 + m_1 \sin^2 \alpha}$ in $\vec{u}_1$ direction

9.41) Proton-Alpha particle collision



Energy conservation: $T_0 + Q = \frac{1}{2} m v_1^2 + \frac{1}{2} (4m) v_2^2$

P_x conservation: $m u_1 = m v_1 \cos \psi + 4m v_2 \cos \phi$

P_y conservation: $0 = m v_1 \sin \psi - 4m v_2 \sin \phi$

We have 3 equations and 3 unknowns (v_1, v_2, ϕ). Let's simplify them first:

$$\begin{aligned} \textcircled{1} \quad \frac{5T_0}{m} &= 3v_1^2 + 12v_2^2 & \textcircled{2} \quad \frac{5T_0}{m} &= 3(32v_2^2 \sin^2 \phi) + 12v_2^2 \\ & & &= 12v_2^2 (1 + 8\sin^2 \phi) \\ \textcircled{3} \quad \frac{\sqrt{2T_0}}{m} &= \frac{\sqrt{2}}{2} v_1 + 4v_2 \cos \phi & \textcircled{4} \quad \frac{\sqrt{2T_0}}{m} &= 4v_2 (\sin \phi + \cos \phi) \end{aligned}$$

Combining the two equations

$$\frac{\frac{2T_0}{m}}{\frac{5T_0}{m}} = \frac{16v_2^2 (\sin \phi + \cos \phi)^2}{12v_2^2 (1 + 8\sin^2 \phi)} = \frac{4(\sin \phi + \cos \phi)^2}{3(9\sin^2 \phi + \cos^2 \phi)} = \frac{4(\tan \phi + 1)^2}{3(9\tan^2 \phi + 1)}$$

$$\Rightarrow 3(9\tan^2 \phi + 1) = 10(\tan \phi + 1)^2$$

$$= 10\tan^2 \phi + 20\tan \phi + 10$$

$$\Rightarrow \boxed{17\tan^2 \phi - 20\tan \phi - 7 = 0}$$

Solving the quadratic:

$$\tan \phi = \frac{20 + \sqrt{200 + 4 \cdot 7 \cdot 17}}{34} = \frac{10 + \sqrt{100 + 119}}{17} = \frac{10 + \sqrt{219}}{17}$$

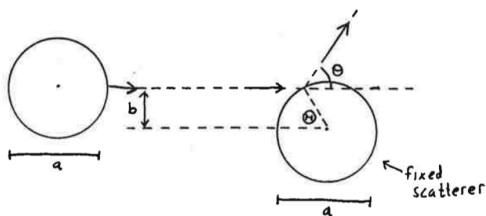
$$\therefore \phi = \tan^{-1} \left(\frac{10 + \sqrt{219}}{17} \right) = \underline{\underline{55.6^\circ}}$$

9-44 Scattering cross-section for hard spheres

$$U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

r = distance between centers of spheres

$$r_{\min} = a$$



$$\begin{aligned} \Theta &= \int_a^\infty \frac{b/r^2}{\sqrt{1 - b^2/r^2}} dr = \int_a^\infty \frac{b dr}{r \sqrt{r^2 - b^2}} \quad [\text{see E.10b pg. 605}] \\ &= \sin^{-1} \left(\frac{-2b^2}{r^2 - b^2} \right) \Big|_a^\infty \\ &= \sin^{-1}(0) - \sin^{-1} \left(-\frac{b}{a} \right) = -\sin^{-1} \left(-\frac{b}{a} \right) \quad (0 \leq b \leq a) \end{aligned}$$

$$\therefore -\frac{b}{a} = \sin(-\Theta) \Rightarrow b = a \sin \Theta$$

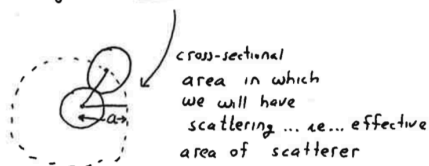
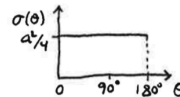
The scattering angle Θ is given by $\Theta = \pi - 2\theta$

$$\begin{aligned} \therefore b &= a \sin \left(\frac{\pi - \Theta}{2} \right) \\ &= \underline{\underline{a \cos(\Theta/2)}} \end{aligned}$$

$$\frac{db}{d\Theta} = -\frac{a}{2} \sin(\Theta/2)$$

$$\begin{aligned} \text{Differential Cross-section: } \sigma(\Theta) &= \frac{b(\Theta)}{\sin \Theta} \left| \frac{db}{d\Theta} \right| = \frac{a \cos \Theta/2}{\sin \Theta} \frac{a}{2} \sin \Theta/2 \\ &= \frac{a^2}{2} \frac{(\frac{1}{2} \sin \Theta)}{\sin \Theta} \\ &= \underline{\underline{\frac{a^2}{4}}} \quad (\text{independent of } \Theta) \end{aligned}$$

$$\text{Total Cross-section: } \sigma_t = \int \sigma(\Theta) d\Omega = \frac{a^2}{4} \int d\Omega = \underline{\underline{\pi a^2}}$$



note: $\Theta = 2 \cos^{-1}(\frac{b}{a})$

$\therefore b=0 \Rightarrow \Theta = 180^\circ$

$b=a \Rightarrow \Theta = 0^\circ$

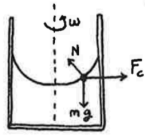
$b=a/\sqrt{2} \Rightarrow \Theta = 90^\circ$

$b < \frac{a}{\sqrt{2}} \Rightarrow \text{back scattering}$

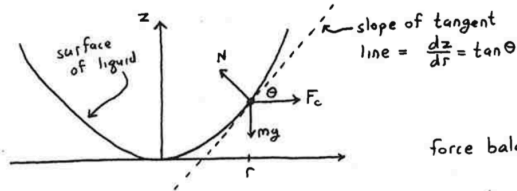
$b > \frac{a}{\sqrt{2}} \Rightarrow \text{forward scattering}$

Assignment 9 - Solutions

10-6 Surface of a rotating liquid



Consider a small mass element @ the surface of the liquid. In the rotating frame there are 3 "forces" acting on the mass: gravity, a normal or buoyancy force (due to the pressure gradient in the liquid), and a centrifugal force. For the mass element to remain @ rest, these forces must balance.



$$\text{force balance: } \left. \begin{array}{l} N \sin \theta = F_c \\ N \cos \theta = mg \end{array} \right\} \tan \theta = \frac{F_c}{mg}$$

Here $F_c = m\omega^2 r$ (since $\vec{r}$ and $\vec{\omega}$ are orthogonal)

$$\therefore \tan \theta = \frac{\omega^2}{g} r$$

From tangent line shown in figure we have :

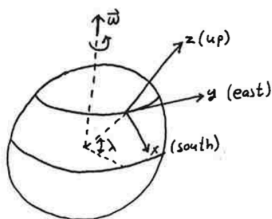
$$\frac{dz}{dr} = \frac{\omega^2}{g} r \quad \Rightarrow \quad dz = \frac{\omega^2}{g} r dr$$

Integrating gives:

$$z = \frac{\omega^2}{2g} r^2 + C$$

Surface assumes a parabolic shape.

10-8 Apparent deflection of a particle projected straight up



Initial velocity: $\vec{v}_0 = v_0 \hat{z}$

max height = h $t_{1/2} = \text{time up} = \text{time down} = \sqrt{\frac{2h}{g}}$

We get v_0 from v @ impact: $v = g t_{1/2} = \sqrt{2gh}$

$\therefore v_z(t) = v_0 - gt = \sqrt{2gh} - gt$

Coriolis force: $\vec{F}_c = -2m\vec{\omega} \times \vec{v}_r$

in local x, y, z frame:

$\omega_x = -\omega \cos \lambda$	$v_x \approx 0$
$\omega_y = 0$	$v_y \approx 0$
$\omega_z = \omega \sin \lambda$	$v_z = \sqrt{2gh} - gt$

$$\vec{\omega} \times \vec{v}_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ 0 & 0 & v_z \end{vmatrix} = \omega \cos \lambda v_z \hat{y}$$

$$\therefore \vec{F}_c = -2mv_z \omega \cos \lambda \hat{y} \quad (\text{note: } -\hat{y} \equiv \text{west})$$

$$a_y = -2v_z \omega \cos \lambda = -2(\sqrt{2gh} - gt)\omega \cos \lambda$$

$$v_y = \int_0^t a_y dt = -2\omega \cos \lambda \left(\sqrt{2gh} t - \frac{1}{2} g t^2 \right)$$

$$\Delta y = \int_0^{\sqrt{\frac{2h}{g}}} v_y dt = -2\omega \cos \lambda \left[\frac{\sqrt{2gh}}{2} t^2 - \frac{g}{6} t^3 \right]_0^{\sqrt{\frac{2h}{g}}}$$

$$= -2\omega \cos \lambda \left\{ \frac{\sqrt{2gh}}{2} \frac{2h}{g} - \frac{g}{6} \left(\frac{2h}{g} \right)^{3/2} \right\}$$

$$= -2\omega \cos \lambda \left\{ 2\sqrt{\frac{8h^3}{g}} - \frac{g}{6} \sqrt{\frac{8h^3}{g}} \right\}$$

$$= \boxed{-\frac{4}{3} \omega \cos \lambda \sqrt{\frac{8h^3}{g}}} \quad (\text{west})$$

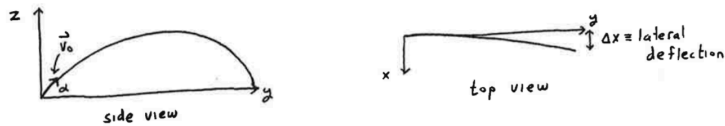
Coriolis is treated as a 1st order correction to the motion. We could construct the 2nd order correction by taking our result for v_y and redoing the Coriolis calculation.

Note that Coriolis effects $\sim \omega$, which is small for motion on the surface of the earth.

Note: max deflection @ the equator ($\lambda = 0$)
no effect @ the poles ($\lambda = 90^\circ$)

(10-9) Apparent deflection of a particle projected due east

Local coordinate system as in 10-8: $x \equiv \text{south}$, $y \equiv \text{east}$, $z \equiv \text{up}$
(Again, we treat Coriolis as a 1st order correction to the motion)



In local x, y, z frame:

$$\begin{aligned} \omega_x &= -\omega \cos \lambda & V_x &\approx 0 \\ \omega_y &= 0 & V_y &= V_0 \cos \alpha \\ \omega_z &= \omega \sin \lambda & V_z &= V_0 \sin \alpha - g t \end{aligned}$$

$$\vec{\omega} \times \vec{v}_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ 0 & V_0 \cos \alpha & V_0 \sin \alpha - g t \end{vmatrix} = \begin{aligned} & -\omega V_0 \sin \lambda \cos \alpha \hat{x} \\ & + \omega \cos \lambda (V_0 \sin \alpha - g t) \hat{y} \\ & - \omega V_0 \cos \lambda \cos \alpha \hat{z} \end{aligned}$$

$$\vec{F}_c = -2m \vec{\omega} \times \vec{v}_r$$

We are only interested in the "lateral" deflection which is caused by the x -component of the Coriolis force:

$$a_x = \frac{(F_c)_x}{m} = 2\omega V_0 \sin \lambda \cos \alpha \quad (+x \equiv \text{south})$$

$$v_x = \int_0^t a_x dt = 2\omega V_0 \sin \lambda \cos \alpha t \quad \leftarrow \begin{cases} \text{Again, 1st order Coriolis effects} \\ \text{are of order } \omega \dots \text{ie, small!} \end{cases}$$

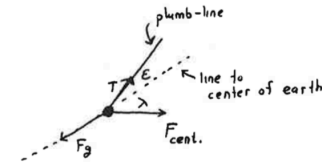
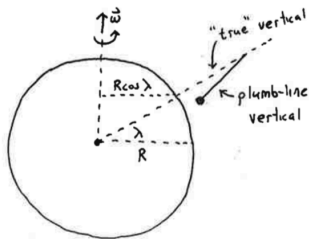
$$\Delta x = \int_0^t v_x dt = \omega V_0 \sin \lambda \cos \alpha t^2$$

$$\frac{1}{2} \text{ time of flight given by: } v_z = 0 = V_0 \sin \alpha - g t_{1/2} \\ \Rightarrow t_{1/2} = \frac{V_0 \sin \alpha}{g}$$

$$\begin{aligned} \therefore \Delta x &= \omega V_0 \sin \lambda \cos \alpha \left(\frac{2 V_0 \sin \alpha}{g} \right)^2 \\ &= \boxed{\frac{4 V_0^3}{g^2} \omega \sin \lambda \sin^2 \alpha \cos \alpha} \end{aligned}$$

Note: max deflection @ poles ($\lambda = 90^\circ$)
No deflection @ equator ($\lambda = 0^\circ$)

(10-12) Plumb-line deviation



$T = \text{string tension}$
 $F_g = mg_0$
 $F_{cent.} = m\omega^2 R \cos \lambda$

Force balance gives:

$$T \cos \epsilon = F_g - F_{cent} \cos \lambda$$

$$T \sin \epsilon = F_{cent} \sin \lambda$$

$$\therefore \tan \epsilon = \frac{m\omega^2 R \cos \lambda \sin \lambda}{mg_0 - m\omega^2 R \cos^2 \lambda}$$

for small ϵ : $\tan \epsilon \approx \epsilon$

$$\therefore \epsilon \approx \frac{\omega^2 R \cos \lambda \sin \lambda}{g_0 - \omega^2 R \cos^2 \lambda} \quad \left(\begin{array}{l} \text{no deviation for} \\ \lambda = 0^\circ, 90^\circ \end{array} \right)$$

Max. deviation when $\frac{d\epsilon}{d\lambda} = 0 = \frac{\omega^2 R (\cos^2 \lambda - \sin^2 \lambda)}{g_0 - \omega^2 R \cos^2 \lambda} + (\omega^2 R \cos \lambda \sin \lambda) \frac{(-2\omega^2 R \cos \lambda \sin \lambda)}{(g_0 - \omega^2 R \cos^2 \lambda)^2}$

$$\Rightarrow (\cos^2 \lambda - \sin^2 \lambda)(g_0 - \omega^2 R \cos^2 \lambda) - 2\omega^2 R \cos^2 \lambda \sin^2 \lambda = 0$$

$$(1 - 2\sin^2 \lambda)(g_0 - \omega^2 R \cos^2 \lambda) - 2\omega^2 R \cos^2 \lambda \sin^2 \lambda = 0$$

$$g_0 - \omega^2 R \cos^2 \lambda - 2g_0 \sin^2 \lambda = 0$$

$$(2g_0 - \omega^2 R) \cos^2 \lambda = g_0$$

$$\cos \lambda = \sqrt{\frac{g_0}{2g_0 - \omega^2 R}}$$

$$\lambda = \arccos \sqrt{\frac{1}{2 - \omega^2 R/g_0}}$$

$$\approx \underline{\underline{44.95^\circ}}$$

$$\epsilon_{max} \approx \frac{\frac{\omega^2 R}{g_0} \left(\frac{1}{2}\right)}{1 - \frac{\omega^2 R}{g_0} \left(\frac{1}{2}\right)} = 1.7 \times 10^{-3} \text{ rad}$$

$$= \underline{\underline{0.099^\circ}}$$

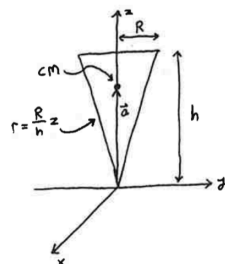
$$\omega = \frac{2\pi}{24 \text{ hr}}$$

$$R = 6.4 \times 10^6 \text{ m}$$

$$g_0 = 9.8 \text{ m/s}^2$$

$$\frac{\omega^2 R}{g_0} = 3.75 \times 10^{-3}$$

11-2 Inertia tensor for a cone: $I_{ij} = \int_V (\delta_{ij} r^2 - x_i x_j) dm$



$$V = \frac{\pi}{3} R^2 h \quad \rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

Cylindrical coordinates: $x = r \cos \theta$
 $y = r \sin \theta$
 $dm = \rho dV = \rho r dr d\theta dz$

$$\begin{aligned} I_{xx} &= \int_V (y^2 + z^2) dm \\ &= \rho \int_0^h dz \int_0^{\frac{R}{h}z} r dr \int_0^{2\pi} d\theta (r^2 \sin^2 \theta + z^2) \\ &= \rho \int_0^h dz \int_0^{\frac{R}{h}z} r dr (\pi r^2 + 2\pi z^2) \\ &= \pi \rho \int_0^h dz \left[\frac{r^4}{4} + z^2 r^2 \right]_0^{\frac{R}{h}z} \\ &= \pi \rho \int_0^h \left(\frac{R^4}{4h^4} + \frac{R^2}{h^2} \right) z^3 dz = \frac{\pi \rho}{5} \left(\frac{R^4}{4} h + R^2 h^3 \right) = \frac{\pi \rho}{20} R^2 h (R^2 + 4h^2) \\ &= \underline{\underline{\frac{3M}{20} (R^2 + 4h^2)}} \end{aligned}$$

$$I_{yy} = \int_V (x^2 + z^2) dm = I_{xx}$$

$$\begin{aligned} I_{zz} &= \int_V (x^2 + y^2) dm = \int_V r^2 dm = 2\pi \rho \int_0^h dz \int_0^{\frac{R}{h}z} r^3 dr = \frac{\pi \rho}{2} \frac{R^4}{h^4} \int_0^h z^2 dz \\ &= \frac{\pi \rho}{10} R^4 h = \underline{\underline{\frac{3M}{10} R^2}} \end{aligned}$$

cm of cone @ $\vec{a}_{cm} = \frac{3}{4} h \hat{z} \quad a^2 = \frac{9}{16} h^2$

For origin @ cm we use Steiner's parallel axis theorem: $I_{ij}^{cm} = I_{ij} - M(a^2 \delta_{ij} - a_i a_j)$

$$I_{xx}^{cm} = I_{xx} - M a^2 = \frac{3M}{20} (R^2 + 4h^2) - \frac{9M}{16} h^2 = \underline{\underline{\frac{3M}{20} (R^2 + \frac{1}{4} h^2)}}$$

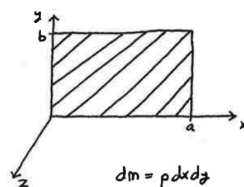
$$I_{yy}^{cm} = I_{yy} - M a^2 = I_{xx}^{cm}$$

$$I_{zz}^{cm} = I_{zz} - M(a^2 - a_z a_z) = I_{zz} = \underline{\underline{\frac{3M}{10} R^2}}$$

(A1) Inertia tensor for a rectangular plate

a) Origin @ plate corner:

$$\text{mass density } \rho = \begin{cases} m/ab & \text{for } 0 \leq x \leq a \\ & 0 \leq y \leq b \\ & z = 0 \\ 0 & \text{otherwise} \end{cases}$$



$$I_{xx} = \int_V (y^2 + z^2) dm = \rho \int_0^a dx \int_0^b y^2 dy = \frac{mb^3}{3}$$

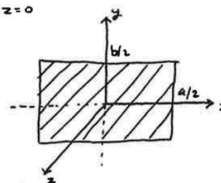
$$I_{yy} = \int_V (x^2 + z^2) dm = \rho \int_0^a x^2 dx \int_0^b dy = \frac{ma^3}{3}$$

$$I_{zz} = \int_V (x^2 + y^2) dm = \rho \int_0^a dx \int_0^b (x^2 + y^2) dy = \frac{m}{ab} \int_0^a \left(x^2 b + \frac{b^3}{3} \right) dx = \frac{m}{3} (a^2 + b^2)$$

$$I_{xy} = I_{yx} = - \int_V xy dm = -\rho \int_0^a x dx \int_0^b y dy = -\frac{m}{4} ab$$

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0 \quad \left(\begin{matrix} \rho \neq 0 \text{ for } \\ z=0 \text{ only} \end{matrix} \right) \quad \therefore I = \begin{pmatrix} \frac{mb^3}{3} & -\frac{mab}{4} & 0 \\ -\frac{mab}{4} & \frac{ma^3}{3} & 0 \\ 0 & 0 & \frac{m}{3}(a^2 + b^2) \end{pmatrix}$$

b) Origin @ plate center: $\rho = \begin{cases} m/ab & \text{for } -a/2 \leq x \leq a/2 \\ & -b/2 \leq y \leq b/2, z=0 \\ 0 & \text{otherwise} \end{cases}$



$$I_{xx} = \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} y^2 dy = \frac{m}{ab} a \frac{2}{3} \left(\frac{b}{2} \right)^3 = \frac{mb^3}{12}$$

$$I_{yy} = \rho \int_{-a/2}^{a/2} x^2 dx \int_{-b/2}^{b/2} dy = \frac{m}{ab} \frac{2}{3} \left(\frac{a}{2} \right)^3 b = \frac{ma^3}{12}$$

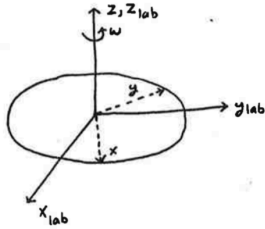
$$I_{zz} = \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} (x^2 + y^2) dy = \frac{m}{ab} \int_{-a/2}^{a/2} dx \left(x^2 b + \frac{b^3}{12} \right) = \frac{m}{12} (a^2 + b^2)$$

$$I_{xy} = I_{yx} = -\rho \int_{-a/2}^{a/2} x dx \int_{-b/2}^{b/2} y dy = 0 \quad (\text{integrals odd})$$

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0 \quad (\rho \neq 0 \text{ for } z=0 \text{ only})$$

$$I_{cm} = \begin{pmatrix} \frac{mb^3}{12} & 0 & 0 \\ 0 & \frac{ma^3}{12} & 0 \\ 0 & 0 & \frac{m}{12}(a^2 + b^2) \end{pmatrix}$$

(A2) Hamiltonian in a rotating system



$$x_{lab} = x \cos \omega t - y \sin \omega t$$

$$y_{lab} = y \cos \omega t + x \sin \omega t$$

$$\dot{x}_{lab} = \dot{x} \cos \omega t - \dot{y} \sin \omega t - x \omega \sin \omega t - y \omega \cos \omega t$$

$$= (\dot{x} - y\omega) \cos \omega t - (\dot{y} + x\omega) \sin \omega t$$

$$\dot{y}_{lab} = \dot{y} \cos \omega t + \dot{x} \sin \omega t - y \omega \sin \omega t + x \omega \cos \omega t$$

$$= (\dot{y} + x\omega) \cos \omega t + (\dot{x} - y\omega) \sin \omega t$$

$$\begin{aligned} \text{a) } L &= \frac{1}{2} m (\dot{x}_{lab}^2 + \dot{y}_{lab}^2) - U(x_{lab}, y_{lab}) \\ &= \frac{1}{2} m \left\{ (\dot{x} - y\omega)^2 + (\dot{y} + x\omega)^2 \right\} - U(x, y) \\ &= \frac{1}{2} m \left\{ \dot{x}^2 + \dot{y}^2 + \omega^2 (x^2 + y^2) + 2\omega (x\dot{y} - y\dot{x}) \right\} - U(x, y) \end{aligned}$$

$$\text{b) } p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - m\omega y = m(\dot{x} - \omega y)$$

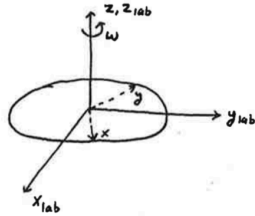
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\omega x = m(\dot{y} + \omega x)$$

$$\begin{aligned} \text{c) } H(x, y, p_x, p_y) &= p_x \dot{x} + p_y \dot{y} - L \\ &= p_x \left(\frac{p_x}{m} + \omega y \right) + p_y \left(\frac{p_y}{m} - \omega x \right) - \frac{1}{2} m \left\{ \left(\frac{p_x}{m} \right)^2 + \left(\frac{p_y}{m} \right)^2 \right\} - U(x, y) \\ &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \omega (p_x y - p_y x) + U(x, y) \end{aligned}$$

$$\begin{aligned} \text{d) } l_z &= x p_y - y p_x \Rightarrow H = \frac{1}{2m} (p_x^2 + p_y^2) - \omega l_z + U(x, y) \\ &= \underline{\underline{H_{\omega=0} - \omega l_z}} \end{aligned}$$

$$\begin{aligned} \text{e) } \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + \omega y & \dot{p}_x &= -\frac{\partial H}{\partial x} = \omega p_y - \frac{\partial U}{\partial x} \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - \omega x & \dot{p}_y &= -\frac{\partial H}{\partial y} = -\omega p_x - \frac{\partial U}{\partial y} \end{aligned}$$

(c1) Force free motion in a rotating system



Acceleration is given by:

$$\vec{a}_r = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \vec{v}_r$$

where $\vec{\omega} = \omega \hat{z}$

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{v}_r = \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$\vec{\omega} \times \vec{v}_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = -\omega\dot{y}\hat{x} + \omega\dot{x}\hat{y}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y\hat{x} + \omega x\hat{y}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} = -\omega^2 x\hat{x} - \omega^2 y\hat{y}$$

$$\therefore \begin{cases} a_x = \omega^2 x + 2\omega\dot{y} \\ a_y = \omega^2 y - 2\omega\dot{x} \end{cases}$$

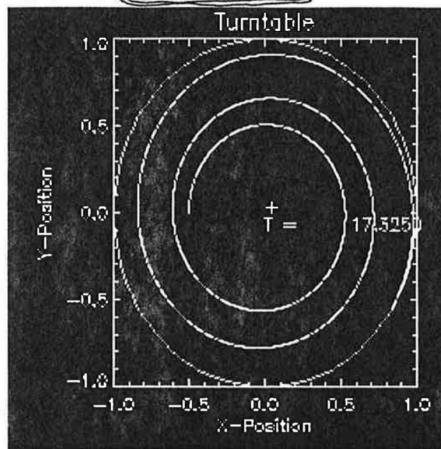
We can also get these equations of motion from Hamilton's equations in the rotating frame: ($U=0$) [see A2, part e]

$$\ddot{x} = \frac{\dot{p}_x}{m} + \omega\dot{y} = \frac{\omega p_y}{m} + \omega\dot{y} = \omega(\dot{y} + \omega x) + \omega\dot{y} = \omega^2 x + 2\omega\dot{y}$$

$$\ddot{y} = \frac{\dot{p}_y}{m} - \omega\dot{x} = \frac{-\omega p_x}{m} - \omega\dot{x} = -\omega(\dot{x} - \omega y) - \omega\dot{x} = \omega^2 y - 2\omega\dot{x}$$

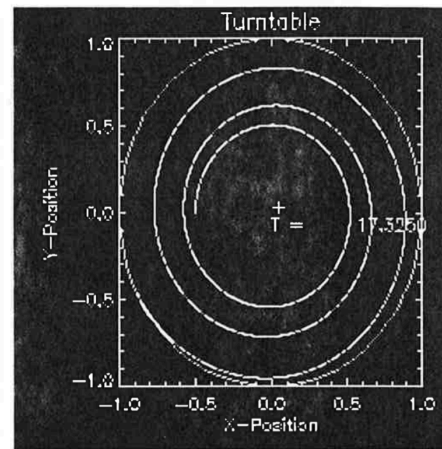
$\omega = 1.0 \text{ rad/s}$ ↻
 $x_0 = -0.5 \text{ m}$
 $y_0 = 0.0 \text{ m}$

release angle
 $\theta = 90^\circ$ ↑



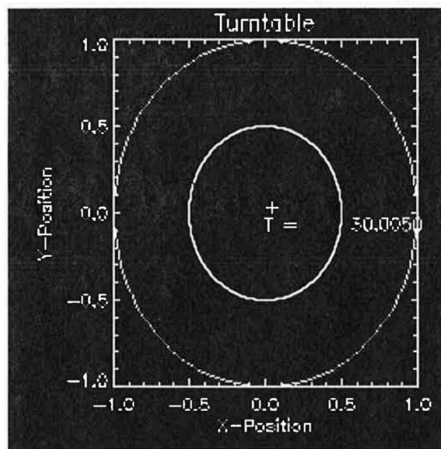
$V_0 = 0.45 \text{ m/s}$
 Fig. 10.4 (c)

speed up



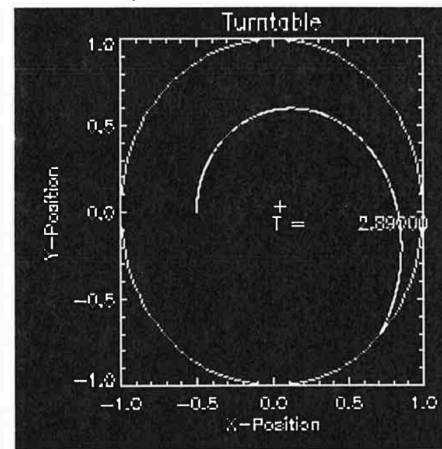
$V_0 = 0.55 \text{ m/s}$

speed up



$V_0 = 0.50 \text{ m/s}$

Particle stays on the table here.

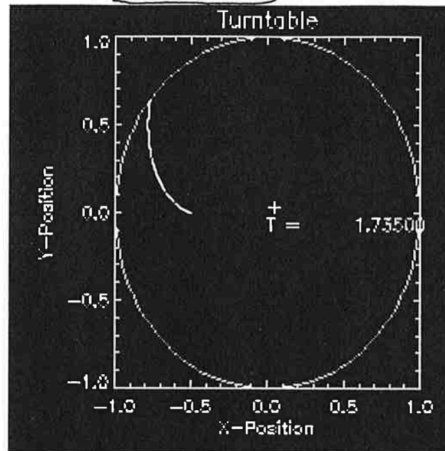


$V_0 = 0.80 \text{ m/s}$

Fig. 10.4 (b)

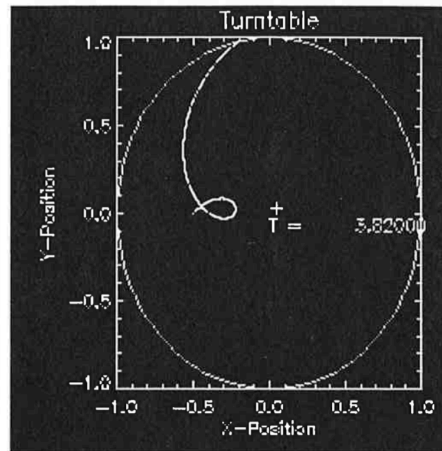
$\omega = 1.0 \text{ rad/s}$
 $x_0 = -0.5 \text{ m}$
 $y_0 = 0.0 \text{ m}$

release angle
 $\theta = 45^\circ$



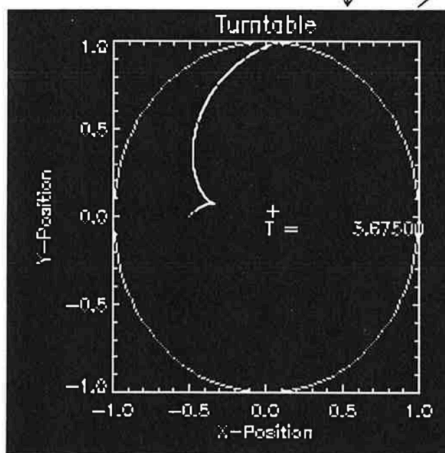
$V_0 = 0.0 \text{ m/s}$
 Release with no velocity
 ... Particle leaves table!!

speed up

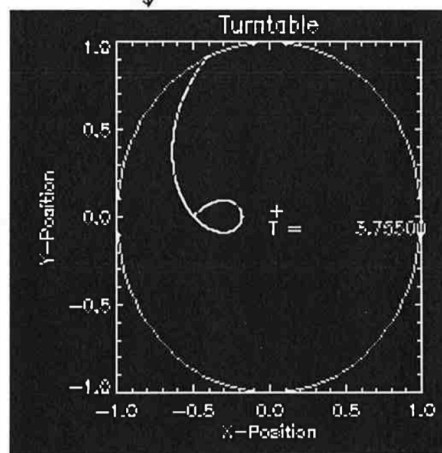


$V_0 = 0.47 \text{ m/s}$
 speed up

Fig. 10.4 (e)



$V_0 = 0.36 \text{ m/s}$

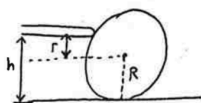


$V_0 = 0.51 \text{ m/s}$

Here the particle returns to the
 release point!

Assignment 10 - Solutions

(11-5) No "English" condition in billiards



a)

At what height $h = R + r$ do we strike the ball so it rolls w/o slipping?

We will assume we shoot w/ the cue parallel to the table surface.

Model shot as an impulsive force $\vec{F}$.

We define the "impulse" as $\vec{J} = \int \vec{F}(t) dt$

$$\text{Since } \vec{F} = \frac{d\vec{p}}{dt} \text{ and } \vec{L} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \text{ we have: } \vec{J} = \Delta\vec{p} \quad \vec{r} \times \vec{J} = \Delta\vec{L}$$

We want to satisfy the rolling condition: $v = \omega R$

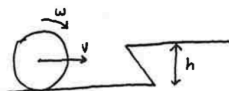
The ball is initially @ rest so $\Delta p = mv$

$$\Delta L = I\omega \quad (\text{where } I = \frac{2}{5}mR^2)$$

$$|\vec{r} \times \vec{J}| = rJ = \Delta L \Rightarrow r = \frac{\Delta L}{J} = \frac{\Delta L}{\Delta p} = \frac{I\omega}{mv} = \frac{I}{mR} = \frac{2}{5}R$$

$$\therefore h = R + r = \frac{7}{5}R$$

b)



This is basically the same as part (a).

We want a rolling ball to rebound off the rail w/o any slipping in the outgoing state.

$$\text{We assume an elastic collision with } p_f = -p_i \Rightarrow \Delta p = 2mv$$

$$L_f = -L_i \Rightarrow \Delta L = 2I\omega$$

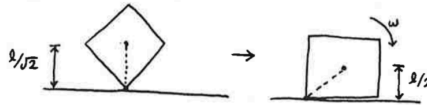
And rolling so $v = \omega R$.

$$\text{As above } h - R = \frac{\Delta L}{\Delta p} = \frac{I\omega}{mv} = \frac{2}{5}R$$

$$h = \frac{7}{5}R$$

11-11 Toppled Cube

a) Contact Edge does not slip



Initial energy:

$$E_i = mg l / \sqrt{2}$$

Final energy (as cube strikes surface):

$$E_f = mg l / 2 + \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$v_{cm} = \omega l / \sqrt{2} \quad I_{cm} = \frac{1}{6} m l^2$$

$$\text{Energy conservation: } mg l / \sqrt{2} = mg l / 2 + \frac{1}{2} m \omega^2 l^2 / 2 + \frac{1}{12} m l^2 \omega^2$$

$$g \left(\frac{\sqrt{2}-1}{2} \right) = \left(\frac{1}{4} l + \frac{1}{12} l \right) \omega^2$$

$$\omega^2 = \frac{3}{2} \frac{g}{l} (\sqrt{2}-1) \quad \text{for no slip}$$

b) Contact Edge slips w/o friction



linear momentum conserved in horizontal direction
 $\Rightarrow$ CM must fall straight down

$$\text{Height of CM given by } z = \frac{l}{\sqrt{2}} \cos \theta$$

$$\text{Speed of CM given by } \dot{z} = -\frac{l}{\sqrt{2}} \sin \theta \dot{\theta} \quad \text{or} \quad v_{cm} = -\frac{l}{\sqrt{2}} \sin \theta \omega$$

$$\text{Cube strikes surface @ } \theta = 45^\circ \Rightarrow v_{cm} = -\frac{l}{2} \omega$$

Same energy conservation equation as in part a:

$$mg l / \sqrt{2} = mg l / 2 + \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

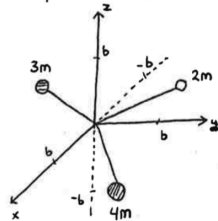
$$= mg l / 2 + \frac{1}{8} m l^2 \omega^2 + \frac{1}{12} m l^2 \omega^2$$

$$g \left(\frac{\sqrt{2}-1}{2} \right) = \frac{5}{24} l \omega^2$$

$$\omega^2 = \frac{12}{5} \frac{g}{l} (\sqrt{2}-1) \quad \text{for slipping w/o friction}$$

Note rotation speed @ impact is much larger for part b.

(11-13) 3 particle system



$$m_1 = 3m \text{ @ } (b, 0, b)$$

$$m_2 = 2m \text{ @ } (b, b, -b)$$

$$m_3 = 2m \text{ @ } (-b, b, 0)$$

$$I_{xx} = \sum_k m_k (y_k^2 + z_k^2) = 3m(b^2) + 4m(2b^2) + 2m(b^2) = 13mb^2$$

$$I_{yy} = \sum_k m_k (x_k^2 + z_k^2) = 3m(2b^2) + 4m(2b^2) + 2m(b^2) = 16mb^2$$

$$I_{zz} = \sum_k m_k (x_k^2 + y_k^2) = 3m(b^2) + 4m(2b^2) + 2m(2b^2) = 15mb^2$$

$$I_{12} = I_{21} = -\sum_k m_k x_k y_k = -[3m(0) + 4m(b^2) + 2m(-b^2)] = -2mb^2$$

$$I_{13} = I_{31} = -\sum_k m_k x_k z_k = -[3m(b^2) + 4m(-b^2) + 2m(0)] = mb^2$$

$$I_{23} = I_{32} = -\sum_k m_k y_k z_k = -[3m(0) + 4m(-b^2) + 2m(0)] = 4mb^2$$

Moment of
Inertia
Tensor:

$$I = mb^2 \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

Principle moments of inertia given by:

$$\begin{vmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{vmatrix} = 0$$

Expanding the determinant will give a cubic equation for λ :

$$(13-\lambda)[(16-\lambda)(15-\lambda)-16] - 2[4+2(15-\lambda)] + [-8-(16-\lambda)] = 0$$

$$(13-\lambda)[224-31\lambda+\lambda^2] - 2[34-2\lambda] + [\lambda-24] = 0$$

$$\lambda^3 - 44\lambda^2 + 622\lambda - 2820 = 0$$

Solving the cubic numerically* gives:

$$\lambda_1 = 10.000 \quad (= 10)$$

$$\lambda_2 = 14.354 \quad (= 17 - \sqrt{5})$$

$$\lambda_3 = 19.646 \quad (= 17 + \sqrt{5})$$

$$\text{Principle moments of inertia: } I_n = \lambda_n mb^2$$

* I used IDL as follows:

$$\text{coefs} = [-2820, 622, -44, 1]$$

$$\text{roots} = \text{fz_roots}(\text{coefs}, 1/\text{double})$$

print, roots

11-13 cont.

To find principle axes we use the fact that if $\vec{\omega}_n$ is directed along the n -th principle axis we get $\vec{L} = \{I\} \cdot \vec{\omega}_n = \lambda_n m b^2 \vec{\omega}_n$

This gives the equation:

$$\begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} \begin{pmatrix} \omega_1^{(n)} \\ \omega_2^{(n)} \\ \omega_3^{(n)} \end{pmatrix} = \lambda_n \begin{pmatrix} \omega_1^{(n)} \\ \omega_2^{(n)} \\ \omega_3^{(n)} \end{pmatrix}$$

where $\vec{\omega}_n = \omega_1^{(n)} \hat{x} + \omega_2^{(n)} \hat{y} + \omega_3^{(n)} \hat{z}$ and thus the n -th eigenvector is $\hat{e}_n = \frac{\vec{\omega}_n}{|\vec{\omega}_n|}$.

For $\lambda_1 = 10.000$:

$$\left. \begin{aligned} 3\omega_1 - 2\omega_2 + \omega_3 &= 0 \\ -2\omega_1 + 6\omega_2 + 4\omega_3 &= 0 \\ \omega_1 + 4\omega_2 + 5\omega_3 &= 0 \end{aligned} \right\} \begin{aligned} 14\omega_2 + 14\omega_3 &= 0 \Rightarrow \frac{\omega_2}{\omega_3} = -1 \\ 3\omega_1 + 3\omega_3 &= 0 \Rightarrow \frac{\omega_3}{\omega_1} = -1 \end{aligned}$$

and thus $\frac{\omega_2}{\omega_1} = 1$

$$\therefore \omega_1 : \omega_2 : \omega_3 \Rightarrow 1 : 1 : -1 \quad N = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{3} \quad \hat{e}_1 = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} - \hat{z})$$

For $\lambda_2 = 14.354$:

$$\left. \begin{aligned} -1.354\omega_1 - 2\omega_2 + \omega_3 &= 0 \\ -2\omega_1 + 1.646\omega_2 + 4\omega_3 &= 0 \\ \omega_1 + 4\omega_2 + 0.646\omega_3 &= 0 \end{aligned} \right\} \begin{aligned} 9.646\omega_2 + 5.292\omega_3 &= 0 \Rightarrow \frac{\omega_2}{\omega_3} = -0.549 \\ -1.354\omega_1 + 2.097\omega_3 &= 0 \Rightarrow \frac{\omega_3}{\omega_1} = 0.646 \end{aligned}$$

and thus $\frac{\omega_2}{\omega_1} = -0.354$

$$\therefore \omega_1 : \omega_2 : \omega_3 \Rightarrow 1 : -0.354 : 0.646 \quad N = 1.242$$

$$\hat{e}_2 = 0.805\hat{x} - 0.285\hat{y} + 0.520\hat{z}$$

For $\lambda_3 = 19.646$:

$$\left. \begin{aligned} -6.646\omega_1 - 2\omega_2 + \omega_3 &= 0 \\ -2\omega_1 - 3.646\omega_2 + 4\omega_3 &= 0 \\ \omega_1 + 4\omega_2 - 4.646\omega_3 &= 0 \end{aligned} \right\} \begin{aligned} 4.354\omega_2 - 5.292\omega_3 &= 0 \Rightarrow \frac{\omega_2}{\omega_3} = 1.215 \\ -6.646\omega_1 - 1.430\omega_3 &= 0 \Rightarrow \frac{\omega_3}{\omega_1} = -4.648 \end{aligned}$$

and thus $\frac{\omega_2}{\omega_1} = -5.647$

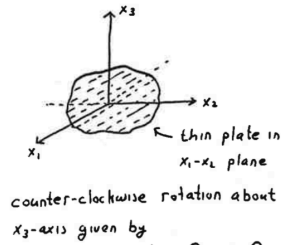
$$\therefore \omega_1 : \omega_2 : \omega_3 \Rightarrow 1 : -5.647 : -4.648 \quad N = 7.382$$

$$\hat{e}_3 = 0.135\hat{x} - 0.765\hat{y} - 0.630\hat{z}$$

[Note that the eigenvectors are orthonormal: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$]

(11-18) Transformation of the inertia tensor for a thin plate

$$I = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$



$$\lambda = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Under rotation the tensor is transformed via the similarity transformation: $I' = \lambda I \lambda^{-1}$

(Note that $\lambda^{-1} = \lambda^T$ for rotation matrices.)

$$I \lambda^{-1} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A \cos \theta - C \sin \theta & -A \sin \theta - C \cos \theta & 0 \\ -C \cos \theta + B \sin \theta & C \sin \theta + B \cos \theta & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

$$I' = \lambda (I \lambda^{-1}) = \begin{pmatrix} A \cos^2 \theta - C \sin \theta \cos \theta - C \sin \theta \cos \theta + B \sin^2 \theta & -A \sin \theta \cos \theta - C \cos^2 \theta + C \sin^2 \theta + B \cos \theta \sin \theta & 0 \\ -A \sin \theta \cos \theta + C \sin^2 \theta - C \cos^2 \theta + B \sin \theta \cos \theta & A \sin^2 \theta + C \sin \theta \cos \theta + C \sin \theta \cos \theta + B \cos^2 \theta & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

$$= \begin{pmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{pmatrix}$$

$$A' = A \cos^2 \theta + B \sin^2 \theta - 2C \sin \theta \cos \theta \quad (A'+B' = A+B)$$

$$B' = A \sin^2 \theta + B \cos^2 \theta + 2C \sin \theta \cos \theta$$

$$-C' = (-A+B) \sin \theta \cos \theta + C (\sin^2 \theta - \cos^2 \theta)$$

note: $2 \sin \theta \cos \theta = \sin 2\theta$
 $\sin^2 \theta - \cos^2 \theta = -\cos 2\theta$

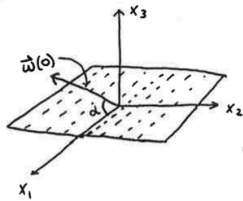
$$\Rightarrow \begin{cases} A' = A \cos^2 \theta + B \sin^2 \theta - C \sin 2\theta \\ B' = A \sin^2 \theta + B \cos^2 \theta + C \sin 2\theta \\ C' = \frac{1}{2}(A-B) \sin 2\theta + C \cos 2\theta \end{cases}$$

x'_1, x'_2 are principle axes if I' is diagonal

$$\therefore C' = 0 \Rightarrow \tan 2\theta = \frac{2C}{B-A}$$

$$\text{or } \theta = \frac{1}{2} \arctan \frac{2C}{B-A}$$

11-31 Freely rotating plate



Initial conditions: $\omega_1(0) = \Omega \cos \alpha$
 $\omega_2(0) = 0$
 $\omega_3(0) = \Omega \sin \alpha$

Plate details: $I_1 = I_2 \cos 2\alpha$
 $I_3 = I_1 + I_2$
 $= I_2 (1 + \cos 2\alpha)$
 $= 2 I_2 \cos^2 \alpha$

$$\begin{bmatrix} \cos 2\alpha = 2 \cos^2 \alpha - 1 \\ = 1 - 2 \sin^2 \alpha \end{bmatrix}$$

Motion described by Euler's equations:

$$\begin{aligned} (I_1 - I_2) \omega_1 \omega_2 - I_3 \dot{\omega}_3 &= 0 \\ (I_2 - I_3) \omega_1 \omega_3 - I_1 \dot{\omega}_1 &= 0 \\ (I_3 - I_1) \omega_2 \omega_3 - I_2 \dot{\omega}_2 &= 0 \end{aligned}$$

here $I_1 - I_2 = I_2 (1 - \cos 2\alpha) = -2 I_2 \sin^2 \alpha$
 $I_2 - I_3 = -I_1$
 $I_3 - I_1 = I_2$

So Euler's equations simplify to:

$$\begin{aligned} \omega_1 \omega_2 \sin^2 \alpha + \dot{\omega}_3 \cos^2 \alpha &= 0 \Rightarrow \dot{\omega}_3 = -\omega_1 \omega_2 \tan^2 \alpha \quad (1) \\ \omega_1 \omega_3 + \dot{\omega}_1 &= 0 \Rightarrow \dot{\omega}_1 = -\omega_1 \omega_3 \quad (2) \\ \omega_3 \omega_1 - \dot{\omega}_2 &= 0 \Rightarrow \dot{\omega}_2 = \omega_1 \omega_3 \quad (3) \end{aligned}$$

We want $\omega_2(t)$, thus we need to get expressions for ω_1 and ω_3 in terms of ω_2 . Then we can write (3) solely in terms of ω_2 .

Combining (2) and (3) gives: $\frac{\dot{\omega}_1}{\dot{\omega}_2} = -\frac{\omega_2}{\omega_1} \Rightarrow \omega_1 d\omega_1 = -\omega_2 d\omega_2$

Integrating $\Rightarrow \frac{1}{2} [\omega_1^2(t) - \omega_1^2(0)] = -\frac{1}{2} [\omega_2^2(t) - \omega_2^2(0)]$

$$\omega_1^2(t) = \Omega^2 \cos^2 \alpha - \omega_2^2(t)$$

Combining (1) and (3) gives: $\frac{\dot{\omega}_3}{\dot{\omega}_2} = -\frac{\omega_2 \tan^2 \alpha}{\omega_3} \Rightarrow \omega_3 d\omega_3 = -\tan^2 \alpha \omega_2 d\omega_2$

Integrating $\Rightarrow \frac{1}{2} [\omega_3^2(t) - \omega_3^2(0)] = -\frac{1}{2} \tan^2 \alpha [\omega_2^2(t) - \omega_2^2(0)]$

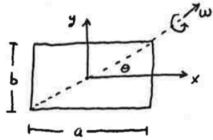
$$\omega_3^2(t) = \Omega^2 \sin^2 \alpha - \omega_2^2(t) \tan^2 \alpha$$

Inserting these two results into (3):

$$\boxed{\dot{\omega}_2 = (\Omega^2 \cos^2 \alpha - \omega_2^2) \tan \alpha} \Rightarrow \frac{d\omega_2}{\omega_2^2 - \Omega^2 \cos^2 \alpha} = \tan \alpha dt \Rightarrow \boxed{\omega_2 = \Omega \cos \alpha \tanh(\Omega t \sin \alpha)}$$

$\leftarrow \text{integral E4.c}$

(A1) Rotating rectangular plate



$$I = \frac{m}{12} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

$$\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$$

$$\omega_x = \omega \cos \theta = \frac{\omega a}{\sqrt{a^2 + b^2}}, \quad \omega_y = \omega \sin \theta = \frac{\omega b}{\sqrt{a^2 + b^2}}, \quad \omega_z = 0$$

$$a) T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j$$

$$= \frac{1}{2} \frac{m}{12} (b^2 \omega_x^2 + a^2 \omega_y^2) = \frac{m}{12} \frac{a^2 b^2}{a^2 + b^2} \omega^2$$

$$b) L_i = \sum_j I_{ij} \omega_j \Rightarrow L_x = \frac{m}{12} \frac{a b^3}{\sqrt{a^2 + b^2}} \omega$$

$$L_y = \frac{m}{12} \frac{a^3 b}{\sqrt{a^2 + b^2}} \omega$$

$$L_z = 0$$

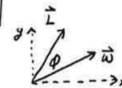
$$\vec{L} = L_x \hat{x} + L_y \hat{y}$$

$$|\vec{L}| = [L_x^2 + L_y^2]^{1/2} = \frac{m}{12} a b \omega$$

$$\text{Angle between } \vec{\omega} \text{ and } \vec{L} \text{ given by } \frac{\vec{\omega} \cdot \vec{L}}{\omega |\vec{L}|} = \cos \phi$$

$$\vec{\omega} \cdot \vec{L} = \omega_x L_x + \omega_y L_y = 2 \frac{m}{12} \frac{a^2 b^2}{a^2 + b^2} \omega^2 \Rightarrow \phi = \arccos \left(\frac{2ab}{a^2 + b^2} \right)$$

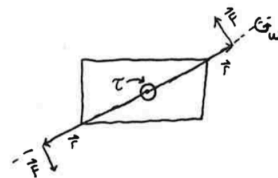
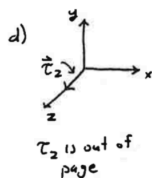
$$\phi = 0 \text{ if } \cos \phi = 1 \Rightarrow a = b$$



$$c) \text{ Constant rotation so } \dot{\vec{\omega}} = 0$$

Since $\omega_z = 0$ we have only one Euler equation to deal with:

$$\tau_z = -(I_x - I_y) \omega_x \omega_y = -\frac{m}{12} (b^2 - a^2) \frac{\omega a}{\sqrt{a^2 + b^2}} \frac{\omega b}{\sqrt{a^2 + b^2}} = \frac{m a b (a^2 - b^2) \omega^2}{12 (a^2 + b^2)}$$

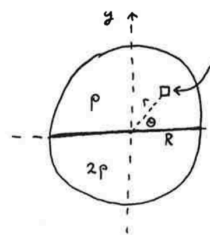


Must supply a pair of forces $\perp$ to the rotation axis, in the plane of the plate such that

$$\tau = 2rF$$

As plate rotates, direction of $\vec{F}$ must also rotate to remain in plane of plate.

(A2) Composite Disk



$dm = \rho(x, y) dx dy = \rho(r, \theta) r dr d\theta$

$$\rho(r, \theta) = \begin{cases} \rho & r < R, \quad 0 < \theta < \pi \\ 2\rho & r < R, \quad \pi < \theta < 2\pi \end{cases}$$

mass of disk = $M = \rho\left(\frac{1}{2}\pi R^2\right) + 2\rho\left(\frac{1}{2}\pi R^2\right) = \frac{3}{2}\rho\pi R^2$

CM will be on the y-axis and is given by:

$$y_{cm} = \frac{1}{M} \int y dm$$

$$= \frac{1}{M} \left\{ \rho \int_0^\pi d\theta \int_0^R (r \sin \theta) r dr + 2\rho \int_\pi^{2\pi} d\theta \int_0^R (r \sin \theta) r dr \right\}$$

$$= \frac{2}{3\rho\pi R^2} \left\{ \rho \frac{R^3}{3} \int_0^\pi \sin \theta d\theta + 2\rho \frac{R^3}{3} \int_\pi^{2\pi} \sin \theta d\theta \right\}$$

$$= \frac{2R}{9\pi} \left\{ 2 + 2(-2) \right\} = \underline{\underline{-\frac{4R}{9\pi}}} \quad (\approx -0.14R)$$

$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

To compute rolling kinetic energy we will need I_{cm} on an axis parallel to the z-axis. Let's compute I_z (through geometric center) first and then use the parallel axis theorem:

$$I_z = \int (x^2 + y^2) dm$$

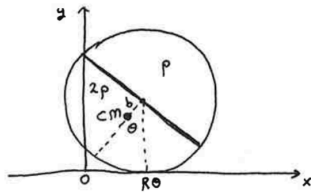
$$= \rho \int_0^\pi d\theta \int_0^R (r^2) r dr + 2\rho \int_\pi^{2\pi} d\theta \int_0^R (r^2) r dr$$

$$= \rho \frac{R^4}{4} \pi + 2\rho \frac{R^4}{4} \pi$$

$$= \frac{3}{4} \pi \rho R^4 = \frac{1}{2} M R^2$$

$$I_{cm} = I_z - M \left(\frac{4R}{9\pi} \right)^2 = \frac{1}{2} M R^2 - M \frac{16R^2}{81\pi^2} = \underline{\underline{\frac{1}{2} M R^2 \left(1 - \frac{32}{81\pi^2} \right)}}$$

A2-cont.



Disk center @ $x = R\theta, y = R$

Disk cm @ $x_{cm} = R\theta - b \sin \theta$ where $b = \frac{4R}{9\pi}$
 $y_{cm} = R - b \cos \theta$

θ = generalized coordinate for Lagrangian

$$T = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \dot{\theta}^2$$

$$U = m g y_{cm}$$

a)

$$\dot{x}_{cm} = R \dot{\theta} - b \cos \theta \dot{\theta}$$

$$\dot{y}_{cm} = b \sin \theta \dot{\theta}$$

$$v_{cm}^2 = \dot{x}_{cm}^2 + \dot{y}_{cm}^2 = b^2 \dot{\theta}^2 + R^2 \dot{\theta}^2 - 2Rb \cos \theta \dot{\theta}^2$$

$$T = \frac{1}{2} M (b^2 + R^2 - 2Rb \cos \theta) \dot{\theta}^2 + \frac{1}{4} M R^2 \left(1 - \frac{32}{81\pi^2}\right) \dot{\theta}^2$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2 \left\{ \frac{16}{81\pi^2} + 1 - \frac{8}{9\pi} \cos \theta + \frac{1}{2} - \frac{16}{81\pi^2} \right\}$$

$$= \frac{1}{2} M R^2 \dot{\theta}^2 \left\{ \frac{3}{2} - \frac{8}{9\pi} \cos \theta \right\}$$

$$U = m g \left(R - \frac{4R}{9\pi} \cos \theta \right)$$

$$L = T - U = \frac{1}{2} M R^2 \dot{\theta}^2 \left\{ \frac{3}{2} - \frac{8}{9\pi} \cos \theta \right\} - m g R \left\{ 1 - \frac{4}{9\pi} \cos \theta \right\}$$

$$b) \frac{\partial L}{\partial \theta} = \frac{1}{2} M R^2 \dot{\theta}^2 \left\{ \frac{8}{9\pi} \sin \theta \right\} - m g R \frac{4}{9\pi} \sin \theta = \frac{4}{9\pi} m R (R \dot{\theta}^2 - g) \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = M R^2 \dot{\theta} \left\{ \frac{3}{2} - \frac{8}{9\pi} \cos \theta \right\}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = M R^2 \ddot{\theta} \left\{ \frac{3}{2} - \frac{8}{9\pi} \cos \theta \right\} + M R^2 \dot{\theta}^2 \left\{ \frac{8}{9\pi} \sin \theta \right\}$$

Euler-Lagrange:

$$\left(\frac{3}{2} - \frac{8}{9\pi} \cos \theta \right) \ddot{\theta} + \left(\frac{8}{9\pi} \sin \theta \right) \dot{\theta}^2 - \left(\frac{4}{9\pi} \sin \theta \right) \dot{\theta}^2 + \frac{4}{9\pi} \frac{g}{R} \sin \theta = 0$$

$$\left(\frac{27\pi}{2} - 8 \cos \theta \right) \ddot{\theta} + 4 \sin \theta \dot{\theta}^2 + 4 \frac{g}{R} \sin \theta = 0$$

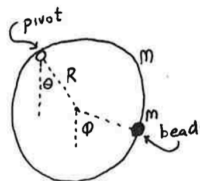
c) For small θ we have:

$$\left(\frac{27\pi}{2} - 8 \right) \ddot{\theta} + 4 \frac{g}{R} \theta = 0 \quad \text{or} \quad \ddot{\theta} + \omega^2 \theta = 0 \quad \text{with}$$

$$\omega^2 = \frac{8}{27\pi - 16} \frac{g}{R}$$

Assignment 11 - Solutions

12-16 Swinging hoop w/ sliding bead



We will take the pivot as the origin.

Generalized coordinates: Θ locates hoop CM relative to pivot

ϕ locates bead relative to hoop CM

$$\text{Hoop: } T_{\text{hoop}} = \frac{1}{2} I_{\text{pivot}} \dot{\Theta}^2 \quad \text{where } I_{\text{pivot}} = I_{\text{cm}} + MR^2 = 2MR^2$$

$$U_{\text{hoop}} = -mgR \cos \Theta$$

$$\text{Bead: } x_b = R \sin \Theta + R \sin \phi \quad \dot{x}_b = R \dot{\Theta} \cos \Theta + R \dot{\phi} \cos \phi$$

$$y_b = -R \cos \Theta - R \cos \phi \quad \dot{y}_b = R \dot{\Theta} \sin \Theta + R \dot{\phi} \sin \phi$$

$$\begin{aligned} T_b &= \frac{1}{2} m (\dot{x}_b^2 + \dot{y}_b^2) \\ &= \frac{1}{2} m \{ R^2 \dot{\Theta}^2 + R^2 \dot{\phi}^2 + 2R^2 \dot{\Theta} \dot{\phi} (\cos \Theta \cos \phi + \sin \Theta \sin \phi) \} \\ &= \frac{1}{2} m R^2 \{ \dot{\Theta}^2 + \dot{\phi}^2 + 2 \dot{\Theta} \dot{\phi} \cos(\Theta - \phi) \} \end{aligned}$$

$$U_b = -mgR (\cos \Theta + \cos \phi)$$

Here we could construct the Lagrangian and derive the equations of motion, but instead we'll use the general results for coupled oscillators.

$$T = T_{\text{hoop}} + T_b$$

$$\begin{aligned} &= \frac{1}{2} \{ 2mR^2 \dot{\Theta}^2 + mR^2 [\dot{\Theta}^2 + \dot{\phi}^2 + 2\dot{\Theta} \dot{\phi} \cos(\Theta - \phi)] \} \\ &\approx \frac{1}{2} mR^2 \{ 3\dot{\Theta}^2 + \dot{\phi}^2 + 2\dot{\Theta} \dot{\phi} \} \quad \leftarrow \text{keeping terms to 2nd order} \end{aligned}$$

for small oscillations
 $\cos(\Theta - \phi) \approx 1 - \frac{1}{2}(\Theta - \phi)^2 + \dots$

$$U = U_{\text{hoop}} + U_b$$

$$= -mgR \cos \Theta - mgR (\cos \Theta + \cos \phi)$$

$$= -mgR \{ 2 \cos \Theta + \cos \phi \}$$

$$= -mgR \left\{ 2 \left(1 - \frac{1}{2} \Theta^2 \right) + \left(1 - \frac{1}{2} \phi^2 \right) \right\} \quad \leftarrow \text{expanding cosines and keeping terms to 2nd order}$$

$$= -mgR \left\{ 3 - \Theta^2 - \frac{1}{2} \phi^2 \right\}$$

$$= U_0 + \frac{1}{2} mgR \{ 2\Theta^2 + \phi^2 \}$$

12-16 cont.

We let $g_i = \theta$, $g_i = \phi$.

(the "coupling constant" matrices)

$$T = \frac{1}{2} \sum_{i,j} m_{ij} \dot{g}_i \dot{g}_j \Rightarrow \{m\} = mR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

once we identify $\{m\}$ and $\{A\}$ everything else is linear algebra

$$U = U_0 + \frac{1}{2} \sum_{i,j} A_{ij} g_i g_j \Rightarrow \{A\} = m_g R \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

eigenfrequencies given by:

$$|A_{ij} - \omega^2 m_{ij}| = mR \begin{vmatrix} 2g - 3\omega^2 R & -\omega^2 R \\ -\omega^2 R & g - \omega^2 R \end{vmatrix} = 0$$

$$\Rightarrow (2g - 3\omega^2 R)(g - \omega^2 R) - \omega^4 R^2 = 0$$

$$2\omega^4 - 5 \frac{g}{R} \omega^2 + 2 \left(\frac{g}{R}\right)^2 = 0$$

$$\omega^2 = \frac{5 \frac{g}{R} \pm \sqrt{25 \frac{g^2}{R^2} - 16 \frac{g^2}{R^2}}}{4} = \frac{g}{4R} \{5 \pm 3\}$$

$$\boxed{\omega_1 = \sqrt{\frac{2g}{R}}, \quad \omega_2 = \sqrt{\frac{g}{2R}}}$$

For normal modes we must determine normal coordinates n_1, n_2 where $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

The coefficients a_{jr} are determined from $\sum_j (A_{jk} - \omega_k^2 m_{jk}) a_{jr} = 0$.

$$\text{For } k=1, r=1: (A_{11} - \omega_1^2 m_{11}) a_{11} + (A_{12} - \omega_1^2 m_{12}) a_{21} = 0$$

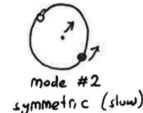
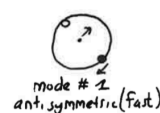
$$(2m_g R - \frac{2g}{R} 3mR^2) a_{11} + (0 - \frac{2g}{R} mR^2) a_{21} = 0 \Rightarrow a_{21} = -2a_{11}$$

$$\text{For } k=1, r=2: (A_{11} - \omega_1^2 m_{11}) a_{12} + (A_{12} - \omega_1^2 m_{12}) a_{22} = 0$$

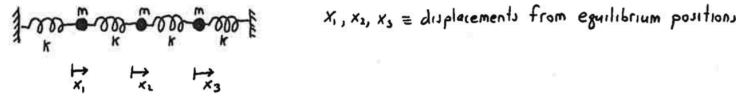
$$(2m_g R - \frac{g}{2R} 3mR^2) a_{12} + (0 - \frac{g}{2R} mR^2) a_{22} = 0 \Rightarrow a_{12} = a_{22}$$

$$\therefore \begin{cases} g_1 = a_{11} n_1 + a_{21} n_2 \\ g_2 = -2a_{11} n_1 + a_{21} n_2 \end{cases} \Rightarrow \begin{cases} n_1 = \frac{g_1 - g_2}{3a_{11}} = \frac{\theta - \phi}{3a_{11}} \\ n_2 = \frac{2g_1 + g_2}{3a_{21}} = \frac{2\theta + \phi}{3a_{21}} \end{cases}$$

Mode #1 given by $\theta_0 = -\frac{1}{2}\phi_0$ (which gives $n_2=0$)
 Mode #2 given by $\theta_0 = \phi_0$ (which gives $n_1=0$)



(12-17) 3 mass, 4 spring coupled system



$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2$$

$$= \frac{1}{2} k \{ 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \}$$

$$\{m\} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = k \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Eigenfrequencies given by secular determinant:

$$\begin{vmatrix} 2k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 m & -k \\ 0 & -k & 2k - \omega^2 m \end{vmatrix} = (2k - \omega^2 m) \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} - k \begin{vmatrix} -k & -k \\ 2k - \omega^2 m & 0 \end{vmatrix}$$

$$= (2k - \omega^2 m) [(2k - \omega^2 m)^2 - k^2] - k^2 (2k - \omega^2 m)$$

$$= (2k - \omega^2 m) [(2k - \omega^2 m)^2 - 2k^2]$$

$$= (2k - \omega^2 m) (2k - \omega^2 m - \sqrt{2}k) (2k - \omega^2 m + \sqrt{2}k) = 0$$

$$\therefore \boxed{\omega_1^2 = \frac{2k}{m} \quad \omega_2^2 = \frac{(2-\sqrt{2})k}{m} \quad \omega_3^2 = \frac{(2+\sqrt{2})k}{m}}$$

intermediate slow fast

For normal modes we need

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

where a_{jr} given by $\sum_j (A_{jk} - \omega_r^2 m_{jk}) a_{jr} = 0$ ← Here r refers to the mode.

12-17 cont.

for $k=1$: $(2k - m\omega_r^2) a_{1r} + (-k) a_{2r} = 0$

$r=1$ $(2k - 2k) a_{11} - k a_{21} = 0 \Rightarrow a_{21} = 0$

$r=2$ $(2k - (2 - \sqrt{2})k) a_{12} - k a_{22} = 0 \Rightarrow \sqrt{2} a_{12} = a_{22}$

$r=3$ $(2k - (2 + \sqrt{2})k) a_{13} - k a_{23} = 0 \Rightarrow -\sqrt{2} a_{13} = a_{23}$

for $k=2$: $-k a_{1r} + (2k - m\omega_r^2) a_{2r} - k a_{3r} = 0$

$r=1$ $-k a_{11} + (2k - 2k) a_{21} - k a_{31} = 0 \Rightarrow a_{11} = -a_{31}$

$r=2$ $-k a_{12} + (2k - (2 - \sqrt{2})k) a_{22} - k a_{32} = 0 \Rightarrow a_{12} - \sqrt{2} a_{22} + a_{32} = 0$

$r=3$ $-k a_{13} + (2k - (2 + \sqrt{2})k) a_{23} - k a_{33} = 0 \Rightarrow a_{13} + \sqrt{2} a_{23} + a_{33} = 0$

for $k=3$: $-k a_{2r} + (2k - m\omega_r^2) a_{3r} = 0$

$r=1$ $-k a_{21} + (2k - 2k) a_{31} = 0 \Rightarrow a_{21} = 0$

$r=2$ $-k a_{22} + (2k - (2 - \sqrt{2})k) a_{32} = 0 \Rightarrow a_{22} = \sqrt{2} a_{32}$

$r=3$ $-k a_{23} + (2k - (2 + \sqrt{2})k) a_{33} = 0 \Rightarrow a_{23} = -\sqrt{2} a_{33}$

$\therefore \begin{aligned} a_{21} &= 0 & a_{12} &= \frac{1}{\sqrt{2}} a_{22} & a_{13} &= a_{33} \\ a_{31} &= -a_{11} & a_{22} &= \frac{1}{\sqrt{2}} a_{22} & a_{23} &= -\sqrt{2} a_{33} \end{aligned}$

$x_1 = a_{11} n_1 + \frac{1}{\sqrt{2}} a_{22} n_2 + a_{33} n_3$

$x_2 = a_{22} n_2 - \sqrt{2} a_{33} n_3$

$x_3 = -a_{11} n_1 + \frac{1}{\sqrt{2}} a_{22} n_2 + a_{33} n_3$

$$\boxed{n_1 = \frac{x_1 - x_3}{2a_{11}} \quad n_2 = \frac{x_1 + \sqrt{2}x_2 + x_3}{2\sqrt{2}a_{22}} \quad n_3 = \frac{x_1 - \sqrt{2}x_2 + x_3}{4a_{33}}}$$



antisymmetric
(intermediate)

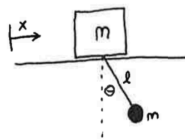


symmetric
(slow)



antisymmetric
(fast)

12-18 Pendulum attached to sliding mass



position of bob: $x_b = x + l \sin \theta$ $\dot{x}_b = \dot{x} + l \dot{\theta} \cos \theta$
 $y_b = -l \cos \theta$ $\dot{y}_b = l \dot{\theta} \sin \theta$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_b^2 + \dot{y}_b^2) = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + l^2 \dot{\theta}^2 + 2l \dot{x} \dot{\theta} \cos \theta)$$

← keeping terms to 2nd order only

$$\approx \frac{1}{2} (M+m) \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m (2l \dot{x} \dot{\theta})$$

$$U = -mgl \cos \theta \approx -mgl (1 - \frac{1}{2} \theta^2) = U_0 + \frac{1}{2} mgl \theta^2$$

← $\cos \theta \approx 1 - \frac{1}{2} \theta^2$ for $\theta \ll 1$

$$\{m\} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix} \quad \{A\} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$$

Eigenfrequencies given by secular determinant:

$$\begin{vmatrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & mgl - \omega^2 ml^2 \end{vmatrix} = -\omega^2(M+m)[mgl - \omega^2 ml^2] - \omega^4 m^2 l^2 = 0$$

$$\boxed{\omega_1^2 = 0, \quad \omega_2^2 = \frac{(M+m)mgl}{(M+m)ml^2 - m^2 l^2} = \left(\frac{M+m}{M}\right) \frac{g}{l}}$$

For normal modes we need $\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. We use $\sum_j (A_{jk} - \omega^2 m_{jk}) a_{jr} = 0$.

for $r=1, k=2$: $A_{22} a_{21} = 0 \Rightarrow a_{21} = 0$

for $r=2, k=1$: $(A_{12} - \omega_2^2 m_{12}) a_{12} + (A_{22} - \omega_2^2 m_{22}) a_{22} = 0$

$$-(M+m) \frac{m}{M} g a_{12} + (mgl - (M+m) \frac{m}{M} g l) a_{22} = 0 \Rightarrow a_{12} = -\frac{ml}{M+m} a_{22}$$

$$x = a_{11} n_1 - \frac{ml}{M+m} a_{22} n_2$$

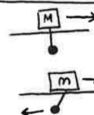
$$\theta = a_{22} n_2$$

$\Rightarrow$

$$\begin{aligned} n_1 &= \frac{1}{a_{11}} \left(x + \frac{ml}{M+m} \theta \right) \\ n_2 &= \frac{\theta}{a_{22}} \end{aligned}$$

Mode #1 produced when $\theta = 0$

Mode #2 produced when $x = -\frac{ml}{M+m} \theta$



no oscillations ($\omega_1 = 0$)

CM fixed

(12-25) 3 particle loaded string



initial displacements: $g_i(0) = a$ $i=1, 2, 3$

initial speeds: $\dot{g}_i(0) = 0$ "

$m \equiv$ particle mass, $\tau \equiv$ string tension, $d \equiv$ particle separation

General solution for displacement of j -th particle

$$g_j(t) = \sum_{r=1}^3 \sin\left(j \frac{r\pi}{n+1}\right) \left(u_r \cos \omega_r t - v_r \sin \omega_r t \right) \quad (\text{this is Eq. 12.155})$$

$$\text{where: } \omega_r = 2 \sqrt{\frac{\tau}{md}} \sin \frac{r\pi}{2(n+1)}$$

here $n=3$

$$u_r = \frac{2}{n+1} \sum_{j=1}^3 g_j(0) \sin\left(j \frac{r\pi}{n+1}\right)$$

note: $j \equiv$ particle number $1 \leq j \leq n$

$$v_r = \frac{2}{n+1} \sum_{j=1}^3 \dot{g}_j(0) \sin\left(j \frac{r\pi}{n+1}\right)$$

$r \equiv$ mode number $1 \leq r \leq n$

Since $\dot{g}_j(0) = 0 \Rightarrow v_r = 0$ for all modes

Using $g_j(0) = a$ we have:

$$u_r = \frac{a}{2} \left\{ \sin\left(\frac{r\pi}{4}\right) + \sin\left(\frac{2r\pi}{4}\right) + \sin\left(\frac{3r\pi}{4}\right) \right\}$$

$$\Rightarrow u_1 = \frac{a}{2} \left\{ \sin \frac{\pi}{4} + \sin \frac{\pi}{2} + \sin \frac{3\pi}{4} \right\} = \frac{a}{2} \left\{ \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} \right\} = \frac{a}{2} (\sqrt{2} + 1)$$

$$u_2 = \frac{a}{2} \left\{ \sin \frac{\pi}{2} + \sin \pi + \sin \frac{3\pi}{2} \right\} = \frac{a}{2} \{ 1 + 0 - 1 \} = 0$$

$$u_3 = \frac{a}{2} \left\{ \sin \frac{3\pi}{4} + \sin \frac{3\pi}{2} + \sin \frac{9\pi}{4} \right\} = \frac{a}{2} \left\{ \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2} \right\} = \frac{a}{2} (\sqrt{2} - 1)$$

for $\sin\left(\frac{jr\pi}{4}\right)$ we have:

$j \backslash r$	1	2	3
1	$\sqrt{2}/2$	1	$\sqrt{2}/2$
2	1	0	-1
3	$\sqrt{2}/2$	-1	$\sqrt{2}/2$

Putting together the pieces:

$$g_1(t) = \frac{\sqrt{2}}{4} a \left\{ (\sqrt{2}+1) \cos \omega_1 t + (\sqrt{2}-1) \cos \omega_3 t \right\}$$

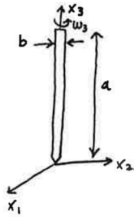
$$g_2(t) = \frac{a}{2} \left\{ (\sqrt{2}+1) \cos \omega_1 t - (\sqrt{2}-1) \cos \omega_3 t \right\}$$

$$g_3(t) = g_1(t)$$

$$\text{where: } \omega_1 = 2 \sqrt{\frac{\tau}{md}} \sin \frac{\pi}{8}$$

$$\omega_3 = 2 \sqrt{\frac{\tau}{md}} \sin \frac{3\pi}{8}$$

(A1) Spinning pencil



pencil length a , diameter b

$$I_3^{cm} = \frac{1}{2} m (b/2)^2 = \frac{1}{8} m b^2$$

$$I_1^{cm} = \frac{1}{4} m (b/2)^2 + \frac{1}{12} m a^2$$

We need I_1 and I_2 about the point of contact which is a distance $a/2$ from the CM.

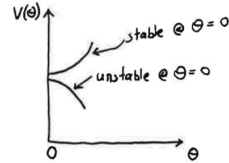
$$\begin{aligned} \therefore I_1 &= I_1^{cm} + m (a/2)^2 \\ &= \frac{m a^2}{3} + \frac{m b^2}{16} \end{aligned}$$

for numerical calculation below:
 $a = 20 \text{ cm}$
 $b = 1 \text{ cm}$
 $g = 980 \text{ cm/s}^2$

a) We start the pencil spinning in the vertical position ($\theta = 0$) so $P\phi = P\psi = I_3 \omega_3$.

Thus we can write the effective potential as:

$$\begin{aligned} V(\theta) &= \frac{P\psi^2 (1 - \cos \theta)^2}{2 I_1 \sin^2 \theta} + mgh \cos \theta \quad (\text{where } h = a/2) \\ &= \frac{P\psi^2}{2 I_1} \frac{1 - \cos \theta}{1 + \cos \theta} + mgh \cos \theta \end{aligned}$$



$$\begin{aligned} \frac{\partial V}{\partial \theta} &= \frac{P\psi^2}{2 I_1} \left[\frac{\sin \theta}{1 + \cos \theta} + (1 - \cos \theta) \frac{-\sin \theta}{(1 + \cos \theta)^2} \right] - mgh \sin \theta \\ &= \frac{P\psi^2}{2 I_1} \left[\frac{(1 + \cos \theta) \sin \theta + (1 - \cos \theta) \sin \theta}{(1 + \cos \theta)^2} \right] - mgh \sin \theta \\ &= \frac{P\psi^2}{I_1} \frac{\sin \theta}{(1 + \cos \theta)^2} - mgh \sin \theta \end{aligned}$$

$$\frac{\partial V}{\partial \theta} = 0 \text{ @ } \theta = 0 \text{ as expected}$$

We need to find the condition on $P\psi$ such that we have a potential minimum in $V(\theta)$ @ $\theta = 0$.

$$\begin{aligned} \frac{\partial^2 V}{\partial \theta^2} &= \frac{P\psi^2}{I_1} \left[\frac{\cos \theta}{(1 + \cos \theta)^2} + \sin \theta \frac{-2 \sin \theta}{(1 + \cos \theta)^3} \right] - mgh \cos \theta \\ &= \frac{P\psi^2}{I_1} \left[\frac{(1 + \cos \theta) \cos \theta - 2 \sin^2 \theta}{(1 + \cos \theta)^3} \right] - mgh \cos \theta \end{aligned}$$

$$\left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta=0} = \frac{P\psi^2}{I_1} \frac{2}{8} - mgh$$

we require $\left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta=0} > 0$ for stability

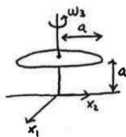
$$\therefore P\psi^2 > 4 I_1 mgh \text{ or } \boxed{\omega_3 > \frac{2}{I_3} \sqrt{I_1 mgh}}$$

this is the spinning speed.

b) For the pencil, the minimum spinning speed is

$$\omega_3 = \frac{2}{I_3} \sqrt{I_1 mgh} = \frac{16}{b^2} \sqrt{\left(\frac{a^2}{3} + \frac{b^4}{16}\right) \frac{g}{2}} = 18,294 \text{ rad/s} \Rightarrow \boxed{2,910 \text{ rev/sec}} \text{ fast!!}$$

(A2) Toy Top



$$a = 2 \text{ cm} \quad M = 100 \text{ g}$$

$$I_3 = \frac{1}{2} M a^2 \left[= \frac{1}{2} (100 \text{ g}) (2 \text{ cm})^2 = 200 \text{ g cm}^2 \right]$$

$$I_1 = I_{cm} + M a^2 = \frac{1}{4} M a^2 + M a^2 = \frac{5}{4} M a^2 \left[= 500 \text{ g cm}^2 \right]$$

a) Sleeping condition: $\omega_3 > \frac{2}{I_3} \sqrt{I_1 m g h}$ where $h = a$

Minimum spinning speed: $\omega_3 = \frac{2}{200} \sqrt{(500)(100)(980)(2)} \text{ rad/s} = 99 \text{ rad/s} \Rightarrow \underline{\underline{15.8 \text{ rev/s}}}$

b) Initial conditions: $\omega_0 = 35 \text{ rev/s} = 220 \text{ rad/s}$

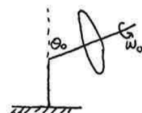
$$\theta_0 = 60^\circ$$

$$\dot{\theta}(0) = 0, \dot{\phi}(0) = 0, \dot{\psi}(0) = \omega_0$$

$$p_\psi = I_3 \omega_0 = \text{constant}$$

$$p_\theta = I_3 \omega_0 \cos \theta_0 = \text{constant}$$

$$E = \frac{1}{2} I_3 \omega_0^2 + M g h \cos \theta_0 = \text{constant} \quad E' = E - \frac{1}{2} I_3 \omega_0^2 = \text{constant}$$



Effective potential:

$$V(\theta) = \frac{(p_\phi - p_\psi \cos \theta)^2}{2 I_1 \sin^2 \theta} + M g h \cos \theta$$

$$= \frac{I_3^2 \omega_0^2 (\cos \theta_0 - \cos \theta)^2}{2 I_1 \sin^2 \theta} + M g h \cos \theta$$

for plotting we have:

$$\frac{V(\theta)}{\text{ergs}} = 1.94 \times 10^6 \frac{(\frac{1}{2} - \cos \theta)^2}{\sin^2 \theta} + 1.96 \times 10^5 \cos \theta$$

Turning points are located by $E' = V(\theta)$

$$\text{where } E' = M g h \cos \theta_0$$

$$\therefore M g h (\cos \theta_0 - \cos \theta) = \frac{I_3^2 \omega_0^2}{2 I_1} \frac{(\cos \theta_0 - \cos \theta)^2}{\sin^2 \theta}$$

Clearly $\theta = \theta_0$ is a root as expected.

← This is a cubic equation for $\cos \theta$. Two roots are the desired turning points. One root is unphysical.

A2 cont.

The 2nd turning point is given by:

$$mgh \sin^2 \theta = \frac{I_3}{2I_1} \omega_0^2 (\cos \theta_0 - \cos \theta) \quad \frac{I_3}{2I_1} = \frac{1/2}{10/4} = \frac{1}{5}$$

$$mgh \cos^2 \theta - \frac{1}{5} I_3 \omega_0^2 \cos \theta + \frac{1}{5} I_3 \omega_0^2 \cos \theta_0 - mgh = 0$$

$$\begin{aligned} \cos \theta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{where} \quad a = mgh = (100)(980)(2) \text{ erg} = 1.96 \times 10^5 \text{ erg} \\ &= \frac{1.936 \pm 1.773}{0.392} \quad b = -\frac{1}{5} I_3 \omega_0^2 = -\frac{1}{5} (200)(220)^2 \text{ erg} = -1.936 \times 10^6 \text{ erg} \\ &= 0.416 \text{ and } 9.46 \quad c = -b \cos \theta_0 - a = 7.72 \times 10^5 \text{ erg} \\ &\quad \swarrow \text{not physical} \end{aligned}$$

$$\theta = \arccos(0.416) = 65.4^\circ$$

$\therefore$ The top nutates between $\theta_1 = 60.0^\circ$ and $\theta_2 = 65.4^\circ$

Increasing ω_0 makes $V(\theta)$ a steeper function and thus decreases $|\theta_2 - \theta_1|$.



Effective potential for toy top:

$$V(\theta) = \frac{I_3 \omega_0^2 (\cos \theta_0 - \cos \theta)}{2 I_1 \sin^2 \theta} + mgh \cos \theta$$

$$\omega_0 = 220 \text{ rad/s}$$

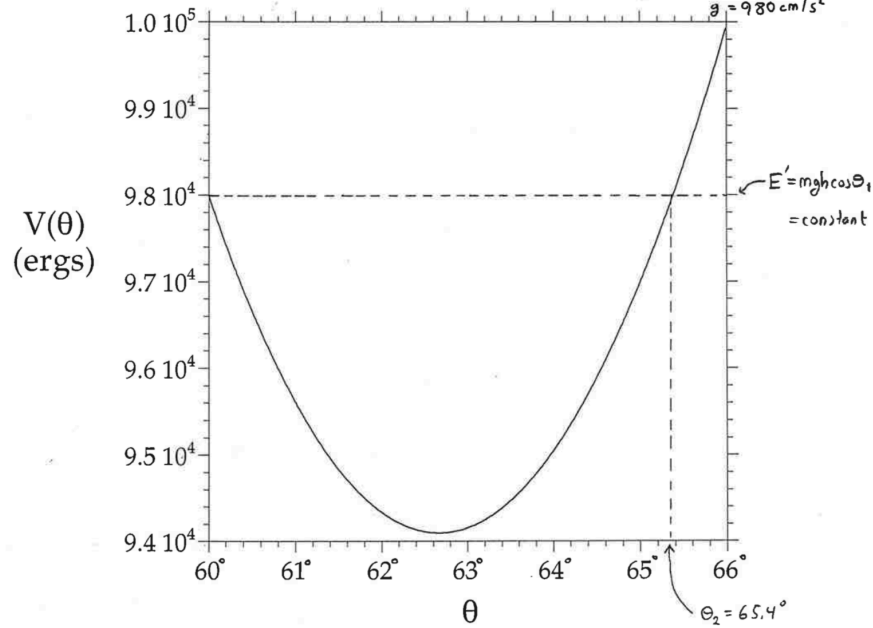
$$I_1 = 500 \text{ g}\cdot\text{cm}^2$$

$$I_3 = 200 \text{ g}\cdot\text{cm}^2$$

$$m = 100 \text{ g}$$

$$h = 2 \text{ cm}$$

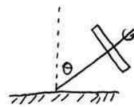
$$g = 980 \text{ cm/s}^2$$



Turning points: $\theta_1 = 60^\circ$

$\theta_2 = 65.4^\circ$

Potential minimum @ $\theta = 62.66^\circ$



c) Nutation:

Energy conservation gives: $\frac{1}{2} I_1 \dot{\Theta}^2 + V(\Theta) = m g a \cos \Theta_0$

Let $\Theta(t) = \Theta_0 + \epsilon(t)$ where $0 \leq \epsilon \ll |\Theta_2 - \Theta_1|$ is small.

$$\cos(\Theta) = \cos(\Theta_0 + \epsilon) = \cos \Theta_0 \cos \epsilon - \sin \Theta_0 \sin \epsilon \approx \cos \Theta_0 - \epsilon \sin \Theta_0$$

$$\sin(\Theta) = \sin(\Theta_0 + \epsilon) = \sin \Theta_0 \cos \epsilon + \cos \Theta_0 \sin \epsilon \approx \sin \Theta_0 + \epsilon \cos \Theta_0$$

$$\begin{aligned} \therefore V(\Theta) &= \frac{I_3^2 \omega_0^2}{2 I_1} \frac{(\cos \Theta_0 - \cos \Theta)^2}{\sin^2 \Theta} + m g a \cos \Theta \\ &\approx \frac{I_3^2 \omega_0^2}{2 I_1} \frac{\epsilon^2 \sin^2 \Theta_0}{(\sin \Theta_0 + \epsilon \cos \Theta_0)^2} + m g a (\cos \Theta_0 - \epsilon \sin \Theta_0) \\ &\approx \frac{I_3^2 \omega_0^2}{2 I_1} \epsilon^2 - m g a \sin \Theta_0 \epsilon + m g a \cos \Theta_0 \quad (\text{keeping terms to } \epsilon^2 \text{ only}) \end{aligned}$$

We can now rewrite our energy equation as:

$$\frac{1}{2} I_1 \dot{\epsilon}^2 + \frac{I_3^2 \omega_0^2}{2 I_1} \epsilon^2 - m g a \sin \Theta_0 \epsilon = 0$$

Taking a time derivative gives:

$$I_1 \dot{\epsilon} \ddot{\epsilon} + \frac{I_3^2 \omega_0^2}{I_1} \epsilon \dot{\epsilon} - m g a \sin \Theta_0 \dot{\epsilon} = 0$$

or

$$\ddot{\epsilon} + \frac{I_3^2 \omega_0^2}{I_1^2} \epsilon = \frac{m g a \sin \Theta_0}{I_1}$$

This is a equation for SHM center about

$$\epsilon = \frac{I_1}{I_3^2 \omega_0^2} m g a \sin \Theta_0 \approx 0.0438 \text{ rad} = 2.51^\circ \quad (\text{in agreement with } \frac{\Theta_2 - \Theta_1}{2})$$

with frequency

$$\omega_{\text{nutation}} = \frac{I_3 \omega_0}{I_1}$$

$$T_{\text{nutation}} = \frac{2\pi}{\omega_{\text{nutation}}} = \frac{2\pi I_1}{\omega_0 I_3} = \frac{5\pi}{\omega_0} = \underline{\underline{0.071 \text{ s}}}$$

d) Precession:

$$\dot{\Phi} = \frac{p\dot{\phi} - p\dot{\psi} \cos \theta}{I_1 \sin^2 \theta} = \frac{I_3 \omega_0 (\cos \theta_0 - \cos \theta)}{I_1 \sin^2 \theta}$$

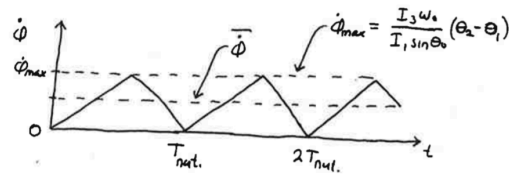
We let $\theta(t) = \theta_0 + \epsilon(t)$ as for our nutation calculation which gives:

$$\dot{\Phi} = \frac{I_3 \omega_0}{I_1} \frac{\epsilon \sin \theta_0}{(\sin \theta_0 + \epsilon \cos \theta_0)^2} \approx \frac{I_3 \omega_0}{I_1 \sin \theta_0} \epsilon \quad \text{to 1st order in } \epsilon$$

Since $\dot{\Phi}$ is linear in ϵ and $\theta_1 \leq \epsilon \leq \theta_2$ the average precession speed is approximately:

$$\bar{\dot{\Phi}} = \frac{I_3 \omega_0}{I_1 \sin \theta_0} \left(\frac{\theta_2 - \theta_1}{2} \right)$$

The precession speed looks like this:



$$T_{\text{precession}} = \frac{2\pi}{\bar{\dot{\Phi}}} = 2\pi \frac{I_1 \sin \theta_0}{I_3 \omega_0} \left(\frac{2}{\theta_2 - \theta_1} \right) \approx \underline{\underline{1.3 \text{ s}}} \quad (18 \text{ nutations/precession})$$

e) Fast and slow precession speeds for a fast spinning top are approximately

$$\dot{\Phi}_+ = \frac{I_3 \omega_3}{I_1 \cos \theta_0} \quad \dot{\Phi}_- = \frac{mgh}{I_3 \omega_3} \quad \text{where } \omega_3 = \dot{\Phi} \cos \theta + \dot{\psi}$$

Here our top has $\dot{\psi} = 220 \text{ rad/s}$ and we push it w/ $\dot{\Phi}$. For slow precession $\omega_3 \approx \dot{\psi}$ is an okay approximation. This gives:

$$\dot{\Phi}_- = \frac{mgh}{I_3 \omega_3} \approx 4.45 \text{ rad/s} \quad (\text{so assuming } \dot{\Phi}_- \ll \dot{\psi} \text{ was reasonable})$$

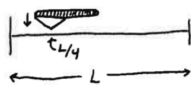
For fast precession we have:

$$\dot{\Phi}_+ = \frac{I_3 (\dot{\Phi}_+ \cos \theta + \dot{\psi})}{I_1 \cos \theta_0} \Rightarrow \dot{\Phi}_+ = \frac{I_3 \dot{\psi}}{(I_1 - I_3) \cos \theta_0} = 293 \text{ rad/s}$$

← this is very fast and would be quite hard to realize w/ our toy.

Assignment 12 - Solutions

(13-6) Hammered String



$$\text{@ } t=0: \quad g(x,0) = 0$$

$$\dot{g}(x,0) = \begin{cases} \frac{4v_0}{L}x & 0 < x < L/4 \\ \frac{4v_0}{L}(\frac{L}{2} - x) & L/4 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

where v_0 = hammer speed

General solution given by:

$$g(x,t) = \sum_r (\mu_r \cos \omega_r t - \nu_r \sin \omega_r t) \sin\left(\frac{r\pi x}{L}\right) \quad r \equiv \text{mode \#} \quad 1 \leq r < \infty$$

$$\text{where } \mu_r = \frac{2}{L} \int_0^L g(x,0) \sin\left(\frac{r\pi x}{L}\right) dx \quad \omega_r = \frac{r\pi}{L} \sqrt{\frac{\tau}{\rho}} \quad \begin{matrix} \tau = \text{string tension} \\ \rho = \text{string density} \end{matrix}$$

$$\nu_r = \frac{-2}{\omega_r L} \int_0^L \dot{g}(x,0) \sin\left(\frac{r\pi x}{L}\right) dx$$

Here $g(x,0) = 0 \Rightarrow \mu_r = 0$

$$\begin{aligned} \nu_r &= -\frac{8v_0}{\omega_r L^2} \left\{ \int_0^{L/4} x \sin\left(\frac{r\pi x}{L}\right) dx + \int_{L/4}^{L/2} \left(\frac{L}{2} - x\right) \sin\left(\frac{r\pi x}{L}\right) dx \right\} \\ &= -\frac{8v_0}{\omega_r L^2} \left\{ \left[\left(\frac{L}{r\pi}\right)^2 \sin\left(\frac{r\pi x}{L}\right) - \frac{xL}{r\pi} \cos\left(\frac{r\pi x}{L}\right) \right]_0^{L/4} \right. \\ &\quad \left. + \left[-\frac{L}{2} \left(\frac{L}{r\pi}\right) \cos\left(\frac{r\pi x}{L}\right) \right]_{L/4}^{L/2} - \left[\left(\frac{L}{r\pi}\right)^2 \sin\left(\frac{r\pi x}{L}\right) - \frac{xL}{r\pi} \cos\left(\frac{r\pi x}{L}\right) \right]_{L/4}^{L/2} \right\} \\ &= -\frac{8v_0}{\omega_r L^2} \left\{ \left(\frac{L}{r\pi}\right)^2 \sin\left(\frac{r\pi}{4}\right) - \frac{L^2}{4r\pi} \cos\left(\frac{r\pi}{4}\right) - \frac{L^2}{2r\pi} \cos\left(\frac{r\pi}{2}\right) + \frac{L^2}{2r\pi} \cos\left(\frac{r\pi}{4}\right) \right. \\ &\quad \left. - \left(\frac{L}{r\pi}\right)^2 \sin\left(\frac{r\pi}{2}\right) + \frac{L^2}{2r\pi} \cos\left(\frac{r\pi}{2}\right) + \left(\frac{L}{r\pi}\right)^2 \sin\left(\frac{r\pi}{4}\right) - \frac{L^2}{4r\pi} \cos\left(\frac{r\pi}{4}\right) \right\} \\ &= -\frac{8v_0}{\omega_r L^2} \left\{ \frac{2}{r^2 \pi^2} \sin\left(\frac{r\pi}{4}\right) - \frac{1}{r^2 \pi^2} \sin\left(\frac{r\pi}{2}\right) \right\} \\ &= \underline{\underline{\frac{8v_0}{\pi^2 r^2 \omega_r} \left\{ \sin\left(\frac{r\pi}{2}\right) - 2 \sin\left(\frac{r\pi}{4}\right) \right\}}}} \end{aligned}$$

Any harmonics with a node @ $x = L/4$ will be absent due to the initial condition $\dot{g}(L/4, 0) \neq 0$.

$\therefore r = 4n$ modes absent
i.e. $(r = 4, 8, 12, \dots)$

13-6 cont.

$$\begin{aligned}
 g(x,t) &= \sum_r -v_r \sin \omega_r t \sin \left(\frac{r\pi x}{L} \right) \\
 &= -\frac{8V_0}{\pi^2} \sum_r \frac{1}{r^2 \omega_r} \left(\sin \frac{r\pi}{2} - 2 \sin \frac{r\pi}{4} \right) \sin \omega_r t \sin \left(\frac{r\pi x}{L} \right) \\
 &= -\frac{8V_0}{\pi^2} \frac{L}{\pi} \sqrt{\frac{\rho}{\epsilon}} \sum_r A_r \sin \omega_r t \sin \left(\frac{r\pi x}{L} \right) \quad \omega_r = \frac{r\pi}{L} \sqrt{\frac{\epsilon}{\rho}}
 \end{aligned}$$

where the mode amplitudes are given by $A_r = \frac{1}{r^3} \left(\sin \frac{r\pi}{2} - 2 \sin \frac{r\pi}{4} \right)$

$$\begin{aligned}
 r=1 \text{ (fundamental)} \quad A_1 &= \sin \frac{\pi}{2} - 2 \sin \frac{\pi}{4} = 1 - \sqrt{2} \\
 r=2 \text{ (2nd)} \quad A_2 &= \frac{1}{8} \left(\sin \pi - 2 \sin \frac{\pi}{2} \right) = -\frac{1}{4} \\
 r=3 \text{ (3rd)} \quad A_3 &= \frac{1}{27} \left(\sin \frac{3\pi}{2} - 2 \sin \frac{3\pi}{4} \right) = -\frac{1}{27} (1 + \sqrt{2})
 \end{aligned}$$

Energy of mode r : $E_r \sim A_r^2$

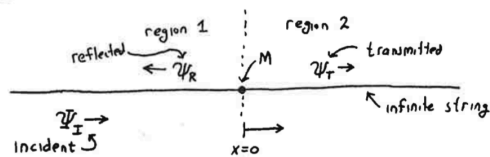
Intensity in decibels defined by: $I \equiv 10 \log (E/E_0)$ where E_0 = energy of reference state

Intensity ratios:

$$\frac{2^{nd}}{1^{st}} = 10 \log \left(\frac{E_2}{E_1} \right) = 20 \log \left(\frac{A_2}{A_1} \right) = 20 \log \left(\frac{-1/4}{1-\sqrt{2}} \right) = \underline{\underline{-4.4 \text{ dB}}}$$

$$\frac{3^{rd}}{1^{st}} = 10 \log \left(\frac{E_3}{E_1} \right) = 20 \log \left(\frac{A_3}{A_1} \right) = 20 \log \left[\frac{-1/27(1+\sqrt{2})}{1-\sqrt{2}} \right] = \underline{\underline{-13.3 \text{ dB}}}$$

13-20 Reflection + Transmission @ mass point on a string



$$\psi_1(x,t) = \psi_I + \psi_R = A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)}$$

$$k = \frac{\omega}{v} = \omega \sqrt{\frac{\mu}{T}}$$

$$\psi_2(x,t) = \psi_T = C e^{i(\omega t - kx)}$$

$$\psi \text{ is continuous @ } x=0 : \psi_1(0,t) = \psi_2(0,t) \Rightarrow \boxed{A+B=C}$$

The derivative of ψ will not be continuous @ $x=0$ since we have a point mass discontinuity there. We determine the resulting discontinuity in

$$\frac{d\psi}{dx} \text{ from the wave equation } \frac{\partial^2 \psi}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 \psi}{\partial t^2}$$

In the neighborhood of $x=0$ $[-\epsilon, \epsilon]$ we can write:

$$\frac{\partial^2 \psi}{\partial x^2} = \lim_{\epsilon \rightarrow 0} \left[\frac{\frac{\partial \psi}{\partial x}|_{+\epsilon} - \frac{\partial \psi}{\partial x}|_{-\epsilon}}{2\epsilon} \right] = \frac{\mu}{2\epsilon} \frac{1}{T} \frac{\partial^2 \psi}{\partial t^2} \quad \text{where } p = \frac{\mu}{2\epsilon} \text{ in } [-\epsilon, \epsilon] \text{ for } \epsilon \rightarrow 0.$$

cancelling the ϵ 's in the denominators and taking the limit we have:

$$\frac{\partial \psi_2}{\partial x} \Big|_{x=0} - \frac{\partial \psi_1}{\partial x} \Big|_{x=0} = \frac{\mu}{T} \frac{\partial^2 \psi}{\partial t^2} \Big|_{x=0}$$

$$\Rightarrow -ikC - (-ikA + ikB) = \frac{\mu}{T} (-\omega^2 C)$$

$$\text{or} \quad \boxed{ik(A-B) + \left(\frac{\mu\omega^2}{T} - ik\right)C = 0}$$

Solving for B and C in terms of A gives:

$$B = \frac{\mu\omega^2}{2ikT - \mu\omega^2} A$$

$$C = \left[1 + \frac{\mu\omega^2}{2ikT - \mu\omega^2} \right] A = \frac{2ikT}{2ikT - \mu\omega^2} A$$

Note that B and C are complex since they include a phase shift.

13-20 cont.)

Introducing the new variable $\tan \theta = \frac{m\omega^2}{2k\tau}$ we have:

$$B = \frac{\tan \theta}{i - \tan \theta} A = \frac{i + \tan \theta}{-1 - \tan^2 \theta} \tan \theta A = \frac{-(\sin \theta + i \cos \theta) \sin \theta}{1 + \tan^2 \theta} A$$

$$C = \frac{i}{i - \tan \theta} A = \frac{i(\tan \theta + i)}{1 + \tan^2 \theta} A = \frac{(\cos \theta - i \sin \theta) \cos \theta}{1 + \tan^2 \theta} A$$

$\left[1 + \tan^2 \theta = \frac{1}{\cos^2 \theta} \right]$

Reflection coefficient: $R \equiv \frac{ B ^2}{ A ^2} = \sin^2 \theta$	Transmission coefficient: $T \equiv \frac{ C ^2}{ A ^2} = \cos^2 \theta$
--	--

Note that $R+T=1$, Also note that for $m=0 \Rightarrow \theta=0 \Rightarrow R=0, T=1$.

To determine phase shifts in ψ_R and ψ_T we write:

$$B = |B| e^{i\phi_R}$$

$$C = |C| e^{i\phi_T}$$

recall that for $z = a + ib$ we

can also write $z = \sqrt{a^2 + b^2} e^{i\phi}$ with $\phi = \text{atan}(b/a)$

$\therefore \phi_R = \text{atan}\left(\frac{\cos \theta}{\sin \theta}\right) = \text{atan}(\cot \theta)$
$\phi_T = \text{atan}\left(\frac{-\sin \theta}{\cos \theta}\right) = \text{atan}[\tan(-\theta)] = -\theta$

Assignment 13 - Solutions

$$\textcircled{4} \quad \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x + \epsilon x^3 = F \cos \omega t$$

$$\omega_0 \sim 1/s, \quad \epsilon \sim 1/m \cdot s^2, \quad \frac{\sqrt{F}}{\omega_0} \sim \frac{1}{m}$$

$$\text{let } t' = \omega t, \text{ let } x' = \frac{\sqrt{F}}{\omega_0} x$$

$$\text{let } \omega_0 = 1, \quad \epsilon = 1$$

$$\text{try } x(t) = A(\omega) \cos(\omega t - \Theta) \quad \leftarrow 1^{\text{st}} \text{ order approx for response}$$

$$\dot{x} = -\omega A \sin(\omega t - \Theta)$$

$$\ddot{x} = -\omega^2 A \cos(\omega t - \Theta)$$

$$-\omega^2 A \cos(\omega t - \Theta) - 2\gamma \omega A \sin(\omega t - \Theta) + A \cos(\omega t - \Theta) + A^3 \cos^3(\omega t - \Theta) = F \cos \omega t$$

$$\cos^3 \alpha = \frac{1}{4} (3 \cos \alpha + \cos 3\alpha)$$

$$\cos(\alpha \mp \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta \quad [p. 60]$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\left(-\omega^2 A + A + \frac{3}{4} A^3\right) \cos(\omega t - \Theta) - 2\gamma \omega A \sin(\omega t - \Theta) + \frac{1}{4} A^3 \cos(3\omega t - 3\Theta) = F \cos \omega t$$

$$\left(1 - \omega^2 + \frac{3}{4} A^2\right) A \left[\cos \omega t \cos \Theta + \sin \omega t \sin \Theta\right] - 2\gamma \omega A \left[\sin \omega t \cos \Theta - \cos \omega t \sin \Theta\right]$$

$$+ \frac{1}{4} A^3 \left[\cos 3\omega t \cos 3\Theta + \sin 3\omega t \sin 3\Theta\right] = F \cos \omega t$$

$$\left\{ \begin{aligned} &\left(1 - \omega^2 + \frac{3}{4} A^2\right) A \cos \Theta + 2\gamma \omega A \sin \Theta - F \cos \omega t \\ &+ \left(1 - \omega^2 + \frac{3}{4} A^2\right) A \sin \Theta - 2\gamma \omega A \cos \Theta \sin \omega t \\ &+ \left[\frac{1}{4} A^3 \cos 3\Theta\right] \cos 3\omega t + \left[\frac{1}{4} A^3 \sin 3\Theta\right] \sin 3\omega t = 0 \end{aligned} \right.$$

different Fourier components must vanish separately for this to be true at all t .

$$\text{Require: } \begin{cases} \left(1 - \omega^2 + \frac{3}{4} A^2\right) A \cos \Theta + 2\gamma \omega A \sin \Theta = F \\ \left(1 - \omega^2 + \frac{3}{4} A^2\right) A \sin \Theta - 2\gamma \omega A \cos \Theta = 0 \end{cases}$$

Note: terms of order $\cos 3\omega t$, $\sin 3\omega t$ will be generated at next order of approx.

Square above two eqns and add:

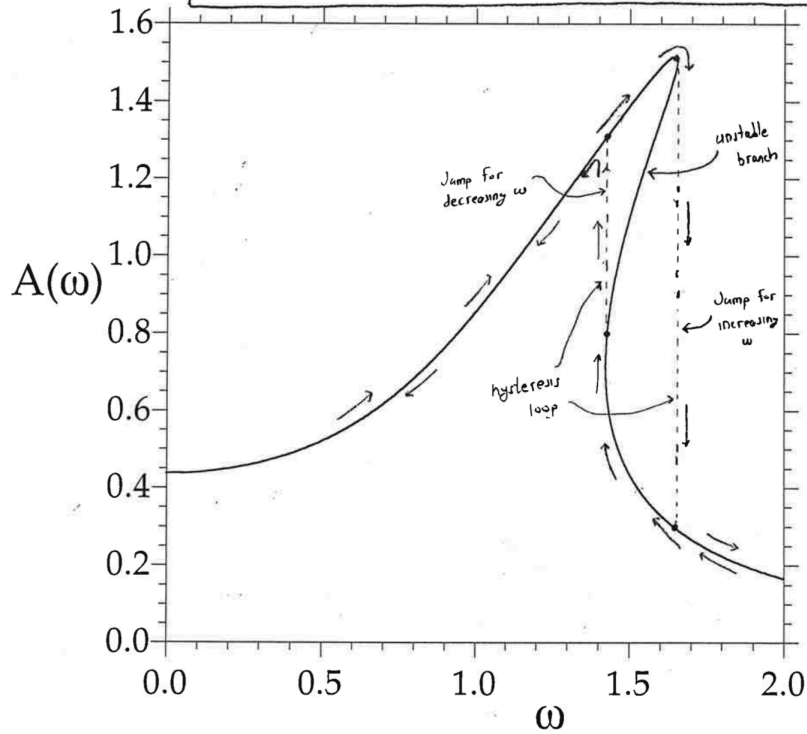
$$\left[1 - \omega^2 + \frac{3}{4} A^2\right]^2 A^2 + 4\gamma^2 \omega^2 A^2 = F^2$$

$$\text{or } F^2 = A^2 \left\{ \left(1 - \omega^2 + \frac{3}{4} A^2\right)^2 + (2\gamma \omega)^2 \right\}$$

Duffing Oscillator - 1st order analytic approximation.

$$F=0.5, \gamma=0.1$$

$A(\omega)$ = response amplitude, F = driving amplitude, ω = driving frequency



Approximate response given by:

$$F^2 = A^2 \left[\left(1 - \omega^2 + \frac{3}{4}A^2 \right)^2 + (2\gamma\omega)^2 \right]$$

or

$$\omega^2 = \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4c} \right) \quad \text{where} \quad b = 4\gamma^2 - 2 - \frac{3}{2}A^2$$

$$c = \left(1 + \frac{3}{4}A^2 \right)^2 - \left(\frac{F}{A} \right)^2$$